

Limits of tangents on a hypersurface

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Let $V := \{z \in \mathbb{C}^{n+1} \mid f(z) = 0\}$ be an analytic complex hypersurface. We shall suppose that $0 \in V$ and that 0 is a singular point of V . We want to know what is the set of all possible limits of sequences of tangent hyperplanes to V at sequences of smooth points of V tending to 0 . More precisely, let us denote by $\check{\mathbb{P}}^n$ the space of hyperplane directions of $\mathbb{C}^{n+1}$, let Σ be the singular locus of V and let $\varphi: V - \Sigma \rightarrow \check{\mathbb{P}}^n$ be the map which makes correspond to a smooth point of V the direction of its tangent hyperplane. Then let us call $\tilde{V}$ the closure of the graph of φ in $V \times \check{\mathbb{P}}^n$. The first projection $V \times \check{\mathbb{P}}^n \rightarrow V$ induces a mapping $\pi: \tilde{V} \rightarrow V$ which is called the Jacobian blowing-up of V . The set we are interested about is $\pi^{-1}(0)$.

Notice that $\dim \pi^{-1}(\Sigma) = \dim V - 1$. Thus if $0 \in V$ is an isolated singularity $\dim \pi^{-1}(0) = \dim V - 1$.

First notice the following lemma:

Lemma 1 ([1]) Let (x_n) be a sequence of smooth points of V such that the lines Ox_n tend to l and the hyperplanes $T(x_n, V)$ tend to T , then l is contained in T .

Recall that all possible limits of lines Ox_n for sequences of points of V tending to 0 define the tangent cone of V at 0 and the corresponding set in the projective space $\mathbb{P}^n$ of lines through 0 is the Proj of this tangent cone.

Now we have the following lemma which was indicated to us by Professor O. Zariski:

Lemma 2 Let us suppose that the tangent cone of V at O is reduced. Then consider (x_n) a sequence of ^{smooth} points of V which tends to O and such that $T(x_n, V)$ tends to T and Ox_n tends to l . Moreover suppose that l gives a smooth point of the Proj of the tangent cone of V at O , then T is tangent to the tangent cone of V at O along l .

Such a lemma has led our interests to compare the limits of tangents ~~of~~ at O , say $\pi^{-1}(O)$ defined above, and the limits of secants at O , say the Proj of the tangent cone at O .

We obtain the following result:

Theorem 3 Suppose $n=2$ and $O \in V$ is an isolated singularity, then the limits of tangents of V at O is the union of the dual curve of the curve, Proj of the reduced tangent cone at O , and a finite number of lines which corresponds to ^{the} pencils of hyperplanes going through ^{the} singular lines of the ^{reduced} tangent cone and a finite number of non-singular lines of this reduced tangent cone.

Remark: The non-singular lines of the theorem are specified in the proof (cf theorem 6)

The proof of the theorem 3 is based on some results of B. Teissier from [4] and a geometrical study involving some results ^{about} equisingularity of O. Zariski [5].

First let us define:

Definition We shall say that a hyperplane H cuts V generically at O if $V \cap H$ has an isolated singularity at O and the Milnor number of $V \cap H$ at O is minimum among all ^{the} hyperplane sections with an isolated singularity at O .

Let us remind that such a hyperplane section has a well-defined topology ^{at least} (when $n \neq 3$) because of the results of [2].

The result of B. Teissier can be stated as ^{it} follows:

Theorem 4 (cf [4]) ^{(If $O \in V$ is an isolated singularity,} The hyperplane H cuts V generically at O if and only if H is not a limit of tangents of V at O , i.e. $H \notin \pi^{-1}(O)$.

Now let $n=2$. Call $\tilde{Z}_1 \xrightarrow{p} \mathbb{C}^3$ the blowing-up of the point O . Let V_1 be the strict transform of V by p . Let H be a hyperplane of $\mathbb{C}^3$ and H_1 be its strict transform by p . Call $C = H \cap V$ and C_1 the strict transform of C . Thus $C_1 = H_1 \cap V_1$.

Then C is reduced if and only if C_1 is reduced. Thus C is reduced if and only if $p^{-1}(O) \cap H_1$ cuts transversally the reduced curve $p^{-1}(O) \cap V_1$ and near each point of intersection H_1 cuts the smooth part of V_1 transversally.

But we have the following lemma:

Lemma 5 (cf [3]) Let C be a plane curve of multiplicity n at O , let C_1 be the strict transform of C after blowing-up O .

Let $O_1, \dots, O_k$ be the points of C_1 over O ,

then:

$$\mu(C, O) = \mu(C_1, O_1) + \dots + \mu(C_1, O_k) + n(n-1) - (k-1)$$

Thus if we apply it to our above situation we find that at each point where $p^{-1}(O) \cap H_1$ cuts $p^{-1}(O) \cap V_1$ the local Milnor number of $H_1 \cap V_1$ must be the minimum one to get the minimum one for $H \cap V$ at O .

Then consider the components $\Gamma_1, \dots, \Gamma_r$ of $p^{-1}(O) \cap V_1$. We have the following case:

- V_1 is singular along Γ_i . Thus V_1 is equisingular along Γ_i outside a finite number of exceptional ^{defines lines} points we shall call exceptional secants of V at O . Remark that ^{the} singular points of Γ_i and points of $\Gamma_i \cap \Gamma_j$ ($j \neq i$) are among these exceptional points;
- V_1 is not singular along Γ_i except maybe at a finite number of points. The ^{corresponding lines of these points} will be exceptional secants of V at O , too.

Then using Zariski's theory of equisingularity we can prove:

Theorem 6 A hyperplane H cuts V generically at O if and only if it does not contain any exceptional secants.

Bibliography

- [1] Lê Dũng Tráng Singularités des hypersurfaces complexes, Acta Sc. Vietn., Hanoi, 1971.
- [2] Lê Dũng Tráng - C.P. Ramanujam The invariance of Milnor's number implies the invariance of the topological type, to appear in Amer. J. Math.
- [3] J. J. Risler Sur l'Idéal Jacobien, Bull. Soc. Mat. Fr., ^{t.99} 1971, 30
- [4] B. Teissier Cycles évanescents, Sections planes et conditions de Whitney, in Singularités à Cargèse, Asterisque 7 et 8, France.
- [5] O. Zariski Studies in Equisingularities I, II Amer. J. Math., t.87 (1965), 507-536 et 972-1006