

Mass Normalization of Collapses in the Theory of Self-Interacting Particles

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1 Introduction

This paper is concerned with the elliptic-parabolic system of cross-diffusion,

$$\left. \begin{aligned} u_t &= \nabla \cdot (\nabla u - u \nabla v) \\ 0 &= \Delta v - av + u \end{aligned} \right\} \quad \text{in} \quad \Omega \times (0, T)$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0, T)$$

$$u|_{t=0} = u_0(x) \quad \text{in} \quad \Omega, \quad (1)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $a > 0$ is a constant, and ν is the outer unit normal vector on $\partial\Omega$. It is proposed by Nagai [8] as a simplified form of the ones given by Jäger and Luckhaus [6], Nanjundiah [12], Keller and Segel [7], and Patlak [14] to describe the chemotactic feature of cellular slime molds. It is also a description of the non-equilibrium mean field of self-attractive particles subject to the second law of thermodynamics. Actually, this physical principle is realized by introducing the friction and fluctuations of particles. See Bavaud [1] and Wolansky [23], [24]. On the other hand, the mathematical study has a long history, and we refer to [21] for the background, known results, and standard arguments.

Actually, it follows from Yagi [25] and Biler [2] that the unique classical solution exists locally in time if the initial value is smooth, and that the solution becomes positive if the initial value is non-negative and not identically

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zero. Letting $T_{\max} > 0$ to be the supremum of the existence time of the solution, we say that the solution blows-up in finite time if $T_{\max} < +\infty$. Then, it is proven in Senba and Suzuki [16] that in the case of $T_{\max} < +\infty$ there exists a finite set $\mathcal{S} \subset \overline{\Omega}$ and a non-negative function $f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$ such that

$$u(x, t)dx \rightarrow \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \quad \text{in } \mathcal{M}(\overline{\Omega}) \quad (2)$$

as $t \uparrow T_{\max}$ with

$$m(x_0) \geq m_*(x_0) \quad (x_0 \in \mathcal{S}), \quad (3)$$

where $\mathcal{M}(\overline{\Omega})$ denotes the set of measures on $\overline{\Omega}$, $\rightarrow$ the $*$ -weak convergence there, and

$$m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

It follows from $T_{\max} < +\infty$ that

$$\lim_{t \rightarrow T_{\max}} \|u(t)\|_{\infty} = +\infty$$

and $\mathcal{S}$ is actually the blowup set of u . That is, $x_0 \in \overline{\Omega}$ belongs to $\mathcal{S}$ if and only if there exist $x_k \rightarrow x_0$ and $t_k \uparrow T_{\max}$ such that $u(x_k, t_k) \rightarrow +\infty$. Because

$$\|u(t)\|_1 = \|u_0\|_1 \quad (4)$$

holds for $t \in [0, T_{\max})$, inequality (2) with (3) implies that

$$2 \cdot \#(\Omega \cap \mathcal{S}) + \#(\partial\Omega \cap \mathcal{S}) \leq \|u_0\|_1 / (4\pi). \quad (5)$$

We have, furthermore, that $\mathcal{S} \neq \emptyset$ if $T_{\max} < +\infty$, and therefore, $\|u_0\|_1 < 4\pi$ implies $T_{\max} = +\infty$. This fact on the existence of the solution globally in time was proven independently by Nagai, Senba, and Yoshida [11], Biler [2], and Gajewski and Zacharias [4], while relation (2) was conjectured by Nanjundiah [12]. It is referred to as the formation of *chemotactic collapses*, and each collapse

$$m(x_0)\delta_{x_0}(dx)$$

is regarded as a spore created by the cellular slime molds.

In 1996, Herrero and Velázquez [5] constructed a family of radially symmetric blowup solutions by the method of matched asymptotic expansion,

where it holds that $m(x_0) = m_*(x_0)$ with $x_0 = 0 \in \Omega \cap \mathcal{S}$. Also, Nagai [9] and Senba and Suzuki [17] showed that if

$$\|u_0\|_1 > 4\pi \quad \text{and} \quad \int_{\Omega} |x - x_0|^2 u_0(x) dx \ll 1$$

hold for $x_0 \in \partial\Omega$, then it follows that $T_{\max} < +\infty$. This means that the mass of collapses made by those solutions can be close to 4π as we like. However, it may be always 4π , and under those considerations it was suspected that $m(x_0) = m_*(x_0)$ for any $x_0 \in \mathcal{S}$.

This problem, referred to as the *mass normalization* in the present paper, is related to the blowup rate, and we say that $x_0 \in \mathcal{S}$ is of type (I) if

$$\limsup_{t \rightarrow T} \sup_{|x - x_0| \leq Cr(t)} r(t)^2 u(x, t) < +\infty$$

holds for any $C > 0$, and that it is of type (II) for the other case that

$$\limsup_{t \rightarrow T} \sup_{|x - x_0| \leq Cr(t)} r(t)^2 u(x, t) = +\infty$$

holds with some $C > 0$, where $T = T_{\max} < +\infty$ and $r(t) = (T - t)^{1/2}$. It is expected that type (I) blowup point never arises. Here, we shall show the following.

Theorem 1 *If $x_0 \in \mathcal{S}$ is of type (II), then the mass normalization $m(x_0) = m_*(x_0)$ occurs.*

2 Preliminaries

We suppose that $T = T_{\max} < +\infty$, and take the standard backward self-similar transformation

$$z(y, s) = (T - t)u(x, t)$$

for $y = (x - x_0)/(T - t)^{1/2}$ and $s = -\log(T - t)$, where $x_0 \in \mathcal{S}$ denotes the blowup point in consideration. The zero extension of $z(y, s)$ is always taken to the region where it is not defined.

The following fact is proven similarly to [20] concerning Jäger - Luckhaus model, where

$$\{m_*(y_0)\delta_{y_0}(dy) \mid y_0 \in \mathcal{B}\}$$

and $F(y)dy = \mu_{0a.c.}(dy)$ are called the sub-collapses and the residual term, respectively. It is referred to as the *formation of sub-collapses*, and the proof is quite similar to the one given in [19] concerning the blowup in infinite time for the pre-scaled system. Here and henceforth, $\mu_s(dy)$ and $\mu_{a.c.}(dy)$ denote the singular and the absolutely continuous parts of $\mu(dy) \in \mathcal{M}(\mathbf{R}^2)$ relative to the Lebesgue measure dy , respectively.

Lemma 2 *Any $s_n \rightarrow +\infty$ admits $\{s'_n\} \subset \{s_n\}$ such that*

$$z(y, s'_n)dy \rightharpoonup \mu_0(dy)$$

as $n \rightarrow \infty$ in $\mathcal{M}(\mathcal{R}^2)$, where $\text{supp } \mu_0(dy) \subset \bar{L}$ and

$$\mu_0(dy) = \sum_{y_0 \in \mathcal{B}} m_*(y_0) \delta_{y_0}(dy) + F(y)dy \quad (6)$$

with

$$m_*(y_0) = \begin{cases} 8\pi & (y_0 \in L) \\ 4\pi & (y_0 \in \partial L), \end{cases}$$

$0 \leq F \in L^1(L) \cap C(\bar{L} \setminus \mathcal{B})$, and

$$L = \begin{cases} \mathbf{R}^2 & (x_0 \in \Omega) \\ H & (x_0 \in \partial\Omega). \end{cases}$$

Here, H denotes the half space in $\mathbf{R}^2$ with ∂H containig the origin and parallel to the tangent line of $\partial\Omega$ at x_0 , and the case $\mathcal{B} = \emptyset$ is admitted.

On the other hand, the following fact is referred to as the existence of the *parabolic envelop*.

Lemma 3 *We have*

$$m(x_0) = \mu_0(\bar{L}) = \sum_{y_0 \in \mathcal{B}} m_*(y_0) + \int_L F(y)dy. \quad (7)$$

Proof: First, we take

$$\varphi = \varphi_{x_0, R', R}$$

for $x_0 \in \mathcal{S}$ and $0 < R' < R$ satisfying $0 \leq \varphi \leq 1$, $\text{supp } \varphi \subset B(x_0, R)$, $\varphi = 1$ on $B(x_0, R')$, and $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial \Omega$. Then, we set

$$M_R(t) = \int_{\Omega} \psi(x) u(x, t) dx$$

for $\psi = \varphi_{x_0, R, 2R}^4$. Relation (2) implies that

$$\lim_{R \downarrow 0} \lim_{t \rightarrow T} M_R(t) = m(x_0).$$

On the other hand, in [16] it is proven that

$$\left| \frac{d}{dt} M_R(t) \right| \leq C (\lambda^2 R^{-2} + \lambda R^{-1})$$

with a constant $C > 0$ determined by Ω , and hence we obtain

$$|M_R(T) - M_R(t)| \leq C (\lambda^2 R^{-2} + \lambda R^{-1}) (T - t).$$

Putting

$$R = br(t) = b(T - t)^{1/2}$$

to this inequality with a constant $b > 0$, we get that

$$|M_{br(t)}(T) - M_{br(t)}(t)| \leq C (\lambda^2 b^{-2} + \lambda b^{-1} (T - t)^{1/2}),$$

and therefore, for

$$\overline{m}_b(x_0) = \limsup_{t \rightarrow T} M_{br(t)}(t) \quad \text{and} \quad \underline{m}_b(x_0) = \liminf_{t \rightarrow T} M_{br(t)}(t)$$

it holds that

$$m(x_0) - C\lambda^2 b^{-2} \leq \underline{m}_b(x_0) \leq \overline{m}_b(x_0) \leq m(x_0) + C\lambda^2 b^{-2}$$

by $m(x_0) = \lim_{t \rightarrow T} M_{br(t)}(T)$. We note that this inequality is indicated as

$$\overline{m}_b(x_0) - C\lambda^2 b^{-2} \leq m(x_0) \leq \underline{m}_b(x_0) + C\lambda^2 b^{-2}. \quad (8)$$

Here, we have

$$\int_{B(x_0, R) \cap \Omega} u(x, t) dx \leq M_R(t) \leq \int_{B(x_0, 2R) \cap \Omega} u(x, t) dx$$

and hence it follows that

$$\int_{B(0,b)} z(y, s) dy \leq M_{br(t)}(t) \leq \int_{B(0,2b)} z(y, s) dy.$$

Thus we obtain

$$\mu_0(B(0, b-1)) \leq \underline{m}_b(x_0) \leq \overline{m}_b(x_0) \leq \mu_0(B(0, 2b+1)),$$

and hence it follows that

$$\lim_{b \rightarrow +\infty} \underline{m}_b(x_0) = \lim_{b \rightarrow +\infty} \overline{m}_b(x_0) = \mu_0(\mathbf{R}^2) = \mu_0(\overline{L}).$$

Then, (7) is obtained by (8).

3 Movement of Sub-collapses

Similarly to the pre-scaled system treated in [18], Lemma 2 is refined in the following way. Namely, any $s_n \rightarrow +\infty$ admits $\{s'_n\} \subset \{s_n\}$ such that

$$z(y, s + s'_n) dy \rightarrow \mu(dy, s)$$

in $C_*((-\infty, +\infty), \mathcal{M}(\mathbf{R}^2))$, where $\text{supp } \mu(dy, s) \subset \overline{L}$, $m(x_0) = \mu(\overline{L}, s)$, and

$$\mu_s(dy, s) = \sum_{y_0 \in B_s} m_*(y_0) \delta_{y_0}(dy)$$

with

$$8\pi \cdot \#(L \cap B_s) + 4\pi \cdot \#(\partial L \cap B_s) + \mu_{a.c.}(L, s) = m(x_0).$$

This $\mu(dy, s)$ becomes a weak solution to

$$\begin{aligned} z_s &= \nabla \cdot (\nabla z - z \nabla p) & \text{in } L \times (-\infty, \infty) \\ \frac{\partial z}{\partial s} &= 0 & \text{on } \partial L \times (-\infty, \infty), \end{aligned} \tag{9}$$

where $p = w + \frac{|y|^2}{4}$ and

$$\nabla_y w(y, s) = \int_L \nabla_y G_0(y, y') z(y', s) dy$$

$$G_0(y, y') = \begin{cases} \frac{1}{2\pi} \log \frac{1}{|y-y'|} & (x_0 \in \Omega) \\ \frac{1}{2\pi} \log \frac{1}{|y-y'|} + \frac{1}{2\pi} \log \frac{1}{|y-y'^*|} & (x_0 \in \partial\Omega) \end{cases}$$

for the reflection y'^* of y' with respect to ∂H . The proof is similar to the one for the pre-scaled case ([18]), and the precise notion of weak solution is not necessary for later arguments. However, let us note that the zero extension of $\mu(dy, s)$ to $\mathbf{R}^2 \setminus L$ is always taken in the case of $x_0 \in \partial\Omega$, following the agreement for $z(y, s)$, and furthermore, that if $\eta \in C_0(\bar{L}) \cap C^2(\bar{L})$ satisfies $\frac{\partial \eta}{\partial \nu}|_{\partial L} = 0$, then the mapping

$$s \in [0, \infty) \mapsto \int_{\bar{L}} \eta(y) \mu(dy, s)$$

is locally absolutely continuous, where $C_0(\bar{L})$ is the set of continuous functions on $\bar{L}$ taking the value zero at infinity.

If $F(y, s)dy = \mu_{a.c.}(dy, s)$, then $F(y, s) \geq 0$ is smooth in

$$\mathcal{D} = \bigcup_{s \in \mathbf{R}} (\bar{L} \setminus \mathcal{B}_s) \times \{s\}.$$

Actually, this is a consequence of the parabolic and elliptic regularity, and $F(y, s)$ satisfies there that

$$F_s = \nabla \cdot (\nabla F - F \nabla p) \quad (10)$$

with smooth p . As a consequence, if $G \subset \bar{L}$ is relatively open, if $\eta \in C^2(G) \cap C(\bar{G})$ satisfies $\eta|_{\partial G} = 0$ and $\frac{\partial \eta}{\partial \nu}|_{\partial L} = 0$, and if $\text{supp } \mu_s(dy, s) \cap \partial G = \emptyset$ holds for $s \in J$ with the time interval $J \neq \emptyset$, then

$$s \in J \mapsto \int_{\bar{L}} \eta(y) \mu(dy, s)$$

is locally absolutely continuous.

First, we study a special case of Theorem 1, making use of

$$\left[\int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \mu(dy, s') \right]_{s'=s}^{s'=s+\Delta s} \geq \int_s^{s+\Delta s} ds' \cdot \left\{ \int_{B_R} (-4 - |y|^2) \mu(dy, s') + \frac{4}{m_*(y_0)} \mu(B_R, s')^2 \right\}, \quad (11)$$

where $R > 0$, $B_R = B(0, R)$, and $0 \leq s < s + \Delta s$.

In fact, in use of the standard backward self-similar transformation given in the previous section,

$$z(y, s) = (T - t)u(x, t) \quad \text{and} \quad w(y, s) = v(x, t)$$

with $y = x/(T - t)^{1/2}$ and $s = -\log(T - t)$, it follows that

$$\left. \begin{aligned} z_s &= \nabla \cdot (\nabla z - z \nabla w - yz/2) \\ 0 &= \Delta w + z - ae^{-s}w \end{aligned} \right\} \quad \text{in } \mathcal{O}$$

$$\frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma$$

$$z|_{s=-\log T} = z_0 \quad (12)$$

for $z_0(y) = Tu_0(x)$,

$$\mathcal{O} = \bigcup_{s > -\log T} e^{s/2} (\Omega - \{x_0\}) \times \{s\},$$

and

$$\Gamma = \bigcup_{s > -\log T} e^{s/2} (\partial\Omega - \{x_0\}) \times \{s\}.$$

Here, we have

$$\begin{aligned} w(y, s) &= v(x, t) = \int_{\Omega} G(x, x') u(x', t) dx' \\ &= \int_{\mathcal{O}(s)} G(e^{-s/2}y + x_0, e^{-s/2}y' + x_0) z(y', s) dy', \end{aligned}$$

and therefore, system (12) is reduced to

$$\left. \begin{aligned} z_s &= \nabla \cdot (\nabla z - z \nabla p) \\ \frac{\partial z}{\partial \nu} &= 0 \end{aligned} \right\} \quad \begin{array}{l} \text{in } \mathcal{O} \\ \text{on } \Gamma \end{array}$$

with $p = w + \frac{|y|^2}{4}$, where $G = G(y, y')$ denotes the Green's function for $-\Delta + a$ in Ω with $\frac{\partial}{\partial \nu} \Big|_{\partial\Omega} = 0$.

Letting

$$\varphi = (R^2 - |y|^2)_+,$$

we have

$$\varphi|_{\partial B_R} = 0, \quad \frac{\partial \varphi}{\partial \nu} \Big|_{\partial B_R} < 0, \quad \text{and} \quad \frac{\partial \varphi}{\partial \nu} \Big|_{\partial H} = 0$$

with the last case valid only for $x_0 \in \partial\Omega$. Let us note that

$$B_R = B(0, R) = \{y \in \mathbf{R}^2 \mid \varphi(y) > 0\}.$$

Then, from (12) we can deduce that

$$\begin{aligned} \frac{d}{ds} \int_{\mathbf{R}^2} \varphi(y) z(y, s) dy &\geq \int_{B_R} (\Delta\varphi + \frac{y}{2} \cdot \nabla\varphi) z(y, s) dy \\ &+ \frac{1}{2} \int \int_{B_R \times B_R} \rho_\varphi^s(y, y') z(y, s) z(y', s) dy dy' \end{aligned} \quad (13)$$

with

$$\rho_\varphi^s(y, y') = \nabla\varphi(y) \cdot \nabla_y G^s(y, y') + \nabla\varphi(y') \cdot \nabla_{y'} G^s(y, y')$$

and $G^s(y, y') = G(e^{-s/2}y + x_0, e^{-s/2}y' + x_0)$.

Here, we have

$$\Delta\varphi + \frac{y}{2} \cdot \nabla\varphi = -4 - |y|^2$$

in B_R . Also we have for $\theta \in (0, 1)$ that

$$G(y, y') = G_0(y, y') + K_1(y, y')$$

with $K_1 \in C_{loc}^{1+\theta}(\Omega \times \bar{\Omega}) \cap C_{loc}^{1+\theta}(\bar{\Omega} \times \Omega)$. In the case of $x_0 \in \Omega$, those relations imply the continuity of ρ_φ^s as well as the uniform convergence $\rho_\varphi^s \rightarrow \rho^0$ as $s \rightarrow +\infty$ on $\bar{B}_R \times \bar{B}_R$, where

$$\rho^0(y, y') = \nabla\varphi(y) \cdot \nabla_y G_0(y, y') + \nabla\varphi(y') \cdot \nabla_{y'} G_0(y, y') = \frac{1}{\pi}. \quad (14)$$

In the case of $x_0 \in \partial\Omega$, on the other hand, we can make use of

$$G(y, y') = G_0(X(y), X(y')) + G_0(X(y), X(y')^*) + K_2(y, y')$$

with $K_2 \in C^{\theta, 1+\theta}(\Omega \cup \gamma \times \bar{\Omega}) \cap C^{1+\theta, \theta}(\bar{\Omega} \times \Omega \cup \gamma)$, where $X : \bar{\hat{\Omega}} \rightarrow \bar{\mathbf{R}}_+^2$ is the conformal mapping satisfying $X(x_0) = 0$, γ is the connected component of $\partial\Omega$ containing x_0 , and $\hat{\Omega}$ is the domain defined by $\partial\hat{\Omega} = \gamma$. Then, the above conclusion follows similarly, with (14) replaced by

$$\rho^0(y, y') = \frac{2}{\pi}.$$

Now, inequality (11) follows from (13) with $z(y, s)$ replaced by $z(y, s + s'_n)$ and sending $n \rightarrow \infty$. Here, we refer to [16], [22] for those facts on the Green's function.

In terms of $\nu(dy, s) = \mu(dy, s) - m_*(y_0)\delta_0(dy)$, inequality (11) reads;

$$\begin{aligned} & \left[\int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s') \right]_{s'=s}^{s'=s+\Delta s} \\ & \geq \int_s^{s+\Delta s} ds' \left\{ \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s') + I_R(s') \right\} \end{aligned} \quad (15)$$

with

$$I_R(s) = m_*(y_0)R^2 - (R^2 + 4)\mu(B_R, s) + \frac{4}{m_*(y_0)}\mu(B_R, s)^2.$$

Here, $0 < R \leq 2$ and

$$\mu(B_R, s) > m_*(y_0) \quad (16)$$

imply $I_R(s) > 0$. On the other hand, (16) follows from

$$\int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s) > 0.$$

We now show that

$$0 < R \leq 2 \quad \text{with} \quad \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, 0) > 0 \quad (17)$$

gives a contradiction. In fact, applying (15) with $s = 0$, we see that

$$\left\{ s \in [0, \infty) \mid \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s') > 0 \text{ on } s' \in [0, s] \right\}$$

is right-closed from the above consideration. Its right-openness follows from $\mu(dy, s) \in C_*((-\infty, \infty), \mathcal{M}(\mathbf{R}^2))$, so that (17) induces

$$\int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s) > 0$$

for any $s \in [0, \infty)$. Simultaneously, it also holds that $I_R(s) > 0$ for $s \in [0, \infty)$, and again (15) assures the monotone increasing of the mapping

$$s \in [0, \infty) \mapsto \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s).$$

Therefore, for $n = 1, 2, \dots$ we have

$$\begin{aligned} \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, n+1) &\geq \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, n) \\ &\quad + \int_n^{n+1} ds' \cdot \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s') \\ &\geq 2 \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, n), \end{aligned}$$

which implies that

$$\int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, n) \geq 2^n \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, 0).$$

However, this is impossible by $\mu(\mathbf{R}^2, s) = m(x_0) < +\infty$.

We have shown that (17) does not occur. If $0 \in \text{supp } \mu_s(dy, 0)$, then $\nu(dy, 0) \geq 0$ holds and this means that

$$\nu(dy, 0) = 0 \quad \text{on} \quad B(0, 2),$$

or equivalently, $\text{supp } \mu_s(dy, 0) \cap B(0, 2) = \{0\}$ and

$$F(y, 0) = 0 \quad \text{for a.e. } y \in B(0, 2).$$

Recall the notation that $F(y, s)dy = \mu_{a.c.}(dy, s)$. Because $F(y, s) \geq 0$ satisfies the parabolic equation (10) with smooth coefficient p in $\mathcal{D} = \cup_{s \in \mathbf{R}} (\bar{L} \setminus B_s) \times \{s\}$, the strong maximum principle guarantees $F(y, s) = 0$ there. Hence $\mu_{0.a.c.}(dy) = 0$ follows.

To treat the general case, we note that if $s \in [0, \infty) \mapsto y_0(s) \in \mathbf{R}^2$ is locally absolutely continuous, then inequality (11) is replaced by

$$\begin{aligned} \left[\int_{\mathbf{R}^2} (R^2 - |y - y_0(s')|^2)_+ \mu(dy, s') \right]_{s'=s}^{s'=s+\Delta s} &\geq \int_s^{s+\Delta s} ds' \cdot \\ &\quad \left\{ \int_{B(y(s), R)} (2y'_0(s') \cdot (y - y_0(s')) - 4 - y \cdot (y - y_0(s'))) \mu(dy, s') \right. \\ &\quad \left. - \frac{4}{m_*(y_0)} \mu(B(y_0(s'), R), s')^2 \right\}. \end{aligned}$$

In terms of $\mu'(dy, s)$ defined by $\mu'(A, s) = \mu(A + \{y_0(s)\}, s)$, it is represented as

$$\left[\int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \mu'(dy, s') \right]_{s'=s}^{s'=s+\Delta s} \geq \int_s^{s+\Delta s} ds'.$$

$$\left\{ \int_{B_R} \left(-4 - |y|^2 + (2y'_0(s) - y_0(s)) \cdot y \right) \mu'(dy, s') \right. \\ \left. + \frac{4}{m_*(y_0)} \mu'(B_R, s')^2 \right\}.$$

If we take

$$y_0(s) = y_0 e^{s/2},$$

then it is reduced to (11):

$$\left[\int_{\mathbf{R}^2} \left(R^2 - |y|^2 \right)_+ \mu'(dy, s') \right]_{s'=s}^{s'=s+\Delta s} \geq \int_s^{s+\Delta s} ds' \cdot \\ \left\{ \int_{B_R} \left(-4 - |y|^2 \right) \mu'(dy, s') + \frac{4}{m_*(y_0)} \mu'(B_R, s')^2 \right\}.$$

We see that $0 \in \text{supp } \mu'_s(dy, 0)$, or equivalently $y_0 \in \text{supp } \mu_s(dy, 0)$, implies $\mu_{a.c.}(dy, 0) = 0$ and

$$\text{supp } \mu_s(dy, 0) \cap B(y_0, 2) = \{y_0\}.$$

If $x_0 \in \mathcal{S}$ is of type (II), then there is $s_n \rightarrow +\infty$ such that $z(y, s_n)dy \rightharpoonup \mu_0(dy)$ in $\mathcal{M}(\mathbf{R}^2)$ with $\text{supp } \mu_{0s}(dy) \neq \emptyset$. We now take $\{s'_n\} \subset \{s_n\}$ such that $z(y, s + s'_n)dy \rightharpoonup \mu(dy, s)$ in $C_*((-\infty, \infty), \mathcal{M}(\mathbf{R}^2))$ with $\mu(dy, s)$ being the weak solution to (9). Because of $\mu_s(dy, 0) = \mu_{0s}(dy) \neq 0$, it follows from the above argument that $\mu_{a.c.}(dy, s) \equiv 0$. We also have $\mu(dy, s) \in C_*((-\infty, \infty), \mathcal{M}(\bar{L}))$ and $\mu(\{y_0\}, s) = m_*(y_0)$ for any $y_0 \in \text{supp } \mu_s(dy, s)$, and therefore, it holds that

$$\mu(dy, s) = \sum_{i=1}^n m_i^* \delta_{y_i(s)}(dy),$$

with $s \in (-\infty, \infty) \mapsto y_i(s) \in \bar{L}$ being continuous, $y_i(s) \in L$ or $y_i(s) \in \partial L$ exclusively in $s \in \mathbf{R}$, and

$$m_i^* = \begin{cases} 8\pi & (y_i(s) \in L) \\ 4\pi & (y_i(s) \in \partial L). \end{cases}$$

Then, again the above argument guarantees that

$$|y_i(s) - y_j(s)| \geq 2 \quad (i \neq j, s \in \mathbf{R}). \quad (18)$$

We also have

$$m(x_0) = \sum_{i=1}^n m_*^i.$$

Now, we take $i = 1, \dots, n$, $R \in (0, 2)$, and the interval

$$J_i = \left\{ s \in [0, \infty) \mid \text{supp } \mu_s(dy, s') \cap \overline{B(y_i(0)e^{s'/2}, R)} = \{y_i(s')\} \right. \\ \left. \text{for any } s' \in [0, s] \right\},$$

which is a right neighbourhood of 0. Then, we repeat the same argument for $\nu(dy, s) = \mu'(dy, s) - m_*^i \delta_0(dy)$ with $\mu'(A, s) = \mu\left(A + \{y_i(0)e^{s/2}\}, s\right)$. This time, we have $I'_R(s) = 0$ for $s \in J_i$, where

$$I'_R(s) = m_*^i R^2 - (R^2 + 4)\mu'(B_R, s) + \frac{4}{m_*^i} \mu'(B_R, s)^2.$$

Furthermore,

$$s \in J_i \quad \mapsto \quad \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s)$$

is locally absolutely continuous, and it holds by (15) that

$$\frac{d}{ds} \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s) \geq \int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s)$$

for a.e. $s \in J_i$. Therefore, because of

$$\int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, 0) = 0$$

we obtain

$$\int_{\mathbf{R}^2} (R^2 - |y|^2)_+ \nu(dy, s) \geq 0,$$

or equivalently

$$R^2 - |y_i(s) - y_i(0)e^{s/2}|^2 \geq R^2,$$

and hence $y_i(s) = y_i(0)e^{s/2}$ follows for $s \in J_i$.

This relation holds for each $i = 1, \dots, n$, so that

$$d_i(s) = \min_{j \neq i} |y_i(s) - y_j(s)|$$

is increasing in s . We have $J_i = [0, \infty)$ and the relation $y_i(s) = y_i(0)e^{s/2}$ continues to hold for every $s \in [0, \infty)$. Now, we translate the time variable

as $s \mapsto s - s_0$, repeat the same argument, and see that $y_i(s - s_0) = y_i(-s_0)e^{s/2}$ holds for any $s_0 \geq 0$. This implies $y_i(-s)e^s = y_i(0)$ for $s \geq 0$, so that

$$y_i(s) = y_i(0)e^{s/2} \quad (s \in \mathbf{R})$$

holds. Consequently,

$$\lim_{s \rightarrow -\infty} y_i(s) = 0$$

follows for $i = 1, \dots, n$. However, this contradicts to (18) in the case of $n \geq 2$. We get $n = 1$, $m(x_0) = m_*(x_0)$, and

$$\mu(dy, s) = m_*(x_0)\delta_{y_0e^{s/2}}(dy) \quad (s \in \mathbf{R}),$$

and the proof is complete.

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