

Regularity of Weak Solutions of the Compressible Navier-Stokes Equations

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Abstract

We prove regularity of weak solutions of the Navier-Stokes equations for compressible, isentropic flow in three space dimension. We allow the presence of vacuum region for the initial data. The pressure law satisfies the general relation $P(\rho) = a\rho^\gamma, \gamma \geq 1$. As was found by Hoff[2], Lions[7] and Desjardins[1], the effective viscosity G plays an important role.

keywords: Navier-Stokes equations, isentropic, weak solution, regularity

1 Introduction

The isothermal gases are governed by isentropic compressible Navier-Stokes equations. Although there are many important results, the existence of solutions under general condition remains still open. When the initial velocity has small norm in sufficiently regular space, say H^3 , and the initial density is near constant, the global existence of classical solution was obtained by Matsumura and Nishida[9]. Then, Hoff[2] extended the global existence of small solutions to more weaker spaces which allow discontinuity of the initial

For the weak solutions, Lions[7] obtained the global existence when the pressure law satisfies $P(\rho) = a\rho^\gamma$, $\gamma \geq 9/5$ for three space dimension, $\gamma \geq 3/2$ for two space dimension and $\gamma > N/2$ for N -space dimension with $N \geq 4$. Now, the remaining question will be the extension of the range of the parameter γ .

On the contrary, Solonnikov[13] showed the local existence of strong solutions if there is no vacuum region for the initial density in the context of classical. Also, Desjardins[1] proved local regularity for the weak solutions when $\gamma \geq 1$ for two space dimension and $\gamma > 3$ for three space dimension.

In this paper, we prove the a priori regularity of weak solutions under the general law $P(\rho) = a\rho^\gamma$, $\gamma \geq 1$ for three space dimension. We allow vacuum region and do not assume any smallness for the initial data. The compactness and local existence of strong solution will be discussed in a forthcoming paper.

First, we consider the isentropic compressible Navier-Stokes equations in periodic domain $\mathbf{T}^3$ with periodicity one to each coordinate direction:

$$\rho_t + \operatorname{div}(\rho u) = 0 \quad \text{in } (0, T) \times \mathbf{T}^3$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div}(u) + \nabla P(\rho) = \rho f \quad \text{in } (0, T) \times \mathbf{T}^3,$$

where the pressure satisfies for a positive constant a

$$P(\rho) = a\rho^\gamma, \quad \gamma \geq 1.$$

The viscosity constants satisfy $\mu > 0$ and $\lambda + \mu \geq 0$ and the external force f belongs to $L^2((0, T) \times \mathbf{T}^3)$. We need to find the unknown velocity $u \in \mathbf{R}^3$ and the unknown density $\rho \in \mathbf{R}$. The velocity and pressure are to satisfy the initial condition

$$\rho(0, x) = \rho_0(x), u(0, x) = u_0(x).$$

Although we do not know yet the global existence of weak solution under the general pressure law, we introduce definition of a weak solution. In fact the estimates of local smoothness of the weak solution will lead to the existence of strong solution and we will discuss the existence in different places. $(\rho, u) \in L^1((0, T_0) \times \mathbf{T}^3)$ is a weak solution if it satisfies

$$\begin{aligned} \int \rho_0 \psi(0, x) dx + \int_0^{T_0} \int \rho \psi_t + \rho u \cdot \nabla \psi dx dt &= 0 \\ \int \rho_0 u_0 \psi(x, 0) dx + \int_0^{T_0} \int \rho u \otimes u \nabla u + P \operatorname{div} \psi dx dt &= 0 \end{aligned}$$

$$= \int_0^{T_0} \int \mu \nabla u \nabla \psi + (\lambda + \mu) \operatorname{div} u \operatorname{div} \psi dx dt + \int_0^{T_0} \int \rho f \psi dx dt$$

for all $\psi \in C_0^\infty[0, T_0 : C^\infty(\mathbf{T}^3))$ which is periodic. Moreover (ρ, u) satisfies

$$\sup_{0 \leq t \leq T_0} |\rho|_\gamma(t) + |\sqrt{\rho}u|_2(t) + \int_0^{T_0} |\nabla u|_2 dt \leq C.$$

We denote $|u|_p = (\int |u|^p dx)^{1/p}$ and c is constant depending only exterior data.

Theorem 1.1 *Suppose that $\rho_0 \in L^\infty$ and $u_0 \in H^1$. Then, there is T such that the weak solution (ρ, u) satisfies $\rho \in L^\infty([0, T) \times \mathbf{T}^3)$ and $u \in L^\infty(0, T : H^1(\mathbf{T}^3))$. Furthermore we have*

$$\sup_{0 \leq t \leq T} |\rho|_\infty(t) + |\nabla u|_2(t) + \left(\int_0^T |\sqrt{\rho}u_t|_2^2(t) dt \right)^{1/2} \leq c.$$

For our simplicity of presentation, we assume zero external force.

2 Estimate of integral norm of density

We define our objective function h by

$$h(t) = |\rho|_\infty(t) + |\nabla u|_2(t).$$

For computational convenience we introduce two universal Lipschitz function Φ and Ψ which could be different in each appearance. $\Phi(h(s))$ depends only on $h(s)$ and $\Psi(\int_0^t \Phi ds)$ depends only on $\int_0^t \Phi(h(s)) ds$. But, after overall computations, they will be decided in natural way.

First, we estimate the Averages. We denote $\bar{u} = \int u dx$. The initial mass is positive so that

$$\int \rho_0 dx = M > 0,$$

otherwise the problem is trivial. From mass conservation and momentum conservation,

$$\bar{\rho}(t) = M \quad \text{and} \quad \bar{\rho u}(t) = \int \rho_0 u_0 dx$$

for all t . From Poincaré inequality we have

$$|\int \rho(u - \bar{u}) dx(t)| \leq |\rho|_\infty \left(\int |u - \bar{u}|^2 dx \right)^{1/2}$$

$$\leq c|\rho|_{\infty}|\nabla u|_2(t)$$

and hence we obtain

$$|\overline{u}|(t) \leq \frac{1}{M} \left| \int \rho u dx(t) \right| + \frac{|\rho|_{\infty}(t)}{M} |\nabla u|_2(t) \leq \Phi(h(t))$$

for some Lipschitz function Φ . We also have

$$\begin{aligned} \left| \int \rho |u|^2 dx(t) - M \overline{|u|^2}(t) \right| &\leq \int \rho ||u|^2 - \overline{|u|^2}| dx(t) \\ &\leq |\rho|_{\infty}(t) \int |u| |\nabla u| dx(t) \leq \frac{1}{4} M \overline{|u|^2}(t) + 4 \left(\frac{|\rho|_{\infty}(t)}{M} \right)^2 |\nabla u|_2^2(t) \end{aligned}$$

and hence it follows that

$$\overline{|u|^2}(t) \leq \frac{2}{M} \int \rho |u|^2 + 6 \left(\frac{|\rho|_{\infty}(t)}{M} \right)^2 |\nabla u|_2^2(t) \leq \Phi(h(t)).$$

Now we estimate the integral norms of density. We apply ρ^{k-1} as a test function to mass conservation. Then, we obtain

$$(\rho^k)_t + \operatorname{div}(\rho^k u) + (k-1)\rho^k \operatorname{div}(u) = 0$$

for any positive constant k and hence integrating in time and space

$$\begin{aligned} \int \rho^k dx(t) &= \int \rho_0^k dx - (k-1) \int_0^t \int \rho^k(s, x) \operatorname{div}(u(s, x)) dx ds \\ &\leq \int \rho_0^k dx + \int_0^t |\nabla u|_2^2(s) ds + c(k) |\rho|_{\infty}^{2k}(s) ds \\ &\leq c + \int_0^t \Phi(h(s)) ds. \end{aligned}$$

Therefore we conclude

$$|\rho|_k(t) \leq \Psi\left(\int_0^t \Phi(h(s)) ds\right)$$

for all fixed positive constant and for some Lipschitz functions Φ and Ψ . We decide appropriate k later.

3 Estimate of velocity

To handle the nonlinear convection term $\rho u \cdot \nabla u$, we first estimate

$$\sup_{0 \leq s \leq t} \int \rho |u|^4 dx(s) + \int_0^t \int |u|^2 |\nabla u|^2 dx ds.$$

For our convenience we define effective pressure Q and effective viscosity flux G by

$$Q = -(\lambda + \mu) \operatorname{div}(u) + P(\rho)$$

$$G = (\lambda + 2\mu) \operatorname{div}(u) - P(\rho) = \mu \operatorname{div}(u) - Q.$$

Taking $|u|^2 u$ as test function for momentum conservation equation, we have

$$\begin{aligned} \frac{1}{4} \int \rho (|u|^4)_t dx + \frac{1}{4} \int \rho u \cdot \nabla (|u|^4) dx + \mu \int |u|^2 |\nabla u|^2 dx \\ + \frac{\mu}{8} \int |\nabla (|u|^2)|^2 dx = \int Q \operatorname{div}(|u|^2 u) dx. \end{aligned}$$

We note that

$$\int \rho (|u|^4)_t dx + \int \rho u \cdot \nabla (|u|^4) dx = \frac{d}{dt} \int \rho |u|^4 dx.$$

Hence, integrating in time, we have

$$\begin{aligned} \int \rho |u|^4 dx(t) + \mu \int_0^t \int |u|^2 |\nabla u|^2 dx ds \\ \leq \int \rho_0 |u_0|^4 dx + c \int_0^t \int |Q| |u| |u \nabla u| dx ds. \end{aligned}$$

It is important to find right exponent to derive closed estimates. From Hölder inequality and Sobolev inequality, we have

$$\begin{aligned} \int_0^t \int |Q| |u| |u \nabla u| dx ds &\leq \int_0^t \left[\int |Q|^{12/5} dx \right]^{5/12} \left[\int (|u|^2)^6 dx \right]^{1/12} \left[\int |u|^2 |\nabla u|^2 dx \right]^{1/2} ds \\ &\leq \int_0^t \left[\int |Q|^{12/5} dx \right]^{5/12} \left[\int (|u|^2 - \overline{|u|^2}(s))^6 dx \right]^{1/12} \left[\int |u|^2 |\nabla u|^2 dx \right]^{1/2} ds \\ &\quad + \int_0^t (\overline{|u|^2})^{1/2} \left[\int |Q|^{12/5} dx \right]^{5/12} \left[\int |u|^2 |\nabla u|^2 dx \right]^{1/2} ds \end{aligned}$$

$$\begin{aligned}
&\leq c \int_0^t \left[\int |Q|^{12/5} dx \right]^{5/12} \left[\int |u|^2 |\nabla u|^2 dx \right]^{3/4} ds \\
&+ \int_0^t (\overline{|u|^2})^{1/2} \left[\int |Q|^{12/5} dx \right]^{5/12} \left[\int |u|^2 |\nabla u|^2 dx \right]^{1/2} ds \\
&\leq c \int_0^t \left[\int |Q|^{12/5} dx \right]^{5/3} + \overline{|u|^2} \left[\int |Q|^{12/5} dx \right]^{5/6} ds \\
&\quad + \frac{\mu}{4} \int_0^t \int |u|^2 |\nabla u|^2 dx ds.
\end{aligned}$$

The estimates for generalized pressure Q can be replaced by effective viscosity flux G so that

$$\int |Q(s, x)|^{12/5} dx \leq c \int |G(s, x)|^{12/5} dx + \Phi(h(s)).$$

From the definition of control variable h and G , we also have

$$|\overline{G}(s)| \leq \Phi(h(s)).$$

Thus from Sobolev inequality and, we find that

$$\begin{aligned}
&\left(\int |G(s, x)|^{12/5} dx \right)^{5/3} \leq \left(\int |G(s, x) - \overline{G}(s)|^{12/5} dx \right)^{5/3} + \Phi(h(s)) \\
&\leq \left(\int |G(s, x) - \overline{G}(s)|^2 dx \right)^{5/4} \left(\int |G(s, x) - \overline{G}(s)|^{18/5} dx \right)^{5/12} + \Phi(h(s)) \\
&\leq \varepsilon_0 \left(\int |G(s, x) - \overline{G}(s)|^{18/5} dx \right)^{1/2} + c \left(\int |G(s, x) - \overline{G}(s)|^2 dx \right)^{15/2} + \Phi(h(s)).
\end{aligned}$$

We note that

$$c \left(\int |G(s, x) - \overline{G}(s)|^2 dx \right)^{15/2} \leq c |\nabla u|_2^{15}(s) + |P(\rho)|_2^1 \leq \Phi(h(s))$$

and

$$\left(\int |G(s, x) - \overline{G}(s)|^{18/5} dx \right)^{1/2} \leq c |\nabla G|_{15/8}^{9/5}.$$

Here important fact is the exponent $9/5$ is less than 2 and $15/8$ is less also less than 2. Therefore combining all the previous estimates, we conclude

$$\sup_{0 \leq s \leq t} \int \rho |u|^4 dx(s) + \int_0^t \int |u|^2 |\nabla u|^2 dx ds$$

$$\leq \left(\int_0^t |\nabla G|_{15/8}^2(s) ds \right)^{9/10} + \int_0^t \Phi(h(s)) ds.$$

We let $\mathbf{P}$ be the projection operator to divergence free vector space. Then, from the definition of G and $\mathbf{P}u$, we have

$$\Delta G = \operatorname{div}(\rho u_t) + \operatorname{div}(\rho u \cdot \nabla u)$$

$$\Delta \mathbf{P}u = \mathbf{P}(\rho u_t + \rho u \cdot \nabla u).$$

For a given nonnegative constant $\delta \in [0, 1)$, we have

$$\begin{aligned} |\nabla G|_{2-\delta}^2 + |\Delta \mathbf{P}u|_{2-\delta}^2 &\leq c \left(|\rho u_t|_{2-\delta}^2 + |\rho u \cdot \nabla u|_{2-\delta}^2 \right) \\ &\leq c(|\rho|_m^2 + 1) \left(|\sqrt{\rho} u_t|_2^2 + |u \nabla u|_2^2 \right) \end{aligned}$$

for some m depends only on δ and integrating with respect to time we obtain

$$\begin{aligned} &\int_0^t |\nabla G|_{2-\delta}^2 + |\Delta \mathbf{P}u|_{2-\delta}^2 ds \\ &\leq c \sup_{0 \leq s \leq t} (|\rho|_m^2 + 1) \int_0^t \int \rho |u_t|^2 + |u \nabla u|^2 dx ds \\ &\leq \Psi \left(\int_0^t \Phi(h(s)) ds \right) \int_0^t \int \rho |u_t|^2 + |u \nabla u|^2 dx ds. \end{aligned}$$

Moreover, the Sobolev inequality implies that

$$|\nabla u|_5 \leq c |\nabla G|_{15/8} + c |\Delta \mathbf{P}u|_{15/8} + \Phi(h(s)).$$

Finally we estimate $|\nabla u|_2(t)$. We multiply u_t to our momentum conservation equation and integrate. Consequently, we have

$$\begin{aligned} &\int_0^t \int \rho |u_t|^2 dx ds + \mu \int |\nabla u|^2 dx(t) + (\lambda + \mu) \int |\operatorname{div} u|^2 dx(t) \\ &\leq \mu \int |\nabla u_0|^2 dx + (\lambda + \mu) \int |\operatorname{div} u_0|^2 dx + \int_0^t \int \rho |u \nabla u|^2 dx ds - \int_0^t \int \nabla p \cdot u_t dx ds. \end{aligned}$$

Again from Hölder inequality, for a given constant $0 < \varepsilon < 1$, we have

$$\int_0^t \int \rho |u \nabla u|^2 dx ds \leq \left(\int_0^t \int |u \nabla u|^2 dx ds \right)^{1-\varepsilon} \left(\int_0^t \int \rho^{1/\varepsilon} |u \nabla u|^2 dx ds \right)^\varepsilon$$

$$\begin{aligned}
\int_0^t \int \rho^{1/\varepsilon} |u \nabla u|^2 dx ds &\leq \int_0^t \left(\int \rho^{10/\varepsilon-5} dx \right)^{1/10} \left(\int \rho |u|^4 dx \right)^{1/2} \left(\int |\nabla u|^5 dx \right)^{2/5} \\
&\leq \sup_{0 \leq s \leq t} |\rho|_{10/\varepsilon-5}^{1/\varepsilon-1/2}(s) \left(\sup_{0 \leq s \leq t} \int \rho |u|^4 dx(s) \right)^{1/2} \int_0^t |\nabla u|_5^2(s) ds \\
&\leq \Psi_\varepsilon \left(\int_0^t \Phi(h(s)) ds \right) \left(\left(\int_0^t |\nabla G|_{15/8}^2 ds \right)^{9/20} + \Psi \right) \\
&\quad \left(\int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P} u|_{15/8}^2 ds + \int_0^t \Phi(h(s)) ds \right),
\end{aligned}$$

where Ψ_ε is a Lipschitz function depending on ε and we choose $\varepsilon = \frac{1}{11}$. From the estimate for $\int_0^t \int |u \nabla u|^2 dx ds$, we have

$$\begin{aligned}
\int_0^t \int \rho |u \nabla u|^2 &\leq \Psi_\varepsilon \left(\int_0^t \Phi(h(s)) ds \right) \left(\int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P} u|_{15/8}^2 ds \right)^{\frac{9}{10} + \frac{11\varepsilon}{20}} \\
&\quad + \Psi \left(\int_0^t \Phi(h(s)) ds \right).
\end{aligned}$$

Now if we choose $\varepsilon = \frac{1}{11}$, then

$$\frac{9}{10} + \frac{11\varepsilon}{20} = \frac{19}{20} < 1$$

and

$$\begin{aligned}
\int_0^t \int \rho |u \nabla u|^2 &\leq \Psi \left(\int_0^t \Phi(h(s)) ds \right) \left(\int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P} u|_{15/8}^2 ds \right)^{19/20} \\
&\quad + \Psi \left(\int_0^t \Phi(h(s)) ds \right).
\end{aligned}$$

From integration by parts,

$$-\int \nabla P \cdot u_s dx = \int P \operatorname{div} u_s dx = \frac{d}{ds} \int P \operatorname{div} u dx + \int P_s \operatorname{div} u dx.$$

Integrating in time, we have

$$-\int_0^t \int \nabla P \cdot u_s dx ds = -\int P \operatorname{div} u dx(t) + \int P_0 \operatorname{div} u_0 dx$$

$$+ \int_0^t \int P' \rho_s \operatorname{div} u dx ds.$$

We find that

$$\begin{aligned} \left| \int P(\rho) \operatorname{div} u dx(t) \right| &\leq \frac{\mu}{4} \int |\nabla u|^2 dx(t) + \frac{4}{\mu} \int P^2 dx \\ &\leq \frac{\mu}{4} \int |\nabla u|^2 dx(t) + \Psi\left(\int_0^t \Phi(h(s)) ds\right). \end{aligned}$$

Since $\rho_t = -\operatorname{div}(\rho u)$, we find that

$$\begin{aligned} \int_0^t \int P' \rho_s \operatorname{div} u dx ds &= - \int_0^t \int P' \operatorname{div} u (\rho \operatorname{div} u + u \cdot \nabla \rho) dx ds \\ &\quad - \int_0^t \int P' \rho |\operatorname{div} u|^2 dx ds - \int_0^t \int \nabla P \cdot u \operatorname{div} u dx ds \\ &\quad \int_0^t \int (P - P' \rho) |\operatorname{div} u|^2 dx ds + \int_0^t \int P u \cdot \nabla \operatorname{div} u dx ds. \end{aligned}$$

Clearly we have

$$\begin{aligned} &\left| \int_0^t \int (P - P' \rho) |\operatorname{div} u|^2 dx ds \right| \\ &\leq \int_0^t |P - P' \rho|_\infty(s) |\nabla u|_2^2(s) ds \\ &\leq \int_0^t \Phi(h) ds. \end{aligned}$$

Since

$$\operatorname{div} u = \frac{1}{\lambda + \mu} (G + P),$$

we have

$$\begin{aligned} \left| \int_0^t \int \nabla P \cdot u \operatorname{div} u dx ds \right| &= \frac{1}{\lambda + \mu} \left| \int_0^t \int P u \nabla (G + P) dx ds \right| \\ &\leq c \int_0^t \int P^2 |\operatorname{div} u| dx ds + c \int_0^t \int P |u| |\nabla G| dx ds \\ &\leq \left(\int_0^t |\nabla G|_{15/8}^2 ds \right)^{19/20} + \Psi\left(\int_0^t \Phi ds\right). \end{aligned}$$

Therefore combining all the estimates, we have

$$\int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t)$$

$$\begin{aligned}
&\leq \Psi \left(\int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P}u|_{15/8}^2 ds \right)^{19/20} + \Psi \\
&\leq \Psi \left(\int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t) \right)^{19/20} + \Psi \\
&\leq \frac{1}{2} \int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t) + \Psi
\end{aligned}$$

and we conclude that

$$\int_0^t \int \rho |u_t|^2 dx ds + \int |\nabla u|^2 dx(t) \leq \Psi.$$

4 L^∞ -bound of density

From the mass conservation law, we have

$$(\log \rho)_t + u \cdot \nabla (\log \rho) + \operatorname{div} u = 0$$

and from momentum conservation law,

$$\begin{aligned}
&(\Delta^{-1} \operatorname{div}(\rho u)) + u \cdot \nabla (\Delta^{-1} \operatorname{div}(\rho u)) \\
&+ [u_j, R_i R_j](\rho u_i) - (\lambda + 2\mu) \operatorname{div} u + P = 0.
\end{aligned}$$

Thus, if we define $F = (\lambda + 2\mu) \log \rho + \Delta^{-1} \operatorname{div}(\rho u)$, F satisfies

$$F_t + u \cdot \nabla F + P = [u, , R_i R_j](\rho u_i).$$

Next we define the Lagrange flow X of u so that

$$(X(t, s, x))_t = u(t, X(t, s, x)), \quad X(s, s, x) = x$$

and derive

$$\begin{aligned}
F(t, X(t, 0, x)) &= F_0 - \int_0^t P(\rho(s, X(s, 0, x))) ds \\
&+ \int_0^t [u, , R_i R_j](\rho u_i)(s, X(s, 0, x)) ds.
\end{aligned}$$

Using the fact that ρ_0 is nonnegative, we have

$$F(t, X(t, 0, x)) \leq F_0 + \int_0^t [u, , R_i R_j](\rho u_i)(s, X(s, 0, x)) ds$$

$$\begin{aligned} \log \rho(t, x) &\leq \log(|\rho_0|_\infty) + c|\Delta^{-1} \operatorname{div}(\rho_0 u_0)|_\infty \\ &+ c|\Delta^{-1} \operatorname{div}(\rho u)|_\infty(t) + c \int_0^t |[u, \cdot, R_i R_j](\rho u_i)|_\infty(s) ds. \end{aligned}$$

In view of Sobolev embedding, we have

$$|\Delta^{-1} \operatorname{div}(\rho_0 u_0)|_\infty \leq |\rho_0 u_0|_{7/2}$$

and

$$|\Delta^{-1} \operatorname{div}(\rho u)|_\infty(t) \leq c|\rho u|_{7/2} \leq |\rho|_{21}^6(t) + \left(\int \rho |u|^4 dx(t) \right)^{2/7} \leq \Psi.$$

Again, from Sobolev embedding, we obtain

$$\begin{aligned} |[u, \cdot, R_i R_j](\rho u_i)|_\infty(s) &\leq |[u, \cdot, R_i R_j](\rho u_i)|_{W^{1,7/2}} \\ &\leq c|\nabla u|_5 |\rho u|_{20} \leq c|\nabla u|_5 |\rho|_{39}^{39/40} |u|_\infty^{9/10} \left(\int \rho |u|^4 dx \right)^{1/40}. \end{aligned}$$

we know that

$$\begin{aligned} |u|_\infty(s) &\leq |u - \bar{u}|_\infty(s) + |\bar{u}|_\infty(s) \leq |\nabla u|_5(s) + \Phi(h(s)) \\ |\rho|_{39}(s) &\leq \Psi \left(\int_0^t \Phi(h(s)) ds \right) \\ \sup_{0 \leq s \leq t} \int \rho |u|^4 dx(s) &\leq \Psi \left(\int_0^t \Phi(h(s)) ds \right). \end{aligned}$$

Hence, we get

$$\begin{aligned} \int_0^t |[u, \cdot, R_i R_j](\rho u_i)|_\infty(s) ds &\leq \int_0^t |\nabla u|_5^2(s) ds + \Psi \\ &\leq c \int_0^t |\nabla G|_{15/8}^2 + |\Delta \mathbf{P} u|_{15/8}^2 ds + \Psi \leq \Psi \end{aligned}$$

and this implies

$$\rho(t, x) \leq \Psi.$$

With the estimate of $|\nabla u|_2(t)$, we conclude that

$$h(t) \leq \Psi \left(\int_0^t \Phi(h(s)) ds \right)$$

for some Lipschitz functions Ψ and Φ . Since Ψ and Φ are Lipschitz, there is T_0 such that

$$h(t) \leq C \quad \text{for all } 0 \leq t \leq T_0.$$

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