

Quasi-trees, Quasimorphisms and Central Extensions

Bingxue Tao

A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics
Kyoto University
2026

ACKNOWLEDGEMENTS

I wish to thank my supervisor, Koji Fujiwara, for his continuous guidance and support throughout my Ph.D. studies. I also wish to thank Ryokichi Tanaka for acting as my official advisor during the final stage of my Ph.D. and for his kind assistance. I am furthermore grateful to Wen Yuan Yang, who hosted me at Peking University during the COVID-19 pandemic and introduced me to geometric group theory.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS	ii
ABSTRACT	v
CHAPTER	
1 Introduction	1
1.1 Property (QT) of relatively hierarchically hyperbolic groups	1
1.2 Property (PH)	3
1.3 Central extensions	5
1.4 Extension of quasimorphisms	7
2 Preliminaries	10
2.1 δ -hyperbolic spaces and group actions	10
2.2 (Relatively) hierarchically hyperbolic groups	13
2.3 Quasimorphisms	16
2.4 Central extensions and mapping class groups	17
3 Property (QT) of relatively hierarchically hyperbolic groups	20
3.1 BBF Projection systems	21
3.2 Thick distance formula	23
3.3 Estimation of thick distance via quasi-axes	26
3.4 Construction of quasi-trees	29
3.4.1 Quasi-trees from domains of Type I	30
3.4.2 Quasi-trees from domains of Type II	32
3.4.3 Proof of Theorem 3.0.2	33
3.5 A criterion for having separable quasi-axes	34
3.6 Applications	35
3.6.1 Mapping class groups	35
3.6.2 Admissible graphs of groups	36
3.6.3 Hyperbolic–2–decomposable groups	37
3.6.4 Artin groups and extensions of lattice Veech groups	37
3.6.5 Graph products	38
4 Property (PH)	39
4.1 Hyperbolic structures on groups	39
4.2 Quasimorphisms and quasi-lines	41
4.3 Property (PH) from lineal actions	42
4.4 First examples	45
4.4.1 The Heisenberg group	45
4.4.2 The crystallographic group	46
4.4.3 The Baumslag-Solitar group	47
4.4.4 Fundamental groups of Anosov mapping tori	48

4.5	Relative HHGs and graph products	48
4.6	Open questions	49
5	Central extensions	51
5.1	Central extensions and bounded Euler classes	51
5.1.1	Geometric properties of central elements	51
5.1.2	Characterizations of central extensions with bounded Euler classes . . .	52
5.2	Property (PH) under quotients	56
5.2.1	Construction of hyperbolic quotient space	56
5.2.2	Proof of Theorem 1.3.1	57
5.3	Applications	61
5.3.1	Mapping class groups	61
5.3.2	Outer automorphism groups of hyperbolic groups	61
6	Extension of quasimorphisms	63
6.1	Quasi-invariant quasimorphisms	63
6.2	Relative presentations	63
6.3	Controlled quasimorphisms	65
6.4	Extending controlled quasimorphisms	68
6.4.1	Auxiliary lemmata	68
6.4.2	The Cayley graph	69
6.5	Normal subgroups	72
6.5.1	A criterion for being controlled	72
6.5.2	Group-theoretic Dehn filling	75
6.5.3	Mapping class groups	76
6.5.4	Stable mixed commutator length	77

ABSTRACT

The main purpose of this thesis is to study several concepts describing the “non-positive curvature” features of groups. First, we identify groups that have property (QT). Under the framework of hierarchically hyperbolic groups, we provide new examples for property (QT), such as residually finite Artin groups of large and hyperbolic type. Then we introduce a weaker notion of property (QT), which we call property (PH), and study the invariance of these properties under central extensions. Through this, we provide new examples of property (QT), including outer automorphism groups of torsion-free one-ended hyperbolic groups. Finally, the study of central extensions leads us to the question about when a quasimorphism on a subgroup extends to the whole group. We establish a sufficient condition for a quasimorphism to be extendable, which generalizes several known results and leads to many new corollaries. Among them, one corollary shows that certain quotients of mapping class groups are hierarchically hyperbolic.

CHAPTER 1

Introduction

Over the past few decades, geometric methods have become central in understanding infinite groups. A paradigm of this idea is Bass–Serre theory [Ser77] back in the 1970s, which connects the algebraic properties of a group to the geometry of its action on a tree. In the 1980s, Gromov [Gro87] introduced the notion of δ -hyperbolic spaces, emphasizing the large-scale negative curvature feature of a metric space. This broad class of spaces encompasses real hyperbolic spaces $\mathbb{H}^n$ and simplicial trees. Groups acting metrically properly and coboundedly on such spaces exhibit many desirable properties, such as Tits alternative and solvable word problem. This class of groups is called *hyperbolic groups* and is “generic” among finitely presented groups in some sense. For example, fundamental groups of closed hyperbolic manifolds, free groups, and $SL(2, \mathbb{Z})$ are all hyperbolic. However, many groups of interest—such as mapping class groups (of surfaces), $\text{Out}(F_n)$, and various cubical groups—are not hyperbolic but still display hyperbolic-like features in some sense. This observation led to the introduction of *acylindrically hyperbolic groups* and *hierarchically hyperbolic groups*.

Complementary to Gromov’s idea of “coarse geometry”, there is a large body of research on “coarse homomorphisms” from a group to $\mathbb{R}$, known as *quasimorphisms* or *quasi-characters*. In classical character theory, homomorphisms to $\mathbb{R}$ detect abelian quotients and play a fundamental role in understanding group representations. Quasimorphisms extend this perspective and reflect deep geometric or dynamical features of the group, even when the group has trivial abelianization. In particular, Poincaré’s rotation number for group actions on the circle is an example motivating the definition of a quasimorphisms. Moreover, quasimorphisms are closely connected to bounded cohomology, another theme advocated by Gromov [Gro82].

This thesis focuses on the above two closely related directions, and can be divided into four parts.

1.1 Property (QT) of relatively hierarchically hyperbolic groups

As a fundamental concept in geometric group theory, a *quasi-isometry* is a map between two metric spaces that respects large-scale geometry of these spaces but ignores their small-scale details. For a finitely generated group G , quasi-isometries arise naturally since two Cayley graphs of G with respect to two different finite generating sets are quasi-isometric to each other.

One of the remarkable features that makes Gromov’s hyperbolicity so powerful is its invariance under quasi-isometries. In particular, any geodesic space quasi-isometric to a tree is hyperbolic. Such spaces are known as *quasi-trees*. A finitely generated group G is said to have *property (QT)* if G acts on a finite product of quasi-trees such that the orbit map is a quasi-isometric embedding. Since a quasi-tree has asymptotic dimension at most 1, property (QT) is a stronger form of finite asymptotic dimension. Since Bestvina, Bromberg, and Fujiwara [BBF21] introduced it, the study of property (QT) has received a great deal of interest. The first goal of this thesis is to study which groups have (QT). Examples of groups with property (QT) include

- Coxeter groups [DJ99];
- Residually finite hyperbolic groups [BBF21];
- Mapping class groups of finite-type surfaces [BBF21];
- Virtually colorable hierarchically hyperbolic groups whose associated hyperbolic spaces are all quasi-trees [HP22] (including virtually compact special groups [BHS17b], the genus 2 handlebody group [Che22], fundamental groups of non-geometric graph manifolds [HRSS24], etc.);
- Fundamental groups of compact orientable 3-manifolds whose sphere-disk decomposition does not support either Sol or Nil geometry [HNY25];

along with their undistorted subgroups.

The last four examples in the above list are proved to have property (QT) with the help of the projection complex techniques developed in [BBF15; BBFS19]. In particular, property (QT) of mapping class groups strengthens [BBF15, Theorem C], which says that mapping class groups equivariantly quasi-isometrically embeds in a finite product of hyperbolic graphs (of finite asymptotic dimension). Counterexamples of property (QT) include certain special linear groups [Man06; Man08], generalized Baumslag-Solitar groups with infinite monodromy [But22] and groups with Property hereditary (NL) [BFG24]. For some basic corollaries of property (QT), see [HNY25, §2.1&2.2]. Recently, Vergara [Ver24] proves that any finitely generated group with property (QT) has a proper uniformly Lipschitz affine action on ℓ^1 with quasi-isometrically embedded orbits.

As a generalization of the Masur–Minsky machinery [MM99; MM00], *hierarchically hyperbolic groups* [BHS17b; BHS19], abbreviated as HHGs, have become an important bridge between mapping class groups, cubical groups, and many other nonpositively-curved groups. A list of papers in this field can be found in [HRSS24]. Coarsely speaking, an HHG is a finitely generated group G whose geometry can be recovered from G -equivariant projections to a specified (infinite) collection of hyperbolic spaces. As shown in [BHS17a], HHGs have finite asymptotic dimension. A natural question is

Question 1.1.1. *Which HHGs have property (QT)?*

The following is a collection of applications in Section 3.6. These results are new except for mapping class groups.

Theorem 1.1.2. *The following groups have property (QT).*

- *Mapping class groups of finite-type surfaces;*
- *Residually finite admissible graphs of groups;*
- *Residually finite hyperbolic-2-decomposable groups with no distorted elements;*
- *Residually finite Artin groups of large and hyperbolic type;*

Moreover, the (QT) embeddings for the above groups are quasi-median.

In application, we also consider a slightly stronger property, which we call *property (QT')*. A group G has property (QT') if G has property (QT) with respect to a finite product of quasi-trees $\prod_{i=1}^n T_i$ and the G -action on $\prod_{i=1}^n T_i$ is diagonal. By [HNY25, Theorem 1.5], if a residually finite group G is relatively hyperbolic such that every peripheral subgroup has property (QT'), then G has property (QT). Without ambiguity, we also say a G -action on a metric space X has *property (QT')* if G acts diagonally on a finite product of quasi-trees $\prod_{i=1}^n T_i$ and X admits a G -equivariant quasi-isometric embedding into $\prod_{i=1}^n T_i$. In particular, if X itself is a finite product of quasi-trees, then any diagonal action on X has property (QT'). We prove the following invariance of property (QT') under graph products.

Theorem 1.1.3. *[Theorem 3.6.10] Any graph product of groups with every vertex group having property (QT') still has property (QT').*

1.2 Property (PH)

Similar to property (QT'), a (not necessarily finitely generated) group G is said to have *property (PH')* (standing for proper diagonal actions on finite products of hyperbolic spaces) if there exist finitely many hyperbolic spaces $X_1, \dots, X_n$ on which G acts coboundedly such that the diagonal action of G on the product $\prod_{i=1}^n X_i$ equipped with ℓ^1 -metric is proper. When $n = 1$, this notion coincides with the definition of word-hyperbolicity. To obtain a commensurability invariant, we say a group G has *property (PH)* if it virtually has property (PH'), i.e. G contains a finite-index subgroup G_0 which has property (PH'). With a result of Button [But22], it is not hard to see that a finitely generated group with property (QT) has also property (PH) (cf. Lemma 4.0.2).

Alternatively, there is another equivalent definition of property (PH'). For every group G , Abbott-Balasubramanya-Osin [ABO19] defined the set of *hyperbolic structures* on G , denoted by $\mathcal{H}(G)$, which consists of equivalence classes of (usually infinite) generating sets of G such that the corresponding Cayley graph is hyperbolic; two generating sets of G are *equivalent* if the corresponding word metrics on G are bi-Lipschitz equivalent. We denote by $[S]$ the equivalence class of a generating set S of G and $\mathcal{G}(G, S)$ the Cayley graph of G with respect to S . Then by the classical Svarc-Milnor lemma, a group G has property (PH') if there exist finitely many hyperbolic structures $[S_1], \dots, [S_n] \in \mathcal{H}(G)$ such that the diagonal action of G on the product $\prod_{i=1}^n \mathcal{G}(G, S_i)$ equipped with ℓ^1 -metric is proper. One reason to relate property (PH') with hyperbolic structures is that only the set of hyperbolic structures cannot distinguish

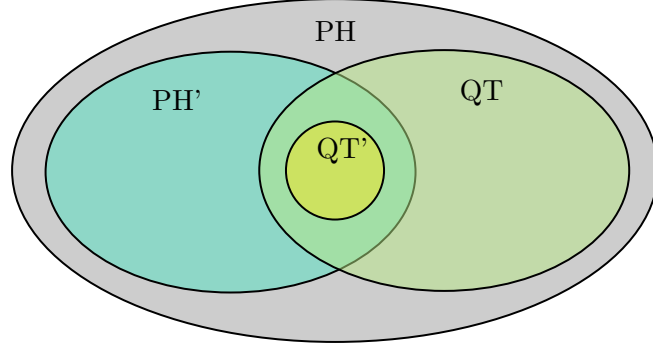


Figure 1.1: The inclusion relationship between properties (PH), (PH'), (QT) and (QT').

the Heisenberg group from the free abelian group $\mathbb{Z}^n (n \geq 2)$ while property (PH') can (cf. §4.4.1).

Another motivation of property (PH) comes from the study of uniform exponential growth by Wan and Yang in [WY25]. They proved that if a group G has property (PH) and there is a well-behaved system of quasi-geodesics, then G has locally uniform exponential growth. If we only assume property (PH), then there is a weaker conclusion stating that G is either virtually abelian or has exponential growth (cf. Corollary 4.3.4).

The second goal of this thesis is to initiate the study of property (PH) of groups. As applications, we will give a characterization of 3-manifold groups with property (PH) and derive more examples of finitely generated groups with property (QT). We say that a virtually abelian group G has *rank* r if G contains a finite-index subgroup G_0 which is isomorphic to $\mathbb{Z}^r$. A group is called *FZ* if it has a finite-index center. First, we characterize the “simplest” class of groups with property (PH').

Theorem 1.2.1. *Suppose a group G has property (PH') from orientable lineal actions $G \curvearrowright L_1, \dots, G \curvearrowright L_m$. Then G is a finitely generated FZ group of rank $r \leq m$. Moreover, a group has property (PH) from lineal actions if and only if it is finitely generated and virtually abelian.*

As we know, word-hyperbolicity of finitely generated groups is a quasi-isometry invariant, and property (QT) is a commensurability invariant [BBF21] but not a quasi-isometry invariant [But22]. By a result of Button [But22], a finitely generated group with property (QT) virtually has property (QT'). See Figure 1.1 for an illustration of relations between properties (PH), (PH'), (QT) and (QT'). The following proposition provides examples that help to clarify this illustration.

Proposition 1.2.2.

1. *The Heisenberg group H has an isomorphic set of hyperbolic structures to the free abelian group $\mathbb{Z}^n (n \geq 2)$. However, $\mathbb{Z}^n$ has property (PH') while H does not.*
2. *The $(3,3,3)$ -triangle group $T = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ac)^3 = 1 \rangle$ has property (QT) but no property (PH'). In particular, neither property (PH') nor (QT') is a commensurability invariant.*

3. The solvable Baumslag-Solitar group $BS(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle$ has property (PH') but no property (QT) for $n \geq 2$. In particular, the class of finitely generated groups with property (QT) (resp. (QT')) is strictly contained in the class of groups with property (PH) (resp. (PH')).

Finally, we consider the invariance of property (PH') under graph products. Recall that any graph product of groups with every vertex group having property (QT') has property (QT') (Theorem 1.1.3). By using a similar method, we will obtain an analogous result for property (PH'). Without ambiguity, we say that an action of a group G on a metric space X has *property (PH')* if G acts diagonally on a finite product of hyperbolic spaces $\prod_{i=1}^n X_i$ and X admits a G -equivariant coarsely controlled map to $\prod_{i=1}^n X_i$. Note that if X itself is a finite product of hyperbolic spaces, then any diagonal action on X has property (PH').

Theorem 1.2.3. *Let $(G, \mathfrak{S})$ be a colorable relative HHG. If for every domain $U \in \mathfrak{S}$ that is nesting-minimal, the action of $\text{Stab}_G(U)$ on $\mathcal{C}U$ has property (PH'), then G has property (PH').*

Any graph product of finitely generated groups is hierarchically hyperbolic relative to its vertex groups by [BR22]. Moreover, it is easy to see from the definition that this relative HHG structure is colorable. Similarly to Theorem 1.1.3, we obtain the following corollary.

Corollary 1.2.4. *Any graph product of finitely generated groups with every vertex group having property (PH') still has property (PH').*

Let M be a connected, compact, orientable 3-manifold. As shown in [HNY25, Theorem 1.1], the fundamental group $\pi_1(M)$ has property (QT) if and only if no summand in the sphere-disk decomposition of M supports either Sol or Nil geometry. We will see later that a nilpotent group cannot have property (PH') unless it is virtually abelian (cf. Corollary 4.3.4) and the fundamental group of Anosov mapping tori has property (PH') (cf. Proposition 4.4.6). Together with Corollary 1.2.4, we have

Theorem 1.2.5. *Let M be a connected, compact, orientable 3-manifold. If no summand in the sphere-disk decomposition of M supports Nil geometry, then $\pi_1(M)$ has property (PH).*

1.3 Central extensions

The next goal of this thesis is to investigate the invariance of property (PH'), (PH), (QT') or (QT) under central extensions.

Let

$$1 \rightarrow Z \xrightarrow{\iota} E \xrightarrow{\pi} G \rightarrow 1. \quad (1.1)$$

be a central extension of groups, where Z is finitely generated. Any such extension defines a cohomology class $[\omega] \in H^2(G, Z)$, called the *Euler class* (see Section 2.4). We say that a class $[\omega] \in H^2(G, Z)$ is *bounded* if it lies in the image of the comparison map $H_b^2(G, Z) \rightarrow H^2(G, Z)$, i.e. if it can be described by a bounded cocycle. Our main results are:

Theorem 1.3.1. *For the central extension (1.1), E has property (PH') (resp. (PH)) if and only if G has property (PH') (resp. (PH)) and the Euler class is bounded.*

Theorem 1.3.2. *For the central extension (1.1) with G (hence, E) being finitely generated, E has property (QT') (resp. (QT)) if and only if G has property (QT') (resp. (QT)) and the Euler class is bounded.*

Recently, Fournier-Facio, Mangioni, and Sisto obtained an analogous result in [FMS25] which states that the hierarchical hyperbolicity of groups is preserved under central extensions with bounded Euler classes.

Since any central extension of a hyperbolic group has a bounded Euler class [NR97, Theorem 3.1], it follows directly from Theorem 1.3.1 that

Corollary 1.3.3. *Let E be a central extension of a hyperbolic group. Then E has property (PH') .*

Moreover, in [BBF15], Bestvina-Bromberg-Fujiwara show that residually finite hyperbolic groups have property (QT) . Thus, as a result of Theorem 1.3.2,

Corollary 1.3.4. *Let E be a central extension of a residually finite hyperbolic group. Then E has property (QT) .*

We remind the readers that it is still an open conjecture that all hyperbolic groups are residually finite.

Let Σ be a finite-type surface. In [BBF21], Bestvina-Bromberg-Fujiwara show that the mapping class group $\text{MCG}(\Sigma)$ has property (QT) . Considering the capping sequence of mapping class groups, which is a central extension with a bounded Euler class, we can prove

Corollary 1.3.5 (Corollary 5.3.1). *$\text{MCG}(\Sigma)$ has property (QT) for any finite-type surface Σ with boundary. In particular, braid groups have property (QT) .*

Corollary 1.3.6 (Corollary 5.3.2). *Let Σ be a finite-type surface. The stabilizer in $\text{MCG}(\Sigma)$ of any multicurve on Σ has property (QT) .*

Another corollary is about the outer automorphism group $\text{Out}(G)$ of a one-ended hyperbolic group G . The group $\text{Out}(G)$ was initially studied by Rips and Sela. In particular, Sela [Sel97] gave a precise description of $\text{Out}(G)$ using mapping class groups and central extensions, assuming G is torsion-free. As a corollary, we show that

Corollary 1.3.7 (Corollary 5.3.5). *$\text{Out}(G)$ has property (QT) for any one-ended torsion-free hyperbolic group G .*

In the end of this section, we remark that there are many counterexamples of property (PH) and (PH') . In [BFG24], a group G is said to have *property (NL)* if no isometric actions of G on a hyperbolic space admits a loxodromic element. Moreover, if all of the finite-index subgroups of G have property (NL) , G is said to have *property hereditary (NL)* . Any infinite group G with property (NL) does not have property (PH') . Indeed, the orbit of G on any hyperbolic space is bounded, which shows that G cannot act properly on a finite product of hyperbolic spaces. As a corollary, any infinite group with property hereditary (NL) does not have property (PH) . We refer the readers to [BFG24] for examples of groups satisfying property (hereditary) (NL) .

1.4 Extension of quasimorphisms

Let G be a group. A function $\varphi : G \rightarrow \mathbb{R}$ is called a *quasimorphism* if its *defect*

$$D(\varphi) := \sup_{g, h \in G} |\varphi(g) + \varphi(h) - \varphi(gh)|$$

is finite. Quasimorphisms are closely connected to bounded cohomology of groups and have been studied for a long time within geometric group theory and symplectic geometry. They also have substantial connections to low-dimensional topology and group actions on the circle (see [Cal09; PR14; Fri17; KKM19]). Many fundamental constructions arise as extensions of quasimorphisms on certain subgroups to the whole group (e.g. Brooks quasimorphisms on mapping class groups and surface diffeomorphism groups [BF02; BHW22], Calabi quasimorphisms on Hamiltonian diffeomorphism groups of symplectic manifolds [EP03; Py06]). The last goal of this thesis is to study the following natural but delicate problem:

Question 1.4.1. *Given a subgroup $K < G$ and a quasimorphism φ on K , when and how can φ be extended to a quasimorphism on G ?*

A remarkable answer to Question 1.4.1 was obtained by Hull and Osin.

Theorem 1.4.2. [HO13] *If K is a hyperbolically embedded subgroup of G , then any antisymmetric quasimorphism on K extends to G .*

In fact, Hull and Osin proved the theorem for 1-quasicycles with more general coefficients. Later, their result has been generalized to higher-dimensional quasicycles by Frigerio, Pozzetti, and Sisto [FPS15]. As an application, they prove that the third bounded cohomology of an acylindrically hyperbolic group is infinite-dimensional.

On the other hand, Kawasaki, Kimura, Maruyama, Matsushita, and Mimura have a series of work studying Question 1.4.1 for K being a normal subgroup of G . We refer to the survey [KKMMM24] and the more recent paper [KKMMM25a] as an update. As a notable extrinsic application, four of them [KKMM23] showed the vanishing of the cup product of the fluxes of commuting symplectomorphisms for any closed oriented surface of genus at least two by examining the extendability of certain Calabi quasimorphisms. Note that an infinite normal subgroup cannot be hyperbolically embedded [DGO17, Proposition 4.33]. Therefore, their work addresses entirely different situations from Theorem 1.4.2. As we will see, one advantage of our result is that Theorem 1.4.3 applies to both.

Let X be a (symmetric) relative generating set of G with respect to K . Let $\Gamma(G, K \sqcup X)$ be the Cayley graph of G with respect to the generating set $K \sqcup X$. In this thesis, we introduce a new concept, which we call (G, X) -controlled quasimorphisms. Roughly speaking, a quasimorphism φ on K is (G, X) -controlled if the φ -value of the product of all K -letters appearing in a cycle of length L in $\Gamma(G, K \sqcup X)$ is bounded by a function with respect to L . We refer the reader to Definition 6.3.1 for the precise definition.

Let $Q_{as}(K)$ be the set of all antisymmetric quasimorphisms on K , and let $Q_{as}(K)^G$ be its subset of all G -quasi-invariant ones. Our main theorem is as follows.

Theorem 1.4.3. *Assume that $\Gamma(G, K \sqcup X)$ is δ -hyperbolic for some constant $\delta \geq 0$. Let $\varphi \in Q_{as}(K)$. Then φ extends to a quasimorphism $\Phi \in Q_{as}(G)$ if φ is (G, X) -controlled. Moreover, if φ is linearly (G, X) -controlled with a constant $C_0 > 0$, then $D(\Phi) \leq MC_0 + MD(\varphi)$, where M is a constant only depending on δ .*

By definition of hyperbolically embedded subgroups (Definition 6.2.4) and Proposition 6.3.6, if K is hyperbolically embedded in G , then there exists a relative generating set X such that the assumption of Theorem 1.4.3 is satisfied. Thus, Theorem 1.4.3 implies Theorem 1.4.2 directly. Furthermore, the proof we give in this thesis is much simpler than that in [HO13]. Being (G, X) -controlled is an appropriate assumption for extendability, because it is necessary in many situations.

Lemma 1.4.4 (Lemma 6.3.3). *If G is generated by K and a finite set X , then an extendable quasimorphism on K must be linearly (G, X) -controlled.*

In [KKMMM25b, Theorem 1.9], Kawasaki, Kimura, Maruyama, Matsushita, and Mimura give a sufficient condition on extendability of invariant quasimorphisms on a normal subgroup. It is required that the quotient group has trivial bounded cohomology in lower degree. In contrast to their result, Theorem 1.4.3 gives the following corollary, which is a special case of Corollary 6.5.5.

Corollary 1.4.5. *Let G be a group and K a normal subgroup. If G/K is hyperbolic, then any $\varphi \in Q_{as}(K)^G$ extends to G .*

For group-theoretic Dehn filling, we obtain the following.

Theorem 1.4.6 (Theorem 6.5.8). *Let H be a hyperbolically embedded subgroup of G . There exists a finite subset $F \subset H - \{1\}$ such that the following holds. Let $N \triangleleft H$ be a subgroup such that $N \cap F = \emptyset$. Let K be the normal closure of N in G . Let $\varphi \in Q_{as}(K)^G$. If there exists a transversal L of N in H such that the restricted quasimorphism $\varphi|_N$ is (H, L) -controlled, then φ extends to G .*

By applying Theorem 1.4.6 to mapping class groups, we obtain new examples of hierarchically hyperbolic groups, and thus answer a question asked by Fournier-Facio, Mangioni, and Sisto [FMS25, Question 4.2].

Corollary 1.4.7 (Corollary 6.5.11). *Let Σ be a finite-type surface with boundary. Let g be a pseudo-Anosov mapping class of Σ . There exists $N \geq 1$ such that the following holds. For any $k \in \mathbb{N}$, let K be the normal closure of g^{kN} in G . The quotient group $\text{MCG}(\Sigma)/K$ is hierarchically hyperbolic.*

Motivated by [HO13] and [FPS15], we ask the following question.

Question 1.4.8. *Can Theorem 1.4.3 be generalized to quasicocycles with more general coefficients or of higher degree in some sense?*

Structure of the thesis

In Chapter 2, we recall some basic concepts in geometric group theory that we need. Chapters 3 to 6 correspond to the above Sections 1.1 to 1.4, respectively.

Chapter 3 is based on the paper [Tao24]. We introduce the BBF projection system and provide a sufficient condition for a hierarchically hyperbolic group to have property (QT). We apply it to recover some known examples and also obtain new examples of property (QT).

Chapters 4 and 5 are based on the paper [TW25] joint with Wan. In Chapter 4, we introduce property (PH) and use quasimorphisms to characterize groups with property (PH) from lineal actions. We also study some basic examples and non-examples. In Chapter 5, we obtain the invariance of (QT) and (PH) under certain central extensions. We use it to provide new examples of (QT).

Chapter 6 is based on the paper [Tao25] and is devoted to the extension problem of quasimorphisms. We obtain a sufficient condition for a quasimorphism to be extendable and show how this problem relates with central extensions. We also apply our result to obtain new examples of hierarchically hyperbolic groups.

CHAPTER 2

Preliminaries

2.1 δ -hyperbolic spaces and group actions

Let (X, d_X) and (Y, d_Y) be two metric spaces. Throughout this thesis, a “map” between X and Y may refer to a *coarse map*, which sends each point to a uniformly bounded subset rather than a single point. All statements involving such maps are meant in the strongest sense, i.e. they hold for all choices of representatives. We will also denote $d_X(x_1, x_2)$ by $|x_1 - x_2|_X$, or simply $|x_1 - x_2|$ if X can be inferred from the context.

Given constants $\lambda \geq 1$, $\mu \geq 0$, we say that a coarse map $f : X \rightarrow Y$ is

- (λ, μ) -*coarsely Lipschitz* if

$$d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \mu$$

for any $x_1, x_2 \in X$.

- a (λ, μ) -*quasi-isometric embedding* if

$$\frac{1}{\lambda} d_X(x_1, x_2) - \mu \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \mu$$

for any $x_1, x_2 \in X$.

- μ -*coarsely surjective* if every point of Y is within the μ -neighborhood of the image of f .
- a (λ, μ) -*quasi-isometry* if f is a (λ, μ) -quasi-isometric embedding and is μ -coarsely surjective.

Alternatively, f is a (λ, μ) -quasi-isometry if f is a (λ, μ) -quasi-isometric embedding and there exists a *quasi-inverse* $f^{-1} : Y \rightarrow X$ such that $|f \circ f^{-1}(y) - y| \leq \mu$ for all $y \in Y$. When such a map exists, the two spaces X and Y are said to be *quasi-isometric*.

Let $I \subset \mathbb{R}$ be a (possibly unbounded) interval. We write I^- and I^+ to mean its left endpoint and right endpoint, respectively. A (λ, μ) -quasi-isometric embedding $\gamma : I \rightarrow X$ is called a (parametrized) (λ, μ) -*quasi-geodesic* in X . A coarse map $\gamma : I \rightarrow X$ is called an *unparametrized* (λ, μ) -quasi-geodesic if there is a strictly increasing function $f : I \rightarrow I$ with $f(I^-) = I^-$, $f(I^+) = I^+$ such that $\gamma \circ f$ is a (λ, μ) -quasi-geodesic. We will not distinguish between a quasi-geodesic and its image in X . For simplicity, a (λ, λ) -quasi-geodesic will be called a λ -*quasi-geodesic*.

Definition 2.1.1. A geodesic metric space is called δ -hyperbolic (or simply, *hyperbolic*) for a constant $\delta \geq 0$ if for any geodesics α, β, γ that form a triangle, α is contained in the δ -neighborhood of $\beta \cup \gamma$.

Example 2.1.2. The following metric spaces are hyperbolic.

- Real hyperbolic spaces $\mathbb{H}^n$.
- Simplicial trees with the graph metric.

If X is δ -hyperbolic and Y is (λ, μ) -quasi-isometric to X , then it is known that Y is δ' -hyperbolic, where δ' depends on δ, λ, μ . A *quasi-tree* (resp. *quasi-line*) is a geodesic space quasi-isometric to a simplicial tree (resp. real line). In particular, a quasi-tree is a hyperbolic space and has asymptotic dimension at most 1. A subspace $Y \subset X$ is σ -quasi-convex for some constant $\sigma \geq 0$ if any geodesic between two points in Y stays within the σ -neighborhood of Y . It is known that any σ -quasi-convex subspace of a δ -hyperbolic space is δ'' -hyperbolic, where δ'' depends on δ, σ .

Let X be a hyperbolic space. Two quasi-geodesic rays in X are said to be *asymptotic* if the Hausdorff distance between them is finite. Being asymptotic is an equivalence relation on quasi-geodesic rays. The *Gromov boundary* of X , denoted by ∂X , is defined to be the set of asymptotic classes of quasi-geodesic rays in X .

By Gromov [Gro87], the isometries of a hyperbolic space X can be divided into three classes. An element $g \in \text{Isom}(X)$ is called *elliptic* if some $\langle g \rangle$ -orbit is bounded. Otherwise, it is called *loxodromic* (resp. *parabolic*) if it has exactly two fixed points (resp. one fixed point), denoted by g^+, g^- , on the Gromov boundary ∂X of X . Alternatively, an isometry $g : X \rightarrow X$ is loxodromic if and only if the $\langle g \rangle$ -orbit $n \mapsto g^n o$ is a quasi-geodesic for some (equivalently, for any) $o \in X$.

Throughout this thesis, group actions on metric spaces are always by isometries if not otherwise specified. Let X be a hyperbolic space and G be a group acting on X . Let $o \in X$ be a basepoint. The *limit set* of G is defined by $\Lambda_X G = \overline{Go} \cap \partial X$, and is independent of the choice of the basepoint. When X can be inferred from the context, we can simplify the notation as ΛG .

According to Gromov [Gro87], all isometric group actions on hyperbolic spaces can be classified into the following five types:

1. *elliptic* if the orbit is bounded;
2. *horocyclic* if it has no loxodromic element;
3. *lineal* if it has a loxodromic element and any two loxodromic elements have the same fixed points in Gromov boundary;
4. *focal* if it has a loxodromic element, is not lineal and any two loxodromic elements have one common fixed point;
5. *of general type* if it has two loxodromic elements with no common fixed point.

The first three actions are called *elementary*. It is easy to see that the orbit of a lineal action is a quasi-line. A lineal action is called *orientable* if no group element permutes the two limit points of this action.

Assume that there exists a loxodromic element $g \in G$. Given constants $\lambda \geq 1$, $\mu \geq 0$, a (λ, μ) -quasi-geodesic $\gamma \subset X$ is called a (λ, μ) -*quasi-axis* for g if γ is g -invariant. The *elementary closure* of g in G , $EC_G(g)$, is defined as the subgroup of G that stabilizes γ up to bounded Hausdorff distance. If G can be inferred from the context, we can simplify the notation as $EC(g)$. Equivalently, $EC(g)$ is the stabilizer of the set $g^\pm$. Thus, the elementary closure does not depend on the choice of γ . Every element that commutes with g is contained in $EC(g)$ (including powers and roots), but there may be other elements.

Definition 2.1.3. Let (X, d) be a metric space. A group action $G \curvearrowright X$ is said to be

- *(metrically) proper* if for any $x \in X$ and any $R > 0$, the set $\{g \in G \mid d(x, gx) \leq R\}$ is finite.
- *cobounded* if there exists a bounded subspace $B \subset X$ such that $G \cdot B = X$.
- *geometric* if it is proper and cobounded.
- *acylindrical* if for any $r \geq 0$, there exist constants $R, N \geq 0$ such that for any pair $a, b \in X$ with $d(a, b) \geq R$, we have

$$\#\{g \in G \mid d(ga, a) \leq r \text{ and } d(gb, b) \leq r\} \leq N.$$

Example 2.1.4. Let G be a group that has a finite generating set S . Then the natural G -action on the Cayley graph $\Gamma(G, S)$ is geometric.

It is easy to see that a geometric action must be acylindrical. A *hyperbolic group* is defined as a group that acts geometrically on a hyperbolic space. Hyperbolic groups satisfy a particular *Tits alternative*, which means that a hyperbolic group G is either virtually cyclic (i.e. G has a finite-index subgroup that is cyclic) or contains a non-abelian free subgroup. We say that a group is *elementary* if it is virtually abelian. As a generalization of non-elementary hyperbolic groups, an *acylindrically hyperbolic group* is defined as a group that admits a non-elementary acylindrical action on a hyperbolic space.

Let X be a hyperbolic space and G be a group acting acylindrically on X with a loxodromic element g . Some basic properties of this kind of actions can be found in [Osi16]. In this case, the elementary closure $EC(g)$ is the unique maximal virtually cyclic subgroup of G that contains g [DGO17, Lemma 6.5]. Moreover, $EC(g)$ has a subgroup of index at most 2 that is a centralizer of a large power of g in G [DGO17, Corollary 6.6].

In this thesis, we will consider group actions with a large kernel, in which case the action cannot be acylindrical. As in [BBF21], an action $G \curvearrowright X$ is said to have *acylindrical image* if the image of G in the isometry group of X is acylindrical.

Let X, Y be metric spaces. We say a map $f : X \rightarrow Y$ is *coarsely controlled* if f is metrically proper and there exists a function $\rho_+ : [0, \infty) \rightarrow [0, \infty)$ such that

$$d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y))$$

for any $x, y \in X$. This is the same as the “coarse map” defined by [NY23b, Definition 1.4.1], but we change the name to avoid the ambiguity. For example, if a finitely generated group G acts properly on a metric space X , then the orbit map is coarsely controlled.

2.2 (Relatively) hierarchically hyperbolic groups

In this thesis, we deal with *(relatively) hierarchically hyperbolic spaces* and *(relatively) hierarchically hyperbolic groups*. For simplicity, we abbreviate them as (relative) HHSes and (relative) HHGs. Briefly speaking, a relative HHS $(\mathcal{X}, \mathfrak{S})$ consists of a quasi-geodesic space $\mathcal{X}$, an index set $\mathfrak{S}$, a collection of geodesic spaces $\{(\mathcal{C}U, d_U) \mid U \in \mathfrak{S}\}$, and a constant $E > 0$ that satisfy several axioms. A full definition can be found in [BHS19, Definition 1.1, 1.21] (see also [BHS17a] for a further study). For simplicity, we collect only necessary information that we need in the following.

- An element $U \in \mathfrak{S}$ is called a *domain* of $\mathcal{X}$. $\mathfrak{S}$ has a partial order $\sqsubseteq$ (called *nesting*) and a symmetric relation $\perp$ (called *orthogonality*). If $U \sqsubseteq V$ (resp. $U \perp V$), we say that U is *nested* in V (resp. U is *orthogonal* to V). These two relations are required to be mutually exclusive. For any two elements that are neither comparable under the partial order nor mutually orthogonal, they are defined to be mutually *transversal* and we denote this relation by $\pitchfork$. By $\mathfrak{S}_U$ (respectively, $\mathfrak{S}_U^\circ$) we denote the set of all domains nested (respectively, properly nested) in U .
- There is a unique $\sqsubseteq$ -maximal element S in $\mathfrak{S}$ and a uniform bound on the length of $\sqsubseteq$ -chains in $\mathfrak{S}$, called the *complexity* of $(\mathcal{X}, \mathfrak{S})$. The *level* $\ell(V)$ of $V \in \mathfrak{S}$ is defined inductively as follows. If V is $\sqsubseteq$ -minimal then we define $\ell(V) = 1$. The element V has level $k + 1$ if k is the maximal integer such that there exists $U \sqsubset V$ with $\ell(U) = k$.
- For HHSes, there is a set $\{(\mathcal{C}U, d_U) : U \in \mathfrak{S}\}$ of uniformly hyperbolic spaces and a set of uniformly coarsely Lipschitz and coarsely surjective maps $\pi_U : \mathcal{X} \rightarrow \mathcal{C}U$ for all $U \in \mathfrak{S}$. For relative HHSes, the complexity is at least 2. If U is $\sqsubseteq$ -minimal, $\mathcal{C}U$ is not required to be hyperbolic. This is the only difference between HHSes and relative HHSes in definition.
- For $U \sqsubset V$ or $U \pitchfork V$, there is a E -bounded set $\rho_V^U \subset \mathcal{C}V$.
- For $U \sqsubset V$, there is a coarse map $\rho_U^V : \mathcal{C}V \rightarrow \mathcal{C}U$.
- Whenever $V \sqsubseteq W$ and $W \perp U$, we require that $V \perp U$.
- (Orthogonal containers) For each $T \in \mathfrak{S}$ and each $U \in \mathfrak{S}_T$ for which $\{V \in \mathfrak{S}_T \mid V \perp U\} \neq \emptyset$, there exists $W \in \mathfrak{S}_T^\circ$, so that whenever $V \perp U$ and $V \sqsubseteq T$, we have $V \sqsubseteq W$. We say that W is an *orthogonal container* of U in T if W is a $\sqsubseteq$ -minimal element satisfying the above property. Let $\text{cont}_T^\perp U$ denote the set of all orthogonal containers of U in T . If T is the maximal element of $\mathfrak{S}$, then we suppress it from the notation and write $\text{cont}^\perp U$. We denote $\mathfrak{S}_U^\perp = \{V \in \mathfrak{S} \mid V \perp U\} \cup \{A\}$, where A is an arbitrary element of $\text{cont}^\perp U$.

- (Consistency) For every $x \in X$, the tuple $(\pi_U(x))_{U \in \mathfrak{S}}$ is κ_0 -consistent (defined below). If $U \sqsubseteq V$, then $d_W(\rho_W^U, \rho_W^V) \leq \kappa_0$ whenever $W \in \mathfrak{S}$ satisfies either $V \sqsubset W$ or $V \pitchfork W$ and $U \not\sqsubseteq W$.
- (Bounded geodesic image) There exists $E > 0$ such that for all $W \in \mathfrak{S}$, all $V \in \mathfrak{S}_W^\circ$, and all $x, y \in \mathcal{X}$ so that some geodesic from $\pi_W(x)$ to $\pi_W(y)$ stays E -far from ρ_W^V , we have $d_V(\pi_V(x), \pi_V(y)) \leq E$. We will refer this property as BGI in this thesis.

Definition 2.2.1 (κ -consistent tuple). For a number $\kappa \geq 0$, let $\vec{b} = (b_U)_{U \in \mathfrak{S}} \in \prod_{U \in \mathfrak{S}} 2^{\mathcal{C}^U}$ be a tuple such that every set b_U has diameter at most κ . We say that $\vec{b}$ is κ -consistent if

$$\min \{d_U(b_U, \rho_U^V), d_V(b_V, \rho_V^U)\} \leq \kappa \quad \text{whenever } U \pitchfork V, \text{ and}$$

$$\min \{d_V(b_V, \rho_V^U), \text{diam}_U(b_U \cup \rho_U^V(b_V))\} \leq \kappa \quad \text{whenever } U \sqsubset V.$$

In application, we always take E to be the greatest constant in all coarseness from the above list (see [BHS19, Remark 1.6] for discussions on these constants). For the rest of this section, let $(\mathcal{X}, \mathfrak{S})$ be a relative HHS.

Notation. Given $x, y \in \mathcal{X}$, we write $d_U(x, y)$ to mean $d_U(\pi_U(x), \pi_U(y))$. If $U \pitchfork V$ or $U \sqsubset V$, then we write $d_V(x, \rho_V^U)$ to mean $d_V(\pi_V(x), \rho_V^U)$.

Notation. Given $A, B \geq 0$, we define a threshold function by

$$\{A\}_B = \begin{cases} A & \text{if } A \geq B \\ 0 & \text{otherwise.} \end{cases}$$

Notation. Given two functions $f, g : X \rightarrow \mathbb{R}$ and $A, B > 0$, we write $f \leq_{(A,B)} g$ to mean

$$f(x) \leq Ag(x) + B$$

for any $x \in X$. We write $f \asymp_{(A,B)} g$ to mean

$$\frac{1}{A}g(x) - B \leq f(x) \leq Ag(x) + B$$

for any $x \in X$. Sometimes we omit the constants to mean that the inequality holds for some constants.

The powerful Masur–Minsky distance formula [MM00] shows that the distance between points in a mapping class group is coarsely the sum of the distances between the projections of these points to the curve graphs of all subsurfaces. Like mapping class groups, relative HHSes also satisfy a Masur–Minsky-style distance formula.

Theorem 2.2.2 (Distance formula). *[BHS19, Thm. 6.10] There exists s_0 such that for all $s \geq s_0$ there exists a constant $C > 0$ such that for all $x, y \in \mathcal{X}$,*

$$d_{\mathcal{X}}(x, y) \asymp_{(C,C)} \sum_{W \in \mathfrak{S}} \{d_W(x, y)\}_s.$$

Closely related to the distance formula is the existence of *hierarchy paths*.

Definition 2.2.3 (Hierarchy path). A (D, D) -quasi-geodesic $\gamma \subset \mathcal{X}$ is a D -*hierarchy path* if $\pi_U(\gamma)$ is an unparametrized (D, D) -quasi-geodesic for each $U \in \mathfrak{S}$.

Theorem 2.2.4 (Existence of hierarchy paths). [BHS19, Thm. 6.11] *There exists D_0 such that any two points in $\mathcal{X}$ are joined by a D_0 -hierarchy path.*

Remark 2.2.5. Let γ be a D -hierarchy path connecting x and y . By the construction of hierarchy paths in [BHS19], $\pi_U(\gamma)$ is contained in the D -neighborhood of a geodesic connecting $\pi_U(x)$ and $\pi_U(y)$. If $\mathcal{CU}$ is hyperbolic, this is easy to see from Morse Lemma. Otherwise, this deserves its own mention.

There is an important class of subspaces in relative HHSes. We will consider them in Section 3.3.

Definition 2.2.6 (Standard product region, standard nested region, standard orthogonal region). Given $U \in \mathfrak{S}$. Fix $\kappa \geq \kappa_0$. Let $\mathbf{F}_U$ be the set of κ -consistent tuples in $\prod_{V \in \mathfrak{S}_U} 2^{\mathcal{C}V}$. Let $\mathbf{E}_U$ be the set of κ -consistent tuples in $\prod_{V \in \mathfrak{S}_U^\perp - \{A\}} 2^{\mathcal{C}V}$. Let $\mathbf{P}_U = \mathbf{F}_U \times \mathbf{E}_U$. We can define a coarse map $\phi_U : \mathbf{P}_U \rightarrow \mathcal{X}$ as follows.

For each $(\vec{a}, \vec{b}) \in \mathbf{F}_U \times \mathbf{E}_U$, and each $V \in \mathfrak{S}$, define the coordinate $(\phi_U(\vec{a}, \vec{b}))_V$ as follows. If $V \subseteq U$, then $(\phi_U(\vec{a}, \vec{b}))_V = a_V$. If $V \perp U$, then $(\phi_U(\vec{a}, \vec{b}))_V = b_V$. If $V \pitchfork U$, then $(\phi_U(\vec{a}, \vec{b}))_V = \rho_V^U$. Finally, if $U \sqsubset V$, let $(\phi_U(\vec{a}, \vec{b}))_V = \rho_V^U$. We can check that the tuple $\phi_U(\vec{a}, \vec{b})$ is κ -consistent, and thus the realization theorem [BHS19, Theorem 3.1] supplies the map $\phi_U : \mathbf{P}_U \rightarrow \mathcal{X}$ (see [BHS19, §5B] for more details).

For convenience, we do not distinguish between $\mathbf{P}_U$ and its image in $\mathcal{X}$. We call $\mathbf{P}_U$ the *standard product region*. By choosing any copy of $\mathbf{F}_U$ in the direct product, ϕ_U restricts to a coarse map $\phi^\sqsubseteq : \mathbf{F}_U \rightarrow \mathcal{X}$. We also define $\phi^\perp : \mathbf{E}_U \rightarrow \mathcal{X}$ in the same way. We call $\mathbf{F}_U$ and $\mathbf{E}_U$ the *standard nested region* and the *standard orthogonal region*, respectively.

Remark 2.2.7. By definition, $\mathbf{F}_U$, $\mathbf{E}_U$ and $\mathbf{P}_U$ depend on the constant κ . In this thesis, we simply fix any $\kappa \geq \kappa_0$ and do not mention it again.

It is known that $(\mathbf{F}_U, \mathfrak{S}_U)$, $(\mathbf{E}_U, \mathfrak{S}_U^\perp)$ are both relatively hierarchically hyperbolic. By definition of $\mathbf{F}_U$, $\mathbf{E}_U$ and $\mathbf{P}_U$, there are natural retractions from $\mathcal{X}$ to these subspaces. We call such a map a *gate map*. Take $\mathbf{F}_U$ for example. We denote the gate map to $\mathbf{F}_U$ by $\mathbf{g}_{\mathbf{F}_U}$. For all $x \in \mathcal{X}$ and all $V \in \mathfrak{S}$ such that $\mathcal{C}V$ is hyperbolic, $\pi_V(\mathbf{g}_{\mathbf{F}_U}(x))$ uniformly coarsely coincides with the closest point projection of $\pi_V(x)$ to the quasi-convex subset $\pi_V(\mathbf{F}_U)$. In fact, gate maps can be defined for all “hierarchically quasi-convex” subspaces, which is a larger class of subspaces of relative HHSes (see [BHS19, §5] for HHSes).

For any (relative) HHS $(\mathcal{X}, \mathfrak{S})$, an automorphism is a bijection g from $(\mathcal{X}, \mathfrak{S})$ to itself such that the following hold.

1. The relations $\sqsubseteq, \perp, \pitchfork$ are preserved by g .
2. For every $U \in \mathfrak{S}$, there is an isometry $g_U^* : \mathcal{CU} \rightarrow \mathcal{C}gU$ such that $\pi_{gU}(gx) = g_U^*(\pi_U(x))$ for any $U \in \mathfrak{S}$ and $x \in \mathcal{X}$.

3. $\rho_{gV}^{gU} = g_V^*(\rho_V^U)$ for any $U, V \in \mathfrak{S}$ with $U \sqsubset V$ or $U \triangleleft V$.

All automorphisms of $(\mathcal{X}, \mathfrak{S})$ form a group $\text{Aut}(\mathfrak{S})$, which we call the *automorphism group* of $(\mathcal{X}, \mathfrak{S})$.

Definition 2.2.8 ((Relatively) hierarchically hyperbolic groups). A finitely generated group G is *(relatively) hierarchically hyperbolic* if there exists a (relatively) hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ and an action $G \rightarrow \text{Aut}(\mathfrak{S})$ such that the action $G \curvearrowright X$ is geometric and the induced action on $\mathfrak{S}$ is cofinite.

Note that if G is (relatively) hierarchically hyperbolic by virtue of its action on the (relatively) hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$, then $(G, \mathfrak{S})$ is a (relatively) hierarchically hyperbolic structure with respect to any word-metric on G .

Let $\text{Aut}(\mathfrak{S}; V)$ be the group of automorphisms $g \in \text{Aut}(\mathfrak{S})$ such that $g \cdot V = V$. Then there is a *restriction homomorphism* $\theta_V : \text{Aut}(\mathfrak{S}; V) \rightarrow \text{Aut}(\mathfrak{S}_V)$ defined as follows. Given $g \in \text{Aut}(\mathfrak{S}; V)$, let $\theta_V(g)$ act like g on the substructure $\mathfrak{S}_V$. For a group $G < \text{Aut}(\mathfrak{S})$, we write $\text{Stab}_G(V)$ to mean $G \cap \text{Aut}(\mathfrak{S}; V)$ and write G_V to mean the image of $\text{Stab}_G(V)$ under θ_V .

For many HHGs (for example, the case of mapping class groups), every G_V acts acylindrically on $\mathcal{C}V$. However, not all HHGs have this property [DHS20].

Definition 2.2.9 (colorability). Let $(G, \mathfrak{S})$ be a relative HHG. Let $\mathfrak{S}' \subset \mathfrak{S}$ be a G -invariant subset. We say $\mathfrak{S}'$ is *colorable* if, $\mathfrak{S}'$ admits a decomposition $\mathfrak{S}' = \bigsqcup_{i=1}^X \mathfrak{S}'_i$ into finitely many G -invariant families $\mathfrak{S}'_i$ such that any two domains in the same family are transverse. Such a decomposition is called a *coloring* of $\mathfrak{S}'$. We say a relative HHG $(G, \mathfrak{S})$ is *colorable* if $\mathfrak{S}$ is colorable.

The notion of colorability is formalized in [DMS23; HP22]. There are many classes of (virtually) colorable HHGs, as listed in the above papers. In particular, a coloring is constructed for (a finite-index subgroup of) a mapping class group in [BBF15, §5], from which the notion comes. However, one cannot expect that all HHGs are virtually colorable [Hag23]. Nevertheless, [HP22, Prop. 3.2] provides a sufficient condition for an HHG to be virtually colorable.

Remark 2.2.10. In this thesis, we only concern unbounded domains, i.e. domains with unbounded associated hyperbolic spaces. As an abuse of terminology, we say a relative HHG $(G, \mathfrak{S})$ is colorable if the collection of unbounded domains is colorable.

2.3 Quasimorphisms

Let G be a group. Recall from Section 1.4 that a function $\phi : G \rightarrow \mathbb{R}$ is called a *quasimorphism* if its *defect*

$$D(\phi) := \sup_{g, h \in G} |\phi(g) + \phi(h) - \phi(gh)|$$

is finite. A quasimorphism ϕ on G is

- *antisymmetric* if $\phi(g^{-1}) = -\phi(g)$ for any $g \in G$.
- *homogeneous* if $\phi(g^n) = n\phi(g)$ for any $n \in \mathbb{Z}$ and any $g \in G$.

Denote the set of all quasimorphisms on G by $\hat{Q}(G)$. We write $Q_{as}(G)$ (resp. $Q(G)$) to mean the set of all antisymmetric (resp. homogeneous) quasimorphisms on G .

Remark 2.3.1. It is well-known [Cal09, Lemma 2.21] that the homogenization of a quasimorphism ϕ defined by

$$\varphi(g) = \lim_{n \rightarrow \infty} \frac{\phi(g^n)}{n}$$

is a homogeneous quasimorphism, so that $\|\phi - \varphi\|_\infty$ bounded above by $D(\phi)$. Moreover, we have an estimate $D(\varphi) \leq 2D(\phi)$ (see [Cal09, Corollary 2.59]). A homogeneous quasimorphism φ is a class function: $\varphi(g) = \varphi(hgh^{-1})$ for any $h, g \in G$. This implies that $|\varphi([g, h])| \leq D(\varphi)$ for any $g, h \in G$.

The following notion is a natural generalization of quasimorphisms.

Definition 2.3.2. [FK16] Let G be a group and H be a group equipped with a proper left-invariant metric. Let $\phi : G \rightarrow H$ be a set-theoretic map. Define $\Delta(\phi) \subseteq H$, the *defect set* of ϕ , by

$$\Delta(\phi) = \{\phi(h)^{-1}\phi(g)^{-1}\phi(gh) : g, h \in G\}.$$

The map $\phi : G \rightarrow H$ is called a *quasi-homomorphism* if its defect set $\Delta(\phi)$ is finite.

Remark 2.3.3. One may also define the defect set of ϕ by $\bar{\Delta}(\phi) = \{\phi(g)\phi(h)\phi(gh)^{-1} : g, h \in G\}$. This change does not affect the definition of quasi-homomorphisms (see [Heu20, Proposition 2.3]).

A useful fact about quasi-homomorphisms is the following result, which can be proved by the same arguments in [Ish14, Lemma 3.1].

Lemma 2.3.4. *Let H be a finite-index normal subgroup of G . Then any quasi-homomorphism $\phi : H \rightarrow \mathbb{R}^n$ induces a quasi-homomorphism $\hat{\mathcal{T}}(\phi) : G \rightarrow \mathbb{R}^n$ whose restriction on $H \cap Z(G)$ is equal to ϕ .*

2.4 Central extensions and mapping class groups

Consider a short exact sequence of groups

$$1 \rightarrow Z \xrightarrow{\iota} E \xrightarrow{\pi} G \rightarrow 1. \tag{2.1}$$

A map $s : G \rightarrow E$, not necessarily a homomorphism, is called a (*set-theoretic*) *section* if $\pi \circ s = 1_G$. A section s is called *normalized* if $s(1) = 1$. A normalized section s is called a *quasi-splitting* if s is a quasi-homomorphism. Following [FK16], we say that (2.1) *quasi-splits* if it admits a quasi-splitting. For example, every section $s : G \rightarrow E$ is a quasi-splitting if G is finite. We also say that (2.1) *virtually quasi-splits* if there exists a finite-index subgroup $E' \leq E$ such that the short exact sequence

$$1 \rightarrow Z' \xrightarrow{\iota|_{Z'}} E' \xrightarrow{\pi|_{E'}} G' \rightarrow 1$$

quasi-splits, where $Z' = Z \cap E'$ and $G' = \pi(E')$. Following [Wan25], a map $\phi : E \rightarrow Z$ is called a *quasi-retraction*¹ if ϕ is a quasi-homomorphism and $\phi|_Z = 1_Z$.

When Z is a central subgroup of E , (2.1) is called a *central extension* of G . In what follows, we assume Z to be finitely generated. Let $s : G \rightarrow E$ be a section. For $g_1, g_2 \in G$, the element $s(g_1)s(g_2)s(g_1g_2)^{-1}$ lies in the kernel of π , hence in the image of ι . Up to identifying $\iota(Z)$ with Z we may thus define the 2-cochain $\omega_s \in C^2(G, Z)$ by

$$\omega_s(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}.$$

Let us recall the following well-known facts (see e.g. [Bro82, Chapter 4]):

- (1) The cochain ω_s is a cocycle;
- (2) If s' is another section of π , then $\omega_{s'}$ is cobordant to ω_s ; therefore, the class $[\omega_s] \in H^2(G, Z)$ does not depend on the choice of s , and is called the *Euler class* of the extension;
- (3) If ω' is any representative of the Euler class, there exists a section $s' : G \rightarrow E$ such that $\omega' = \omega_{s'}$;
- (4) The cocycle ω_s is normalized, in the sense that $\omega_s(1, g) = 1 = \omega_s(g, 1)$, if and only if s is normalized.

Let $\omega : G^2 \rightarrow Z$ be 2-cochain. We say that ω is

- *bounded* if the image $\omega(G, G)$ is bounded.
- *weakly bounded* if $\omega(g, G)$ and $\omega(G, g)$ are both bounded for all $g \in G$.

Correspondingly, a 2-coclass in $H^2(G, \mathbb{Z})$ is called *bounded* (resp. *weakly bounded*) if it is represented by a bounded (resp. weakly bounded) 2-cocycle. For more background on bounded and weakly bounded Euler classes, we refer to [FS23].

Central extensions arise naturally when studying mapping class groups. A *finite-type* surface is a closed surface with finitely many punctures. Let Σ be a finite-type surface, possibly with boundary $\partial\Sigma$. The *mapping class group* $\text{MCG}(\Sigma)$ is the group of homeomorphisms of Σ restricting to the identity on $\partial\Sigma$, modulo isotopy. If there are punctures $P = \{p_1, \dots, p_n\}$ on Σ , we define $\text{MCG}(\Sigma, P)$ to be the finite-index subgroup of $\text{MCG}(\Sigma)$ that fixes each puncture.

Let Σ be a finite-type surface. Let $V \subset \Sigma$ be a (possibly disconnected) closed subsurface. Let $V^\perp$ be the closure of $\Sigma - V$ in Σ . We say that V is *essential* if

1. No component of V or $V^\perp$ is a closed disk.
2. Annular components of V are pairwise non-homotopic, and the same holds for $V^\perp$.

In the following, let $V \subset \Sigma$ be an essential subsurface, and let $\hat{V}$ be the *capped* surface obtained by gluing a once-punctured disk to each boundary component of V . There is an induced homomorphism $\eta_V : \text{MCG}(V) \rightarrow \text{MCG}(\Sigma)$. If V is a closed annulus, then η_V is always injective. The following theorem determines the kernel of η_V when V is not a closed annulus.

¹This definition is different from that in [Alo94], where a quasi-retraction is defined metrically.

Theorem 2.4.1. *[FM12, Theorem 3.18] Assume that V is non-annular. Let $\alpha_1, \dots, \alpha_m$ denote the boundary components of V that bound once-punctured disks in $V^\perp$, and let $\{\beta_1^+, \beta_1^-\}, \dots, \{\beta_n^+, \beta_n^-\}$ denote the pairs of boundary components of V that bound annuli in $V^\perp$. Then $\ker(\eta_V)$ is the free abelian group generated by $T_{\alpha_1}, \dots, T_{\alpha_m}$ and $T_{\beta_1^+} T_{\beta_1^-}^{-1}, \dots, T_{\beta_n^+} T_{\beta_n^-}^{-1}$.*

By taking $\Sigma = \hat{V}$, we obtain the following capping sequence of mapping class groups.

Example 2.4.2. Let V be a finite-type surface with nonempty boundary components $\gamma_1, \dots, \gamma_n$. Let $\hat{V}$ be the capped surface obtained by gluing a once-punctured disk with puncture p_i to each γ_i , and let $P = \{p_1, \dots, p_n\}$. There is a central extension called the *capping sequence*

$$1 \rightarrow T_\Gamma \rightarrow \mathrm{MCG}(V) \xrightarrow{\pi} \mathrm{MCG}(\hat{V}, P) \rightarrow 1,$$

where T_Γ is the free abelian subgroup generated by Dehn twists T_{γ_i} around γ_i for $i = 1, \dots, n$. By [FK16, Proposition 6.4], the capping sequence has a bounded Euler class.

CHAPTER 3

Property (QT) of relatively hierarchically hyperbolic groups

Recall from Chapter 1.1 that a finitely generated group G has *property (QT)* if G equivariantly quasi-isometrically embeds in a finite product of quasi-trees $\prod_{i=1}^n T_i$ equipped with ℓ^1 -metric. In this chapter, we give a sufficient condition for relative HHGs to have property (QT). Our result unifies and generalizes those of [BBF21], [HP22], and [HNY25] and produces many new examples with property (QT). Furthermore, we provide a sufficient condition for the existence of a quasi-median (QT) embedding in the sense of [HP22]. This stronger property can be used to prove the existence of globally stable cylinders (see [PSZ25]), which connects to a long-standing question of Rips and Sela [RS95] about canonical representatives of elements in hyperbolic groups.

Definition 3.0.1. Let $(G, \mathfrak{S})$ be a relative HHG. For any $U \in \mathfrak{S}$, We write $G_U < \text{Aut}(\mathfrak{S}_U)$ to mean the image of $\text{Stab}_G(U)$ under the restriction homomorphism.

1. We say a domain $U \in \mathfrak{S}$ is of *Type I* if the following holds.

- (a) (Hyperbolicity) $\mathcal{CU}$ is hyperbolic.
- (b) (Acylindrical image) G_U acts on $\mathcal{CU}$ acylindrically.
- (c) (Cobounded nested region) G_U acts on $\mathbf{F}_U$ coboundedly.
- (d) (Separable quasi-axes) For any element $g \in \text{Stab}_G(U)$ that acts loxodromically on $\mathcal{CU}$, the elementary closure $EC(g)$ is separable in G .

2. We say a domain $U \in \mathfrak{S}$ is of *Type II* if the action $G_U \curvearrowright \mathcal{CU}$ has property (QT').

For any $U \in \mathfrak{S}$ of Type II, property (QT') provides quasi-trees T_U^i along with G_U -equivariant maps $\iota_U^i : \mathcal{CU} \rightarrow T_U^i$ for $i = 1, \dots, n_U$ such that

$$\prod_{i=1}^{n_U} \iota_U^i : \mathcal{CU} \rightarrow \prod_{i=1}^{n_U} T_U^i$$

is a quasi-isometric embedding.

Our main theorem is as follows.

Theorem 3.0.2. *Let $(G, \mathfrak{S})$ be a relative HHG that is virtually colorable. If every $U \in \mathfrak{S}$ is of Type I or Type II, then G has property (QT).*

Moreover, if for any $D \geq 1$, there exists $D' \geq 1$ such that for every $U \in \mathfrak{S}$ of Type II and for each $i = 1, \dots, n_U$, the map $\iota_U^i : \mathcal{CU} \rightarrow T_U^i$ sends (D, D) -quasi-geodesics to unparametrized (D', D') quasi-geodesics, then G is coarse median and the (QT) embedding of G is quasi-median.

Sketch of proof. We roughly explain how to prove Theorem 3.0.2 in the case that $(G, \mathfrak{S})$ is an HHG with only Type I domains excluding the “moreover” part. This case contains most of the key ideas.

First, we introduce a class of “thick” distances on G each of which is defined using a class of “thick” segments of hierarchy paths on G . We prove in Section 3.2 a thick distance formula saying that the word metric of G can be recovered by summing up these thick distances. This is an analogue of the distance formula for HHGs.

Then we show that any class of thick segments is cofinite up to the group action. Furthermore, these thick segments can be extended to a cofinite collection of quasi-axes. Using projections to these quasi-axes, we can estimate the thick distance. This step is done in Section 3.3

Finally, we take a finite-index subgroup of G , say H , such that a collection of quasi-axes as above is divided into finitely many H -orbits. Each H -orbit satisfies the Bestvina-Bromberg-Fujiwara projection axioms so it gives us a quasi-tree with an H -action. We prove that H equivariantly quasi-isometrically embeds in the product of these finitely many quasi-trees in Section 3.4. Since property (QT) is commensurably invariant, G has property (QT) . $\square$

As stated in the above proof, property (QT) is commensurably invariant [BBF21, §2.2]. It follows that the conclusion of Theorem 3.0.2 also holds for any group that is virtually a relative HHG that satisfies our condition, even though such a group may not be a relative HHG itself [PS23].

It is clear that residually finite hyperbolic groups and virtually colorable hierarchically hyperbolic groups whose associated hyperbolic spaces are all quasi-trees satisfy the assumption of Theorem 3.0.2. In fact, for most examples of HHGs that emerged from the study, every domain satisfies the first three conditions of Type I. Virtual colorability is also common in practice. Therefore, the biggest restriction of our theorem comes from the assumption of separable quasi-axes. We will further discuss it in Section 3.5 and give an easy-to-use criterion for having separable quasi-axes.

3.1 BBF Projection systems

In this section, we review the construction of a quasi-tree of spaces in [BBF15] with improvements from [BBFS19].

Let $\mathbf{Y}$ be a collection of geodesic metric spaces, and $\pi_Y(X) \subset Y$ be specified subsets whenever $X \neq Y$ are elements of $\mathbf{Y}$. Write $d_Y^\pi(X, Z)$ to mean $\text{diam}(\pi_Y(X) \cup \pi_Y(Z))$ for $X \neq Y \neq Z$. We say that $(\mathbf{Y}, \{\pi_Y\})$ is a *projection system* with *projection constant* $\xi \geq 0$ if it satisfies the following *projection axioms*.

(P0) $\text{diam}(\pi_Y(X)) \leq \xi$ when $X \neq Y$; (Bounded projection)

(P1) if X, Y, Z are distinct and $d_Y^\pi(X, Z) > \xi$, then $d_X^\pi(Y, Z) \leq \xi$; (Behrstock inequality)

(P2) for $X \neq Z$ the set

$$\{Y \in \mathbf{Y} \mid d_Y^\pi(X, Z) > \xi\}$$

is finite. (Finiteness)

Moreover, we say that $(\mathbf{Y}, \{\pi_Y\})$ is a *G-projection system* if a group G acts on the set $\mathbf{Y}$ in such a way that every $g \in G$ acts as an isometry from Y to gY and the projections π_Y are G -equivariant, that is, $\pi_{gY}(gX) = g\pi_Y(X)$.

If we replace (P1) with

(P1)' if X, Y, Z are distinct and $d_Y^\pi(X, Z) > \xi$, then $\pi_X(Y) = \pi_X(Z)$,

then we say that $(\mathbf{Y}, \{\pi_Y\})$ satisfies the *strong projection axioms*. While there are many natural situations where the projection axioms hold, the strong projection axioms are not as natural. However, we can modify the projections so that they do hold.

Theorem 3.1.1. [BBFS19, Theorem 4.1] *If $(\mathbf{Y}, \{\pi_Y\})$ is a projection system with constant ξ , then there are projections $\{\pi'_Y\}$ such that $(\mathbf{Y}, \{\pi'_Y\})$ satisfies the strong projections axioms with constant ξ' , where $\pi'_Y(X)$ and $\pi_Y(X)$ are apart from each other within a uniform Hausdorff distance ϵ , and ϵ and ξ' only depend on ξ . Moreover, if $(\mathbf{Y}, \{\pi_Y\})$ is a G -projection system, then $(\mathbf{Y}, \{\pi'_Y\})$ is still a G -projection system.*

Let $\mathcal{C}_K \mathbf{Y}$ denote the space obtained from the disjoint union

$$\bigsqcup_{Y \in \mathbf{Y}} Y$$

by joining points in $\pi_X(Z)$ with points in $\pi_Z(X)$ by an edge of length one whenever $d_Y(X, Z) < K$ for all $Y \in \mathbf{Y} - \{X, Z\}$. When the spaces are graphs and projections are subgraphs, we can join just the vertices in these projections. Moreover, if $\mathbf{Y}$ is a G -projection system, then G acts isometrically on $\mathcal{C}_K \mathbf{Y}$.

Theorem 3.1.2. [BBF15, §4] *If $(\mathbf{Y}, \{\pi_Y\})$ satisfies the strong projection axioms with constant ξ , then for all $K > 2\xi$*

- $\mathcal{C}_K \mathbf{Y}$ is hyperbolic if all $Y \in \mathbf{Y}$ are δ -hyperbolic;
- $\mathcal{C}_K \mathbf{Y}$ is a quasi-tree if all $Y \in \mathbf{Y}$ are quasi-trees with uniform QI constants.

There is a very useful distance formula in $\mathcal{C}_K \mathbf{Y}$. Let $X, Z \in \mathbf{Y}$ and $x \in X, z \in Z$. We define $\pi_Y(x) = \pi_Y(X)$ if $Y \neq X$ and define $\pi_X(x) = x$. Then define $d_Y(x, z) = \text{diam}(\pi_Y(x) \cup \pi_Y(z))$.

Theorem 3.1.3. [BBFS19, Thm. 6.3] *Let $(\mathbf{Y}, \{\pi_Y\})$ satisfy the strong projection axioms with constant ξ . Let $x \in X$ and $z \in Z$ be two points of $\mathcal{C}_K(\mathbf{Y})$ with $X, Z \in \mathbf{Y}$. Then*

$$\frac{1}{4} \sum_{Y \in \mathbf{Y}} \{d_Y(x, z)\}_K \leq d_{\mathcal{C}_K \mathbf{Y}}(x, z) \leq 2 \sum_{Y \in \mathbf{Y}} \{d_Y(x, z)\}_K + 3K$$

for all $K \geq 4\xi$.

Next we recall a theorem that allows us to pass projection axioms from a projection system to a collection of certain subspaces. Let $\mathbf{Y}$ be a collection of δ -hyperbolic spaces and $(\mathbf{Y}, \{\pi_Y\})$ be a projection system with constant ξ . For each $Y \in \mathbf{Y}$, let $\mathcal{A}_Y$ be a collection of quasi-geodesics in Y . Let $\mathcal{A}$ be the disjoint union of all $\mathcal{A}_Y$'s. We also make the following assumptions.

- As a collection of quasi-geodesics, $\mathcal{A}$ has uniform coarse constants.
- For $\alpha, \beta \in \mathcal{A}_Y$, we define $\pi_\alpha(\beta)$ to be the closest-point projection of β to α .
- For $\alpha \in \mathcal{A}_X$ and $\beta \in \mathcal{A}_Y$ where $X \neq Y$, we define $\pi_\alpha(\beta)$ to be the closest point projection of $\pi_X(Y)$ to α .

Theorem 3.1.4. [BBF21, Thm. 4.17] *For any $\theta > 0$, there exists $\xi' > 0$, depending only on θ, δ, ξ and coarse constants of $\mathcal{A}$, such that the following holds. If $\text{diam}(\pi_\alpha(\beta)) \leq \theta$ whenever α and β are distinct elements in the same $\mathcal{A}_Y$, then $(\mathcal{A}, \{\pi_\alpha\})$ is a projection system with constant ξ' .*

3.2 Thick distance formula

In this section, we will prove a *thick distance formula* that is similar to [BBF21, Theorem 4.13]. This allows us to estimate the distance in a relative HHS by counting only “thick” segments of a hierarchy path instead of the whole hierarchy path. The readers should be aware that the definitions in this section are different from those [BBF21]. In particular, we do not have tight geodesics in a general HHG.

Let $(\mathcal{X}, \mathfrak{S})$ be a relative HHS and fix $T > 100E + 10D_0$ (see Section 2.2 for constants associated with a relative HHS). As in [BHS19, §2B], we say a domain $U \in \mathfrak{S}$ is *T-relevant* for $x, y \in \mathcal{X}$ if $d_U(x, y) > T$. We write $\text{Rel}_T(x, y)$ for the set of *T-relevant* domains for x, y , and define $\text{Rel}_T(V; x, y) := \text{Rel}_T(x, y) \cap \mathfrak{S}_V^\circ$. We write $\text{Rel}_T^m(V; x, y)$ for the set of $\sqsubseteq$ -maximal elements in $\text{Rel}_T(V; x, y)$. When x and y are fixed, we often omit them from these notations.

Lemma 3.2.1. *Given $x, y \in \mathcal{X}$ and $U \in \text{Rel}_T(x, y)$, there exist at most two domains $V_1, V_2 \in \text{Rel}_T(x, y)$ such that $U \in \text{Rel}_T^m(V_i; x, y)$ for $i = 1, 2$.*

Proof. Suppose there exist three such domains V_1, V_2, V_3 . Since U is maximal in each $\text{Rel}_T(V_i; x, y)$, we know that V_i 's are not $\sqsubseteq$ -comparable. Moreover, any two of them cannot be orthogonal since $U \sqsubsetneq V_i$. Thus, V_i 's must be pairwise transverse.

By [BHS19, Proposition 2.8], any set of pairwise transverse elements in $\text{Rel}_T(x, y)$ has a total order $<$, obtained by setting $U < V$ whenever $d_U(y, \rho_U^V) \leq E$. We assume that $V_1 < V_2 < V_3$.

On the one hand, $d_{V_2}(\rho_{V_2}^{V_1}, \rho_{V_2}^{V_3}) \geq d_{V_2}(x, y) - 2E > T - 2E$ by the triangle inequality. On the other hand, $d_{V_2}(\rho_{V_2}^{V_1}, \rho_{V_2}^U) \leq \kappa_0$ and $d_{V_2}(\rho_{V_2}^{V_3}, \rho_{V_2}^U) \leq \kappa_0$ by consistency, which gives $d_{V_2}(\rho_{V_2}^{V_1}, \rho_{V_2}^{V_3}) \leq 2\kappa_0 < T - 2E$. This gives a contradiction. $\square$

Definition 3.2.2 (*T-thickness*). Given $\mathfrak{S}' \subset \mathfrak{S}$, we say a pair of points $(x, y) \in \mathcal{X} \times \mathcal{X}$ is *T-thick for $\mathfrak{S}'$* if $\text{diam}(\pi_U(x) \cup \pi_U(y)) \leq T$ for all $U \in \mathfrak{S}'$. We define $\mathcal{P}_T(\mathfrak{S}')$ to be the set of all *T-thick* pairs of points for $\mathfrak{S}'$. If $\mathfrak{S}' = \{U\}$, we also say (x, y) is *T-thick for U* and write $(x, y) \in \mathcal{P}_T(U)$.

Note that $(x, y) \in \mathcal{P}_T(U)$ if and only if $U \notin \text{Rel}_T(x, y)$. Also note that $(x, y) \in \mathcal{P}_T(\mathfrak{S}_V^\circ)$ if and only if $\text{Rel}_T(V; x, y) = \emptyset$.

Lemma 3.2.3. *Let D_0 be the constant provided by Theorem 2.2.4. For any $x, y \in \mathcal{X}$, let γ be a D_0 -hierarchy path between x, y . Given any $U \in \mathfrak{S}$ and any $x', y' \in \gamma$, then*

$$d_U(x', y') \leq d_U(x, y) + 2D_0.$$

In particular, if $(x, y) \in \mathcal{P}_T(U)$, then $(x', y') \in \mathcal{P}_{T+2D_0}(U)$.

Proof. By Remark 2.2.5, $\pi_U(\gamma)$ lies in the D_0 -neighborhood of a geodesic connecting $\pi_U(x)$ and $\pi_U(y)$. The conclusion then follows from the triangle inequality. $\square$

Notation. Throughout this chapter, let $\hat{T} = T + 2D_0$ and $\check{T} = T - 2D_0$ for every constant $T > 2D_0$.

Notation. Given two points x, y in a hyperbolic space, we write $[x, y]$ to mean a geodesic segment between x, y , which is coarsely unique. For an interval I or a path γ , we write I^-, I^+ or γ^-, γ^+ to mean their endpoints.

Definition 3.2.4 ((T, R) -thick distance). Fix sufficiently large constants T, R . Let γ be a D_0 -hierarchy path between x and y . Let $\gamma_1, \dots, \gamma_n \subset \gamma$ be disjoint subpaths occurring in this order such that $(\gamma_i^-, \gamma_i^+) \in \mathcal{P}_T(\mathfrak{S}_V^\circ)$ for each i .

The (T, R) -thick distance in V is denoted by $d_V^{T,R}(x, y)$ and is defined to be the supremum of $\sum_{i=1}^n \{\{d_V(\gamma_i^-, \gamma_i^+)\}\}_R$ over all such choices for γ_i 's, and for all D_0 -hierarchy paths from x to y .

It is always true that $d_V^{T,R}(x, y) \leq \{\{d_V(x, y)\}\}_R$. This becomes an equality if V is $\sqsubseteq$ -minimal. For the opposite direction, we have the following estimate.

Lemma 3.2.5. *Fix constants $T, R > 100E$. For any $x, y \in \mathcal{X}$ and $W \in \mathfrak{S}$, we have*

$$\{\{d_W(x, y)\}\}_R \leq d_W^{T,R}(x, y) + (6E + 2R)|\text{Rel}_T^m(W; x, y)|.$$

Proof. If $\text{Rel}_T^m(W) = \emptyset$, then $(x, y) \in \mathcal{P}_T(\mathfrak{S}_W^\circ)$. Thus, both sides of the above inequality are equal.

Now we assume that $\text{Rel}_T^m(W) \neq \emptyset$ and $d_W(x, y) \geq R$. In particular, CW is not $\sqsubseteq$ -minimal so it is hyperbolic. Let $\gamma : I \rightarrow \mathcal{X}$ be a D_0 -hierarchy path realizing $d_W^{T,R}(x, y)$, where I is an interval of $\mathbb{R}$. For any $V \sqsubset W$, we define

$$s_V^- := \inf\{s \in I \mid \exists U \sqsubseteq V \text{ such that } d_W(\gamma(s), \rho_W^U) \leq 2E\},$$

$$s_V^+ := \sup\{s \in I \mid \exists U \sqsubseteq V \text{ such that } d_W(\gamma(s), \rho_W^U) \leq 2E\}.$$

For any $U \sqsubseteq V \sqsubset W$, we know that $d_W(\rho_W^U, \rho_W^V) \leq \kappa_0$ by consistency. Thus,

$$d_W(\gamma(s_V^-), \rho_W^V) \leq 2E + \kappa_0 \leq 3E,$$

$$d_W(\gamma(s_V^+), \rho_W^V) \leq 2E + \kappa_0 \leq 3E.$$

Therefore,

$$d_W(\gamma(s_V^-), \gamma(s_V^+)) \leq d_W(\gamma(s_V^-), \rho_W^V) + d_W(\gamma(s_V^+), \rho_W^V) \leq 6E.$$

Let $J_0, \dots, J_n$ be the collection of maximal intervals in $I - \bigcup_{V \in \text{Rel}_T^m(W)} (s_V^-, s_V^+)$. Note that $n \leq |\text{Rel}_T^m(W)|$. Now we are going to prove that $(\gamma(J_i^-), \gamma(J_i^+)) \in \mathcal{P}_T(\mathfrak{S}_W^\circ)$.

On the one hand, $d_W(\rho_W^U, \gamma(J_i)) \geq 2E$ for any $U \in \text{Rel}_T(W)$ by definition of J_i . Then by Morse Lemma,

$$d_W(\rho_W^U, [\gamma(J_i^-), \gamma(J_i^+)]) \geq d_W(\rho_W^U, \gamma(J_i)) - E \geq E.$$

Therefore, $d_U(\gamma(J_i^-), \gamma(J_i^+)) \leq E < T$ by BGI. On the other hand, (x, y) is $\tilde{T}$ -thick for $\mathfrak{S}_W^\circ - \text{Rel}_T(W)$ by definition. It follows from Lemma 3.2.3 that $(\gamma(J_i^-), \gamma(J_i^+))$ is T -thick for $\mathfrak{S}_W^\circ - \text{Rel}_T(W)$. In summary, $(\gamma(J_i^-), \gamma(J_i^+)) \in \mathcal{P}_T(\mathfrak{S}_W^\circ)$.

Finally, we estimate that

$$\begin{aligned} d_W(x, y) &\leq \sum_{i=0}^n d_W(\gamma(J_i^+), \gamma(J_i^-)) + 6E|\text{Rel}_T^m(W)| \\ &\leq d_W^{T,R}(x, y) + R(n+1) + 6E|\text{Rel}_T^m(W)| \\ &\leq d_W^{T,R}(x, y) + (6E + 2R)|\text{Rel}_T^m(W)|. \end{aligned}$$

□

Let S denote the unique maximal domain in $\mathfrak{S}$. Recall that the level $\ell(S)$ of S is equal to the complexity of $(\mathcal{X}, \mathfrak{S})$.

Theorem 3.2.6. *Fix the constants T, R with $\tilde{T} \geq R > 100E$. Let $x, y \in \mathcal{X}$. Then, for each n*

$$\sum_{\ell(W) \leq n} \{d_W(x, y)\}_R \leq \sum_{\ell(W)=n} d_W^{T,R}(x, y) + 7 \sum_{\ell(W) < n} \{d_W(x, y)\}_R.$$

Note that each sum is over finitely many W since there are only finitely many W such that $d_W(x, y) \geq R$ for given x, y by the distance formula (Theorem 2.2.2).

Proof. If $\ell(W) = n$ then by Lemma 3.2.5,

$$\begin{aligned} \{d_W(x, y)\}_R &\leq d_W^{T,R}(x, y) + (6E + 2R)|\text{Rel}_T^m(W)| \\ &\leq d_W^{T,R}(x, y) + (6E + 2R) \sum_{V \in \text{Rel}_T^m(W)} \frac{\{d_V(x, y)\}_{\tilde{T}}}{\tilde{T}} \\ &\leq d_W^{T,R}(x, y) + 3 \sum_{V \in \text{Rel}_T^m(W)} \{d_V(x, y)\}_{\tilde{T}} \\ &\leq d_W^{T,R}(x, y) + 3 \sum_{V \in \text{Rel}_T^m(W)} \{d_V(x, y)\}_R. \end{aligned}$$

By Lemma 3.2.1, any V appears in at most two $\text{Rel}_T^m(W)$. Therefore, if we sum up the left

hand side over all W with $\ell(W) = n$, we have

$$\sum_{\ell(W)=n} \{d_W(x, y)\}_R \leq \sum_{\ell(W)=n} d_W^{T,R}(x, y) + 6 \sum_{\ell(W)<n} \{d_W(x, y)\}_R.$$

Adding $\sum_{\ell(W)<n} \{d_W(x, y)\}_R$ to both sides gives the desirable inequality. $\square$

Corollary 3.2.7. *Fix the constants T, R with $\check{T} \geq R > 100E$. Let $x, y \in \mathcal{X}$. Then*

$$\frac{1}{D_0} \sum_{W \in \mathfrak{S}} d_W^{T,R}(x, y) - D_0 \leq \sum_{W \in \mathfrak{S}} \{d_W(x, y)\}_R \leq 7^{\ell(S)-1} \sum_{W \in \mathfrak{S}} d_W^{T,R}(x, y).$$

Proof. The first inequality is trivial since $\frac{1}{D_0} d_W^{T,R}(x, y) - D_0 \leq \{d_W(x, y)\}_R$ for all W . By inductively applying Theorem 3.2.6, with base case $n = \ell(S)$, we have

$$\sum_{W \in \mathfrak{S}} \{d_W(x, y)\}_R \leq 7^{\ell(S)-n} \left(\sum_{n \leq \ell(W) \leq \ell(S)} d_W^{T,R}(x, y) + 7 \sum_{\ell(W) < n} \{d_W(x, y)\}_R \right).$$

When $n = 1$, the last term on the right is zero, and the result follows. $\square$

Combining the distance formula (Theorem 2.2.2) with Corollary 3.2.7, we obtain our thick distance formula.

Theorem 3.2.8 (Thick distance formula). *There exists R_0 such that for all T, R with $\check{T} \geq R > R_0$, there exists a constant $L > 0$ such that for all $x, y \in \mathcal{X}$,*

$$d_{\mathcal{X}}(x, y) \asymp_{(L,L)} \sum_{W \in \mathfrak{S}} d_W^{T,R}(x, y).$$

3.3 Estimation of thick distance via quasi-axes

Our proof in this section is inspired by [NY23a]. The main technique in the proof of [NY23a, Lemma 5.5] that is different from [BBF21] is the use of the Extension Lemma [Yan19, Lemma 2.14]. Lemma 3.3.1 below is a trimmed version for acylindrical actions on hyperbolic spaces. In the original statement of Lemma 3.3.1 in [NY23a], the group action is required to be cobounded, but it is easy to see from the proof that this condition can be removed. The readers could compare it with [BBF21, Theorem 4.19] for mapping class groups.

Lemma 3.3.1 (Extension Lemma). *[NY23a, Lem. 4.13] Let H be a group acting non-elementarily and acylindrically on a δ -hyperbolic space Y . Fix a base point $o \in Y$. There exists a set $F \subset H$ of three loxodromic elements and constants $\lambda \geq 1, c \geq 0$ with the following property.*

For any $h \in H$ there exists $f \in F$ such that hf is a loxodromic element and the bi-infinite path

$$\gamma_h = \bigcup_{i \in \mathbb{Z}} (hf)^i([o, ho][ho, hfo])$$

is a (λ, c) -quasi-geodesic.

Let $(\mathcal{X}, \mathfrak{S})$ be a relative HHS with a coarse constant E , and let G be a relative HHG by virtue of its action on $\mathcal{X}$. Corresponding to Definition 3.0.1, we say that a domain $U \in \mathfrak{S}$ has

1. *hyperbolicity* if $\mathcal{C}U$ is hyperbolic.
2. *acylindrical image* if G_U acts on $\mathcal{C}U$ acylindrically.
3. *cobounded nested region* if G_U acts on $\mathbf{F}_U$ coboundedly.
4. *separable quasi-axes* if for any element $g \in \text{Stab}_G(U)$ that acts loxodromically on $\mathcal{C}U$, the elementary closure $EC(g)$ is *separable* in G , i.e. $EC(g)$ equals the intersection of all finite-index subgroups of G that contain $EC(g)$.

Till the end of this section, let $V \in \mathfrak{S}$ be an unbounded domain that has hyperbolicity, cobounded nested region and acylindrical image.

Lemma 3.3.2 (Extension of thick segments). *There exist constants $\lambda \geq 1, c \geq 0, B \geq 0$ such that the following holds. For any $T, R > 0$, there exists a G_V -finite collection $\mathcal{A}_V = \mathcal{A}_V^{T,R}$ of (λ, c) -quasi-axes in $\mathcal{C}V$ such that for any pair of points $(x, y) \in \mathcal{P}_{\max\{\hat{T}, R\}}(\mathfrak{S}_V)$, there exists $\gamma \in \mathcal{A}_V$ such that $[\pi_V(x), \pi_V(y)] \subset \mathcal{N}_B(\gamma)$.*

Proof. Fix a base point $o \in \mathbf{F}_V$ and project it to a base point in $\mathcal{C}V$. If the action $G_V \curvearrowright \mathcal{C}V$ is elementary, $\mathcal{C}V$ itself is a quasi-axis of a loxodromic element, and thus satisfies the requirement. Now assume that the action $G_V \curvearrowright \mathcal{C}V$ is non-elementary. Lemma 3.3.1 provides a finite set $F \subset G_V$ and constants $\lambda \geq 1, c \geq 0$. Recall that $(\mathbf{F}_V, \mathfrak{S}_V)$ is a relative HHS. $\mathbf{F}_V$ is proper because $\mathcal{X}$ is proper.

Since G_V acts coboundedly on $\mathbf{F}_V$, there exists $\epsilon > 0$ such that $\mathbf{F}_V$ is covered by the G_V -translates of any ϵ -ball. Let $T' = \max\{\hat{T}, R\}$. By Theorem 2.2.2, there exists $r > 0$, depending only on E, T', ϵ , such that the distance between any pair of points in $\mathbf{F}_V$ that is $(T' + \epsilon)$ -thick for $\mathfrak{S}_V$ is bounded above by r . Fix any base point $o \in \mathbf{F}_V$. Since $\mathbf{F}_V$ is proper, there exists a finite subset $S \subset G_V$ such that $\mathcal{N}_{r+\epsilon}(o)$ is covered by $\bigcup_{s \in S} s \cdot \mathcal{N}_\epsilon(o)$.

Lemma 3.3.1 tells us that for each $s \in S$, there exists $f \in F$ such that sf is a loxodromic element acting on $\mathcal{C}V$. Let $\mathcal{A}_V = \mathcal{A}_V^{T,R}$ be the collection of G_V -translates of the (λ, c) -quasi-axes provided by Lemma 3.3.1 for all loxodromic elements of the form sf .

Now we verify that $\mathcal{A}_V$ meets our requirements. Let $(x, y) \in \mathcal{P}_{T'}(\mathfrak{S}_V)$. We can choose $g \in G_V$ such that $d_{\mathbf{F}_V}(x, go) < \epsilon$. Then $d_{\mathbf{F}_V}(o, g^{-1}y) \leq d_{\mathbf{F}_V}(x, y) + d_{\mathbf{F}_V}(x, go) < r + \epsilon$. By our choice of S , there exists $s \in S$ such that $d_{\mathbf{F}_V}(g^{-1}y, so) < \epsilon$. Thus, we have

$$d_V(x, go) < E\epsilon + E,$$

$$d_V(y, gso) < E\epsilon + E,$$

because π_V is E -coarsely Lipschitz. Since $\mathcal{C}V$ is E -hyperbolic, we can find $B > 0$ by fellow-traveller property such that $[\pi_V(x), \pi_V(y)] \subset \mathcal{N}_B([g \cdot \pi_V(o), gs \cdot \pi_V(o)])$. By construction, $[g \cdot \pi_V(o), gs \cdot \pi_V(o)]$ is contained in some $\gamma \in \mathcal{A}_V$ so we are done. $\square$

Notation. Assume that γ is a quasi-geodesic in a hyperbolic space Y . We write $\pi_\gamma : Y \rightarrow \gamma$ to mean the closest point projection. For $x, y \in Y$, we write $d_\gamma(x, y)$ to mean $\text{diam}(\pi_\gamma(x) \cup \pi_\gamma(y))$.

Notation. Assume that γ is a quasi-geodesic in $\mathcal{CV}$. We write $\pi_\gamma^\mathcal{X}$ to mean $\pi_\gamma \circ \pi_V$. For $x, y \in \mathcal{X}$, we write $d_\gamma^\mathcal{X}(x, y)$ to mean $\text{diam}(\pi_\gamma^\mathcal{X}(x) \cup \pi_\gamma^\mathcal{X}(y))$.

The following lemma is a well-known corollary of the Morse Lemma for quasi-geodesics in δ -hyperbolic spaces so we omit the proof.

Lemma 3.3.3. *Let γ and α be two (λ, c) -quasi-geodesics in a δ -hyperbolic space. Then for any $B > 0$, there exists a constant $C = C(\lambda, c, B, \delta) > 0$ such that*

$$d_\gamma(\alpha^-, \alpha^+) \geq \text{diam}(\alpha \cap \mathcal{N}_B(\gamma)) - C.$$

The main result of this section is the following estimation that generalizes [BBF21, Proposition 4.18].

Proposition 3.3.4. *For any $K > 0$, there exists $R > 0$ such that the following holds. Given any $T > 0$, let $\mathcal{A}_V = \mathcal{A}_V^{T,R}$ be the collection of (λ, c) -quasi-axes provided by Lemma 3.3.2. Then for any two points $x, y \in \mathcal{X}$,*

$$d_V^{T,R}(x, y) \leq 2(D_0 + 1) \sum_{\gamma \in \mathcal{A}_V} \{ \{ d_\gamma^\mathcal{X}(x, y) \} \}_K$$

where D_0 is the constant provided by Theorem 2.2.4.

Proof. Let $C = C(\lambda + D_0, c + D_0, B, E)$ be the constant provided by Lemma 3.3.3. Let $R > 2D_0(C+1)+K$. We will show that projections to quasi-axes $\mathcal{A}_V = \mathcal{A}_V^{T,R}$ bound the (T, R) -thick distance in V from above.

For any two points $x, y \in \mathcal{X}$, let β be a (D_0, D_0) -hierarchy path connecting x and y realizing $d_V^{T,R}(x, y)$. Let $\{\alpha_1, \dots, \alpha_n\}$ be the collection of disjoint subpaths of β with $d_V(\alpha_i^-, \alpha_i^+) \geq R$ and $(\alpha_i^-, \alpha_i^+) \in \mathcal{P}_T(\mathfrak{S}_V^\circ)$ such that

$$d_V^{T,R}(x, y) = \sum_{i=1}^n d_V(\alpha_i^-, \alpha_i^+).$$

By definition of gate maps, $\pi_V(z)$ is coarsely $\pi_V(\mathbf{g}_{\mathbf{F}_V}(z))$ for any $z \in \mathcal{X}$. By substituting with the image under the gate map $\mathbf{g}_{\mathbf{F}_V}$, we can assume that α_i is contained in $\mathbf{F}_V$ up to a bounded error of the thick distance in V . We divide each α_i into several consecutive subpaths $\{\tilde{\alpha}_{i,j} \mid 1 \leq j \leq m_i\}$ with $d_V(\tilde{\alpha}_{i,j}^-, \tilde{\alpha}_{i,j}^+) = R$ for $j = 1, \dots, m_i - 1$ and $d_V(\tilde{\alpha}_{i,m_i}^-, \tilde{\alpha}_{i,m_i}^+) \leq R$. By Lemma 3.2.3, we already know that $(\tilde{\alpha}_{i,j}^-, \tilde{\alpha}_{i,j}^+) \in \mathcal{P}_{\hat{T}}(\mathfrak{S}_V^\circ)$ for every pair (i, j) . Thus,

$$d_U(\tilde{\alpha}_{i,j}^-, \tilde{\alpha}_{i,j}^+) \leq \max\{\hat{T}, R\}$$

for all $U \in \mathfrak{S}_V$. By Lemma 3.3.2, there exists $\gamma_{i,j} \in \mathcal{A}_V$ such that $\pi_U(\tilde{\alpha}_{i,j}) \subset \mathcal{N}_B(\gamma_{i,j})$ (with an increased B by a uniform constant), which yields

$$\text{diam}(\pi_V(\alpha_i) \cap \mathcal{N}_B(\gamma_{i,j})) \geq R.$$

Let $\mathcal{A}'_V$ be the collection of all distinct $\gamma_{i,j}$'s. We see that

$$\pi_V(\alpha_i \setminus \tilde{\alpha}_{i,m_i}) \subset \bigcup_{\gamma \in \mathcal{A}'_V} \pi_V(\alpha_i) \cap \mathcal{N}_B(\gamma).$$

Thus, we have

$$\begin{aligned} d_V(\alpha_i^-, \alpha_i^+) &\leq \sum_{\gamma \in \mathcal{A}'_V} \text{diam}(\pi_V(\alpha_i) \cap \mathcal{N}_B(\gamma)) + R \\ &\leq 2 \sum_{\gamma \in \mathcal{A}'_V} \text{diam}(\pi_V(\alpha_i) \cap \mathcal{N}_B(\gamma)). \end{aligned}$$

Summing up from $i = 1$ to n yields that

$$d_V^{T,R}(x, y) \leq 2 \sum_{\gamma \in \mathcal{A}'_V} (D_0 \text{diam}(\pi_V(\beta) \cap \mathcal{N}_B(\gamma)) + D_0). \quad (3.1)$$

Note that $R > 2D_0(C + 1) + K$. Thus, Lemma 3.3.3 tells us that $d_\gamma^{\mathcal{X}}(x, y) \geq \text{diam}(\pi_V(\beta) \cap \mathcal{N}_B(\gamma)) - C \geq R - C > D_0(C + 1) + K$ for each $\gamma \in \mathcal{A}'_V$. We now estimate by Lemma 3.3.3 that

$$\begin{aligned} D_0 \text{diam}(\pi_V(\beta) \cap \mathcal{N}_B(\gamma)) + D_0 &\leq D_0(d_\gamma^{\mathcal{X}}(x, y) + C) + D_0 \\ &< (D_0 + 1)d_\gamma^{\mathcal{X}}(x, y). \end{aligned}$$

Combining it with Equation (3.1), we obtain that

$$\begin{aligned} d_V^{T,R}(x, y) &\leq 2(D_0 + 1) \sum_{\gamma \in \mathcal{A}'_V} d_\gamma^{\mathcal{X}}(x, y) \\ &= 2(D_0 + 1) \sum_{\gamma \in \mathcal{A}'_V} \{\!\!\{d_\gamma^{\mathcal{X}}(x, y)\}\!\!\}_K \\ &\leq 2(D_0 + 1) \sum_{\gamma \in \mathcal{A}_V} \{\!\!\{d_\gamma^{\mathcal{X}}(x, y)\}\!\!\}_K. \end{aligned}$$

□

3.4 Construction of quasi-trees

This section devotes to the proof of Theorem 3.0.2. Let $(G, \mathfrak{S})$ be a relative HHG that is virtually colorable and assume that every domain in $\mathfrak{S}$ is of Type I or Type II. The index set $\mathfrak{S}$ admits a G -invariant decomposition $\mathfrak{S} = \mathfrak{S}^I \sqcup \mathfrak{S}^{II}$, where $\mathfrak{S}^I$ (respectively, $\mathfrak{S}^{II}$) only contains domains of Type I (respectively, Type II). Note that Type I and Type II are not mutually exclusive, but for those domains of both Type I and Type II, we can simply put them in $\mathfrak{S}^{II}$.

Before starting the proof, we summarize the dependency of some important constants that will be used in the proof as follows.

$$(E, D_0, A) \xrightarrow{\text{Corollary 3.4.3}} \theta \xrightarrow{\text{Theorem 3.1.4}} \xi \xrightarrow{\text{Lemma 3.4.4}} K \xrightarrow{\text{Proposition 3.3.4}} R \xrightarrow{\text{Theorem 3.2.8}} T$$

Here A stands for the acylindrical constants. We draw an arrow from a constant M to N if N depends on M . Remember that the dependency shown above is incomplete, but we hope it is helpful to the reader.

3.4.1 Quasi-trees from domains of Type I

In this subsection, we are going to prove the following proposition.

Proposition 3.4.1. *There exists a finite-index subgroup $H < G$ satisfying the following. For any sufficiently large constant R and any $T > 0$, there exist quasi-trees $\mathcal{T}_1, \dots, \mathcal{T}_n$ such that H acts on $\prod_{j=1}^n \mathcal{T}_j$ diagonally and for any choice of base points $o_j \in \mathcal{T}_j$*

$$\sum_{V \in \mathfrak{S}^I} d_V^{T,R}(1, h) \leq \sum_{j=1}^n d_{\mathcal{T}_j}(o_j, ho_j)$$

for any $h \in H$.

First, we recall the following lemma.

Lemma 3.4.2. [BBF21, Prop. 3.4] *Let a group H act on a δ -hyperbolic space Y . Assume that the image of H in $\text{Isom}(Y)$ is acylindrical. Consider a loxodromic element $g \in H$ and the collection $\mathbb{A}$ of all H -translates of a fixed (λ, c) -quasi-axis of g . Then there exists a constant $\theta > 0$ depending only on λ, c, δ and the acylindrical constants such that for any $\gamma \in \mathbb{A}$, the set*

$$\{h \in H \mid \text{diam}(\pi_\gamma(h\gamma)) \geq \theta\}$$

is a finite union of double $EC(g)$ -cosets.

Corollary 3.4.3. *Let $U \in \mathfrak{S}^I$. Consider a (λ, c) -quasi-axis $\gamma \subset \mathcal{CU}$ for some loxodromic element of the acylindrical action of G_U . Then there exists $\theta > 0$, only depending on λ, c, E and the acylindrical constants, and a finite-index subgroup $G_\gamma < G$ such that every translate of γ by an element of $G_\gamma \cap \text{Stab}_G(U)$ either has finite Hausdorff distance with γ or has θ -bounded projection to γ .*

Proof. It is clear by Lemma 3.4.2 and separability of quasi-axes. $\square$

By definition of relative HHGs, $\mathfrak{S}^I$ consists of finitely many G -orbits, so acylindrical constants for $U \in \mathfrak{S}^I$ can be chosen uniformly. Thus, Lemma 3.3.2 provides uniform constants $\lambda \geq 1, c \geq 0$ for every $U \in \mathfrak{S}^I$. This further gives a uniform constant $\theta > 0$ by Corollary 3.4.3.

Let $\mathcal{U}$ be a G -representative set of $\mathfrak{S}^I$ such that $1 \in \mathbf{P}_U$ for any $U \in \mathcal{U}$. Let $U \in \mathcal{U}$. Let $T > 0$ and let $K > 0$ be a sufficiently large constant that will be decided by Lemma 3.4.4. Lemma 3.3.2 provides a G_U -finite collection $\mathcal{A}_U = \mathcal{A}_U^{T,R}$ of (λ, c) -quasi-axes, where R is provided by Proposition 3.3.4. By Corollary 3.4.3, we can find a finite-index subgroup $H_U < G$ such that for any $\gamma \in \mathcal{A}_U$ and $h \in H_U \cap \text{Stab}_G(U)$, either $d_{\text{Haus}}(h\gamma, \gamma) < \infty$ or $\text{diam}\pi_\gamma(h\gamma) < \theta$. For any $g \in G$, we define $\mathcal{A}_{gU} := \{g\gamma \mid \gamma \in \mathcal{A}_U\}$.

Let $\mathcal{A} := \bigsqcup_{U \in \mathfrak{S}} \mathcal{A}_U$ and $H := \bigcap_{U \in \mathcal{U}} H_U$. Since $\mathcal{U}$ is finite, H is of finite index in G . By adding finitely many domains to $\mathcal{U}$ so that there is one representative for each H -orbit on $\mathfrak{S}^I$,

we obtain an H -representative set $\tilde{\mathcal{U}}$ of $\mathfrak{S}^I$. We still assume that $1 \in \mathbf{P}_U$ for any $U \in \tilde{\mathcal{U}}$. Let $\{\gamma_1, \dots, \gamma_n\}$ be an H -representative set of $\mathcal{A}$. We assume that every representative γ_j is contained in $\mathcal{C}U$ for some $U \in \tilde{\mathcal{U}}$. Let $\mathcal{A}_j \subset \mathcal{A}$ be the H -orbit of γ_j .

Without loss of generality, we assume that H is colorable instead of virtually colorable. Thus, the H -orbit of any domain is pairwise transverse. By [HP22, Lemma 3.4], every H -orbit of $\mathfrak{S}^I$ is an H -projection system with constant $s_0 + 4E$, where s_0 is the constant provided by Theorem 2.2.2. Thus, every $\mathcal{A}_j$ is an H -projection system with a uniform projection constant $\xi = \xi(\theta, s_0, \lambda, c, E)$ by Theorem 3.1.4. The projections defined there will be denoted by Π_γ .

Using Theorem 3.1.1, we obtain modified projections Π'_γ such that $(\mathcal{A}_j, \{\Pi'_\gamma\})$ satisfies the strong projection axioms with constant $\xi' = \xi'(\xi)$ and that $\Pi_\gamma(\alpha)$ and $\Pi'_\gamma(\alpha)$ are apart from each other within a uniform Hausdorff distance $\epsilon = \epsilon(\xi)$. For any $K' \geq 4\xi'$, $\mathcal{C}_{K'}\mathcal{A}_j$ is a quasi-tree by Theorem 3.1.2. The following lemma is an estimate via the orbit map between the projections $\pi_\gamma^H = \pi_\gamma \circ \pi_V$ in the relative HHG structure and the projections Π'_γ in the quasi-tree $\mathcal{C}_{K'}\mathcal{A}_j$. Recall from Section 3.3 that $d_\gamma^H(g, h)$ is defined as $\text{diam}(\pi_\gamma^H(g) \cup \pi_\gamma^H(h))$ for any $g, h \in H$.

Lemma 3.4.4. *Fix a base point $o_j \in \gamma_j$ for each $j = 1, \dots, n$. There exists a sufficiently large constant $K' = K'(\xi, \lambda, c, E)$ and a constant $\Delta > 0$ such that if $K \geq 2K'$, then*

$$\sum_{\gamma \in \mathcal{A}_j} \{\{d_\gamma^H(1, h)\}\}_K \leq 8d_{\mathcal{C}_{K'}\mathcal{A}_j}(o_j, ho_j) + \Delta$$

for any $h \in H$ and any $j = 1, \dots, n$.

For the proof of Lemma 3.4.4, we need the following lemma.

Lemma 3.4.5. *For any constants $A, B \geq 0$ and constants $L, M > 0$,*

$$\frac{\{A + B\}_{L+M}}{L + M} \leq \frac{\{A\}_L}{L} + \frac{\{B\}_M}{M}.$$

Proof. Assume that $A + B \geq L + M$. First, if $A < L$ then $B > M$. Thus, $\{B\}_M = B \geq \frac{M}{L+M}(A + B)$. Second, if $B < M$ then $A > L$ and the same argument holds. Finally, if $A \geq L$ and $B \geq M$, then all thresholds are reached and the inequality is obviously true. $\square$

Proof of Lemma 3.4.4. For simplicity, we use $|p - q|$ to mean the distance between two points p, q in the same space. Assume that $\gamma_j \subset \mathcal{C}U$ for some $U \in \tilde{\mathcal{U}}$. For any $g \in H - \text{Stab}_H(U)$, since $1 \in \mathbf{P}_U$ and $U \triangleleft gU$, we have $|\pi_{gU}(1) - \rho_{gU}^U| \leq E$ by definition of standard product regions.

Assume $\gamma = g\gamma_j$. By hyperbolicity, there exists a constant $F = F(E, \lambda, c)$ such that if $\gamma \subset \mathcal{C}gU \neq \mathcal{C}U$ then $|\pi_\gamma^H(1) - \Pi'_\gamma(o_j)| \leq |\pi_{gU}(1) - \rho_{gU}^U| + F + \epsilon \leq E + F + \epsilon$. Let $M > \xi' + E + F + \epsilon$ and define $\delta_\gamma(h) = |\pi_\gamma^H(h) - \Pi'_\gamma(ho_j)|$. We see that if $\delta_\gamma(h) = \delta_{h^{-1}\gamma}(1) \geq M$ then $\gamma \subset \mathcal{C}hU$. Thus, for a fixed $h \in H$, there are only finitely many $\gamma \in \mathcal{A}_j$ such that $\delta_\gamma(h) \geq M$ by projection axiom (P2).

Let $K' > 2M + 4\xi'$. Define $D_\gamma(1, h) = |\Pi'_\gamma(o_j) - \Pi'_\gamma(ho_j)|$. By the triangle inequality and Lemma 3.4.5, we obtain that

$$\{\{d_\gamma^H(1, h)\}\}_{K'+2M} \leq \frac{K' + 2M}{K'} \{D_\gamma(1, h)\}_{K'} + \frac{K' + 2M}{2M} \{\delta_\gamma(1) + \delta_\gamma(h)\}_{2M}.$$

Therefore,

$$\{\{d_\gamma^H(1, h)\}\}_K \leq 2 \{\{D_\gamma(1, h)\}\}_{K'} + \frac{K' + 2M}{M} \{\{\delta_\gamma(1)\}\}_M.$$

Summing over $\gamma \in \mathcal{A}_j$, we obtain that

$$\sum_{\gamma \in \mathcal{A}_j} \{\{d_\gamma^H(1, h)\}\}_K \leq 2 \sum_{\gamma \in \mathcal{A}_j} \{\{D_\gamma(1, h)\}\}_{K'} + \frac{K' + 2M}{M} \sum_{\gamma \in \mathcal{A}_j} \{\{\delta_\gamma(1)\}\}_M.$$

By discussions above, $\Delta_j = \frac{K' + 2M}{M} \sum_{\gamma \in \mathcal{A}_j} \{\{\delta_\gamma(1)\}\}_M$ is a finite constant. Let $\Delta = \max_{1 \leq j \leq n} \Delta_j$. We conclude by Theorem 3.1.3. □

Now we finish the proof of Proposition 3.4.1.

Proof of Proposition 3.4.1. Let K and K' be the constant provided by Lemma 3.4.4. Let $T > 0$. From discussions before Lemma 3.4.4, we know that the collection $\mathcal{A}$ of quasi-axes provided by Proposition 3.3.4 forms n quasi-trees $\mathcal{C}_{K'}\mathcal{A}_j$, $j = 1, \dots, n$. Moreover, Proposition 3.3.4 tells us that there exists $R > 0$ such that

$$\sum_{V \in \mathfrak{S}^I} d_V^{T, R}(1, h) \leq \sum_{\gamma \in \mathcal{A}} \{\{d_\gamma^H(1, h)\}\}_K.$$

Finally, we conclude by Lemma 3.4.4. □

3.4.2 Quasi-trees from domains of Type II

In this subsection, we prove an analogue of Proposition 3.4.1 for domains of Type II.

Let $H < G$ be the subgroup provided by Proposition 3.4.1. Let $\mathcal{V}$ be an H -representative set of $\mathfrak{S}^H$ such that $1 \in \mathbf{P}_V$ for any $V \in \mathcal{V}$. For $V \in \mathcal{V}$ we write $[V]$ to mean its H -orbit. By colorability and [HP22, Lemma 3.4], every $[V]$ is an H -projection system with constant $s_0 + 4E$.

Fix any $V \in \mathcal{V}$. By property (QT') of the action $\text{Stab}_G(V) \curvearrowright \mathcal{C}V$, there exist quasi-trees T_V^i along with $\text{Stab}_H(V)$ -equivariant maps $\iota_V^i : \mathcal{C}V \rightarrow T_V^i$ for $i = 1, \dots, n_V$ such that

$$\prod_{i=1}^{n_V} \iota_V^i : \mathcal{C}V \rightarrow \prod_{i=1}^{n_V} T_V^i$$

is a (λ', c') -quasi-isometric embedding. In particular, ι_V^i is (λ', c') -coarsely Lipschitz. Fix $i \in \{1, \dots, n_V\}$. It is conventional to extend the map ι_V^i on $[V]$ in an H -equivariant way. It means that we can construct a collection of quasi-trees $\mathbf{T}_{[V]}^i = \{T_U^i \mid U \in [V]\}$ with an H -action and a collection of coarsely Lipschitz maps $\{\iota_U^i : \mathcal{C}U \rightarrow T_U^i \mid U \in [V]\}$ such that the following diagram commutes for any $h \in H$ and $U \in [V]$.

$$\begin{array}{ccc} \mathcal{C}U & \xrightarrow{h} & ChU \\ \iota_U^i \downarrow & & \downarrow \iota_{hU}^i \\ T_U^i & \xrightarrow{h} & T_{hU}^i \end{array}$$

Define $\Pi_{T_{hU}^i}(T_U^i) := \iota_{hU}^i(\rho_{hU}^U)$ for any $U \in [V]$ and $hU \neq U$. Clearly, these projections are H -equivariant and the projection axioms pass to the collection $(\mathbf{T}_{[V]}^i, \{\Pi_{T_U^i}\})$ under coarsely Lipschitz maps $\{\iota_U^i\}$. We modify the projections within an error ϵ such that $(\mathbf{T}_{[V]}^i, \{\Pi'_{T_U^i}\})$ satisfies the strong projection axioms with constant $\zeta = \zeta(s_0, \lambda', c', E)$. For any $K'' \geq 4\zeta$, $\mathcal{C}_{K''}\mathbf{T}_{[V]}^i$ is a quasi-tree by Theorem 3.1.2. Define $d_{T_U^i}(1, h) := |\iota_U^i(\pi_U(1)) - \iota_U^i(\pi_U(h))|$ for any $U \in [V]$. For any $V \in \mathcal{V}$ and $i \in \{1, \dots, n_V\}$, fix a base point $o_V^i \in T_V^i$. The following proposition is an analogue of Proposition 3.4.1.

Proposition 3.4.6. *There exists a constant $K'' = K''(\zeta, \lambda', c', E)$ such that if $R \geq 2K''$ then*

$$\sum_{U \in \mathfrak{S}^{II}} \{d_U(1, h)\}_R \leq \sum_{V \in \mathcal{V}} \sum_{i=1}^{n_V} d_{\mathcal{C}_{K''}\mathbf{T}_{[V]}^i}(o_V^i, ho_V^i).$$

for any $h \in H$.

Proof. Fix $V \in \mathcal{V}$ and $i \in \{1, \dots, n_V\}$. For any $U \in [V] - \{V\}$, we have $|\iota_U^i(\pi_U(1)) - \Pi'_{T_U^i}(o_V^i)| \leq \lambda'|\pi_U(1) - \rho_U^V| + c' + \epsilon \leq \lambda'E + c' + \epsilon$. Let $K'' > 4\zeta + 2(\lambda'E + c' + \epsilon)$. Similarly to the proof of Lemma 3.4.4, we estimate that

$$\begin{aligned} \sum_{U \in [V]} \{d_{T_U^i}(1, h)\}_R &\leq \left\{ \left| \Pi'_{T_U^i}(o_i) - \Pi'_{T_U^i}(ho_i) \right| \right\}_{K''} \\ &\leq d_{\mathcal{C}_{K''}\mathbf{T}_{[V]}^i}(o_V^i, ho_V^i). \end{aligned}$$

Here the first inequality follows from the triangle inequality and Lemma 3.4.5, and the second inequality is by Theorem 3.1.3. Since the map $\prod_{i=1}^{n_U} \iota_U^i : \mathcal{C}U \rightarrow \prod_{i=1}^{n_U} T_U^i$ is a quasi-isometric embedding for any $U \in \mathfrak{S}^{II}$, we conclude by summing the inequality over $1 \leq i \leq n_V$ for all $V \in \mathcal{V}$. $\square$

3.4.3 Proof of Theorem 3.0.2

Proof. Let $R > 0$ be sufficiently large that satisfies Proposition 3.4.1, Proposition 3.4.6 and Theorem 3.2.8. Let $T \geq R + 2D_0$. By Proposition 3.4.1 and Proposition 3.4.6, there exists quasi-trees $\mathcal{T}_1, \dots, \mathcal{T}_m$ such that H acts on $\prod_{k=1}^m \mathcal{T}_k$ diagonally and for any choice of base points $o_k \in \mathcal{T}_k$ and any $h \in H$,

$$\sum_{V \in \mathfrak{S}^I} d_V^{T,R}(1, h) + \sum_{U \in \mathfrak{S}^{II}} \{d_U(1, h)\}_R \leq \sum_{k=1}^m d_{\mathcal{T}_k}(o_k, ho_k).$$

By definition of thick distance, $d_V^{T,R}(1, h) \leq \{d_U(1, h)\}_R$. Thus, by Theorem 3.2.8,

$$\sum_{V \in \mathfrak{S}} d_V^{T,R}(1, h) \leq \sum_{k=1}^m d_{\mathcal{T}_k}(o_k, ho_k)$$

for any $h \in H$.

On the other hand, the orbit map from H to $\prod_{k=1}^m \mathcal{T}_k$ is coarsely Lipschitz since H is finitely generated. Therefore, H embeds quasi-isometrically into $\prod_{k=1}^m \mathcal{T}_k$, which means that H has

property (QT'). Finally, we know that G has property (QT) since property (QT) passes to any finite-index supergroup.

For the “moreover” part, first note that G is coarse median for the same reason as [BHS19, Theorem 7.3]. The rest of the proof is just a combination of [HP22; Pet21]. The proof of [HP22, Proposition 3.9] can be naturally generalized to quasi-trees from domains of Type II. The proof in [Pet21, §3] for mapping class groups can be repeated verbatim to deal with quasi-trees from domains of Type I. $\square$

For relative HHGs with only Type II domains, we obtain the following stronger theorem.

Theorem 3.4.7. *Let $(G, \mathfrak{S})$ be a relative HHG that is colorable. If every $U \in \mathfrak{S}$ is of Type II, then G has property (QT').*

Proof. The proof is just a simplified version of that of Theorem 3.0.2. Note that G is colorable and that every domain is of Type II, so the finite-index subgroup H in the above proof can be replaced with G itself. $\square$

3.5 A criterion for having separable quasi-axes

In this section, we provide a criterion for a relative HHG to have separable quasi-axes that is easy to use in application. For an acylindrical action on a hyperbolic space, we have seen in Section 2.1 that the elementary closure of any loxodromic element is a *virtual centralizer*. Inspired by the discussion in [BBF21, §4.3], the following lemma generalizes this fact. In general, we cannot expect the elementary closure to be a centralizer as in [BBF21, §4.3], because it may contain a flip.

Lemma 3.5.1. *Let G be a group acting on a δ -hyperbolic space X with an acylindrical image. Let K be the kernel of this action. Assume that there is a subgroup $H < G$ such that $[H, K] = 1$ and H is mapped to a finite-index subgroup of G/K . Then for any loxodromic element $g \in G$, the elementary closure $EC_G(g)$ is a virtual centralizer in G of some loxodromic element in H .*

Proof. Let $\bar{G} = G/K$, and let $\theta : G \rightarrow \bar{G}$ be the quotient map. For any $g \in G$, denote the image $\theta(g) \in \bar{G}$ by $\bar{g}$.

Let $g \in G$ be a loxodromic element. Since $\theta(H)$ is a finite-index subgroup of $\bar{G}$, we can choose $h \in H$ and $n > 0$ such that $\bar{h} = \bar{g}^n$. Since $\bar{G}$ acts acylindrically, $EC_{\bar{G}}(\bar{g}) = EC_{\bar{G}}(\bar{h})$, which contains the cyclic subgroup $\langle \bar{h} \rangle$ as a finite-index subgroup. By definition, $EC_G(g) = \theta^{-1}(EC_{\bar{G}}(\bar{g}))$. Thus, the preimage $\theta^{-1}(\langle \bar{h} \rangle) = K \cdot \langle h \rangle$ is a finite-index subgroup of $EC_G(g)$.

On the other hand, any element in G that commutes with h has an image in $\bar{G}$ that commutes with $\bar{h}$. Thus, $C_G(h) < EC_G(g)$. Note that $K \cdot \langle h \rangle < C_G(h)$ since $[H, K] = 1$. In conclusion, $C_G(h)$ is a finite-index subgroup of $EC_G(g)$. $\square$

Definition 3.5.2. For a relative HHG $(G, \mathfrak{S})$, we say a domain $V \in \mathfrak{S}$ has *neat kernel* if there exists a subgroup $H_V < \text{Stab}_G(V)$ such that $[H_V, \ker(\theta_V)] = 1$ and θ_V maps H_V to a finite-index subgroup of G_V .

Proposition 3.5.3. *Let $(G, \mathfrak{S})$ be a relative HHG that is residually finite. Let $V \in \mathfrak{S}$. If V has hyperbolicity, acylindrical image and neat kernel, then V has separable quasi-axes.*

Proof. Let $g \in \text{Stab}_G(V)$ be a loxodromic element. Since V has neat kernel, $EC(g)$ is a virtual centralizer in $\text{Stab}_G(V)$ of some loxodromic element $h \in H_V$ by Lemma 3.5.1. Every element that commutes with h stabilizes V . Therefore, $EC(g)$ is a virtual centralizer of h in G . It is known that a centralizer in a residually finite group is separable (see [BBF21, Lemma 2.1] or the proof of [Lon87, Proposition]). It is also known that a finite-index supergroup of a separable subgroup is still separable (easy to see from the profinite topology). Therefore, V has separable quasi-axes. $\square$

Corollary 3.5.4. *Let $(G, \mathfrak{S})$ be a relative HHG that is residually finite. Let $S \in \mathfrak{S}$ be the unique maximal domain. Then S is of Type I.*

Proof. By [BHS17b, Theorem 14.3], G acts on $\mathcal{CS}$ acylindrically. Now $\ker(\theta_V)$ is trivial so S has neat kernel. Thus, S has separable quasi-axes by Proposition 3.5.3. Moreover, S has cobounded nested region because G acts on $\mathcal{X}$ coboundedly. In conclusion, S is of Type I. $\square$

3.6 Applications

3.6.1 Mapping class groups

In this subsection, we explain how Theorem 3.0.2 applies to mapping class groups to recover the following theorem. For definitions about mapping class groups, see the end of Section 2.4.

Theorem 3.6.1. *[BBF21, Theorem 1.2] Mapping class groups of finite-type surfaces have property (QT).*

Proof. Let Σ be a finite-type surface, and $\mathcal{M}(\Sigma)$ be its marking complex [MM00]. Let $\mathfrak{S}$ be the collection of isotopy classes of essential non-pants subsurfaces of Σ , where disconnected subsurfaces are also allowed. Given any $V \in \mathfrak{S}$, let $\mathcal{CV}$ be the curve graph of the capped surface $\hat{V}$. Here the curve graph of a disconnected surface is defined as the join of the curve graphs of its components, and thus is bounded. It is known that $(\mathcal{M}(\Sigma), \mathfrak{S})$ is an HHS. Moreover, the mapping class group $\text{MCG}(\Sigma)$ is an HHG by virtue of its action on $(\mathcal{M}(\Sigma), \mathfrak{S})$ [BHS19, §11].

For any $V \in \mathfrak{S}^\circ$, let $\text{MCG}(\Sigma; V) < \text{MCG}(\Sigma)$ be the stabilizer of V . Let $\eta_V : \text{MCG}(V) \rightarrow \text{MCG}(\Sigma)$ be the induced homomorphism by the inclusion $V \hookrightarrow \Sigma$. Denote the image of η_V by $\overline{\text{MCG}}(V)$. By Theorem 2.4.1, $\overline{\text{MCG}}(V^\perp)$ is exactly the kernel of the restriction homomorphism $\theta_V : \text{MCG}(\Sigma; V) \rightarrow G_V$, where G_V is a finite-index subgroup of $\text{MCG}(\hat{V})$. Moreover, it is clear that $\overline{\text{MCG}}(V)$ commutes with $\overline{\text{MCG}}(V^\perp)$ and θ_V maps $\overline{\text{MCG}}(V)$ to a finite-index subgroup of G_V . Therefore, V has neat kernel.

It is known that $\text{MCG}(\Sigma)$ is virtually colorable [BBF15, §5] and residually finite [Gro74]. We only need to prove that every unbounded domain $V \in \mathfrak{S}$ is of Type I. First, it is clear that $\mathbf{F}_V$ is coarsely $\mathcal{M}(V)$. Since $\text{MCG}(\hat{V})$ acts coboundedly on $\mathcal{M}(V)$ and acts acylindrically on $\mathcal{CV}$ [Bow08], V has cobounded nested region and acylindrical image. Furthermore, V has separable quasi-axes by the above discussion and Proposition 3.5.3. Therefore, V is of Type I.

In conclusion, mapping class groups of finite-type surfaces have property (QT) by Theorem 3.0.2. $\square$

3.6.2 Admissible graphs of groups

Admissible groups are introduced by Croke-Kleiner in [CK02], which generalize the fundamental groups of non-geometric 3-dimensional graph manifolds.

Definition 3.6.2. Let $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups. We say $\mathcal{G}$ is *admissible* if the following hold:

1. Γ is a finite graph with at least one edge.
2. Each vertex group G_v has center $Z_v \cong \mathbb{Z}$, and $H_v := G_v/Z_v$ is a non-elementary hyperbolic group.
3. Every edge group G_e is isomorphic to $\mathbb{Z}^2$.
4. If e is an edge with $v = e^+$ and $w = e^-$, and $\tau_e, \tau_{\bar{e}}$ are the edge monomorphisms, then the subgroup $\langle \tau_e^{-1}(Z_v), \tau_{\bar{e}}^{-1}(Z_w) \rangle$ has finite index in G_e .
5. Let e_1 and e_2 be distinct edges entering a vertex v , and let $K_i \subset G_v$ be the image of the edge homomorphism τ_{e_i} for $i = 1, 2$. Then
 - for every $g \in G_v$, gK_1g^{-1} is not commensurable with K_2 ;
 - for every $g \in G_v - K_i$, $gK_i g^{-1}$ is not commensurable with K_i .

A group G is *admissible* if it is the fundamental group of an admissible graph of groups.

Every admissible group G has a (combinatorial) HHG structure by [HRSS24, Theorem 1.4]. According to the classification of simplices by [HRSS24, Lemma 6.2], if $\Delta \perp g\Delta$, where Δ corresponds to an unbounded hyperbolic space, then Δ is of Type 8 and g exchanges two adjacent vertices in the Bass-Serre tree (see [BHMS24, Definition 1.11] for definition of orthogonality in a combinatorial HHS). Therefore, it is easy to see that G has a subgroup of index at most 2 that is colorable (see [NY23a, Lemma 4.6] for example). This shows the virtual colorability of G . In particular, every non-geometric graph manifold group has a virtually colorable HHG structure with all associated hyperbolic spaces being quasi-trees. Thus, non-geometric graph manifold groups have property (QT) by [HP22, Theorem 3.1] or Theorem 3.0.2. However, in the HHG structure of an admissible group, associated hyperbolic spaces are not necessarily quasi-trees. As an application of Theorem 3.0.2, we show that property (QT) still holds true in this case if we assume G to be residually finite.

Theorem 3.6.3. *Let $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\})$ be an admissible graph of groups, and let $G = \pi_1\mathcal{G}$. If G is residually finite, then G has property (QT).*

Proof. According to the classification of simplices by [HRSS24, Lemma 6.2], any simplex that is not of Type 7 corresponds to a quasi-tree so it is a domain of Type II. Thus, we only need to check that simplices of Type 7 are of Type I. The stabilizer of such a simplex Δ is exactly a vertex group G_v that acts on $\mathcal{C}(\Delta)$ with image $H_v = G_v/Z_v$. Now $\mathcal{C}(\Delta)$ is coarsely the hyperbolic space obtained by coning off H_v as a relatively hyperbolic group, and $\mathbf{F}_\Delta$ is coarsely H_v itself. Therefore, acylindrical image and cobounded nested region hold true (see [Osi16, Proposition 5.2] for acylindricity). Since G_v is a central extension of H_v by Z_v , Δ has neat kernel. Therefore, we conclude by Corollary 3.5.3 and Theorem 3.0.2. $\square$

There is another approach to property (QT) of non-geometric graph manifold groups in [HNY25]. For graph manifolds with nonempty boundary, they actually prove in a more general setting. A *Croke-Kleiner admissible group* (abbreviated as CKA group) is an admissible group that admits a geometric action on a complete proper CAT(0) space. As a corollary of Theorem 3.6.3, we recover the following theorem.

Corollary 3.6.4. *[HNY25, Theorem 1.3] Let G be a CKA group where for every vertex v the central extension $1 \rightarrow Z_v \rightarrow G_v \rightarrow H_v \rightarrow 1$ has an omnipotent hyperbolic quotient group H_v . Then G has property (QT).*

For definition of omnipotence, we refer the reader to [Wis00]. Note that if every hyperbolic group is residually finite, then every hyperbolic group is omnipotent by [Wis00, Remark 3.4]. Under the assumption of Corollary 3.6.4, the central extension associated with any vertex virtually splits by [BH99, Theorem II.7.1]. Therefore, Theorem 3.6.3 implies Corollary 3.6.4 due to the following lemma.

Lemma 3.6.5. *Let G be an admissible group where for every vertex v the central extension $1 \rightarrow Z_v \rightarrow G_v \rightarrow H_v \rightarrow 1$ virtually splits and the hyperbolic quotient group H_v is omnipotent. Then G is residually finite.*

We omit the proof of Lemma 3.6.5 since it is almost the same as the proof of residual finiteness for graph manifold groups by Hempel [Hem87]. The reader can also see [Ngu25] for an improved result.

3.6.3 Hyperbolic–2–decomposable groups

We say a group G is *hyperbolic–2–decomposable* if G splits as a graph of hyperbolic groups with 2-ended edge groups.

Theorem 3.6.6. *Let G be a residually finite hyperbolic–2–decomposable group. Then G has property (QT) if and only if G does not contain any distorted element.*

Proof. If G has property (QT), then G does not contain any distorted element by [HNY25, Lemma 2.5]. Now assume that G does not contain any distorted element. Let $\mathfrak{S}$ be the HHG structure of G given by [RS20]. By construction, there is no orthogonality in $\mathfrak{S}$. Thus, G is colorable. Let $U \in \mathfrak{S}$. Then $\mathcal{C}U$ is either a quasi-tree or a hyperbolic space obtained by coning off a vertex group G_v as a relatively hyperbolic group. In the former case, U is of Type II. Now we only need to consider the latter case. Similarly to Theorem 3.6.3, we have $\text{Stab}_G(U) = G_v$ and $\mathbf{F}_U$ is coarsely G_v itself. It is easy to see that acylindrical image, cobounded nested region and neat kernel hold true. Therefore, we conclude by Corollary 3.5.3 and Theorem 3.0.2. $\square$

Similarly to Lemma 3.6.5, if G is a hyperbolic–2–decomposable group without any distorted element such that every vertex group is omnipotent, then G is residually finite (see [Wis00, §4]).

3.6.4 Artin groups and extensions of lattice Veech groups

Let G be either of the following groups.

- An Artin group of large and hyperbolic type;
- The $\pi_1(\Sigma)$ –extension group of a lattice Veech group in the mapping class group $\text{MCG}(\Sigma)$ of a closed surface Σ .

As shown in [HMS24] and [DDLS24] respectively, G is a virtually colorable HHG. Moreover, the associated hyperbolic spaces of G are all quasi-trees except the maximal one. By Corollary 3.5.4 and Theorem 3.0.2, G has property (QT) if G is residually finite. In particular, we obtain that

Theorem 3.6.7. *Every residually finite Artin group of large and hyperbolic type has property (QT).*

It is proved in [Jan22] that any 3-generator Artin groups with labels ≥ 4 except for $(2m + 1, 4, 4)$ for any $m \geq 2$ is residually finite. Therefore, any 3-generator Artin group with labels ≥ 4 except for $(2m + 1, 4, 4)$ for any $m \geq 2$ has property (QT). On the other hand, we ask

Question 3.6.8. *When is the $\pi_1(\Sigma)$ –extension group of a lattice Veech group residually finite?*

3.6.5 Graph products

Definition 3.6.9 (Graph product). Let Γ be a finite simplicial graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Each vertex $v \in V(\Gamma)$ is labeled by a group G_v . The *graph product* G_Γ is the group

$$G_\Gamma = \left(\bigast_{v \in V(\Gamma)} G_v \right) / \langle\langle [g_v, g_w] \mid g_v \in G_v, g_w \in G_w, \{v, w\} \in E(\Gamma) \rangle\rangle.$$

We call G_v the *vertex groups* of the graph product G_Γ .

Theorem 3.6.10. *Let G_Γ be a graph product of groups with every vertex group having property (QT'). Then G_Γ has property (QT').*

Proof. Any graph product G_Γ has a relative HHG structure $\mathfrak{S}_\Gamma$ by [BR22]. By definition of $\mathfrak{S}_\Gamma$, any G_Γ –orbit on $\mathfrak{S}_\Gamma$ corresponds to a unique subgraph of Γ and is pairwise transversal. Thus, G_Γ is colorable. By [BR22, Theorem 4.4], for each domain $[g\Lambda] \in \mathfrak{S}_\Gamma$, either $[g\Lambda]$ is $\sqsubseteq$ –minimal or $\mathcal{C}g\Lambda$ is a quasi-tree. Since each $\sqsubseteq$ –minimal domain corresponds to a vertex group, this means that every domain is of Type II. By Theorem 3.4.7, G_Γ has property (QT). $\square$

CHAPTER 4

Property (PH)

Definition 4.0.1. A (not necessarily finitely generated) group G has *property (PH')* if there exist finitely many hyperbolic spaces $X_1, \dots, X_n$ on which G acts coboundedly such that the diagonal action of G on $\prod_{i=1}^n X_i$ equipped with ℓ^1 -metric is proper. In this case, we also say G has *property (PH')* from actions $G \curvearrowright X_1, \dots, G \curvearrowright X_n$.

A group has *property (PH)* if it virtually has property (PH').

To begin with, we show that a finitely generated group with property (QT) has property (PH).

Lemma 4.0.2. *Let G be a finitely generated group. If G has property (QT') or (QT), then G has property (PH') or (PH), respectively.*

Proof. By [But22, Theorem 6.1], a finitely generated group with property (QT) virtually has property (QT'). It suffices for us to show that property (QT') implies property (PH'). Let G be a finitely generated group with property (QT'). By definition, there exist finitely many quasi-trees $T_i (1 \leq i \leq n)$ on which G acts isometrically so that for any finite generating set F of G and $o \in X = \prod_{i=1}^n T_i$, the orbit map $\mathcal{G}(G, F) \rightarrow X, g \mapsto go$ is a quasi-isometric embedding. This quasi-isometric embedding of the orbit map implies that the diagonal action $G \curvearrowright X = \prod_{i=1}^n T_i$ is proper. By [Man06, Remark 3.2], any isometric group action of a finitely generated group on a quasi-tree can be assumed to be cobounded. This shows that G has property (PH') in the sense of Definition 4.0.1. $\square$

By Lemma 4.0.2, it is easy to see that property (PH) is the weakest while property (QT') is the strongest among these four properties (PH'), (PH), (QT'), (QT). We refer the readers to Figure 1.1 for an illustration.

4.1 Hyperbolic structures on groups

In this section, we recall some basic material about hyperbolic structures on groups and give an equivalent definition of property (PH). For more details about hyperbolic structures on groups, see [ABO19].

Definition 4.1.1. [ABO19, Definition 1.1] Let S and T be two generating sets of a group G . We say that S is *dominated* by T , written $S \leq T$, if the identity map on G induces a Lipschitz

map between metric spaces $\mathcal{G}(G, T) \rightarrow \mathcal{G}(G, S)$. It is clear that $\leq$ is a preorder on the set of generating sets of G and therefore it induces an equivalence relation in the standard way:

$$S \sim T \iff S \leq T \text{ and } T \leq S.$$

We denote by $[S]$ the equivalence class of a generating set S of G . The preorder $\leq$ induces an order relation $\leqslant$ on all equivalence classes of generating sets by the rule:

$$[S] \leqslant [T] \iff S \leq T. \quad (4.1)$$

Definition 4.1.2. [ABO19, Definition 1.2] A *hyperbolic structure* on G is an equivalence class $[S]$ such that $\mathcal{G}(G, S)$ is hyperbolic. We denote the set of hyperbolic structures by $\mathcal{H}(G)$ and endow it with the order induced from (4.1).

Since hyperbolicity of a geodesic metric space is a quasi-isometry invariant, the definition above is independent of the choice of a particular representative in the equivalence class $[S]$.

Using the standard argument from the proof of the Svarc-Milnor lemma, it is easy to show that elements of $\mathcal{H}(G)$ are in one-to-one correspondence with equivalence classes of cobounded actions of G on hyperbolic spaces considered up to a natural equivalence: two actions $G \curvearrowright X$ and $G \curvearrowright Y$ are *equivalent* if there is a coarsely G -equivariant quasi-isometry $X \rightarrow Y$. This gives another equivalent definition of property (PH').

Definition 4.1.3 (Second definition of property (PH')). A (not necessarily finitely generated) group G has property (PH') if there exist finitely many hyperbolic structures $[S_1], \dots, [S_n] \in \mathcal{H}(G)$ such that the diagonal action of G on $\prod_{i=1}^n \mathcal{G}(G, S_i)$ equipped with ℓ^1 -metric is proper.

Remark 4.1.4. In the above definition of property (PH'), one can require that $[S_1], \dots, [S_n]$ forms a finite antichain in $\mathcal{H}(G)$. A brief reason is as follows: suppose that $[S_1] \leqslant [S_2]$. By definition, there exists a constant $M > 0$ such that $\sup_{s \in S_2} |s|_{S_1} \leq M$. Then for any $g \in G$, $|g|_{S_1} \leq M|g|_{S_2}$. Therefore, for any $R > 0$, the set $\{g \in G : \sum_{i=2}^n |g|_{S_i} \leq R\}$ is contained in $\{g \in G : \sum_{i=1}^n |g|_{S_i} \leq (M+1)R\}$ and thus is finite. This implies that removing $[S_1]$ does not affect the properness of the G -action. By repeating this procedure, one finally gets a finite antichain of hyperbolic structures which gives the (PH') property of G .

In [ABO19], a hyperbolic structure $[S] \in \mathcal{H}(G)$ is called

- *elliptic* if $|\partial\mathcal{G}(G, S)| = 0$ (equivalently, $\mathcal{G}(G, S)$ is bounded);
- *lineal* if $|\partial\mathcal{G}(G, S)| = 2$ (equivalently, $\mathcal{G}(G, S)$ is quasi-isometric to a line);
- *quasiparabolic* or *focal* if $|\partial\mathcal{G}(G, S)| = \infty$ and G fixes a point of $\partial\mathcal{G}(G, S)$;
- *of general type* if $|\partial\mathcal{G}(G, S)| = \infty$ and G does not fix any point of $\partial\mathcal{G}(G, S)$.

The sets of elliptic, lineal, quasiparabolic and general type hyperbolic structures on G are denoted by $\mathcal{H}_e(G)$, $\mathcal{H}_\ell(G)$, $\mathcal{H}_{qp}(G)$ and $\mathcal{H}_{gt}(G)$, respectively. For any hyperbolic structure $[S] \in \mathcal{H}(G)$, it is easy to see that the type of action $G \curvearrowright \mathcal{G}(G, S)$ coincides with the type of $[S] \in \mathcal{H}(G)$.

Since a horocyclic action cannot be cobounded, the classification of unbounded group actions on hyperbolic spaces gives the following result.

Theorem 4.1.5. [ABO19, Theorem 2.2] For every group G , we have

$$\mathcal{H}(G) = \mathcal{H}_e(G) \sqcup \mathcal{H}_\ell(G) \sqcup \mathcal{H}_{qp}(G) \sqcup \mathcal{H}_{gt}(G).$$

The subsets $\mathcal{H}_e(G) \sqcup \mathcal{H}_\ell(G)$ and $\mathcal{H}_e(G) \sqcup \mathcal{H}_\ell(G) \sqcup \mathcal{H}_{qp}(G)$ are initial segments of the poset $\mathcal{H}(G)$.

4.2 Quasimorphisms and quasi-lines

The following two notions will be used in the proof of Lemma 4.2.5.

Definition 4.2.1. A (λ, μ) -quasi-action of a group G on a metric space X is a map $\rho : G \times X \rightarrow X$, denoted $\rho(g, x) \mapsto gx$, so that the following hold:

- (i) for each g , $\rho(g, -) : X \rightarrow X$ is a (λ, μ) -quasi-isometry;
- (ii) for each $x \in X$ and $g, h \in G$, we have

$$|g(hx) - (gh)x| = |\rho(g, \rho(h, x)) - \rho(gh, x)| \leq \mu.$$

(Note that λ and μ must be independent of g and h .) We call a quasi-action *cobounded* if, for every $x \in X$, the map $\rho(-, x) : G \rightarrow X$ is μ' -coarsely onto for some $\mu' \geq 0$.

Definition 4.2.2. Suppose that $\rho_X : G \times X \rightarrow X$ and $\rho_Y : G \times Y \rightarrow Y$ are quasi-actions. A map $f : X \rightarrow Y$ is called *coarsely equivariant* if there is some $\epsilon \geq 0$ so that $|f \circ \rho_X(g, x) - \rho_Y(g, f(x))| \leq \epsilon$ for all $g \in G$ and $x \in X$. A coarsely equivariant quasi-isometry is called a *quasi-conjugacy*.

Example 4.2.3. Let $\phi : G \rightarrow \mathbb{R}$ be a quasimorphism with $D(\phi) = \epsilon$. A $(1, \epsilon)$ -quasi-action of G on $\mathbb{R}$ is given by $\rho(g, x) = x + \phi(g)$.

Lemma 4.2.4. [Man06, Lemma 3.7] Any quasi-line is $(1, \epsilon)$ -quasi-isometric to $\mathbb{R}$ for some $\epsilon > 0$.

The following two lemmas will be very useful in the later proofs. Recall that a lineal action of a group G on a quasi-line L is orientable if no element of G permutes the two points of the Gromov boundary ∂L . Clearly the property of being orientable is invariant under the equivalence of actions and thus we can speak of orientable lineal hyperbolic structures on G . We denote the set of such structures by $\mathcal{H}_\ell^+(G)$.

Lemma 4.2.5. Let G be a group. The following are equivalent:

1. There exists a nonzero homogeneous quasimorphism $\varphi : G \rightarrow \mathbb{R}$. This induces a quasi-action $\rho : G \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $\rho : (g, x) \mapsto x + \varphi(g)$.
2. G has a cobounded quasi-action on $\mathbb{R}$ fixing $\pm\infty$.
3. There exists a generating set S of G such that $[S] \in \mathcal{H}_\ell^+(G)$, i.e. G acts lineally and orientably on $\mathcal{G}(G, S)$.

4. G has an orientable lineal action on a quasi-line L .

Moreover, if G satisfies one of the above conditions, which gives a quasi-action of G , then we can prove the others under quasi-actions quasi-conjugate to this one.

Proof. “(1) $\Rightarrow$ (2)” and “(3) $\Rightarrow$ (4)” are obvious. “(2) $\Rightarrow$ (3)” follows from [Man05, Proposition 4.4] and the Svarc-Milnor lemma. Thus, we only need to verify “(4) $\Rightarrow$ (1)”. Let $\xi \in \partial L$ be a boundary point. The Busemann homogeneous quasimorphism $\beta_\xi : G \rightarrow \mathbb{R}$ associated with ξ given by [CCMT15, Proposition 3.7] (see also [Man08, Proposition 4.9]) satisfies (1). Note that the assumption of local compactness in [CCMT15, Proposition 3.7] is only used to prove continuity, which we do not need here.

To show the “Moreover” part, we start with Item (4). The Svarc-Milnor lemma gives a generating set S of G satisfying (3) and gives a G -coarsely equivariant quasi-isometry between L and $\mathcal{G}(G, S)$. Next, Lemma 4.2.4 shows that there exists a $(1, \epsilon)$ -quasi-isometry $f : \mathcal{G}(G, S) \rightarrow \mathbb{R}$ for some $\epsilon \geq 0$. Via this quasi-isometry, one can define a quasimorphism $\phi : G \rightarrow \mathbb{R}$ by $\phi(g) = f \circ g \circ f^{-1}(0)$ satisfying (1) and a quasi-action $\rho : G \times \mathbb{R} \rightarrow \mathbb{R}$ by $\rho(g, x) = x + f \circ g \circ f^{-1}(0)$ satisfying (2). It is easy to see that this quasi-isometry f is coarsely equivariant with respect to the action $G \curvearrowright \mathcal{G}(G, S)$ and the quasi-action $\rho : G \times \mathbb{R} \rightarrow \mathbb{R}$ and gives the other two desired quasi-conjugations. $\square$

Lemma 4.2.6. *Suppose that a group G has an orientable lineal action on a hyperbolic space X with a basepoint $o \in X$. Then there exists $C > 0$ such that for any $g, h \in G$, $|o - [g, h]o| \leq C$.*

Proof. When restricted to the orbit of the G -action in X , we know that G has an orientable lineal action on a quasi-line. According to Lemma 4.2.5, there exists a homogeneous quasimorphism $\phi : G \rightarrow \mathbb{R}$. As shown in Remark 2.3.1, $|\phi([g, h])| \leq D(\phi)$ for any $g, h \in G$. The proof of Lemma 4.2.5 shows that the quasi-action $\rho : G \times \mathbb{R} \rightarrow \mathbb{R}, (g, x) \mapsto x + \phi(g)$ is quasi-conjugate to the action $G \curvearrowright G \cdot o$. The conclusion then follows. $\square$

Remark 4.2.7. We caution the readers that the conclusion of Lemma 4.2.6 depends heavily on the orientability. For example, let $G = \langle r, s \mid s^2 = (rs)^2 = 1 \rangle$ be the infinite dihedral group and $S = \{r, s\}$. Then $[S] \in \mathcal{H}_\ell(G) \setminus \mathcal{H}_\ell^+(G)$. However, it is direct to compute that $||r^n, s||_S = |r^{2n}|_S \rightarrow \infty$ as $n \rightarrow \infty$.

4.3 Property (PH) from lineal actions

A group G is said to have *property (PH') from (orientable) lineal actions* if there exist finitely many quasi-lines $L_1, \dots, L_n$ on which G acts (orientably) lineally such that the diagonal action of G on the product $\prod_{i=1}^n L_i$ is proper. In this case, we also say that G has property (PH') from (orientable) lineal actions $G \curvearrowright L_1, \dots, G \curvearrowright L_n$. Similarly, a group G is said to have *property (PH) from lineal actions* if some finite-index subgroup of G has property (PH') from lineal actions. Since horocyclic actions cannot be cobounded and elliptic actions can be removed in a proper diagonal action, the simplest objects for us to study are those groups with property (PH) or (PH') from lineal actions. In this section, we are going to characterize the class of groups with property (PH') or (PH) from lineal actions.

A group G is called *FC* if every conjugacy class of G contains only finitely many elements. Note that a group with finitely many commutators is FC. Recall that a group G is an *FZ group of rank r* if $G/Z(G)$ is finite and $Z(G)$ has rank r as an abelian group.

Lemma 4.3.1. *Assume a group G has property (PH') from orientable lineal actions $G \curvearrowright L_1, \dots, G \curvearrowright L_m$. Then G is a finitely generated FZ group of rank $r \leq m$. Moreover, there exists a subset $\{L'_1, \dots, L'_r\} \subseteq \{L_1, \dots, L_m\}$ so that the diagonal action $G \curvearrowright \prod_{i=1}^r L'_i$ is geometric.*

Proof. By Lemma 4.2.5, there exist nonzero homogeneous quasimorphisms φ_i ($i = 1, \dots, m$) on G such that the induced quasi-action $\rho_i : G \times \mathbb{R} \rightarrow \mathbb{R}, (g, x) \mapsto x + \varphi_i(g)$ is quasi-conjugate to $G \curvearrowright L_i$ for each $1 \leq i \leq m$. Properness of the diagonal action $G \curvearrowright \prod_{i=1}^m L_i$ is equivalent to say that the set

$$T_k := \{g \in G \mid |\varphi_i(g)| \leq k, \quad \forall 1 \leq i \leq m\}$$

is finite for any $k \geq 0$.

If there exist constants $c_1, \dots, c_{m-1} \in \mathbb{R}$ such that $\varphi_m = \sum_{i=1}^{m-1} c_i \varphi_i$, then for any $g \in G$ with $|\varphi_i(g)| \leq k$ for $1 \leq i \leq m-1$, we have $|\varphi_m(g)| \leq k \cdot \max\{|c_i| \mid 1 \leq i \leq m-1\}$. This implies that the set

$$\{g \in G \mid |\varphi_i(g)| \leq k, \quad \forall 1 \leq i \leq m-1\}$$

is finite. Equivalently, the diagonal action $G \curvearrowright \prod_{i=1}^{m-1} L_i$ is still proper. By choosing $\{\varphi_i \mid 1 \leq i \leq r\} \subseteq \{\varphi_i \mid 1 \leq i \leq m\}$ to be a maximal linearly independent subset, the above arguments show that the diagonal action $G \curvearrowright \prod_{i=1}^r L_i$ is still proper.

Since T_k is finite for any $k \geq 0$ and $G = \bigcup_{k=1}^{\infty} T_k$, we can arrange the group elements of G in a sequence $(g_n)_{n \geq 1}$ such that T_k is exactly the set of the first n_k elements, where $n_k = |T_k| \in \mathbb{N}$ is non-decreasing with respect to k . Define a map $\Phi : G \rightarrow \mathbb{R}^r$ by $\Phi(g) = (\varphi_1(g), \dots, \varphi_r(g))^T$ for any $g \in G$. For any $k \in \mathbb{N}$, define an (r, n_k) -matrix M_k by $M_k = (\Phi(g_1), \dots, \Phi(g_{n_k}))$. Note that for any $k < k'$, M_k is contained in $M_{k'}$ as the first n_k columns. Therefore, $\text{rk}(M_k)$ is non-decreasing with respect to k . Moreover, $\text{rk}(M_k) \leq r$ by definition. Thus, there exists $r' \leq r$ and $k_0 \in \mathbb{N}$ such that $\text{rk}(M_k) = r'$ for any $k \geq k_0$.

Claim. $r' = r$.

Proof of Claim. Suppose that $r' < r$. After passing to a subsequence $(M_{k_l})_{l \in \mathbb{N}}$ and reordering the quasimorphisms if necessary, we can assume that the first r' row vectors of M_{k_l} are linearly independent for any $l \in \mathbb{N}$. Thus, there exist constants $c_i \in \mathbb{R}$ ($i = 1, \dots, r'$) such that $\varphi_r(g_n) = \sum_{i=1}^{r'} c_i \varphi_i(g_n)$ for any $n \geq 1$. This contradicts with the linear independency of φ_i ($i = 1, \dots, r$) on G . $\square$

Since $\text{rk}(M_{k_0}) = r$, we can choose r column vectors of M_{k_0} that are linearly independent. WLOG, let $\Phi(g_1), \dots, \Phi(g_r)$ be linearly independent. For any $g \in G$, there exist b_n ($n = 1, \dots, r$) such that

$$\Phi(g) = \sum_{n=1}^r b_n \Phi(g_n).$$

For any number $x \in \mathbb{R}$, we use $[x]$ to denote the maximal integer which does not exceed x . By

definition of homogeneous quasimorphisms,

$$|\varphi_i(g \prod_{n=1}^r g_n^{-[b_n]}) - \varphi_i(g) + \sum_{n=1}^r [b_n] \varphi_i(g_n)| \leq rD(\varphi_i),$$

for any $1 \leq i \leq r$. Thus,

$$|\varphi_i(g \prod_{n=1}^r g_n^{-[b_n]})| \leq \sum_{n=1}^r |\varphi_i(g_n)| + rD(\varphi_i) \leq r(k_0 + D(\varphi_i)), \quad (4.2)$$

for any $1 \leq i \leq r$. This shows that $g \in \langle T_k \rangle$, where $k = r(k_0 + \max_{1 \leq i \leq m} D(\varphi_i))$. In particular, G is finitely generated.

Lemma 4.2.6 and properness of $G \curvearrowright \prod_{i=1}^r L_i$ imply that G is an FC group. Since a finitely generated FC group is FZ (see [Neu51, §2]), G is FZ. Moreover, (4.2) also shows that the diagonal action $G \curvearrowright \prod_{i=1}^r L_i$ is cobounded, and thus geometric. Therefore, the rank of G equals r . $\square$

Now we can give a characterization of groups that have property (PH') from lineal actions.

Lemma 4.3.2. *If a group G has property (PH') from lineal actions $G \curvearrowright L_1, \dots, G \curvearrowright L_m$, then G fits into the following short exact sequence*

$$1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z}_2^s \rightarrow 1$$

where $0 \leq s \leq m$ and H is a finitely generated FZ group of rank $r \leq m$. Moreover, there exists a subset $\{L'_1, \dots, L'_r\} \subseteq \{L_1, \dots, L_m\}$ so that $G \curvearrowright \prod_{i=1}^r L'_i$ is geometric.

Proof. The induced action of G on $\{\partial L_i : 1 \leq i \leq m\}$ gives a homomorphism $\rho : G \rightarrow \mathbb{Z}_2^m$. Thus, $H := \ker \rho$ acts orientably on each L_i . By applying Lemma 4.3.1 to H , we have that H is a finitely generated FZ group of rank $r \leq m$ and there exists a subset $\{L'_1, \dots, L'_r\} \subseteq \{L_1, \dots, L_m\}$ so that $H \curvearrowright \prod_{i=1}^r L'_i$ is geometric. Since H is of finite index in G , the diagonal action $G \curvearrowright \prod_{i=1}^r L'_i$ is also geometric. $\square$

Corollary 4.3.3. *A group has property (PH) from lineal actions if and only if it is finitely generated and virtually abelian.*

Proof. “ $\Leftarrow$ ” is clear and “ $\Rightarrow$ ” follows from Lemma 4.3.2 and the fact that any group containing a finitely generated finite-index subgroup is also finitely generated. $\square$

By combining Lemma 4.3.1 and Corollary 4.3.3, we complete the proof of Theorem 1.2.1.

Corollary 4.3.4. *A finitely generated group with sub-exponential growth has property (PH) if and only if it is virtually abelian.*

Proof. We only need to show “ $\Rightarrow$ ” since the other direction is clear. Let G be a finitely generated group with sub-exponential growth that has property (PH). By passing to a finite-index subgroup, we can assume that G has property (PH'). By definition, there exist finite unbounded hyperbolic spaces $X_1, \dots, X_n$ on which G acts coboundedly such that the diagonal

action $G \curvearrowright X = \prod_{i=1}^n X_i$ is proper. Since any group admitting a focal or general type action on a hyperbolic space contains a non-abelian free sub-semigroup and thus has exponential growth, G must act lineally on each X_i for $1 \leq i \leq n$. This implies that G has property (PH') from lineal actions. Then the conclusion follows from Corollary 4.3.3. $\square$

4.4 First examples

In this section, we are going to give some examples and non-examples of groups with property (PH).

4.4.1 The Heisenberg group

Let $H = \langle a, b, c \mid [a, c] = [b, c] = 1, [a, b] = c \rangle$ be the Heisenberg group. [ABO19, Example 4.23] shows that $\mathcal{H}(\mathbb{Z}^n) = \mathcal{H}_e(\mathbb{Z}^n) \sqcup \mathcal{H}_\ell^+(\mathbb{Z}^n)$ for $n \geq 2$ where $\mathcal{H}_\ell^+(\mathbb{Z}^n)$ is an antichain of cardinality continuum. The first result here is to show that H and $\mathbb{Z}^n$ have the same hyperbolic structures.

Lemma 4.4.1. $\mathcal{H}(H) = \mathcal{H}_e(H) \sqcup \mathcal{H}_\ell^+(H)$ where $\mathcal{H}_\ell^+(H)$ is an antichain of cardinality continuum.

Proof. Since H has polynomial growth, it is clear that $\mathcal{H}_{qp}(H) = \mathcal{H}_{gt}(H) = \emptyset$ and thus $\mathcal{H}(H) = \mathcal{H}_e(H) \sqcup \mathcal{H}_\ell(H)$. Note that there are two natural surjective homomorphisms $\phi_a : H \rightarrow \langle a \rangle, a \mapsto a, b \mapsto 1, c \mapsto 1$ and $\phi_b : H \rightarrow \langle b \rangle, a \mapsto 1, b \mapsto b, c \mapsto 1$. By the Svarc-Milnor lemma, these two homomorphisms give two non-equivalent generating sets in $\mathcal{H}_\ell^+(H)$. Thus $|\mathcal{H}_\ell^+(H)| \geq 2$. Since [ABO19, Theorem 4.22 (b)] shows that for every group G , the cardinality of $\mathcal{H}_\ell^+(G)$ is 0, 1 or at least continuum. Hence, the cardinality of $\mathcal{H}_\ell^+(H)$ is continuum. It remains to show that $\mathcal{H}_\ell(H) = \mathcal{H}_\ell^+(H)$.

Pick an arbitrary element $[S] \in \mathcal{H}_\ell(H)$ and an arbitrary element $g \in H$ such that g is loxodromic on $\mathcal{G}(H, S)$. Note that $\partial\mathcal{G}(H, S) = \{g^+, g^-\}$. Since $c \in Z(H)$, one has that $cg = gc$ and thus c fixes pointwise $\{g^+, g^-\}$. Moreover, it follows from another relation $ab = bac$ that g can be written as $a^m b^n c^k$ for some $m, n, k \in \mathbb{Z}$ and

$$aga^{-1} = a^{m+1}b^n a^{-1}c^k = a^m b^n c^{k+n} = gc^n, \quad bgb^{-1} = ba^m b^{n-1}c^k = a^m b^n c^{k-m} = gc^{-m}.$$

Since both g and c fix pointwise $\{g^+, g^-\}$, the above equations show that $a \cdot g^\pm = g^\pm, b \cdot g^\pm = g^\pm$. Since $\{a, b, c\}$ generates H , one gets that any element in H fixes pointwise $\partial\mathcal{G}(H, S)$. This means that $H \curvearrowright \mathcal{G}(H, S)$ is orientable. Since $[S]$ is arbitrary, we get that $\mathcal{H}_\ell(H) = \mathcal{H}_\ell^+(H)$. $\square$

Since H is a non-elementary group with polynomial growth, Corollary 4.3.4 shows that H does not have property (PH). Therefore, having property (PH) or not gives a distinction between $\mathbb{Z}^n (n \geq 2)$ and H . Another way to prove that H does not have property (PH) will be given by Corollary 5.1.4.

It is well-known that H fits into a central extension

$$1 \rightarrow \mathbb{Z} \rightarrow H \rightarrow \mathbb{Z}^2 \rightarrow 1.$$

As a corollary of Theorem 1.3.1, this central extension has an unbounded Euler class. We will discuss central extensions and their Euler classes more detailedly in the next chapter.

4.4.2 The crystallographic group

In this subsection, we give an equivalent condition for a crystallographic group to have property (PH'). As a corollary, we show that the (3,3,3)-triangle group has property (QT) but no (PH').

Recall that a *crystallographic group* is a discrete subgroup of $\text{Isom}(\mathbb{R}^r)$ that acts properly cocompactly on $\mathbb{R}^r$. By the first Bieberbach Theorem (see [Szc25]), any such group Γ fits into a short exact sequence of the form

$$1 \rightarrow A \rightarrow \Gamma \rightarrow F \rightarrow 1 \quad (4.3)$$

where A , called the *translation subgroup*, is isomorphic to $\mathbb{Z}^r$ and F , the *point group*, is a finite subgroup of the orthogonal group $O_r(\mathbb{R})$. The above short exact sequence right-splits. This amounts to saying that Γ can be written as a semi-direct product $A \rtimes_{\phi} F$ where ϕ is a homomorphism from F to $\text{Aut}(\mathbb{Z}^r) = \text{GL}_r(\mathbb{Z})$.

Lemma 4.4.2. *Let $\Gamma \leq \text{Isom}(\mathbb{R}^r)$ be a crystallographic group. Γ has property (PH') if and only if the corresponding homomorphism $\phi : F \rightarrow \text{GL}_r(\mathbb{Z})$ factors through $\prod_{i=1}^r \text{Aut}(\mathbb{Z}) = \mathbb{Z}_2^r$.*

Proof. “ $\Rightarrow$ ”: Since Γ is a virtually abelian group of rank r and has property (PH'), Lemma 4.3.2 shows Γ admits a geometric diagonal action on a finite product of quasi-lines $\prod_{i=1}^r L_i$. Since $[\Gamma : A] < \infty$, the diagonal action $A \curvearrowright \prod_{i=1}^r L_i$ is also geometric. Moreover, the induced action of Γ on $\{\partial L_i : 1 \leq i \leq r\}$ gives a homomorphism $\rho : \Gamma \rightarrow \mathbb{Z}_2^r$. Since A is a torsion-free abelian group of rank r , the geometric action $A \curvearrowright \prod_{i=1}^r L_i$ implies that no generating element of A can swap the two points of ∂L_i ($i = 1, \dots, r$). Hence, $A \leq \ker \rho$ and this induces a quotient homomorphism $\bar{\rho} : F = \Gamma/A \rightarrow \mathbb{Z}_2^r$, which completes the proof.

“ $\Leftarrow$ ”: Since $\Gamma \leq \text{Isom}(\mathbb{R}^r)$ is a crystallographic group, the translation subgroup A is isomorphic to a lattice $\mathbb{Z}^r$ in $\mathbb{R}^r$ and A acts by translation on $\mathbb{Z}^r$. When the corresponding homomorphism $\phi : F \rightarrow \text{GL}_r(\mathbb{Z})$ factors through $\prod_{i=1}^r \text{Aut}(\mathbb{Z}) = \mathbb{Z}_2^r$, the point group F preserves each factor of $\mathbb{Z}^r$ and acts by permutation on the boundary of each factor. Since $\Gamma = A \rtimes_{\phi} F$, one gets an isometric action of Γ on $\mathbb{Z}^r$ which preserves each $\mathbb{Z}$ -coordinate. Since $[\Gamma : A] < \infty$, the action $\Gamma \curvearrowright \mathbb{Z}^r$ is also geometric, which gives the (PH') property of Γ . $\square$

Let $T = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ac)^3 = 1 \rangle$ be the (3,3,3)-triangle group which acts properly and cocompactly on $\mathbb{R}^2$. Clearly T is a crystallographic group with translation subgroup isomorphic to $\mathbb{Z}^2$. Recall that two groups are commensurable if they contain isomorphic finite-index subgroups. Petyt-Spriano [PS23, Corollary 4.5] used this triangle group to show that being an HHG is not a commensurability invariant.

Corollary 4.4.3. *The (3,3,3)-triangle group T does not have property (PH'). In particular, neither property (PH') nor (QT') is a commensurability invariant.*

Proof. From the short exact sequence (4.3), we see that T is virtually $\mathbb{Z}^2$. Since property (QT) is a commensurability invariant (see [BBF21]), T has property (QT) and thus property (PH) by

Lemma 4.0.2. According to Lemma 4.4.2, if T has property (PH'), then its point group embeds in $\mathbb{Z}_2^2$. However, T contains an element of order 3. This shows that T does not have property (PH'). In particular, neither property (PH') nor (QT') is a commensurability invariant. $\square$

4.4.3 The Baumslag-Solitar group

Recall that a Baumslag-Solitar group is defined by $BS(m, n) = \langle a, t \mid ta^mt^{-1} = a^n \rangle$. In this subsection, we prove that any solvable Baumslag-Solitar group $BS(1, n)$ has property (PH') but does not have an HHG structure or property (QT) for $n \geq 2$.

The reason why $BS(1, n)$ has neither an HHG structure nor property (QT) is due to the distortion of some infinite-order element in $BS(1, n)$; see [DHS17, Theorem 7.1] (of which the proof is corrected in [DHS20]) and [HNY25, Lemma 2.5].

Lemma 4.4.4. *[HNY25, Corollary 2.7] The Baumslag-Solitar group $BS(1, n)$ for $n \geq 2$ does not have property (QT).*

From now on, let us focus on $G_n = BS(1, n) = \langle a, t \mid tat^{-1} = a^n \rangle$ for $n \geq 2$.

It is well-known and easy to prove that G_n is isomorphic to the subgroup of $SL_2(\mathbb{R})$ generated by

$$t = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & 1/\sqrt{n} \end{pmatrix} \text{ and } a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus we obtain an action of G_n on $\mathbb{H}^2$ that factors through the action of $SL_2(\mathbb{R})$. It is easy to see that this action is focal and cobounded. Indeed, the generator t is a loxodromic element and the other generator a is a parabolic element and both generators have a common fixed point ∞ in $\partial\mathbb{H}^2$. Moreover, a bounded fundamental domain of this action is given by the ‘rectangle’ with vertices $i, ni, n + i, n + ni$.

Another G_n -action on a hyperbolic space comes from the HNN-extension structure of G_n . Let T_n be the corresponding Bass-Serre tree with a path metric d_T . Then $G_n \curvearrowright T_n$ is also focal with the unique fixed point $t^- \in \partial T_n$ and cobounded.

It is not hard to show that these two focal actions of G_n are not equivalent. Indeed there are no non-trivial elliptic elements with respect to the first action, while all elements from $[G_n, G_n]$ are elliptic with respect to the second action. As shown in [FSY04, Proposition 5.2(4)], the diagonal action $G_n \curvearrowright \mathbb{H}^2 \times T_n$ is proper. Thus, we obtain

Proposition 4.4.5. *The solvable Baumslag-Solitar group $G_n = BS(1, n)$ has property (PH') from focal actions $G_n \curvearrowright \mathbb{H}^2$ and $G_n \curvearrowright T_n$.*

As a direct corollary of Lemma 4.4.4 and Proposition 4.4.5, one gets that the class of finitely generated groups with property (QT) (resp. (QT')) is strictly contained in the class of groups with property (PH) (resp. (PH')). This is a complement to Lemma 4.0.2.

By combining Lemma 4.4.1, Corollary 4.4.3 and Proposition 4.4.5, we complete the proof of Proposition 1.2.2.

4.4.4 Fundamental groups of Anosov mapping tori

Fix an element $\varphi \in \mathrm{SL}_2(\mathbb{Z})$ that is Anosov; that is, φ has distinct irrational eigenvalues $\lambda > 1$ and $0 < \lambda^{-1} < 1$. The map φ is an element of the mapping class group of the torus T^2 , and the group $G_\varphi = \mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$ is the fundamental group of the mapping torus of φ .

As shown in [AR24, §7], G_φ admits two non-equivalent focal actions on $\mathbb{H}^2$. These two focal actions may not be cobounded. However, [ABO19, Lemma 4.24] tells us that any focal action is weakly equivalent to a cobounded focal action. Hence, we can drop the requirement of coboundedness for focal actions in the definition of property (PH'). Moreover, [FSY04, Proposition 5.3] shows that the induced diagonal action of G_φ on $\mathbb{H}^2 \times \mathbb{H}^2$ is proper. In summary, we obtain

Proposition 4.4.6. *The fundamental group of Anosov mapping tori has property (PH') from two non-equivalent focal actions on $\mathbb{H}^2$.*

4.5 Relative HHGs and graph products

This section is devoted to the proof of Theorem 1.2.3, which is an analogue of [HP22, Theorem 3.1], [HNY25, Theorem 1.5] and Theorem 3.0.2. Let $(G, \mathfrak{S})$ be a colorable relative HHG with a constant E . Let $\mathcal{V}$ be a representing set of $\mathfrak{S}$ under the G -action such that $1 \in \mathbf{P}_V$ for any $V \in \mathcal{V}$, where $\mathbf{P}_V$ is the standard product region associated with V . Note that $\mathcal{V}$ is finite by the definition of relative HHGs. We write $[V]$ to mean the G -orbit of V on $\mathfrak{S}$.

Theorem 4.5.1 (Theorem 1.2.3). *Let $(G, \mathfrak{S})$ be a colorable relative HHG. If for every domain $U \in \mathfrak{S}$ that is nesting-minimal, the action of $\mathrm{Stab}_G(U)$ on $\mathcal{C}U$ has property (PH'), then G has property (PH').*

Proof. By assumption, for every $V \in \mathcal{V}$, there exist hyperbolic spaces $X_V^i (1 \leq i \leq n_V)$ along with $\mathrm{Stab}_G(V)$ -equivariant maps $\iota_V^i : \mathcal{C}V \rightarrow X_V^i$ for $i = 1, \dots, n_V$ such that the map

$$\prod_{i=1}^{n_V} \iota_V^i : \mathcal{C}V \rightarrow \prod_{i=1}^{n_V} X_V^i$$

is coarsely controlled. Thus, there exists a function $\rho_+ : [0, \infty) \rightarrow [0, \infty)$ such that

$$\sum_{i=1}^{n_V} d_{X_V^i}(\iota_V^i(x), \iota_V^i(y)) \leq \rho_+(d_{\mathcal{C}V}(x, y)) \quad (4.4)$$

for any $x, y \in \mathcal{C}V$.

Fix any $i \in \{1, \dots, n_V\}$. Let $\mathbf{X}_{[V]}^i = \{X_U^i \mid U \in [V]\}$ be a collection of copies of X_V^i . For any $g \in G$ and $U \in \mathfrak{S}$, let $g_U^i : X_U^i \rightarrow X_{gU}^i$ be the canonical isometry. It is conventional to define a collection of maps $\{\iota_U^i : \mathcal{C}U \rightarrow X_U^i \mid U \in [V]\}$ such that the diagram

$$\begin{array}{ccc} \mathcal{C}U & \xrightarrow{g_U^*} & \mathcal{C}gU \\ \downarrow \iota_U^i & & \downarrow \iota_{gU}^i \\ X_U^i & \xrightarrow{g_U^i} & X_{gU}^i \end{array}$$

commutes for any $g \in G$ and $U \in [V]$.

Fix any $V \in \mathcal{V}$. By colorability and [HP22, Lemma 3.4], the set $\{\mathcal{CU} \mid U \in [V]\}$ equipped with $\pi_{gU}(U) = \rho_{gU}^U$ is a G -projection system in the sense of Bestvina-Bromberg-Fujiwara [BBF15]. By (4.4), it is easy to verify that every $\mathbf{X}_{[V]}^i$ equipped with $\Pi_{X_{gU}^i}(X_U^i) := \iota_{gU}^i(\rho_{gU}^U)$ is also a G -projection system. We also define $\Pi_{X_{gU}^i}(x)$ for any $x \in \mathcal{CU}$ and $g \in G$ by

$$\Pi_{X_{gU}^i}(x) = \begin{cases} \iota_U^i(x) & \text{if } gU = U \\ \iota_{gU}^i(\rho_{gU}^U) & \text{if } gU \neq U. \end{cases}$$

By the construction in [BBF15, §4], we obtain a quasi-tree of metric spaces $\mathcal{CX}_{[V]}^i$ for every $V \in \mathcal{V}$ and $1 \leq i \leq n_V$. Since X_V^i is hyperbolic, $\mathcal{CX}_{[V]}^i$ is also hyperbolic by [BBF15, Theorem 4.17]. We only need to show that the diagonal action

$$G \curvearrowright \prod_{V \in \mathcal{V}} \prod_{i=1}^{n_V} \mathcal{CX}_{[V]}^i$$

is proper. For simplicity, we will use $|p - q|$ to mean the distance between two points p, q in the same space.

Fix any $V \in \mathcal{V}$ and any $i \in \{1, \dots, n_V\}$. Fix a base point $o_V^i \in X_V^i$. Since $1 \in \mathbf{P}_V$, we have $|\pi_U(1) - \rho_U^V| \leq E$ for any $U \in [V] - \{V\}$ by definition of standard product regions. Thus, by (4.4), we have $|\iota_U^i(\pi_U(1)) - \Pi_{X_U^i}(o_V^i)| \leq \rho_+(|\pi_U(1) - \rho_U^V|) \leq \rho_+(E)$ for any $U \in [V] - \{V\}$. Therefore,

$$\begin{aligned} |\iota_U^i(\pi_U(1)) - \iota_U^i(\pi_U(g))| &\leq |\iota_U^i(\pi_U(1)) - \Pi_{X_U^i}(o_V^i)| + |\Pi_{X_U^i}(o_V^i) - \Pi_{X_U^i}(go_V^i)| \\ &\quad + |\Pi_{X_U^i}(go_V^i) - \iota_U^i(\pi_U(g))| \\ &\leq |\Pi_{X_U^i}(o_V^i) - \Pi_{X_U^i}(go_V^i)| + 2\rho_+(E) \end{aligned}$$

for any $g \in G$ and $U \in [V]$.

Suppose that $d_{\mathcal{CX}_{[V]}^i}(o_V^i, go_V^i) \leq C$ for any $V \in \mathcal{V}$ and $1 \leq i \leq n_V$. By the distance formula for $\mathcal{CX}_{[V]}^i$ (see [BBF15, Theorem 4.13]), $|\Pi_{X_U^i}(o_V^i) - \Pi_{X_U^i}(go_V^i)| \leq C'$ for any $U \in [V]$, where C' is a constant independent of g, V, i . Thus,

$$|\iota_U^i(\pi_U(1)) - \iota_U^i(\pi_U(g))| \leq C' + 2\rho_+(E).$$

Since $\prod_{i=1}^{n_V} \iota_U^i$ is proper, $|\pi_U(1) - \pi_U(g)|$ is bounded for all $U \in \mathfrak{S}$. Therefore, by Theorem 2.2.2, the word length of g is bounded. In conclusion, G has property (PH'). $\square$

4.6 Open questions

Note that Theorem 1.2.1 shows that the property of being finitely generated is implied by the property (PH) from lineal actions. We wonder whether this phenomenon occurs for all groups with property (PH). So we ask that

Question 4.6.1. *Does property (PH) imply finite generation?*

If each hyperbolic space in the above definition of property (PH') is assumed to be proper, then we say G has *property (PPH)*, which is studied by Cui and Wan in [CW25]. In particular, they obtained the following Tits Alternative for finitely generated groups with property (PPH): a finitely generated group with property (PPH) from focal and lineal actions must be amenable. We think the additional assumption that each hyperbolic space is proper is not essential. So we raise the following question.

Question 4.6.2. *Are (finitely generated) groups with property (PH) from focal and lineal actions amenable?*

Let $\mathcal{L} = \langle t, a, b \mid [a, b] = 1, ta^2b^{-1}t^{-1} = a^2b, tab^2t^{-1} = a^{-1}b^2 \rangle$ be the Leary-Minasyan group defined in [LM21]. It is also shown in [LM21] that $\mathcal{L}$ is a CAT(0) group quasi-isometric to $F_5 \times \mathbb{Z}^2$. In [But22], Button utilized this group to show that property (QT) is not a quasi-isometry invariant. As a generalization of both word-hyperbolicity and property (QT), we wonder that

Question 4.6.3. *Is property (PH) on finitely generated groups a quasi-isometry invariant? In particular, does the Leary-Minasyan group $\mathcal{L}$ have property (PH)?*

We remark that Item (3) in Proposition 1.2.2 reveals an essential difference between property (PH) and (QT). That is, a finitely generated group with property (PH) may contain distorted elements, while a finitely generated group with property (QT) cannot (cf. [HNY25, Lemma 2.5]). By studying existing examples with or without property (PH), we ask that

Question 4.6.4. *Is every element in a finitely generated group with property (PH) either undistorted or at least exponentially distorted?*

Motivated by the work of Wan and Yang in [WY25], another question is that

Question 4.6.5. *Does the shadowing property defined in [WY25] have an invariance similar to Theorem 1.3.1 and 1.3.2?*

CHAPTER 5

Central extensions

5.1 Central extensions and bounded Euler classes

In this chapter, we will show that all of properties (PH'), (PH), (QT') and (QT) are preserved under a central extension with a bounded Euler class (Theorems 1.3.1, 1.3.2). This gives us more interesting examples of these properties in a unified way. In this section, we briefly introduce some notations about central extensions and then give some equivalent conditions for bounded Euler classes.

5.1.1 Geometric properties of central elements

Lemma 5.1.1. *Let E be a group acting isometrically on a hyperbolic space X and $Z = Z(E)$.*

1. *If the action $E \curvearrowright X$ is cobounded, then any element $c \in Z$ is either elliptic or loxodromic on X .*
2. *If there exists an element $c \in Z$ that is loxodromic on X , then the E -action on X is lineal and orientable.*

Proof. (1) Assume that c is parabolic on X . Let $\xi \in \partial X$ be the unique fixed point of c . If there exists a loxodromic element $g \in E$ on X , then the equality $cgc^{-1} = g$ shows that $cg^{\pm} = g^{\pm}$ which contradicts with $|\text{Fix}(c)| = 1$. Hence, there is no loxodromic elements in E on X . Since a horocyclic action cannot be cobounded, one gets that $E \curvearrowright X$ is a bounded action. This implies that c is an elliptic element, which is also a contradiction.

(2) Since c is central in E and loxodromic on X , E leaves $\Lambda\langle c \rangle = \{c^+, c^-\}$ invariant. This means the action of E on X is lineal. Suppose that there exists an element $b \in E$ that swaps the two points c^+, c^- . Then for any $o \in X$, it is clear that $|bc^n o - c^n b o| \rightarrow \infty$ as $n \rightarrow \infty$. This shows that $bc^n b^{-1} \neq c^n$ for $n \gg 0$, which is a contradiction to $c \in Z$. Hence, the action of E on X is lineal and orientable. $\square$

Lemma 5.1.2. *Let E be a group that has property (PH') from actions $E \curvearrowright X_1, \dots, E \curvearrowright X_n$. Assume $Z \leq Z(E)$ is an abelian group of rank r . Then $r \leq n$ and there exists a subset of quasi-lines $\{L_1, \dots, L_r\} \subseteq \{X_1, \dots, X_n\}$ such that the diagonal action of Z on $\prod_{i=1}^r L_i$ is geometric.*

Proof. From Lemma 5.1.1, an important observation is that each X_i on which Z acts non-elliptically is a quasi-line and E has an orientable lineal action on it. Hence, by removing all

X_i 's on which Z acts elliptically, we get some quasi-lines $L'_1, \dots, L'_m$ in $X_1, \dots, X_n$ and Z has property (PH') from orientable lineal actions $Z \curvearrowright L'_1, \dots, Z \curvearrowright L'_m$. The remaining proof is completed by Lemma 4.3.1. $\square$

Next we give a necessary condition for a group to have property (PH).

Lemma 5.1.3. *Let G be a group and $c \in Z(G)$ be a central element. Suppose that a subgroup $G' \leq G$ of finite index has a cobounded action on a hyperbolic space X . If c is a commutator in G , then c^k acts as an elliptic element under the action $G' \curvearrowright X$ for some $k > 0$.*

Proof. If c is of finite order, then there is nothing to prove. Suppose c is of infinite order. Up to taking a finite-index subgroup, we can assume that $G' \triangleleft G$. Denote $[G : G'] = k \geq 1$. Then it follows from the normality of G' that $g^k \in G'$ for any $g \in G$.

Claim. $c^{k^2 m^2}$ is a commutator in G' for any $m \geq 1$.

Proof of Claim. Since c is a commutator in G , there exist $g, h \in G$ such that $ghg^{-1} = ch$. Since $c \in Z(G)$, $gh^{km}g^{-1} = c^{km}h^{km}$ for all $m \geq 1$. Reformulating this equality gives that $h^{-km}gh^{km} = c^{km}g$ and then $h^{-km}g^{km}h^{km} = c^{k^2 m^2}g^{km}$. Note that $g^{km}, h^{km} \in G'$ for each $m \geq 1$. Thus, $[g^{-km}, h^{-km}] = c^{k^2 m^2}$ is a commutator in G' . $\square$

Suppose that c^k is not elliptic on X . By Lemma 5.1.1, the action $G' \curvearrowright X$ is orientable lineal. Hence, Lemma 4.2.6 shows that the orbit of $\langle c^{k^2} \rangle$ on X is bounded, which is a contradiction to the non-ellipticity of c^k . $\square$

Corollary 5.1.4. *Let G be a group. If there exists an infinite-order element $c \in Z(G)$ such that c is a commutator in G , then G does not have property (PH).*

Proof. Let $G' \leq G$ be a finite-index subgroup that has property (PH') from actions $G' \curvearrowright X_1, \dots, G' \curvearrowright X_n$. By Lemma 5.1.3, some power of c is elliptic on each X_i for $1 \leq i \leq n$. This contradicts the properness of the diagonal action $G' \curvearrowright \prod_{i=1}^n X_i$ since c is of infinite order. $\square$

We remark that Heisenberg groups satisfy the condition of Corollary 5.1.4.

5.1.2 Characterizations of central extensions with bounded Euler classes

Consider a central extension of groups

$$1 \rightarrow Z \xrightarrow{\iota} E \xrightarrow{\pi} G \rightarrow 1 \quad (5.1)$$

where Z is finitely generated. Let $[\omega] \in H^2(G, Z)$ be the Euler class of this central extension. In what follows, we will identify Z with $\iota(Z)$. For any $g \in E$, we write $\bar{g}$ to mean $\pi(g) \in G$.

According to the structure theorem of finitely generated abelian groups, there exists $r \geq 0$ and a finite abelian group F such that $Z \cong \mathbb{Z}^r \oplus F$. Let $\{c_1, \dots, c_r\}$ be a standard generating set for $\mathbb{Z}^r$. Part of the following lemma is also proved in the recent work of Fournier-Facio, Mangioni, and Sisto (see [FMS25, Proposition 2.9]).

Lemma 5.1.5. *For the central extension (5.1), the following are equivalent:*

1. $[\omega]$ is bounded.
2. The central extension (5.1) quasi-splits.
3. The central extension (5.1) admits a quasi-retraction.
4. There exist quasimorphisms $\phi_i : E \rightarrow \mathbb{Z}$ for $i = 1, \dots, r$, such that ϕ_i is unbounded on $\langle c_i \rangle$ and bounded on $\langle c_j \rangle$ whenever $i \neq j$.
5. There exist $[S_1], \dots, [S_r] \in \mathcal{H}_\ell^+(E)$ such that c_i is loxodromic on $\mathcal{G}(E, S_i)$ and elliptic on $\mathcal{G}(E, S_j)$ whenever $i \neq j$.
6. There exist quasi-lines $L_1, \dots, L_r$ on which E acts lineally such that the diagonal action of Z on $\prod_{i=1}^r L_i$ is geometric.
7. The central extension (5.1) virtually quasi-splits.

Proof. The equivalence of (1) and (2) is given by [FK16, Lemma 2.9]. The equivalence of “(2) and (3)” is given by [Wan25, Theorem 1.5]. “(2) $\Rightarrow$ (7)” is obvious. “(3) $\Rightarrow$ (4)” is also obvious because a composition of quasi-homomorphisms is still a quasi-homomorphism. The equivalence of (4) and (5) is easy to see from Lemma 4.2.5. “(5) $\Rightarrow$ (6)” is also obvious. Thus, we only need to prove “(6) $\Rightarrow$ (1)” and “(7) $\Rightarrow$ (2)”.

(6) $\Rightarrow$ (1): We divide the proof into three steps.

Step 1: we construct a quasi-homomorphism φ from E to Z .

Since the action of Z on $\prod_{i=1}^r L_i$ is geometric, there exists at least one loxodromic element in Z on each L_i . It follows from Lemma 5.1.1 that E acts lineally and orientably on each L_i . By Lemma 4.2.5, each orientable lineal action $E \curvearrowright L_i$ gives a homogeneous quasimorphism $\phi_i : E \rightarrow \mathbb{R}$. Define $\Phi : E \rightarrow \mathbb{R}^r$ by $\Phi(g) := (\phi_1(g), \dots, \phi_r(g))^T$ for any $g \in E$. As a product of homogeneous quasimorphisms, Φ is a homogeneous quasi-homomorphism. Denote $K = \sup_{g, h \in E} \|\Phi(gh) - \Phi(g) - \Phi(h)\|_1$. Then for any $c = c_1^{n_1} \cdots c_r^{n_r} \in Z$,

$$\|\Phi(c) - (\sum_{i=1}^r \Phi(c_i^{n_i}))\|_1 = \|\Phi(c) - (\sum_{i=1}^r n_i \Phi(c_i))\|_1 = \|\Phi(c) - \Theta(n_1, \dots, n_r)^T\|_1 \leq (r-1)K \quad (5.2)$$

where $\Theta := (\Phi(c_1), \dots, \Phi(c_r))$ is a matrix in $M_r(\mathbb{R})$. Since the action of Z on $\prod_{i=1}^r L_i$ is geometric, $\Theta \in \text{GL}_r(\mathbb{R})$. To obtain a quasi-homomorphism from E to Z , we need to composite another two quasi-homomorphisms.

For any $\alpha = (x_1, \dots, x_r)^T \in \mathbb{R}^r$, denote $[\alpha] := ([x_1], \dots, [x_r])^T \in \mathbb{Z}^r$. Define two quasi-homomorphisms $\eta : \mathbb{R}^r \rightarrow \mathbb{Z}^r$ and $\rho : \mathbb{Z}^r \rightarrow Z$ by $\eta(\alpha) := [\Theta^{-1}\alpha]$ for any $\alpha \in \mathbb{R}^r$ and $\rho(n_1, \dots, n_r) := c_1^{n_1} \cdots c_r^{n_r}$ for any $n_1, \dots, n_r \in \mathbb{Z}$. Note that ρ is a group isomorphism between $\mathbb{Z}^r$ and a finite-index subgroup of Z . Let $\varphi := \rho \circ \eta \circ \Phi : E \rightarrow Z$. As a composition of quasi-homomorphisms, φ is a quasi-homomorphism from E to Z .

Step 2: we show $\varphi|_Z$ is close to the identity map.

For any $c = c_1^{n_1} \cdots c_r^{n_r} \in Z$, we denote $\alpha := \Phi(c) - \Theta(n_1, \dots, n_r)^T$. By compositing η on the left, we have

$$\eta(\alpha) = [\Theta^{-1}\alpha] = \eta \circ \Phi(c) - (n_1, \dots, n_r)^T.$$

Next, by compositing ρ on the left, we have

$$\rho([\Theta^{-1}\alpha]) = c^{-1}\varphi(c).$$

Note that Formula (5.2) shows $\|\alpha\|_1 \leq (r-1)K$. Hence, the above equality shows that there exists a finite subset $A \subset Z$ such that $c^{-1}\varphi(c) \in A$ for any $c \in Z$.

Step 3: we construct a bounded representative of $[\omega]$.

For any $g \in E$, $g = \varphi(g) \cdot (\varphi(g)^{-1}g)$. Since φ is a quasi-homomorphism and $\varphi(g)^{-1} \in Z$, one has

$$\varphi(\varphi(g)^{-1}g) \in \varphi(\varphi(g)^{-1})\varphi(g)\Delta(\varphi) \in \varphi(g)^{-1}\varphi(g)A \cdot \Delta(\varphi) = A \cdot \Delta(\varphi).$$

Denote $B = A \cdot \Delta(\varphi)$. The above equality shows that for any $g \in E$, $\varphi(g)^{-1}g \in \varphi^{-1}(B)$. Since $\varphi(g) \in Z = \ker(\pi)$, $\pi(\varphi(g)^{-1}g) = \pi(g)$. This allows us to pick a section $s : G \rightarrow E$ such that $s(G) \subseteq \varphi^{-1}(B)$. Let $\omega_s : G \times G \rightarrow Z$ given by $\omega_s(g, h) = s(g)s(h)s(gh)^{-1}$ be the corresponding cocycle. Since the image of ω_s lies in Z , **Step 2** implies the equivalence between boundedness of ω_s and boundedness of $\varphi(\omega_s)$. The latter is clearly bounded since $s(G) \subseteq \varphi^{-1}(B)$. Therefore, the Euler class $[\omega_s]$ is bounded.

“(7) $\Rightarrow$ (2)”: By definition, there exists a finite-index subgroup $E' \leq E$ such that the following central extension $1 \rightarrow Z' \rightarrow E' \xrightarrow{\pi} G' \rightarrow 1$ quasi-splits where $Z' = Z \cap E'$ and $G' = \pi(E')$. By the equivalence of (2) and (3), there exists a quasi-retraction $\Phi' : E' \rightarrow Z'$. Assume $Z' = \mathbb{Z}^r$ that embeds into $R = \mathbb{R}^r$ canonically. We consider Φ' as a quasi-homomorphism with values in R . By Lemma 2.3.4, $\Phi' : E' \rightarrow R$ induces a quasi-homomorphism $\hat{\mathcal{T}}(\Phi') : E \rightarrow R$ whose restriction on Z' is the identity map.

Denote $\bar{E} = E/Z'$, $\bar{Z} = Z/Z'$. Then $G = E/Z \cong \frac{E/Z'}{Z/Z'} = \bar{E}/\bar{Z}$. The original central extension $1 \rightarrow Z \rightarrow E \xrightarrow{\pi} G \rightarrow 1$ is splitted into the following two central extensions

$$1 \rightarrow Z' \rightarrow E \xrightarrow{\pi_1} \bar{E} \rightarrow 1 \tag{5.3}$$

and

$$1 \rightarrow \bar{Z} \rightarrow \bar{E} \xrightarrow{\pi_2} G \rightarrow 1. \tag{5.4}$$

Since the map $\hat{\mathcal{T}}(\Phi') : E \rightarrow R$ is a quasi-homomorphism whose restriction on Z' is the identity map, the equivalence of (2) and (3) shows that the central extension (5.3) quasi-splits. Note that $\bar{Z}$ is finite since Z' is of finite index in Z . Thus any normalized section of the central extension (5.4) is a quasi-splitting.

Let $s_1 : \bar{E} \rightarrow E$ and $s_2 : G \rightarrow \bar{E}$ be two quasi-splittings. Define a map $s : G \rightarrow E$ as the composite map $s_1 \circ s_2$. Then s is the desired quasi-splitting from G to E . □

Remark 5.1.6. (1) Suppose that $[\omega]$ is bounded. Let $s : G \rightarrow E$ be the quasi-splitting given by (2). From the proof of [Wan25, Theorem 3.15], the map $\phi : E \rightarrow Z$ defined by $\phi(g) = gs(\bar{g})^{-1}$ is a desired quasi-retraction for (3). Note that $\phi(g) = 1$ for any $g \in s(G)$. Since each quasimorphism ϕ_i in (4) is obtained by compositing ϕ with a natural projection, one gets that $\phi_i(s(G)) = 0$

for $1 \leq i \leq r$. According to Lemma 4.2.5, this implies that $s(G)$ has a bounded orbit on each quasi-line L_i obtained in (6).

(2) When the group extension is not a central extension, Items (2), (3), (7) in Lemma 5.1.5 may not be equivalent anymore although the relation “(3) $\Rightarrow$ (2) $\Rightarrow$ (7)” always holds. In [Wan25], Wan shows that a general group extension admits a quasi-retraction if and only if it quasi-splits and the image of this quasi-splitting almost commutes with the normal subgroup. Besides, there exists a group extension which virtually quasi-splits but not quasi-splits. For example, the surjective homomorphism from F_2 to the infinite dihedral group D_∞ gives a short exact sequence of groups $1 \rightarrow H \rightarrow F_2 \rightarrow D_\infty \rightarrow 1$. Since D_∞ is virtually cyclic, it is easy to see this extension virtually quasi-splits. However, it does not quasi-split since any unbounded quasi-homomorphism to F_2 is either a homomorphism or a quasimorphism with images in a cyclic subgroup [FK16, Theorem 4.1].

The following lemma is a complement to Lemma 5.1.5 (6).

Lemma 5.1.7. *If $[\omega]$ is bounded and G admits a proper action on a metric space X , then the induced action of E on $X \times \prod_{i=1}^r L_i$ by $g \cdot (x, y_1, \dots, y_r) := (\bar{g}x, gy_1, \dots, gy_r)$ is proper where $L_1, \dots, L_r$ are provided by Lemma 5.1.5 (6). If in addition that G is generated by a finite set S , then the induced action of E on $\mathcal{G}(G, S) \times \prod_{i=1}^r L_i$ is geometric.*

Proof. By Lemma 5.1.5 (2), there exists a quasi-splitting $s : G \rightarrow E$. Any group element $g \in E$ has a decomposition $g = g_Z g_G$, where $g_Z = g \cdot s(\bar{g})^{-1}$ and $g_G = s(\bar{g})$. Note that $g_Z \in Z$ and $\bar{g} = \bar{g}_G$. Fix a basepoint $(x, o_1, \dots, o_r) \in X \times \prod_{i=1}^r L_i$. For any $C > 0$, we denote $B(C) := \{g \in E : |x - \bar{g}x| + \sum_{i=1}^r |o_i - g o_i| \leq C\}$. Let $g \in B(C)$. There are only finitely many choices of $g_G = s(\bar{g})$ in E since $\{\bar{g} \in G : |x - \bar{g}x| \leq C\}$ is finite. Now we fix such a choice of g_G . By triangle inequality,

$$\sum_{i=1}^r |o_i - g_Z o_i| \leq \sum_{i=1}^r (|o_i - g_Z g_G o_i| + |o_i - g_G o_i|) = \sum_{i=1}^r |o_i - g o_i| + \sum_{i=1}^r |o_i - g_G o_i| \leq C + \sum_{i=1}^r |o_i - g_G o_i|.$$

Since the action $Z \curvearrowright \prod_{i=1}^r L_i$ is geometric, there are only finitely many choices of g_Z . This shows that $B(C)$ is finite for any $C > 0$, which gives the properness of the diagonal action $E \curvearrowright X \times \prod_{i=1}^r L_i$.

Suppose that G is generated by a finite set S . By the above arguments, we only need to show that the induced action $E \curvearrowright \mathcal{G}(G, S) \times \prod_{i=1}^r L_i$ is cobounded. Fix a basepoint $(1, o_1, \dots, o_r) \in \mathcal{G}(G, S) \times \prod_{i=1}^r L_i$ and an arbitrary point $(x, y_1, \dots, y_r) \in \mathcal{G}(G, S) \times \prod_{i=1}^r L_i$. Since $Z \curvearrowright \prod_{i=1}^r L_i$ is geometric, there exists $c \in Z$ such that $c \cdot (o_1, \dots, o_r)$ is uniformly close to $(y_1, \dots, y_r)$. Let $g = s(x)c \in E$. By Remark 5.1.6, $s(G)$ has a bounded orbit on $\prod_{i=1}^r L_i$. Hence, $g \cdot (1, o_1, \dots, o_r)$ is uniformly close to $(x, y_1, \dots, y_r)$, which completes the proof of coboundedness. $\square$

Corollary 5.1.8. *Assume that $[\omega]$ is bounded. If G has property (PH') or (PH), then E has property (PH') or (PH) respectively. If G is finitely generated and has property (QT') or (QT), then E has property (QT') or (QT) respectively.*

Proof. If G has property (PH'), then it is easy to see from Lemma 5.1.7 that E has property (PH'). Now assume that G has property (PH). Then G has a finite-index subgroup G' with

property (PH'). Denote $E' = \pi^{-1}(G')$, which is a finite-index subgroup of E . Note that Z is contained in the center of E' . Hence, one has the following central extension

$$1 \rightarrow Z \rightarrow E' \xrightarrow{\pi'} G' \rightarrow 1$$

where $\pi' = \pi|_{E'}$. The corresponding Euler class $[\omega']$ is bounded since $[\omega]$ is bounded. The conclusion thus follows from the former case.

Next, we assume that G is finitely generated and has property (QT') or (QT). By the Svarc-Milnor lemma, the geometric action $E \curvearrowright \mathcal{G}(G, S) \times \prod_{i=1}^r L_i$ in Lemma 5.1.7 gives a finite generating set F of E such that the orbit map $\mathcal{G}(E, F) \rightarrow \mathcal{G}(G, S) \times \prod_{i=1}^r L_i$ is a quasi-isometry. Via this quasi-isometry, one gets easily that E has also property (QT') or (QT), respectively. $\square$

Corollary 5.1.9. *If E has property (PH), then $[\omega]$ is bounded.*

Proof. By Lemma 5.1.5, it suffices to show the central extension (5.1) virtually quasi-splits.

Let $E' \leq E$ be a finite-index subgroup that has property (PH'). Let $Z' = Z \cap E'$ and $G' = \pi(E')$. Consider the following central extension $1 \rightarrow Z' \rightarrow E' \xrightarrow{\pi} G' \rightarrow 1$. By Lemma 5.1.2, there exist quasi-lines $L_1, \dots, L_r$ on which E' acts linearly such that the diagonal action of Z' on $\prod_{i=1}^r L_i$ is geometric. This shows that the above central extension satisfies Lemma 5.1.5 (6). As a result of Lemma 5.1.5, the above central extension quasi-splits, which amounts to saying that the original central extension (5.1) virtually quasi-splits. $\square$

Note that Corollary 5.1.8 gives one direction of Theorems 1.3.1, 1.3.2, and Corollary 5.1.9 gives part of the other direction. We will complete the remaining proof in Section 5.2.

5.2 Property (PH) under quotients

Keep the setup of Section 5.1.2: $1 \rightarrow Z \xrightarrow{\iota} E \xrightarrow{\pi} G \rightarrow 1$ is a central extension of groups where Z is finitely generated and $[\omega] \in H^2(G, Z)$ is the corresponding Euler class. We identify $\iota(Z)$ with Z and write $\bar{g}$ to mean $\pi(g) \in G$.

The goal of this section is the following proposition, which together with Corollary 5.1.9 almost gives the other direction of Theorem 1.3.1.

Proposition 5.2.1. *If E has property (PH'), then G has property (PH').*

5.2.1 Construction of hyperbolic quotient space

Let X be a connected graph with path metric. Let Γ be a group acting isometrically on X . Let $H \triangleleft \Gamma$. Recall that there is a well-defined quotient metric on X/H , and Γ/H acts isometrically on X/H .

Lemma 5.2.2. *Let Γ be a group acting coboundedly on a hyperbolic space X . Let c be a central element of Γ . If c is elliptic on X , then $X/\langle c \rangle$ is a hyperbolic space quasi-isometric to X . If c is loxodromic on X , then $X/\langle c \rangle$ is a bounded space.*

Proof. If c is elliptic on X , then the conclusion can be deduced from [BFG24, Lemma 4.10]. If c is loxodromic on X , then Lemma 5.1.1 shows that X is a quasi-line. So $X/\langle c \rangle$ is a bounded space. $\square$

From now on, we assume that E has property (PH') from actions $E \curvearrowright X_1, \dots, E \curvearrowright X_n$.

According to the structure theorem of finitely generated abelian groups, there exist $r \in \mathbb{N}$ and a finite abelian group F such that $Z \cong \mathbb{Z}^r \oplus F$. By Lemma 5.2.2, each quotient space X_i/F is still hyperbolic and the induced diagonal action $E/F \curvearrowright \prod_{i=1}^n X/F$ is still proper. Based on this fact, we can assume that Z is infinite (i.e. $r \geq 1$) and torsion-free. Let $\{c_1, \dots, c_r\}$ be a standard generating set of Z .

For simplicity, we denote $\bar{X}_i := X_i/Z$ for each $1 \leq i \leq n$. Up to re-ordering, let $\bar{X}_1, \dots, \bar{X}_m$ ($1 \leq m \leq n$) be all bounded quotient spaces. We collect some basic properties about $\bar{X}_i$ ($1 \leq i \leq n$).

Lemma 5.2.3. *1. For $1 \leq i \leq m$, X_i is a quasi-line and the action of E on X_i is orientable lineal.*

2. For each $1 \leq i \leq n$, $\bar{X}_i$ is hyperbolic and the following are equivalent:

- (i) $m+1 \leq i \leq n$, i.e. $\bar{X}_i$ is unbounded;*
- (ii) Z has an elliptic action on X_i ;*
- (iii) X_i is quasi-isometric to $\bar{X}_i$.*

3. The integer m satisfies $r \leq m \leq n$. Moreover, if $m = n$, then E and G are two finitely generated elementary (i.e. virtually abelian) groups.

Proof. (1) This follows from Lemma 5.1.1.

(2) This follows easily from Lemma 5.2.2.

(3) Since each $\bar{X}_i$ is unbounded for $m+1 \leq i \leq n$, it follows from Item (2) that Z has an elliptic action on $\prod_{i=m+1}^n X_i$. Thus the proper diagonal action $Z \curvearrowright \prod_{i=1}^n X_i$ passes to a proper diagonal action $Z \curvearrowright \prod_{i=1}^m X_i$. By Item (1), these orientable lineal actions $Z \curvearrowright X_1, \dots, Z \curvearrowright X_m$ give the (PH') property of Z . Then it follows from Lemma 4.3.1 that $r \leq m$. Moreover, if $m = n$, then all $\bar{X}_i$ ($1 \leq i \leq n$) are bounded. By Item (1) again, E has property (PH') from orientable lineal actions. As a result of Lemma 4.3.1, E is a finitely generated elementary group. As a quotient of E , G is also finitely generated and elementary. $\square$

5.2.2 Proof of Theorem 1.3.1

Keep all notations in Section 5.2.1. We are in a position to prove Proposition 5.2.1.

Proof of Proposition 5.2.1. We will divide the proof into two steps.

Step 1: we construct auxiliary quasi-lines on which G acts lineally.

Recall from Lemma 5.2.3 that each X_i ($1 \leq i \leq m$) is a quasi-line and the action of E on X_i is orientable lineal. By Lemma 4.2.4, there exists a uniform constant $\epsilon \geq 0$ such that each X_i is $(1, \epsilon)$ -quasi-isometric to $\mathbb{R}$. Denote $f_i : X_i \rightarrow \mathbb{R}$ as a $(1, \epsilon)$ -quasi-isometry. It is easy to

see the map $\chi_i : E \rightarrow \mathbb{R}$ given by $\chi_i(g) := f_i \circ g \circ f_i^{-1}(0)$ is an unbounded quasimorphism for $1 \leq i \leq m$.

Since E has property (PH'), Corollary 5.1.9 shows that $[\omega]$ is bounded. This is equivalent to the existence of a quasi-splitting $s : G \rightarrow E$ by Lemma 5.1.5. Then the composite map $\chi'_i = \chi_i \circ s : G \rightarrow \mathbb{R}$ is also a (possibly bounded) quasimorphism. WLOG, we assume that $\chi'_1, \dots, \chi'_k$ are all unbounded quasimorphisms. If $k = 0$, then there is nothing to supplement in this step. If $1 \leq k \leq m$, then Lemma 4.2.5 shows that for each $1 \leq i \leq k$, there exists a generating set S_i of G such that $[S_i] \in \mathcal{H}_\ell^+(G)$. Moreover, [ABO19, Lemma 4.15] shows that the map χ'_i is also a quasi-isometry between $\mathcal{G}(G, S_i)$ and $\mathbb{R}$. Let $\lambda \geq 1, \epsilon' \geq 0$ be two constants such that $\chi'_i : \mathcal{G}(G, S_i) \rightarrow \mathbb{R}$ is a (λ, ϵ') -quasi-isometry for each $1 \leq i \leq k$.

Step 2: we show the diagonal action $G \curvearrowright \prod_{i=1}^k \mathcal{G}(G, S_i) \times \prod_{i=m+1}^n \bar{X}_i$ is proper.

Fix a basepoint $(x_1, \dots, x_n) \in \prod_{i=1}^n X_i$ with $x_i \in f_i^{-1}(0)$ for $1 \leq i \leq m$. Let $\bar{x}_i \in \bar{X}_i$ be the image of $x_i \in X_i$ under the quotient map.

Claim. There exists a constant $M > 0$ such that

$$\sum_{i=1}^k |\bar{g}|_{S_i} + \sum_{i=m+1}^n |\bar{x}_i - \bar{g}\bar{x}_i| \geq \lambda^{-1} \left(\sum_{i=1}^n |x_i - s(\bar{g})x_i| \right) - M$$

for any $\bar{g} \in G$.

Proof of Claim. We prove the claim in three cases according to the range of i .

Firstly, we estimate $|x_i - s(\bar{g})x_i|$ for $1 \leq i \leq k$. Note that each χ'_i is a (λ, ϵ') -quasi-isometry. Hence, $|\chi'_i(\bar{g})| = |\chi_i(s(\bar{g}))| \leq \lambda|\bar{g}|_{S_i} + \epsilon'$. By definition of χ_i , $\chi_i(s(\bar{g})) = f_i \circ s(\bar{g}) \circ f_i^{-1}(0) = f_i(s(\bar{g})x_i)$. Since each f_i is a $(1, \epsilon)$ -quasi-isometry and $f_i(x_i) = 0$, one gets that

$$|x_i - s(\bar{g})x_i| \leq |f_i(x_i) - f_i(s(\bar{g})x_i)| + \epsilon = |\chi_i(s(\bar{g}))| + \epsilon = |\chi'_i(\bar{g})| + \epsilon \leq \lambda|\bar{g}|_{S_i} + \epsilon + \epsilon'.$$

Secondly, we estimate $|x_i - s(\bar{g})x_i|$ for $k+1 \leq i \leq m$. In this case, each $\chi'_i : G \rightarrow \mathbb{R}$ is a bounded quasimorphism. The same arguments as above show that

$$|x_i - s(\bar{g})x_i| \leq |f_i(x_i) - f_i(s(\bar{g})x_i)| + \epsilon = |\chi'_i(\bar{g})| + \epsilon \leq M_1$$

for some $M_1 > 0$ independent of $\bar{g}$.

Finally, we estimate $|x_i - s(\bar{g})x_i|$ for $m+1 \leq i \leq n$. In this case, Z has an elliptic action on each X_i since $\bar{X}_i(m+1 \leq i \leq n)$ are still unbounded hyperbolic spaces. Thus there exists a constant $M_2 > 0$ such that $|x_i - cx_i| \leq M_2$ for any $c \in Z$. By the definition of quotient metric, one has that $|\bar{x}_i - \bar{g}\bar{x}_i| = |x_i - s(\bar{g})cx_i|$ for some $c \in Z$. Therefore, by triangle inequality,

$$|x_i - s(\bar{g})x_i| \leq |x_i - s(\bar{g})cx_i| + |x_i - cx_i| = |\bar{x}_i - \bar{g}\bar{x}_i| + |x_i - cx_i| \leq |\bar{x}_i - \bar{g}\bar{x}_i| + M_2.$$

In conclusion, we get that

$$\begin{aligned}
& \sum_{i=1}^k |\bar{g}|_{S'_i} + \sum_{i=m+1}^n |\bar{x}_i - \bar{g}\bar{x}_i| \\
& \geq \lambda^{-1} \sum_{i=1}^k |x_i - s(\bar{g})x_i| - k\lambda^{-1}(\epsilon + \epsilon') + \sum_{i=m+1}^n |x_i - s(\bar{g})x_i| - M_2(n-m) \\
& \geq \lambda^{-1} \sum_{i=1}^k |x_i - s(\bar{g})x_i| + \sum_{i=k+1}^n |x_i - s(\bar{g})x_i| - k\lambda^{-1}(\epsilon + \epsilon') - M_2(n-m) - M_1(m-k) \\
& \geq \lambda^{-1} \left(\sum_{i=1}^n |x_i - s(\bar{g})x_i| \right) - M
\end{aligned}$$

where $M = k\lambda^{-1}(\epsilon + \epsilon') + M_2(n-m) + M_1(m-k)$. $\square$

Since the diagonal action $E \curvearrowright \prod_{i=1}^n X_i$ is proper, it follows from the above Claim that the diagonal action $G \curvearrowright \prod_{i=1}^k \mathcal{G}(G, S_i) \times \prod_{i=m+1}^n \bar{X}_i$ is also proper. This gives the (PH') property of G . $\square$

Finally, we can complete the proof of Theorems 1.3.1, 1.3.2 now.

Proof of Theorem 1.3.1. Recall that Corollary 5.1.8 gives one direction of Theorem 1.3.1. As for the other direction, Corollary 5.1.9 and Proposition 5.2.1 give a partial proof. It suffices for us to show that if E has property (PH), then G has also property (PH).

Let E' be a finite-index subgroup of E so that E' has property (PH'). Now we have the following central extension of groups

$$1 \rightarrow Z' \rightarrow E' \rightarrow G' \rightarrow 1$$

where $Z' = Z \cap E'$ and G' , seen as the image of $\pi|_{E'}$, is a finite-index subgroup of G . Let $[\omega']$ be the corresponding Euler class. By Proposition 5.2.1, G' has property (PH'). Hence, G has property (PH) since $[G : G'] < \infty$. $\square$

Proof of Theorem 1.3.2. Recall that Corollary 5.1.8 gives one direction of Theorem 1.3.2. As for the other direction, Corollary 5.1.9 give a partial proof. It suffices for us to show that if E has property (QT') or (QT), then G has also property (QT') or (QT), respectively.

Case I: if E has property (QT'), then G has property (QT').

According to the proof of Lemma 4.0.2, there exist finitely many quasi-trees $T_i (1 \leq i \leq n)$ on which E acts coboundedly and the orbit map from E to the product $X = \prod_{i=1}^n T_i$ equipped with ℓ^1 -metric is a quasi-isometric embedding. Since E has property (QT') and thus (PH), $[\omega]$ is bounded by Corollary 5.1.9. Let $s : G \rightarrow E$ be a quasi-splitting provided by Lemma 5.1.5. Fix a finite generating set F of G and a finite generating set A of Z . It is easy to see that $F' = A \cup s(F)$ is a finite generating set of E . Fix a basepoint $x = (x_1, \dots, x_n) \in \prod_{i=1}^n T_i$. The (QT') property of E shows that the orbit map $\mathcal{G}(E, F') \rightarrow \prod_{i=1}^n T_i, g \mapsto gx = (gx_1, \dots, gx_n)$ is

a (λ_0, ϵ_0) -quasi-isometric embedding for some $\lambda_0 \geq 1, \epsilon_0 \geq 0$. That is,

$$\lambda_0^{-1}|g|_{F'} - \epsilon_0 \leq |x - gx| = \sum_{i=1}^n |x_i - gx_i| \leq \lambda_0|g|_{F'} + \epsilon_0 \quad (5.5)$$

for any $g \in E$.

Let $\bar{T}_i = T_i/Z$ be the quotient space for $1 \leq i \leq n$. Similar to the proof of Proposition 5.2.1, we assume that there exist $1 \leq k \leq m \leq n$ such that $\bar{T}_i (1 \leq i \leq m)$ are all bounded quotient spaces and $\chi'_i = \chi_i \circ s : G \rightarrow \mathbb{R} (1 \leq i \leq k)$ are all unbounded quasimorphisms. Then there exist generating sets S_i of G such that $[S_i] \in \mathcal{H}_\ell^+(G)$ for $1 \leq i \leq k$ and G admits a proper diagonal action on $\prod_{i=1}^k \mathcal{G}(G, S_i) \times \prod_{i=m+1}^n \bar{T}_i$. Moreover, the Claim in the proof of Proposition 5.2.1 shows that there exists $M > 0$ such that

$$\sum_{i=1}^k |\bar{g}|_{S_i} + \sum_{i=m+1}^n |\bar{x}_i - \bar{g}\bar{x}_i| \geq \lambda^{-1} \left(\sum_{i=1}^n |x_i - s(\bar{g})x_i| \right) - M$$

for any $\bar{g} \in G$.

Together with the inequality (5.5), one has that

$$\begin{aligned} \sum_{i=1}^k |\bar{g}|_{S_i} + \sum_{i=m+1}^n |\bar{x}_i - \bar{g}\bar{x}_i| &\geq \lambda^{-1} \left(\sum_{i=1}^n |x_i - s(\bar{g})x_i| \right) - M \\ &\geq (\lambda\lambda_0)^{-1} |s(\bar{g})|_{F'} - \lambda^{-1}\epsilon_0 - M \geq (\lambda\lambda_0)^{-1} |\bar{g}|_F - \lambda^{-1}\epsilon_0 - M \end{aligned}$$

where the last inequality holds since $|s(\bar{g})|_{F'} \geq |\bar{g}|_F$.

On the other hand, it follows from the triangle inequality that

$$\sum_{i=1}^k |\bar{g}|_{S_i} + \sum_{i=m+1}^n |\bar{x}_i - \bar{g}\bar{x}_i| \leq \lambda' |\bar{g}|_F$$

where $\lambda' = \max_{t \in F} \left(\sum_{i=1}^k |t|_{S_i} + \sum_{i=m+1}^n |\bar{x}_i - t\bar{x}_i| \right)$.

By setting $\lambda_1 = \max\{\lambda', \lambda\lambda_0\}$ and $\epsilon_1 = \lambda^{-1}\epsilon_0 + M$, we have shown that the orbit map $\mathcal{G}(G, F) \rightarrow \prod_{i=1}^k \mathcal{G}(G, S_i) \times \prod_{i=m+1}^n \bar{T}_i, \bar{g} \mapsto (\bar{g}, \dots, \bar{g}, \bar{g}\bar{x}_{m+1}, \dots, \bar{g}\bar{x}_n)$ is a (λ_1, ϵ_1) -quasi-isometric embedding. This gives the (QT') property of G since each $\mathcal{G}(G, S_i)$ is a quasi-line.

Case II: if E has property (QT), then G has property (QT).

Suppose that E has property (QT). Then E contains a finite-index subgroup E' that has property (QT'). Denote $Z' = Z \cap E'$ and $G' = \pi(E')$. Since $[E : E'] < \infty$, it follows that $[Z : Z'] < \infty$ and $[G : G'] < \infty$. Now we focus on the following central extension

$$1 \rightarrow Z' \rightarrow E' \xrightarrow{\pi'} G' \rightarrow 1.$$

Let $[\omega']$ be the corresponding Euler class. The same arguments in **Case I** shows that G' has property (QT') and thus (QT). Since property (QT) is a commensurability invariant, we get that G has property (QT). $\square$

5.3 Applications

5.3.1 Mapping class groups

In [BBF21], Bestvina-Bromberg-Fujiwara show that the mapping class group of any finite-type surface has property (QT). Recall from Section 2.4 that the capping sequence of mapping class groups has a bounded Euler class. Therefore, as a corollary of Theorem 1.3.2, we have

Corollary 5.3.1. *MCG(Σ) has property (QT) for any finite-type surface Σ with boundary. In particular, braid groups have property (QT).*

Now let Σ be a finite-type surface, possibly with boundary. A *multicurve* $C \subset \Sigma$ is a collection of pairwise disjoint, homotopically distinct, essential simple closed curves. Let $C = \{\alpha_1, \dots, \alpha_n\}$ be a multicurve on Σ . Let $\text{MCG}(\Sigma; C) < \text{MCG}(\Sigma)$ denote the stabilizer of C . Let N be an open neighborhood of $\bigcup_{i=1}^n \alpha_i$, and let $\{\alpha_i^+, \alpha_i^-\}$ be the boundary components of the neighborhood of α_i . By Theorem 2.4.1, the inclusion $\Sigma - N \hookrightarrow \Sigma$ induces a central extension

$$1 \rightarrow T_{C^\pm} \rightarrow \text{MCG}(\Sigma - N) \rightarrow \text{MCG}(\Sigma; C) \rightarrow F,$$

where $T_{C^\pm}$ is the free abelian group generated by $T_{\alpha_1^+} T_{\alpha_1^-}^{-1}, \dots, T_{\alpha_n^+} T_{\alpha_n^-}^{-1}$ and F is a finite group. By Corollary 5.3.1, $\text{MCG}(\Sigma - N)$ has (QT). Therefore, Theorem 1.3.2 gives the following.

Corollary 5.3.2. *Let Σ be a finite-type surface. The stabilizer in $\text{MCG}(\Sigma)$ of any multicurve on Σ has property (QT).*

5.3.2 Outer automorphism groups of hyperbolic groups

The basic idea for understanding the outer automorphism group $\text{Out}(G)$ of a torsion-free one-ended word hyperbolic group G is due to Rips and Sela [RS94; Sel97]. Since G has a canonical JSJ splitting [Lev05], automorphisms of G may be seen on this splitting: $\text{Out}(G)$ is virtually generated by Dehn twists along edges of the splitting, together with mapping class groups of punctured surfaces (automorphism groups of ‘maximal quadratically hanging’ subgroups).

Theorem 5.3.3. [Lev05, Theorem 1.1][Sel97, Theorem 1.9] *Let G be a torsion-free one-ended hyperbolic group. There is a central extension*

$$1 \rightarrow \mathbb{Z}^n \rightarrow \text{Out}'(G) \rightarrow \prod_{v \in V_2} \text{PMCG}(\Sigma_v) \rightarrow 1$$

where $\text{Out}'(G)$ has finite index in $\text{Out}(G)$, and $n = |\mathcal{E}| - |V_1|$ is the number of edges in the ‘essential JSJ decomposition of G ’.

Theorem 5.3.4. [Lev05, Theorem 1.2] *Let G be a torsion-free one-ended hyperbolic group. Let r be the number of vertices of V_1 connected only to vertices of V_2 , and $q = |\mathcal{E}_3| - |V_1| + r$. The group $\text{Out}(G)$ is virtually a direct product $\mathbb{Z}^q \times M$, where M is the quotient of $\prod_{v \in V_2} \text{PMCG}^\partial(\Sigma_v)$ by a central subgroup isomorphic to $\mathbb{Z}^r$.*

By Corollary 5.3.1 and Theorem 1.3.2, M in Theorem 5.3.4 has property (QT). Since property (QT) is a commensurability invariant, Theorem 5.3.4 implies that

Corollary 5.3.5. *Let G be a torsion-free one-ended hyperbolic group. Then $\text{Out}(G)$ has property (QT) .*

By Theorem 1.3.2 again, together with the above corollary,

Corollary 5.3.6. *The central extension in Theorem 5.3.3 has a bounded Euler class.*

CHAPTER 6

Extension of quasimorphisms

In this chapter, we focus on the study of the extension problem of quasimorphisms (Question 1.4.1).

6.1 Quasi-invariant quasimorphisms

Let $K < G$ and $\varphi \in \hat{Q}(K)$. If φ extends to a quasimorphism $\Phi \in \hat{Q}(G)$, we have

$$|\Phi(ghg^{-1}) - \Phi(h)| \leq |\Phi(g) + \Phi(g^{-1})| + 2D(\Phi) \leq 4D(\Phi)$$

for any $g, h \in G$. Restricting to K , we know that

$$D^G(\varphi) := \sup\{|\varphi(k_1) - \varphi(k_2)| \mid k_1, k_2 \in K \text{ that are conjugate in } G\} < \infty. \quad (6.1)$$

Thus, (6.1) gives a necessary condition for extendability of φ . We say that a quasimorphism $\varphi \in \hat{Q}(K)$ is *G-quasi-invariant* if it satisfies (6.1). Let

$$\hat{Q}(K)^G := \{\text{all } G\text{-quasi-invariant quasimorphisms on } G\}.$$

The above discussion shows that there is a restricting map

$$\hat{r} : \hat{Q}(G) \rightarrow \hat{Q}(K)^G.$$

Therefore, Question 1.4.1 is essentially asking what the image of $\hat{r}$ is. Note that we do not explicitly require quasi-invariance in Theorem 1.4.3. This is because controlled quasimorphisms are always quasi-invariant (see Lemma 6.3.5).

6.2 Relative presentations

For any generating set Z of a group G , we write Z^* to mean the set of finite words on Z . For any $w_1, w_2 \in Z^*$, we write $w_1 =_G w_2$ to mean that w_1 and w_2 represents the same element in G .

A *relative presentation* of G with respect to a subgroup K and a relative generating set X

is a presentation of the form

$$G = \langle K, X \mid R_K, R \rangle, \quad (6.2)$$

where $R_K \subset K^*$ is the set of all words on K that represent the identity of K . A word on $K \sqcup X$ is *reduced* if it does not contain a subword of the form $k_1 k_2$ with $k_1, k_2 \in K - \{1\}$ or a subword of the form xx^{-1} with $x \in X$. Throughout this chapter, we can always assume that R only contains reduced words. Thus, any word in R can be identified with an element in $K * F(X)$.

For any $a \in K * F(X)$, we write $|a|$ to mean the word length of a on the generating set $K \sqcup X$. The unique word on $K \sqcup X$ whose length achieves $|a|$ is called the *normal form* of a .

Let $\theta : K * F(X) \rightarrow G$ be the natural projection. Clearly, $\ker(\theta)$ is the normal closure of R in $K * F(X)$. A function $f : \mathbb{N} \rightarrow \mathbb{R}$ is called a *relative isoperimetric function* of (6.2) if for every element $a \in \ker(\theta)$ with $|a| \leq n$, there exists $m \leq f(n)$ and an expression

$$a = \prod_{i=1}^m h_i r_i h_i^{-1},$$

where $h_i \in K * F(X)$, $r_i \in R$.

Definition 6.2.1. The relative presentation (6.2) is *bounded* if the words from R have uniformly bounded lengths. The relative presentation (6.2) is *strongly bounded* if it is bounded and the set of letters from K appearing in words from R is finite. The relative presentation (6.2) is *finite* if R is finite.

Note that a finite relative presentation is always strongly bounded. We refer to [DGO17] for more discussions. In particular, there is a characterization for weak hyperbolicity using bounded relative presentations as below.

Definition 6.2.2. We say that G is *weakly hyperbolic* relative to K and X if the Cayley graph $\Gamma(G, K \sqcup X)$ is hyperbolic.

Lemma 6.2.3. [DGO17, Lemma 4.9] Let $K < G$. Let X be a relative generating set of G with respect to K . The following are equivalent.

1. G is weakly hyperbolic relative to K and X .
2. There exists a bounded relative presentation $G = \langle K, X \mid R_K, R \rangle$ with a linear relative isoperimetric function.
3. There exists a bounded presentation $G = \langle K, X \mid R'_K, R \rangle$ with a linear isoperimetric function in the classical sense, where $R'_K \subset R_K$ consists of the words of length at most 3 and thus can be seen as the multiplication table of K .

For strongly bounded relative presentations, we recall the following.

Definition 6.2.4. [DGO17, Theorem 4.24] A subgroup $H < G$ is *hyperbolically embedded* with respect to a subset $X \subset G$ if there exists a strongly bounded relative presentation $G = \langle H, X \mid R_H, R \rangle$ with a linear relative isoperimetric function.

We write $H \hookrightarrow_h (G, X)$ to mean that H is a hyperbolically embedded subgroup of G with respect to X . We also write $H \hookrightarrow_h G$ to mean that $H \hookrightarrow_h (G, X)$ for some $X \subset G$.

For finite relative presentations, we recall the following definition of relatively hyperbolic groups given by Osin [Osi06], which is equivalent to that of Farb's [Far98].

Definition 6.2.5. We say that G is *hyperbolic* relative to a subgroup K and a relative generating set X if there exists a finite relative presentation $G = \langle K, X \mid R_K, R \rangle$ with a linear relative isoperimetric function.

6.3 Controlled quasimorphisms

We are going to introduce the core definition of this paper, i.e., controlled quasimorphisms. Throughout this section, let G be a group and K a subgroup. Let X be a (symmetric) relative generating set of G with respect to K . Let $F(X)$ be the free group with the basis X . There are two natural projections

$$K \xleftarrow{\eta} K * F(X) \xrightarrow{\theta} G,$$

where η maps every $x \in X$ to the identity and every $k \in K$ to itself, and θ is given by the assumption that $K \sqcup X$ generates G .

For any $\varphi \in \hat{Q}(K)$, let $\tilde{\varphi}$ be the pull-back of φ via η . Specifically,

$$\tilde{\varphi}(a) := \varphi\left(\prod_{i=1}^n k_i\right),$$

where k_i 's are the K -letters that appear in the normal form of $a \in K * F(X)$ in order. Clearly, $\tilde{\varphi}$ is a quasimorphism on $K * F(X)$ with the same defect as φ .

Definition 6.3.1. We say that a quasimorphism $\varphi \in \hat{Q}(K)$ is (G, X) -*controlled* by a function $f : \mathbb{N} \rightarrow \mathbb{R}$ if for any $a \in \ker(\theta)$ with $|a| \leq n$, we have $|\tilde{\varphi}(a)| \leq f(n)$. We say that φ is *linearly* (G, X) -controlled with a constant C_0 if φ is (G, X) -controlled by a linear function $f(n) = C_0 n + C_0$.

First, we explain the motivation behind the definition of controlled quasimorphisms. Let G be a group and $\omega : G^2 \rightarrow \mathbb{Z}$ be 2-cochain. We say that ω is *bounded* if the image $\omega(G, G)$ is bounded. We say that ω is *weakly bounded* if $\omega(g, G)$ and $\omega(G, g)$ are both bounded for all $g \in G$.

Let

$$1 \rightarrow \mathbb{Z} \xrightarrow{\iota} E \xrightarrow{\pi} G \rightarrow 1. \tag{6.3}$$

be a central extension of groups. Recall from Section 2.4 that a central extension is associated with an Euler class in $H^2(G, \mathbb{Z})$. We have seen in Lemma 5.1.5 that the central extension (6.3) has a bounded Euler class if and only if there is a quasimorphism $\Phi : E \rightarrow \mathbb{R}$ such that $\Phi \circ \iota = 1_{\mathbb{Z}}$.

Back to the 1990s, Neumann and Reeves [NR97] proved that any such central extension with G being hyperbolic has a bounded Euler class. A key step in their proof is to employ a

maximizing technique similar to [EF97] to prove the following, which is somehow implicit in their paper.

Lemma 6.3.2. *If the central subgroup $\mathbb{Z}$ is undistorted, then the Euler class of (6.3) is weakly bounded.*

Note that Lemma 6.3.2 gives an answer to [FS23, Question 1.9]. Definition 6.3.1 is inspired by the use of undistortedness of $\mathbb{Z}$ in their proof. Indeed, assume that E is finitely generated by X . It is easy to see that the identity map $1_{\mathbb{Z}}$ is linearly (E, X) -controlled if and only if the central subgroup $\mathbb{Z}$ is undistorted in E .

The following lemma shows that a linear control is necessary for extendability if X is finite.

Lemma 6.3.3. *If G is generated by K and a finite set X , then an extendable quasimorphism on K must be linearly (G, X) -controlled.*

Proof. Assume that $\varphi \in \hat{Q}(K)$ extends to $\psi \in \hat{Q}(G)$. Let $\tilde{\psi} \in \hat{Q}(K * F(X))$ be the pull-back of ψ via θ . Let $B := \max\{|\psi(x)| \mid x \in X\}$.

Let $a \in \ker(\theta)$. Let $x_1, \dots, x_m$ and $k_1, \dots, k_n$ be the X -letters and the K -letters in the normal form of a in order, respectively. Note that $|a| = m + n$. Since $\tilde{\psi}$ is a quasimorphism and $\psi|_K = \varphi$, we have

$$|\tilde{\psi}(a) - \sum_{i=1}^n \varphi(k_i) - \sum_{j=1}^m \psi(x_j)| \leq (m + n - 1)D(\psi).$$

By definition,

$$\begin{aligned} |\tilde{\varphi}(a) - \sum_{i=1}^n \varphi(k_i)| &= |\varphi(\prod_{i=1}^n k_i) - \sum_{i=1}^n \varphi(k_i)| \\ &\leq (n - 1)D(\varphi). \end{aligned}$$

Thus,

$$\begin{aligned} |\tilde{\psi}(a) - \tilde{\varphi}(a)| &\leq mB + (m + n - 1)D(\psi) + (n - 1)D(\varphi) \\ &\leq |a|(B + D(\psi) + D(\varphi)). \end{aligned}$$

The conclusion follows because $|\tilde{\psi}(a)| = |\varphi(1)| \leq D(\varphi)$. □

When X is allowed to be infinite, it is easy to construct counterexamples of Lemma 6.3.3. Let $\Phi \in \hat{Q}(G)$. Assume that the restriction quasimorphism $\varphi = \Phi|_K$ is unbounded. Take a sequence of elements $\{k_n\}_{n \in \mathbb{N}}$ in K such that $\varphi(k_n) \rightarrow \infty$. Fix any $x \in X$. We can produce a new relative generating set X' from X by adding the elements $x_n := k_n^{-1}x \in G$ for every $n \in \mathbb{N}$. Now any element of the form $k_n x_n x^{-1}$ lies in $\ker(\theta)$ and has word length at most 3, while $\tilde{\varphi}(k_n x_n x^{-1}) \rightarrow \infty$ as $n \rightarrow \infty$. However, the following question does not seem obvious.

Question 6.3.4. *Does there exist a group G , a subgroup K , and an extendable quasimorphism $\varphi \in \hat{Q}(K)$ such that φ is not linearly (G, X) -controlled for every relative generating set X ?*

The following lemma shows that controlled quasimorphisms are quasi-invariant. We will not use it, but we prove it here for completeness.

Lemma 6.3.5. *For any $\varphi \in \hat{Q}(K)$, if φ is (G, X) -controlled, then φ is G -quasi-invariant with $D^G(\varphi) \leq 2D(\varphi)$.*

Proof. Let $k \in K$ and $g \in G$ such that $k' =_G gkg^{-1} \in K$. Choose any $a \in \theta^{-1}(g)$. Consider the element $ak^n a^{-1} k'^{-n} \in \ker(\theta)$, where $n \in \mathbb{Z}$. By definition,

$$\begin{aligned} |\tilde{\varphi}(ak^n a^{-1} k'^{-n})| &= |\varphi(\eta(a)k^n \eta(a)^{-1} k'^{-n})| \\ &\geq |\varphi(k^n) + \varphi(k'^{-n})| - 5D(\varphi) \\ &\geq n|\varphi(k) - \varphi(k')| - (2n + 5)D(\varphi). \end{aligned}$$

On the other hand,

$$\begin{aligned} |\tilde{\varphi}(ak^n a^{-1} k'^{-n})| &\leq f(|ak^n a^{-1} k'^{-n}|) \\ &\leq f(2|a| + 2), \end{aligned}$$

where f is the function that controls φ . Combining the above inequalities and letting n be arbitrarily large, we can conclude that $|\varphi(k) - \varphi(k')| \leq 2D(\varphi)$, and thus finish the proof. $\square$

The following proposition gives a partial characterization of controlled quasimorphisms using relative presentations.

Proposition 6.3.6. *Let*

$$G = \langle K, X \mid R_K, R \rangle \tag{6.4}$$

be a relative presentation. Let $\varphi \in \hat{Q}(K)$. If (6.4) has a relative isoperimetric function f and $\sup\{|\tilde{\varphi}(r)| \mid r \in R\}$ is finite, then φ is (G, X) -controlled by a constant multiple of f . Conversely, if (6.4) is bounded and φ is (G, X) -controlled, then $\sup\{|\tilde{\varphi}(r)| \mid r \in R\}$ is finite.

Proof. Let $B := \sup\{|\tilde{\varphi}(r)| \mid r \in R\}$ and $D := D(\varphi)$. For any $a \in \ker(\theta)$, there exists $m \leq f(|a|)$ and an expression

$$a = \prod_{i=1}^m h_i r_i h_i^{-1},$$

where $f_i \in K * F(X)$, $r_i \in R$. Note that $|\tilde{\varphi}(h_i r_i h_i^{-1}) - \tilde{\varphi}(r_i)| \leq 4D$ since $\tilde{\varphi}$ is a quasimorphism. Therefore,

$$|\tilde{\varphi}(a)| \leq \sum_{i=1}^m |\tilde{\varphi}(r_i)| + 5Dm \leq (B + 5D)f(|a|).$$

This shows that φ is (G, X) -controlled by a constant multiple of f .

Now we need to prove the second statement. By definition, for any $\varphi \in \hat{Q}(K)$ that is (G, X) -controlled by a function f ,

$$|\tilde{\varphi}(r)| \leq f(|r|)$$

for any $r \in R$. The right hand side is bounded because R is bounded. $\square$

Corollary 6.3.7. *Assume that G is weakly hyperbolic relative to K and X . A quasimorphism $\varphi \in \hat{Q}(K)$ is linearly (G, X) -controlled if and only if φ is (G, X) -controlled.*

Proof. By Lemma 6.2.3, there exists a bounded relative presentation $G = \langle K, X \mid R_K, R \rangle$ with a linear relative isoperimetric function. Assume that φ is (G, X) -controlled. By the second part of Proposition 6.3.6, $\sup\{|\tilde{\varphi}(r)| \mid r \in R\}$ is finite. Therefore, φ is linearly (G, X) -controlled by the first part of Proposition 6.3.6. $\square$

6.4 Extending controlled quasimorphisms

Throughout this section, let G be a group and K a subgroup. Let X be a (symmetric) relative generating set of G with respect to K . Let $\varphi \in Q_{as}(K)$ with the defect $D(\varphi) = D$. Assume that φ is linearly (G, X) -controlled with a constant $C_0 > 0$. For any $a \in K * F(X)$, we write $|a|_X$ to mean the number of X -letters (with multiplicity) in the normal form of a .

Now let $a \in \ker(\theta)$. If $|a|_X \geq 1$, since φ is linearly controlled, we have

$$\begin{aligned} |\tilde{\varphi}(a)| &\leq C_0|a| + C_0 \\ &\leq C_0(2|a|_X + 1) + C_0 \\ &\leq 4C_0|a|_X. \end{aligned} \tag{6.5}$$

If $|a|_X = 0$, then $a = 1$, because θ is injective on K . By antisymmetry, $\varphi(1) = 0$, and thus (6.5) also holds. We will use the new inequality (6.5) for linearly controlled quasimorphisms in the rest of this section.

Given a constant $C \geq 20C_0 + 11D$, define a function on $K * F(X)$ by

$$\tilde{\varphi}_C(a) := \tilde{\varphi}(a) + C|a|_X.$$

For any $g \in G$, define

$$\Phi_C(g) := \inf\{\tilde{\varphi}_C(a) \mid \theta(a) = g\}.$$

We will prove that the antisymmetrization of Φ_C is the quasimorphism we want.

6.4.1 Auxiliary lemmata

The following coarse triangle inequalities mix those associated with $\tilde{\varphi}$ and $|\cdot|_X$ and are easy to use.

Lemma 6.4.1. *For any $a, b \in K * F(X)$, we have*

- (i) $|\tilde{\varphi}_C(ab) - (\tilde{\varphi}_C(a) + \tilde{\varphi}_C(b))| \leq C|b|_X + D$
- (ii) $|\tilde{\varphi}_C(ab) - (\tilde{\varphi}(a) + \tilde{\varphi}_C(b))| \leq C|a|_X + D$

Proof. For (i), since $\tilde{\varphi}$ is a quasimorphism and $|\cdot|_X$ satisfies the triangle inequality, we estimate that

$$\begin{aligned} |\tilde{\varphi}_C(ab) - (\tilde{\varphi}_C(a) + \tilde{\varphi}_C(b))| &= |\tilde{\varphi}(ab) + C|ab|_X - \tilde{\varphi}(a) - C|a|_X - \tilde{\varphi}(b)| \\ &\leq |\tilde{\varphi}(ab) - \tilde{\varphi}(a) - \tilde{\varphi}(b)| + C||ab|_X - |a|_X| \\ &\leq D + C|b|_X. \end{aligned}$$

Similarly, the inequality (ii) is also true. $\square$

The following lemma shows that Φ_C is a well-defined real-valued function.

Lemma 6.4.2. *For any $g \in G$, $\Phi_C(g)$ is finite. Moreover, for any $k \in K$, $\Phi_C(k) = \varphi(k)$ and the only element in $\theta^{-1}(k)$ that realizes the infimum is k itself.*

Proof. Fix any $g \in G$. Let $a, b \in \theta^{-1}(g)$. Obviously, $ab^{-1} \in \ker(\theta)$. By (6.5), $\tilde{\varphi}(ab^{-1}) \geq -4C_0|ab^{-1}|_X$. Thus,

$$\tilde{\varphi}_C(ab^{-1}) \geq (C - 4C_0)|ab^{-1}|_X \geq 0.$$

The second equality holds if and only if $a = b$. On the other hand, by Lemma 6.4.1, $\tilde{\varphi}_C(ab^{-1}) \leq \tilde{\varphi}_C(a) - \tilde{\varphi}_C(b) + C|b|_X + D$. Therefore,

$$\tilde{\varphi}_C(a) \geq \tilde{\varphi}_C(b) - C|b|_X - D.$$

It follows that $\Phi_C(g) \geq \tilde{\varphi}_C(b) - C|b|_X - D$, which is finite.

For the “moreover” part, let $a \in \theta^{-1}(k)$ such that $a \neq k$. Note that $\tilde{\varphi}_C(k) = \varphi(k)$ and $|a|_X \geq 1$. By (6.5),

$$\begin{aligned} \tilde{\varphi}_C(a) - \tilde{\varphi}_C(k) &= \tilde{\varphi}(a) + C|a|_X - \varphi(k) \\ &\geq \tilde{\varphi}(ak^{-1}) + C|a|_X - D \\ &\geq -4C_0|ak^{-1}|_X + C|a|_X - D \\ &= (C - 4C_0)|a|_X - D \\ &\geq C - 4C_0 - D \\ &> 0, \end{aligned}$$

and hence the conclusion follows. $\square$

6.4.2 The Cayley graph

Let $a \in K * F(X)$ and $\epsilon \geq 0$. If $\tilde{\varphi}_C(a) \leq \Phi_C(\theta(a)) + \epsilon$, we say that a is an ϵ -minimal element. Note that for any $\epsilon > 0$ and any $g \in G$, there exists an ϵ -minimal element in $\theta^{-1}(g)$. Correspondingly, a path in $\Gamma(G, K \sqcup X)$ labeled by the normal form of an ϵ -minimal element is called an ϵ -minimal path. Clearly, the set of ϵ -minimal paths are G -invariant. We fix any $\epsilon \in (0, C_0)$ in the following.

For any path $\gamma \subset \Gamma(G, K \sqcup X)$, we write $l(\gamma)$ to mean its length and use $\mathbf{Lab}(\gamma)$ to denote its label. For simplicity, we also write $\mathbf{Lab}(\gamma)$ to mean the element in $K * F(X)$ represented by $\mathbf{Lab}(\gamma)$ when $\mathbf{Lab}(\gamma)$ is a normal form. The following lemma shows that a subpath of an ϵ -minimal path is $(4D + \epsilon)$ -minimal.

Lemma 6.4.3. *Let $\gamma \subset \Gamma(G, K \sqcup X)$ be an ϵ -minimal path and $\alpha \subset \gamma$ a subpath. Then $\tilde{\varphi}_C(\mathbf{Lab}(\alpha)) \leq \Phi_C(\theta(\mathbf{Lab}(\alpha))) + 4D + \epsilon$.*

Proof. Assume that $\gamma = \beta_1 \alpha \beta_2$. Let $a = \mathbf{Lab}(\alpha)$, $b_1 = \mathbf{Lab}(\beta_1)$, $b_2 = \mathbf{Lab}(\beta_2)$. Let $a' \in \theta^{-1}(\theta(a))$ be an ϵ -minimal element. By assumptions, $\Delta := \tilde{\varphi}_C(b_1 a' b_2) - \tilde{\varphi}_C(b_1 a b_2) \geq -\epsilon$.

On the other hand, note that $|b_1ab_2|_X = |b_1|_X + |a|_X + |b_2|_X$ since γ is labeled by a normal form. We estimate that

$$\begin{aligned}\Delta &= \tilde{\varphi}(b_1a'b_2) - \tilde{\varphi}(b_1ab_2) + C|b_1a'b_2|_X - C|b_1ab_2|_X \\ &\leq \tilde{\varphi}(a') - \tilde{\varphi}(a) + 4D + C|a'|_X - C|a|_X \\ &= \tilde{\varphi}_C(a') - \tilde{\varphi}_C(a) + 4D.\end{aligned}$$

In conclusion, $\tilde{\varphi}_C(a) \leq \Phi_C(\theta(a)) + 4D + \epsilon$. □

Lemma 6.4.4. *Any ϵ -minimal path is a $(3, 2)$ -quasi-geodesic in $\Gamma(G, K \sqcup X)$.*

Proof. Let $\gamma \subset \Gamma(G, K \sqcup X)$ be an ϵ -minimal path and α a subpath. Let β be a path with the same endpoints as α . Let $a = \mathbf{Lab}(\alpha)$, $b = \mathbf{Lab}(\beta)$.

By Lemma 6.4.3, $\tilde{\varphi}_C(a) \leq \Phi_C(\theta(a)) + 4D + \epsilon \leq \tilde{\varphi}_C(b) + 4D + \epsilon$. On the other hand, we estimate by (6.5) that

$$\begin{aligned}\tilde{\varphi}_C(b) - \tilde{\varphi}_C(a) &= \tilde{\varphi}(b) - \tilde{\varphi}(a) + C|b|_X - C|a|_X \\ &\leq \tilde{\varphi}(ba^{-1}) + D + C|b|_X - C|a|_X \\ &\leq 4C_0|ba^{-1}|_X + D + C|b|_X - C|a|_X \\ &\leq (C + 4C_0)|b|_X - (C - 4C_0)|a|_X + D\end{aligned}$$

Therefore,

$$|a|_X \leq \frac{C + 4C_0}{C - 4C_0}|b|_X + \frac{5D + \epsilon}{C - 4C_0}.$$

Note that $l(\alpha) = |a|$ because $\mathbf{Lab}(\gamma)$ is a normal form. Thus, $l(\alpha) \leq 2|a|_X + 1$. On the other hand, $l(\beta) \geq |b| \geq |b|_X$. Choosing β as a geodesic, we conclude that γ is a (λ, μ) -quasi-geodesic, where $\lambda = \frac{2C+8C_0}{C-4C_0} \leq 3$ and $\mu = \frac{10D+2\epsilon}{C-4C_0} + 1 \leq 2$. □

Lemma 6.4.5. *For any $\delta, m \geq 0$, there exists $M \geq 1$ such that the following holds. Assume that $\Gamma(G, K \sqcup X)$ is δ -hyperbolic. For any $g, h \in G$, if g lies within distance m from a geodesic from 1 to gh , then $|\Phi_C(g) + \Phi_C(h) - \Phi_C(gh)| \leq MC$.*

Proof. Let α_1 be an ϵ -minimal path from 1 to g , let α_2 be an ϵ -minimal path from g to gh , and let β be an ϵ -minimal path from 1 to gh . By projecting g to a closest point g' on β , we divide β into $\beta_1\beta_2$. Let γ be an ϵ -minimal path from g to g' . Let a_1, a_2, b, b_1, b_2, c be the labels of $\alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \gamma$, respectively. By definition, $\tilde{\varphi}_C(a_1) \leq \Phi_C(g) + \epsilon$, $\tilde{\varphi}_C(a_2) \leq \Phi_C(h) + \epsilon$ and $\tilde{\varphi}_C(b) \leq \Phi_C(gh) + \epsilon$.

By Lemma 6.4.3 and Lemma 6.4.1, we estimate for the triangle $\alpha_1\gamma\beta_1^{-1}$ that

$$\begin{aligned}\tilde{\varphi}_C(b_1) &\leq \Phi_C(\theta(b_1)) + 4D + \epsilon \\ &\leq \tilde{\varphi}_C(a_1c) + 4D + \epsilon \\ &\leq \tilde{\varphi}_C(a_1) + \tilde{\varphi}(c) + C|c|_X + 5D + \epsilon.\end{aligned}$$

Similarly, we also have

$$\tilde{\varphi}_C(a_1) \leq \tilde{\varphi}_C(b_1) - \tilde{\varphi}(c) + C|c|_X + 5D + \epsilon.$$

Therefore,

$$|\tilde{\varphi}_C(a_1) + \tilde{\varphi}(c) - \tilde{\varphi}_C(b_1)| \leq C|c|_X + 5D + \epsilon. \quad (6.6)$$

For the triangle $\alpha_2\beta_2^{-1}\gamma^{-1}$, we can estimate in a similar way and obtain

$$|\tilde{\varphi}_C(a_2) - \tilde{\varphi}_C(b_2) - \tilde{\varphi}(c)| \leq C|c|_X + 5D + \epsilon. \quad (6.7)$$

Combining (6.6) with (6.7) and note that $|\tilde{\varphi}_C(b_1) + \tilde{\varphi}_C(b_2) - \tilde{\varphi}_C(b)| \leq D$ and $C \geq 11D + 2\epsilon$, we obtain

$$|\tilde{\varphi}_C(a_1) + \tilde{\varphi}_C(a_2) - \tilde{\varphi}_C(b)| \leq (2|c|_X + 1)C.$$

Since γ is a $(3, 2)$ -quasi-geodesic, we have $|c|_X \leq l(\gamma) \leq 3d(g, g') + 2$. Since β is a $(3, 2)$ -quasi-geodesic, the distance $d(g, g') = d(g, \beta)$ is bounded above by a constant depending only on δ, m by Morse Lemma. Thus, the conclusion follows. $\square$

Proof of Theorem 1.4.3. By Lemma 6.3.7, we can assume that φ is linearly (G, X) -controlled with a constant C_0 . Therefore, we can follow the discussions in this section. For any $g, h \in G$, consider a geodesic triangle with vertices $1, g, gh$. Since $\Gamma(G, K \sqcup X)$ is δ -hyperbolic, there exists $c \in G$ such that c lies within distance δ from each side. Let

$$\Phi'_C(g) := \frac{1}{2}(\Phi_C(g) - \Phi_C(g^{-1}))$$

be the antisymmetrization of Φ_C . Clearly, Φ'_C inherits the property described in Lemma 6.4.5 from Φ_C . Therefore, there exists $M \geq 0$ that only depends on δ such that

$$\begin{aligned} |\Phi'_C(c) + \Phi'_C(c^{-1}g) - \Phi'_C(g)| &\leq MC, \\ |\Phi'_C(g^{-1}c) + \Phi'_C(c^{-1}gh) - \Phi'_C(h)| &\leq MC, \\ |\Phi'_C(c) + \Phi'_C(c^{-1}gh) - \Phi'_C(gh)| &\leq MC. \end{aligned}$$

Combining these inequalities and noting that $\Phi'_C(c^{-1}g) + \Phi'_C(g^{-1}c) = 0$ by antisymmetry, we have

$$|\Phi'_C(g) + \Phi'_C(h) - \Phi'_C(gh)| \leq 3MC + D.$$

This proves that Φ'_C is a quasimorphism with the defect at most $3MC + D$. By choosing $C = 20C_0 + 11D$, we see that $D(\Phi'_C) \leq 60MC_0 + 34MD$. By Lemma 6.4.2, $\Phi'_C|_K = \varphi$. In conclusion, φ extends to $\Phi'_C \in Q_{as}(G)$. $\square$

6.5 Normal subgroups

6.5.1 A criterion for being controlled

In this subsection, we are going to give sufficient conditions for quasimorphisms to be controlled, including a general criterion (Proposition 6.5.3) and its corollaries. Throughout this subsection, let G be a group and K a normal subgroup. Let $\bar{G} = G/K$. Consider the short exact sequence

$$1 \rightarrow K \rightarrow G \xrightarrow{\pi} \bar{G} \rightarrow 1. \quad (6.8)$$

Assume that there are presentations $K = \langle Y \mid S \rangle$ and $\bar{G} = \langle \bar{X} \mid \bar{R} \rangle$. For each $\bar{x} \in \bar{X}$, choose any $x \in \pi^{-1}(\bar{x})$, and let $X := \{x \mid \bar{x} \in \bar{X}\}$. This induces a natural map $s : \bar{X}^* \rightarrow X^*$ called a *lift* of $\bar{X}^*$. The quotient map π induces a natural map from $(X \sqcup Y)^*$ to $\bar{X}^*$. For simplicity, we still use π to denote this map.

For any $\bar{r} \in \bar{R}$, its lift $r := s(\bar{r})$ represents an element in K . Let $k_r \in K$ be the element represented by r . Let $w_r \in Y^*$ be a word that represents k_r , and let

$$R := \{rw_r^{-1} \mid \bar{r} \in \bar{R}\}.$$

For each $x \in X$ and $y \in Y$, the word xyx^{-1} represents an element in K , and thus is represented by a word $w_{xy} \in Y^*$. Let

$$T := \{xyx^{-1}w_{xy}^{-1} \mid x \in X, y \in Y\}.$$

The following lemma seems to be well-known (see, for example [HEO05, Proposition 2.55] or [BW11, Lemma 2.1]).

Lemma 6.5.1. *With the above notations, $\langle X, Y \mid R, S, T \rangle$ is a presentation of G .*

For our interest, the presentation of K will be fixed as $\langle K \mid R_K \rangle$. In this case, each w_r or w_{xy} can be chosen as a unique K -letter, and the presentation provided by Lemma 6.5.1 can be seen as a relative presentation with respect to K . We rewrite R as

$$R = \{rk_r^{-1} \mid \bar{r} \in \bar{R}\}.$$

For each $w \in X^*$ and $k \in K$, the word wkw^{-1} represents an element $k_w \in K$. Thus, we can expand T to

$$T' := \{wkw^{-1}k_w^{-1} \mid w \in X^*, k \in K\}.$$

The following lemma is a variation of the Bryant Park lemma [BBM20].

Lemma 6.5.2. *If the presentation $\bar{G} = \langle \bar{X} \mid \bar{R} \rangle$ has an isoperimetric function $f(n)$, then the relative presentation $G = \langle K, X \mid R_K, R, T' \rangle$ has a relative isoperimetric function $2f(n) + n$.*

Proof. Consider any $a \in \ker(\theta)$ of word length n . Without loss of generality, assume that the normal form w_a of a starts with a K -letter and ends with an X -letter. There is an expression

$$w_a = \prod_{i=1}^m k_i w_i,$$

where $w_i \in X^*$. We need to reduce w_a to the empty word in the group G using relations from $R_K \cup R \cup T'$ and bound the number of times a relation from $R \cup T'$ is used.

It is easy to see that

$$w_a =_{K * F(X)} \prod_{i=1}^m w_i \cdot \prod_{i=1}^m v_i,$$

where $v_i = (\prod_{j=i}^m w_j)^{-1} \cdot k_i \cdot \prod_{j=i}^m w_j$. Thus, we can use at most n relations from T' to reduce w_a to $w'_a = \prod_{i=1}^m w_i \in X^*$.

The word w'_a projects to $\bar{w}'_a \in \bar{X}^*$ under the map π . Since w_a represents the identity of G , $\bar{w}'_a$ represents the identity of $\bar{G}$. Therefore, there is an expression

$$\bar{w}'_a =_{F(\bar{X})} \prod_{j=1}^l \bar{f}_j \bar{r}_j \bar{f}_j^{-1},$$

where $\bar{r}_j \in R$, $\bar{f}_j \in X^*$, and $l \leq f(n)$. By lifting this equality, we have

$$w'_a =_{F(X)} \prod_{j=1}^l f_j r_j f_j^{-1}.$$

Now we can use at most $f(n)$ relations from R and at most $f(n)$ relations from T' to reduce each subword $f_j r_j f_j^{-1}$ to a single K -letter. Till now, w_a has been reduced into a word on K that represents the identity. We are done since it is a relation from R_K .

During the whole process, we use at most $f(n) + n$ relations from T' and $f(n)$ relations from R so the conclusion follows. $\square$

The following criterion is very useful for extending quasimorphisms from normal subgroups.

Proposition 6.5.3. *Let $\varphi \in \hat{Q}(K)^G$. Assume that the presentation $\bar{G} = \langle \bar{X} \mid \bar{R} \rangle$ has a linear isoperimetric function. If $\sup\{|\varphi(k_r)| \mid r \in R\}$ is finite, then φ is linearly (G, X) -controlled.*

Proof. By Lemma 6.5.2, the relative presentation $G = \langle K, X \mid R_K, R, T' \rangle$ has a linear relative isoperimetric function. By Proposition 6.3.6, we only need to prove that $\sup\{|\tilde{\varphi}(v)| \mid v \in R \cup T'\} < \infty$.

First, note that $|\tilde{\varphi}(rk_r^{-1})| = |\varphi(k_r^{-1})|$ for any $rk_r^{-1} \in R$. By assumption, this gives $\sup\{|\tilde{\varphi}(v)| \mid v \in R\} < \infty$. Second, for any $kwk_w^{-1} \in T'$, k and k_w are conjugate in G . Since φ is G -quasi-invariant, $|\varphi(k) - \varphi(k_w)| \leq D^G(\varphi)$ for any $kwk_w^{-1} \in T'$. Thus,

$$\begin{aligned} |\tilde{\varphi}(kwk_w^{-1})| &= |\varphi(kk_w^{-1})| \\ &\leq |\varphi(k) - \varphi(k_w)| + 3D(\varphi) \\ &\leq D^G(\varphi) + 3D(\varphi). \end{aligned}$$

This shows that $\sup\{|\tilde{\varphi}(v)| \mid v \in T'\} < \infty$.

In conclusion, φ is linearly (G, X) -controlled. $\square$

Recall that a section $s : \bar{G} \rightarrow G$ is called a quasi-splitting if the set

$$\Delta(s) := \{s(\bar{g}_2)^{-1} s(\bar{g}_1)^{-1} s(\bar{g}_1 \bar{g}_2) \mid \bar{g}_1, \bar{g}_2 \in \bar{G}\}$$

is finite.

Corollary 6.5.4. *If (6.8) admits a quasi-splitting $s : \bar{G} \rightarrow G$, then any $\varphi \in \hat{Q}(K)^G$ is linearly $(G, s(\bar{G}))$ -controlled.*

Proof. Consider the trivial presentation $\bar{G} = \langle \bar{G} \mid R'_G \rangle$, where $R'_G \subset R_G$ consists of the words of length at most 3. Clearly, this presentation has a linear isoperimetric function. The quasi-splitting s induces a lift map $\tilde{s} : \bar{G}^* \rightarrow G^*$. Let $\bar{r} = \bar{g}_1 \bar{g}_2 \bar{g}_3 \in R'_G$. Since $\bar{g}_3 =_{\bar{G}} \bar{g}_2^{-1} \bar{g}_1^{-1}$, we have

$$\begin{aligned} k_{\tilde{s}(\bar{r})} &= \tilde{s}(\bar{g}_1 \bar{g}_2 \bar{g}_3) \\ &=_G s(\bar{g}_1) s(\bar{g}_2) s(\bar{g}_2^{-1} \bar{g}_1^{-1}) \\ &\subset_G s(\bar{g}_1) s(\bar{g}_1^{-1}) \Delta(s)^{-1} \\ &\subset_G s(1) \Delta(s)^{-2}. \end{aligned}$$

Since $\Delta(s)$ is finite, Proposition 6.5.3 tells us that any $\varphi \in Q_{as}(K)^G$ is linearly $(G, s(\bar{G}))$ -controlled. $\square$

A *transversal* of a normal subgroup K in G is defined to be a set of coset representatives of K in G .

Corollary 6.5.5. *Let G be a group and K a normal subgroup. Assume that there exists $H < G$ such that the quotient group G/K is hyperbolic relative to $H/(K \cap H)$. Let $\varphi \in Q_{as}(K)^G$. If there exists a transversal L of $K \cap H$ in H such that the restricted quasimorphism $\varphi|_{K \cap H}$ is (H, L) -controlled, then φ extends to an antisymmetric quasimorphism on G .*

Proof. Let $\bar{G} := G/K$ and $\bar{H} := H/(K \cap H)$. By Definition 6.2.5, $\bar{G}$ has a presentation

$$\bar{G} = \langle \bar{H}, \bar{X} \mid R'_{\bar{H}}, \bar{R} \rangle$$

with a linear isoperimetric function, where $\bar{R}$ is finite.

Let $\pi : G \rightarrow \bar{G}$ be the quotient map, and let L be the given transversal of $K \cap H$ in H . As before, we also use π to denote the map from $(H \sqcup X)^*$ or $(K \sqcup L \sqcup X)^*$ to $(\bar{H} \sqcup \bar{X})^*$ induced by π . Let $s : (\bar{H} \sqcup \bar{X})^* \rightarrow (L \sqcup X)^*$ be a lift map. By Proposition 6.5.3 and Theorem 1.4.3, we only need to prove that $\sup\{|\varphi(k_{s(\bar{r})})| \mid \bar{r} \in R'_{\bar{H}} \cup \bar{R}\} < \infty$, where $k_{s(\bar{r})} \in K$ is the element represented by the word $s(\bar{r})$.

Let $\bar{r} \in R'_{\bar{H}}$. Suppose that $s(\bar{r})$ has the form $s(\bar{r}) = l_1 l_2 l_3$, where $l_i \in L$. Note that $k_{s(\bar{r})} \in K \cap H$, and thus $l_1 l_2 l_3 k_{s(\bar{r})}^{-1} \in (L \sqcup (K \cap H))^*$. Since $\varphi|_{K \cap H}$ is (H, L) -controlled by a function f , we have $|\varphi(k_{s(\bar{r})})| \leq f(4)$. Since $\bar{R}$ is finite, we know that $\sup\{|\varphi(k_{s(\bar{r})})| \mid \bar{r} \in R'_{\bar{H}} \cup \bar{R}\} < \infty$. $\square$

Clearly, Corollary 1.4.5 follows from Corollary 6.5.5.

Example 6.5.6. Let G be a hyperbolic group, and let K be a normal subgroup which is itself hyperbolic. By [Mos96], the quotient group G/K is also hyperbolic. Thus, by Corollary 1.4.5, any antisymmetric G -quasi-invariant quasimorphism on K extends to G .

6.5.2 Group-theoretic Dehn filling

Let $H \hookrightarrow_h (G, X)$. By Definition 6.2.4, there exists a strongly bounded relative presentation

$$G = \langle H, X \mid R_H, R \rangle$$

with a linear relative isoperimetric function. Given a subgroup $N \triangleleft H$, the quotient group $\bar{G} := G / \langle\langle N \rangle\rangle^G$ is called the *Dehn filling* of G with respect to N .

Let $K := \langle\langle N \rangle\rangle^G$, the normal closure of N in G . Consider the short exact sequence

$$1 \rightarrow K \rightarrow G \xrightarrow{\pi} \bar{G} \rightarrow 1.$$

Let $\bar{H} := \pi(H) \cong H/(K \cap H)$ and $\bar{X} := \pi(X)$. We still use π to denote the map from $(H \sqcup X)^*$ to $(\bar{H} \sqcup \bar{X})^*$ induced by π , where $\bar{H} := \pi(H) \cong H/(K \cap H)$ and $\bar{X} := \pi(X)$.

The following is part of [DGO17, Theorem 7.19].

Theorem 6.5.7. *With the above notations, there exists a finite subset $F \subset H - \{1\}$ such that for any $N \triangleleft H$ satisfying that $N \cap F = \emptyset$, the following hold.*

1. *The natural map from H/N to $\bar{G}$ is injective. In other words, $K \cap H = N$.*
2. *$\bar{G}$ has a strongly bounded relative presentation $\bar{G} = \langle \bar{H}, \bar{X} \mid R_{\bar{H}}, \pi(R) \rangle$ with a linear relative isoperimetric function.*
3. *The restricted map $\pi|_X : X \rightarrow \bar{X}$ is bijective.*

Now we can state the main result of this subsection. It shows that the extendability of a quasimorphism $\varphi \in Q_{as}(K)^G$ only depends on its restriction on N .

Theorem 6.5.8. *Let $H \hookrightarrow_h G$, and let F be the finite subset provided by Theorem 6.5.7. Let $N \triangleleft H$ be a subgroup such that $N \cap F = \emptyset$. Let $K = \langle\langle N \rangle\rangle^G$ and $\varphi \in Q_{as}(K)^G$. If there exists a transversal L of N in H such that the restricted quasimorphism $\varphi|_N$ is (H, L) -controlled, then φ extends to G .*

Proof. Let $X \subset G$ be a relative generating set such that $H \hookrightarrow_h (G, X)$. Let $s : (\bar{H} \sqcup \bar{X})^* \rightarrow (L \sqcup X)^*$ be the lift map induced by the transversal L and the map $(\pi|_X)^{-1}$. By Theorem 6.5.7(2) and Lemma 6.2.3, $\bar{G}$ has a strongly bounded presentation

$$\bar{G} = \langle \bar{H}, \bar{X} \mid R'_{\bar{H}}, \pi(R) \rangle$$

with a linear isoperimetric function. By Proposition 6.5.3 and Theorem 1.4.3, we only need to prove that $\sup\{|\varphi(k_{s(\bar{r})})| \mid \bar{r} \in R'_{\bar{H}} \cup \pi(R)\} < \infty$, where $k_{s(\bar{r})} \in K$ is the element represented by the word $s(\bar{r})$.

Case 1. Let $\bar{r} \in R'_{\bar{H}}$. Suppose that $s(\bar{r})$ has the form $s(\bar{r}) = l_1 l_2 l_3$, where $l_i \in L$. Note that $k_{s(\bar{r})} \in K \cap H = N$, and thus $l_1 l_2 l_3 k_{s(\bar{r})}^{-1} \in (L \sqcup N)^*$. Since $\varphi|_N$ is (H, L) -controlled by a function f , we have $|\varphi(k_{s(\bar{r})})| \leq f(4)$.

Case 2. Let $\bar{r} = \pi(r)$, where $r \in R$. Suppose that r has the form

$$r = \prod_{i=1}^m w_i h_i,$$

where $w_i \in X^*$ and $h_i \in H$. Each h_i can be represented by a unique word $l_i n_i$, where $l_i \in L$ and $n_i \in N$. Specifically, $l_i = s(\pi(h_i))$ and $n_i = {}_H s(\pi(h_i))^{-1} h_i$. By definition,

$$s(\bar{r}) = \prod_{i=1}^m w_i l_i.$$

It is easy to see that

$$r = {}_{K*F(L \sqcup X)} \prod_{i=1}^m w_i l_i \cdot \prod_{i=1}^m v_i,$$

where $v_i = (\prod_{j=i+1}^m w_j l_j)^{-1} \cdot n_i \cdot \prod_{j=i+1}^m w_j l_j$. Therefore,

$$k_{s(\bar{r})} = {}_G (\prod_{i=1}^m v_i)^{-1}.$$

Let $k_i \in K$ be the element represented by v_i . Since φ is G -quasi-invariant, $|\varphi(k_i) - \varphi(n_i)| \leq D^G(\varphi)$. Therefore,

$$\begin{aligned} |\varphi(k_{s(\bar{r})})| &\leq \sum_{i=1}^m |\varphi(k_i)| + (m-1)D(\varphi) \\ &\leq \sum_{i=1}^m |\varphi(n_i)| + mD^G(\varphi) + (m-1)D(\varphi). \end{aligned}$$

Since R is bounded, the number m is uniformly bounded. Since there are only finitely many $h \in H$ that appear in words from R , the set

$$\{n \in N \mid n = s(\pi(h))^{-1} h, \text{ where } h \in H \text{ appear in words from } R\}$$

is finite. Thus, $\sup\{|\varphi(k_{s(\bar{r})})| \mid \bar{r} \in \pi(R)\}$ is finite. $\square$

6.5.3 Mapping class groups

Theorem 6.5.9. *Let Σ be a finite-type surface, and let $G = \text{MCG}(\Sigma)$. Let g be a pseudo-Anosov mapping class. There exists $N \geq 1$ such that the following holds. For any $k \in \mathbb{N}$, any antisymmetric G -quasi-invariant quasimorphism on $\langle\langle g^{kN} \rangle\rangle^G$ extends to G .*

Proof. As shown in [BF02], the given g acts as a WPD element on the curve graph of Σ (see also [PS17] for a simpler proof). This implies that the cyclic subgroup $\langle g \rangle$ is contained in a unique maximal elementary subgroup H by [DGO17, Lemma 6.5]. Moreover, H is hyperbolically embedded in G by [DGO17, Theorem 6.8].

Let $F \subset H - \{1\}$ be the finite subset provided by Theorem 6.5.7. We can choose N sufficiently large such that $\langle g^N \rangle \triangleleft H$ and $\langle g^N \rangle \cap F = \emptyset$. Let $\varphi \in Q_{as}(K)^G$, where $K := \langle\langle g^{kN} \rangle\rangle^G$. Since

$\langle g^N \rangle$ is a normal subgroup of H of finite index, Corollary 6.5.4 tells us that the restricted quasimorphism $\varphi|_{\langle g^N \rangle}$ is linearly (H, L) -controlled for any transversal L . Finally, we can apply Theorem 6.5.8 and conclude. $\square$

Let Σ be a finite-type surface with nonempty boundary components $\gamma_1, \dots, \gamma_n$. Let $\hat{\Sigma}$ be the capped surface obtained by gluing a once-punctured disk with puncture p_i to each γ_i , and let $P = \{p_1, \dots, p_n\}$. Let g be a pseudo-Anosov mapping class of Σ , and let $K := \langle\langle g^{kN} \rangle\rangle^G$, where $N \in \mathbb{N}$ is provided by Theorem 6.5.9 and $k \in \mathbb{N}$ is arbitrary. By [FMS25, Proposition 4.4], $K \cap T_\Gamma = \emptyset$. Thus, from the capping sequence (see Section 2.4), we obtain a quotient central extension

$$1 \rightarrow T_\Gamma \rightarrow \text{MCG}(\Sigma)/K \rightarrow \text{MCG}(\hat{\Sigma}, P)/\pi(K) \rightarrow 1. \quad (6.9)$$

[FMS25, Proposition 2.16] says that if a certain (antisymmetric) quasi-invariant quasimorphism on $\pi(K)$ extends to $\text{MCG}(\hat{\Sigma}, P)$, then the quotient central extension (6.9) has a bounded Euler class. Theorem 6.5.9 shows that it is indeed the case.

Corollary 6.5.10. *The quotient central extension (6.9) has a bounded Euler class.*

The base group $\text{MCG}(\hat{\Sigma}, P)/\pi(K)$ is known to be hierarchically hyperbolic by [BHS17a, Theorem 6.2]. Using [AHPZ23, Theorem 5.14], we have shown that

Corollary 6.5.11. *$\text{MCG}(\Sigma)/K$ is hierarchically hyperbolic.*

This answers [FMS25, Question 4.2] affirmatively.

6.5.4 Stable mixed commutator length

In this subsection, we introduce a direct application to stable mixed commutator length. Let G be a group and K a normal subgroup. In their paper [KKMM22], Kawasaki, Kimura, Matsushita, and Mimura introduce the (stable) mixed commutator length on the subgroup $[G, K]$, generalizing the classical (stable) commutator length defined on commutator subgroups. We briefly review their definitions.

A (G, K) -commutator is an element in $[G, K]$ of the form $[g, k] = gkg^{-1}k^{-1}$ for some $g \in G$ and $k \in K$. Let $[G, K]$ be the subgroup generated by (G, K) -commutators. The (G, K) -commutator length $\text{cl}_{G,K}(g)$ of an element $g \in [G, K]$ is defined to be the word length of g with respect to the generating set consisting of all (G, K) -commutators. It is known that the limit

$$\text{scl}_{G,K}(g) = \lim_{n \rightarrow \infty} \frac{\text{cl}_{G,K}(g^n)}{n}$$

exists, and we call $\text{scl}_{G,K}(g)$ the stable (G, K) -commutator length of g . When K equals G , $\text{cl}_{G,G}$ and $\text{scl}_{G,G}$ equal the commutator length and the stable commutator length, respectively. In this case, we write cl_G and scl_G instead.

The celebrated Bavard's duality theorem [Bav91] (see also [Cal09]) relates stable commutator length to homogeneous quasimorphisms. Recently, this duality is generalized to stable mixed commutator length and invariant quasimorphisms in [KKMM22]. In particular, for any $g \in$

$[G, K]$, $\text{scl}_{G,K}(g) > 0$ if and only if there is a G -invariant quasimorphism φ on K with $\varphi(g) \neq 0$. Here we say that a quasimorphism φ on K is G -invariant if $\varphi(gkg^{-1}) = \varphi(k)$ for any $g \in G$ and $k \in K$. Clearly, every G -invariant quasimorphism on K is antisymmetric and G -quasi-invariant. From this duality, Kawasaki, Kimura, Maruyama, Matsushita, and Mimura [KKMMM25b] prove the following as part of their Theorem 2.1.

Theorem 6.5.12. *If every G -invariant quasimorphism on K extends to G , then scl_G and $\text{scl}_{G,K}$ are bi-Lipschitz equivalent on $[G, K]$.*

Combining Theorem 6.5.12 with Corollary 1.4.5 and Theorem 6.5.9 respectively, we have

Corollary 6.5.13.

1. *If G/K is hyperbolic, then scl_G and $\text{scl}_{G,K}$ are bi-Lipschitz equivalent on $[G, K]$.*
2. *If $G = \text{MCG}(\Sigma)$ and $K = \langle\langle g^{k_N} \rangle\rangle^G$ as in Theorem 6.5.9, then scl_G and $\text{scl}_{G,K}$ are bi-Lipschitz equivalent on $[G, K]$.*

Bibliography

- [ABO19] Carolyn Abbott, Sahana H. Balasubramanya, and Denis Osin. “Hyperbolic structures on groups”. In: *Algebraic & Geometric Topology* 19.4 (2019), pp. 1747–1835. ISSN: 1472-2747,1472-2739. DOI: [10.2140/agt.2019.19.1747](https://doi.org/10.2140/agt.2019.19.1747).
- [AHPZ23] Carolyn Abbott et al. “Uniform undistortion from barycentres, and applications to hierarchically hyperbolic groups”. In: arXiv:2305.09742 (2023). DOI: [10.48550/arXiv.2305.09742](https://doi.org/10.48550/arXiv.2305.09742).
- [Alo94] Juan M. Alonso. “Finiteness conditions on groups and quasi-isometries”. In: *Journal of Pure and Applied Algebra* 95.2 (1994), pp. 121–129. ISSN: 0022-4049,1873-1376. DOI: [10.1016/0022-4049\(94\)90069-8](https://doi.org/10.1016/0022-4049(94)90069-8).
- [AR24] Carolyn R. Abbott and Alexander J. Rasmussen. “Largest hyperbolic actions and quasi-parabolic actions in groups”. In: *Journal of Topology and Analysis* 16.4 (2024), pp. 495–539. ISSN: 1793-5253,1793-7167. DOI: [10.1142/S1793525322500066](https://doi.org/10.1142/S1793525322500066).
- [Bav91] Christophe Bavard. “Longueur stable des commutateurs”. In: *L’Enseignement Mathématique. Revue Internationale. 2e Série* 37.1 (1991), pp. 109–150. ISSN: 0013-8584.
- [BBF15] Mladen Bestvina, Ken Bromberg, and Koji Fujiwara. “Constructing group actions on quasi-trees and applications to mapping class groups”. In: *Publ. Math. Inst. Hautes Études Sci.* 122 (2015), pp. 1–64. ISSN: 0073-8301. DOI: [10.1007/s10240-014-0067-4](https://doi.org/10.1007/s10240-014-0067-4).
- [BBF21] M. Bestvina, K. Bromberg, and K. Fujiwara. “Proper actions on finite products of quasi-trees”. In: *Annales Henri Lebesgue* 4 (2021), pp. 685–709. DOI: [10.5802/ahl.85](https://doi.org/10.5802/ahl.85).
- [BBFS19] Mladen Bestvina et al. “Acylindrical actions on projection complexes”. In: *L’Enseignement Mathématique* 65.1-2 (2019), pp. 1–32. ISSN: 0013-8584. DOI: [10.4171/lem/65-1/2-1](https://doi.org/10.4171/lem/65-1/2-1).
- [BBM20] Gilbert Baumslag, Martin R. Bridson, and Charles F. Miller. “Isoperimetric and isodiametric functions of group extensions”. In: *Elementary Theory of Groups and Group Rings, and Related Topics*. Ed. by Paul Baginski et al. Berlin, Boston: De Gruyter, 2020, pp. 7–10. ISBN: 9783110638387. DOI: [doi:10.1515/9783110638387-002](https://doi.org/doi:10.1515/9783110638387-002).
- [BF02] Mladen Bestvina and Koji Fujiwara. “Bounded cohomology of subgroups of mapping class groups”. In: *Geometry and Topology* 6 (2002), pp. 69–89. ISSN: 1465-3060. DOI: [10.2140/gt.2002.6.69](https://doi.org/10.2140/gt.2002.6.69).
- [BFG24] Sahana H. Balasubramanya, Francesco Fournier-Facio, and Anthony Genevois. “Property (NL) for group actions on hyperbolic spaces (with an appendix by Alessandro Sisto)”. In: *Groups, Geometry, and Dynamics* (2024). published online first. ISSN: 1661-7207. DOI: [10.4171/ggd/806](https://doi.org/10.4171/ggd/806).

- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Vol. 319. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999. ISBN: 9783540643241. DOI: [10.1007/978-3-662-12494-9](https://doi.org/10.1007/978-3-662-12494-9).
- [BHMS24] Jason Behrstock et al. “A combinatorial take on hierarchical hyperbolicity and applications to quotients of mapping class groups”. In: *Journal of Topology* 17.3 (2024), Paper No. e12351, 94. ISSN: 1753-8416,1753-8424. DOI: [10.1112/topo.12351](https://doi.org/10.1112/topo.12351).
- [BHS17a] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. “Asymptotic dimension and small-cancellation for hierarchically hyperbolic spaces and groups”. In: *Proceedings of the London Mathematical Society. Third Series* 114.5 (2017), pp. 890–926. ISSN: 0024-6115. DOI: [10.1112/plms.12026](https://doi.org/10.1112/plms.12026).
- [BHS17b] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. “Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups”. In: *Geom. Topol.* 21.3 (2017), pp. 1731–1804. ISSN: 1465-3060. DOI: [10.2140/gt.2017.21.1731](https://doi.org/10.2140/gt.2017.21.1731).
- [BHS19] J. Behrstock, M. Hagen, and A. Sisto. “Hierarchically hyperbolic spaces II: Combination theorems and the distance formula”. In: *Pacific J. Math.* 299.2 (2019), pp. 257–338. ISSN: 0030-8730. DOI: [10.2140/pjm.2019.299.257](https://doi.org/10.2140/pjm.2019.299.257).
- [BHW22] Jonathan Bowden, Sebastian Wolfgang Hensel, and Richard Webb. “Quasi-morphisms on surface diffeomorphism groups”. In: *Journal of the American Mathematical Society* 35.1 (2022), pp. 211–231. ISSN: 0894-0347,1088-6834. DOI: [10.1090/jams/981](https://doi.org/10.1090/jams/981).
- [Bow08] Brian H. Bowditch. “Tight geodesics in the curve complex”. In: *Inventiones Mathematicae* 171.2 (2008), pp. 281–300. ISSN: 0020-9910. DOI: [10.1007/s00222-007-0081-y](https://doi.org/10.1007/s00222-007-0081-y).
- [BR22] Daniel Berlyne and Jacob Russell. “Hierarchical hyperbolicity of graph products”. In: *Groups, Geometry, and Dynamics* 16.2 (2022), pp. 523–580. ISSN: 1661-7207. DOI: [10.4171/ggd/652](https://doi.org/10.4171/ggd/652).
- [Bro82] Kenneth S. Brown. *Cohomology of groups*. Vol. 87. Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1982. x+306. ISBN: 9780387906881. DOI: [10.1007/978-1-4684-9327-6](https://doi.org/10.1007/978-1-4684-9327-6).
- [But22] J. O. Button. “Generalised Baumslag-Solitar groups and Hierarchically Hyperbolic Groups”. In: (2022). arXiv:2208.12688 [math]. DOI: [10.48550/arXiv.2208.12688](https://doi.org/10.48550/arXiv.2208.12688).
- [BW11] Martin R. Bridson and Henry Wilton. “On the difficulty of presenting finitely presentable groups”. In: *Groups, Geometry, and Dynamics* 5.2 (2011), pp. 301–325. ISSN: 1661-7207,1661-7215. DOI: [10.4171/GGD/129](https://doi.org/10.4171/GGD/129).
- [Cal09] Danny Calegari. *scl*. Vol. 20. MSJ Memoirs. Mathematical Society of Japan, Tokyo, 2009. xii+209. ISBN: 9784931469532. DOI: [10.1142/e018](https://doi.org/10.1142/e018).
- [CCMT15] Pierre-Emmanuel Caprace et al. “Amenable hyperbolic groups”. In: *Journal of the European Mathematical Society (JEMS)* 17.11 (2015), pp. 2903–2947. ISSN: 1435-9855. DOI: [10.4171/JEMS/575](https://doi.org/10.4171/JEMS/575).
- [Che22] Marissa Chesser. “Stable subgroups of the genus 2 handlebody group”. In: *Algebraic & Geometric Topology* 22.2 (2022), pp. 919–971. ISSN: 1472-2747,1472-2739. DOI: [10.2140/agt.2022.22.919](https://doi.org/10.2140/agt.2022.22.919).
- [CK02] C. B. Croke and B. Kleiner. “The geodesic flow of a nonpositively curved graph manifold”. In: *Geometric and Functional Analysis* 12.3 (2002), pp. 479–545. ISSN: 1016-443X. DOI: [10.1007/s00039-002-8255-7](https://doi.org/10.1007/s00039-002-8255-7).

- [CW25] Jiaqi Cui and Renxing Wan. “Proper actions on finite products of proper Gromov-hyperbolic spaces”. In: arXiv:2505.09454 (2025). DOI: [10.48550/arXiv.2505.09454](https://doi.org/10.48550/arXiv.2505.09454).
- [DDLS24] Spencer Dowdall et al. “Extensions of Veech groups II: Hierarchical hyperbolicity and quasi-isometric rigidity”. In: *Commentarii Mathematici Helvetici. A Journal of the Swiss Mathematical Society* 99.1 (2024), pp. 149–228. ISSN: 0010-2571,1420-8946. DOI: [10.4171/cmh/568](https://doi.org/10.4171/cmh/568).
- [DGO17] F. Dahmani, V. Guirardel, and D. Osin. *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*. en. Vol. 245. Memoirs of the American Mathematical Society 1156. American Mathematical Society, 2017. ISBN: 9781470436018;9781470421946. DOI: [10.1090/memo/1156](https://doi.org/10.1090/memo/1156).
- [DHS17] Matthew Durham, Mark Hagen, and Alessandro Sisto. “Boundaries and automorphisms of hierarchically hyperbolic spaces”. en. In: *Geometry & Topology* 21.6 (Aug. 2017), pp. 3659–3758. ISSN: 1364-0380, 1465-3060. DOI: [10.2140/gt.2017.21.3659](https://doi.org/10.2140/gt.2017.21.3659).
- [DHS20] Matthew Durham, Mark Hagen, and Alessandro Sisto. “Correction to the article Boundaries and automorphisms of hierarchically hyperbolic spaces”. en. In: *Geometry & Topology* 24.2 (2020), pp. 1051–1073. ISSN: 1364-0380, 1465-3060. DOI: [10.2140/gt.2020.24.1051](https://doi.org/10.2140/gt.2020.24.1051).
- [DJ99] A. Dranishnikov and T. Januszkiewicz. “Every Coxeter group acts amenably on a compact space”. In: *Topology Proceedings*. Vol. 24. 1999, pp. 135–141.
- [DMS23] Matthew G. Durham, Yair N. Minsky, and Alessandro Sisto. “Stable cubulations, bi-combings, and barycenters”. In: *Geometry & Topology* 27.6 (2023), pp. 2383–2478. ISSN: 1465-3060,1364-0380. DOI: [10.2140/gt.2023.27.2383](https://doi.org/10.2140/gt.2023.27.2383).
- [EF97] David B. A. Epstein and Koji Fujiwara. “The second bounded cohomology of word-hyperbolic groups”. In: *Topology. An International Journal of Mathematics* 36.6 (1997), pp. 1275–1289. ISSN: 0040-9383. DOI: [10.1016/S0040-9383\(96\)00046-8](https://doi.org/10.1016/S0040-9383(96)00046-8).
- [EP03] Michael Entov and Leonid Polterovich. “Calabi quasimorphism and quantum homology”. In: *International Mathematics Research Notices* 30 (2003), pp. 1635–1676. ISSN: 1073-7928,1687-0247. DOI: [10.1155/S1073792803210011](https://doi.org/10.1155/S1073792803210011).
- [Far98] B. Farb. “Relatively hyperbolic groups”. In: *Geometric and Functional Analysis* 8.5 (1998), pp. 810–840. ISSN: 1016-443X,1420-8970. DOI: [10.1007/s000390050075](https://doi.org/10.1007/s000390050075).
- [FK16] Koji Fujiwara and Michael Kapovich. “On quasihomomorphisms with noncommutative targets”. In: *Geometric and Functional Analysis* 26.2 (2016), pp. 478–519. ISSN: 1016-443X,1420-8970. DOI: [10.1007/s00039-016-0364-9](https://doi.org/10.1007/s00039-016-0364-9).
- [FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*. Vol. 49. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 2012. ISBN: 9780691147949.
- [FMS25] Francesco Fournier-Facio, Giorgio Mangioni, and Alessandro Sisto. “Bounded cohomology, quotient extensions, and hierarchical hyperbolicity”. In: arXiv:2505.20462 (2025). DOI: [10.48550/arXiv.2505.20462](https://doi.org/10.48550/arXiv.2505.20462).
- [FPS15] R. Frigerio, M. B. Pozzetti, and A. Sisto. “Extending higher-dimensional quasicocycles”. In: *Journal of Topology* 8.4 (2015), pp. 1123–1155. ISSN: 1753-8416,1753-8424. DOI: [10.1112/jtopol/jtv017](https://doi.org/10.1112/jtopol/jtv017).
- [Fri17] Roberto Frigerio. *Bounded cohomology of discrete groups*. Vol. 227. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017. xvi+193. ISBN: 9781470441463. DOI: [10.1090/surv/227](https://doi.org/10.1090/surv/227).

- [FS23] Roberto Frigerio and Alessandro Sisto. “Central extensions and bounded cohomology”. In: *Annales Henri Lebesgue* 6 (2023), pp. 225–258. ISSN: 2644-9463. DOI: [10.5802/ahl.164](https://doi.org/10.5802/ahl.164).
- [FSY04] Koji Fujiwara, Takashi Shioya, and Saeko Yamagata. “Parabolic isometries of CAT(0) spaces and CAT(0) dimensions”. In: *Algebraic & Geometric Topology* 4 (2004), pp. 861–892. ISSN: 1472-2747. DOI: [10.2140/agt.2004.4.861](https://doi.org/10.2140/agt.2004.4.861).
- [Gro74] Edna K. Grossman. “On the residual finiteness of certain mapping class groups”. In: *Journal of the London Mathematical Society. Second Series* 9 (1974), pp. 160–164. ISSN: 0024-6107. DOI: [10.1112/jlms/s2-9.1.160](https://doi.org/10.1112/jlms/s2-9.1.160).
- [Gro82] Michael Gromov. “Volume and bounded cohomology”. In: *Institut des Hautes Études Scientifiques. Publications Mathématiques* 56 (1982), pp. 5–99. ISSN: 0073-8301,1618-1913.
- [Gro87] M. Gromov. “Hyperbolic groups”. In: *Essays in group theory*. Vol. 8. Math. Sci. Res. Inst. Publ. Springer, 1987, pp. 75–263. DOI: [10.1007/978-1-4613-9586-7_3](https://doi.org/10.1007/978-1-4613-9586-7_3).
- [Hag23] Mark Hagen. “Non-colorable hierarchically hyperbolic groups”. In: *International Journal of Algebra and Computation* 33.02 (2023), pp. 337–350. ISSN: 0218-1967. DOI: [10.1142/S0218196723500170](https://doi.org/10.1142/S0218196723500170).
- [Hem87] John Hempel. “Residual finiteness for 3-manifolds”. In: *Ann. of Math. Stud.* 111 (1987), pp. 379–396. DOI: [10.1515/9781400882083-018](https://doi.org/10.1515/9781400882083-018).
- [HEO05] Derek F. Holt, Bettina Eick, and Eamonn A. O’Brien. *Handbook of computational group theory*. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2005. xvi+514. ISBN: 9781584883722. DOI: [10.1201/9781420035216](https://doi.org/10.1201/9781420035216).
- [Heu20] Nicolaus Heuer. “Low-dimensional bounded cohomology and extensions of groups”. In: *Mathematica Scandinavica* 126.1 (2020), pp. 5–31. ISSN: 0025-5521,1903-1807. DOI: [10.7146/math.scand.a-114969](https://doi.org/10.7146/math.scand.a-114969).
- [HMS24] Mark Hagen, Alexandre Martin, and Alessandro Sisto. “Extra-large type Artin groups are hierarchically hyperbolic”. In: *Mathematische Annalen* 388.1 (2024), pp. 867–938. ISSN: 0025-5831,1432-1807. DOI: [10.1007/s00208-022-02523-4](https://doi.org/10.1007/s00208-022-02523-4).
- [HNY25] Suzhen Han, Hoang Thanh Nguyen, and Wenyuan Yang. “Property (QT) for 3-manifold groups”. In: *Algebraic & Geometric Topology* 25.1 (2025), pp. 107–159. ISSN: 1472-2747,1472-2739. DOI: [10.2140/agt.2025.25.107](https://doi.org/10.2140/agt.2025.25.107).
- [HO13] Michael Hull and Denis Osin. “Induced quasicycles on groups with hyperbolically embedded subgroups”. In: *Algebraic & Geometric Topology* 13.5 (2013), pp. 2635–2665. ISSN: 1472-2747,1472-2739. DOI: [10.2140/agt.2013.13.2635](https://doi.org/10.2140/agt.2013.13.2635).
- [HP22] Mark F. Hagen and Harry Petyt. “Projection complexes and quasimedial maps”. In: *Algebraic & Geometric Topology* 22.7 (2022), pp. 3277–3304. ISSN: 1472-2739. DOI: [10.2140/agt.2022.22.3277](https://doi.org/10.2140/agt.2022.22.3277).
- [HRSS24] Mark Hagen et al. “Equivariant hierarchically hyperbolic structures for 3-manifold groups via quasimorphisms”. In: *Annales de l’Institut Fourier* (2024), pp. 1–60. ISSN: 1777-5310. DOI: [10.5802/aif.3654](https://doi.org/10.5802/aif.3654).
- [Ish14] Tomohiko Ishida. “Quasi-morphisms on the group of area-preserving diffeomorphisms of the 2-disk via braid groups”. In: *Proceedings of the American Mathematical Society. Series B* 1 (2014), pp. 43–51. ISSN: 2330-1511. DOI: [10.1090/S2330-1511-2014-00002-X](https://doi.org/10.1090/S2330-1511-2014-00002-X).

- [Jan22] Kasia Jankiewicz. “Residual finiteness of certain 2-dimensional Artin groups”. In: *Advances in Mathematics* 405 (2022), Paper No. 108487, 37. ISSN: 0001-8708,1090-2082. DOI: [10.1016/j.aim.2022.108487](https://doi.org/10.1016/j.aim.2022.108487).
- [KKM19] Sang-hyun Kim, Thomas Koberda, and Mahan Mj. *Flexibility of group actions on the circle*. Vol. 2231. Lecture Notes in Mathematics. Springer, Cham, 2019. x+134. ISBN: 9783030028541 9783030028558. DOI: [10.1007/978-3-030-02855-8](https://doi.org/10.1007/978-3-030-02855-8).
- [KKMM22] Morimichi Kawasaki et al. “Bavard’s duality theorem for mixed commutator length”. In: *L’Enseignement Mathématique* 68.3 (2022), pp. 441–481. ISSN: 0013-8584,2309-4672. DOI: [10.4171/lem/1037](https://doi.org/10.4171/lem/1037).
- [KKMM23] Morimichi Kawasaki et al. “Commuting symplectomorphisms on a surface and the flux homomorphism”. In: *Geometric and Functional Analysis* 33.5 (2023), pp. 1322–1353. ISSN: 1016-443X,1420-8970. DOI: [10.1007/s00039-023-00644-9](https://doi.org/10.1007/s00039-023-00644-9).
- [KKMMM24] Morimichi Kawasaki et al. “Survey on invariant quasimorphisms and stable mixed commutator length”. In: *Topology Proceedings* 64 (2024), pp. 129–174. ISSN: 0146-4124,2331-1290.
- [KKMMM25a] Morimichi Kawasaki et al. *Non-extendability of Shelukhin’s quasimorphism and non-triviality of Reznikov’s class*. 2025. DOI: [10.48550/arXiv.2503.09455](https://doi.org/10.48550/arXiv.2503.09455).
- [KKMMM25b] Morimichi Kawasaki et al. “The space of nonextendable quasimorphisms”. In: *Algebraic & Geometric Topology* 25.2 (2025), pp. 1169–1226. ISSN: 1472-2739. DOI: [10.2140/agt.2025.25.1169](https://doi.org/10.2140/agt.2025.25.1169).
- [Lev05] Gilbert Levitt. “Automorphisms of hyperbolic groups and graphs of groups”. In: *Geometriae Dedicata* 114 (2005), pp. 49–70. ISSN: 0046-5755,1572-9168. DOI: [10.1007/s10711-004-1492-1](https://doi.org/10.1007/s10711-004-1492-1).
- [LM21] Ian J. Leary and Ashot Minasyan. “Commensurating HNN extensions: nonpositive curvature and biautomaticity”. In: *Geometry & Topology* 25.4 (2021), pp. 1819–1860. ISSN: 1465-3060,1364-0380. DOI: [10.2140/gt.2021.25.1819](https://doi.org/10.2140/gt.2021.25.1819).
- [Lon87] D. D. Long. “Immersions and embeddings of totally geodesic surfaces”. In: *The Bulletin of the London Mathematical Society* 19.5 (1987), pp. 481–484. ISSN: 0024-6093,1469-2120. DOI: [10.1112/blms/19.5.481](https://doi.org/10.1112/blms/19.5.481).
- [Man05] Jason Fox Manning. “Geometry of pseudocharacters”. In: *Geometry and Topology* 9 (2005), pp. 1147–1185. ISSN: 1465-3060,1364-0380. DOI: [10.2140/gt.2005.9.1147](https://doi.org/10.2140/gt.2005.9.1147).
- [Man06] J. F. Manning. “Quasi-actions on trees and property (QFA)”. In: *Journal of the London Mathematical Society. Second Series* 73.1 (2006), pp. 84–108. ISSN: 0024-6107,1469-7750. DOI: [10.1112/S0024610705022738](https://doi.org/10.1112/S0024610705022738).
- [Man08] Jason Fox Manning. “Actions of certain arithmetic groups on Gromov hyperbolic spaces”. In: *Algebraic & Geometric Topology* 8.3 (2008), pp. 1371–1402. ISSN: 1472-2747,1472-2739. DOI: [10.2140/agt.2008.8.1371](https://doi.org/10.2140/agt.2008.8.1371).
- [MM00] H. A. Masur and Y. N. Minsky. “Geometry of the complex of curves. II. Hierarchical structure”. In: *Geometric and Functional Analysis* 10.4 (2000), pp. 902–974. ISSN: 1016-443X. DOI: [10.1007/PL00001643](https://doi.org/10.1007/PL00001643).
- [MM99] H. A. Masur and Y. N. Minsky. “Geometry of the complex of curves. I. Hyperbolicity”. In: *Inventiones Mathematicae* 138.1 (1999), pp. 103–149. ISSN: 0020-9910. DOI: [10.1007/s002220050343](https://doi.org/10.1007/s002220050343).

- [Mos96] Lee Mosher. “Hyperbolic extensions of groups”. In: *Journal of Pure and Applied Algebra* 110.3 (1996), pp. 305–314. ISSN: 0022-4049,1873-1376. DOI: [10.1016/0022-4049\(95\)00081-X](https://doi.org/10.1016/0022-4049(95)00081-X).
- [Neu51] B. H. Neumann. “Groups with finite classes of conjugate elements”. In: *Proceedings of the London Mathematical Society. Third Series* 1 (1951), pp. 178–187. ISSN: 0024-6115,1460-244X. DOI: [10.1112/plms/s3-1.1.178](https://doi.org/10.1112/plms/s3-1.1.178).
- [Ngu25] Hoang Thanh Nguyen. “Separability properties of extended admissible groups”. In: *Forum Mathematicum* (2025). ISSN: 1435-5337. DOI: [10.1515/forum-2024-0194](https://doi.org/10.1515/forum-2024-0194).
- [NR97] Walter D. Neumann and Lawrence Reeves. “Central extensions of word hyperbolic groups”. In: *Annals of Mathematics. Second Series* 145.1 (1997), pp. 183–192. ISSN: 0003-486X,1939-8980. DOI: [10.2307/2951827](https://doi.org/10.2307/2951827).
- [NY23a] Hoang Thanh Nguyen and Wenyuan Yang. “Croke-Kleiner admissible groups: property (QT) and quasiconvexity”. In: *Michigan Mathematical Journal* 73.5 (2023), pp. 971–1019. ISSN: 0026-2285,1945-2365. DOI: [10.1307/mmj/20216045](https://doi.org/10.1307/mmj/20216045).
- [NY23b] Piotr W. Nowak and Guoliang Yu. *Large scale geometry*. Second. EMS Textbooks in Mathematics. EMS Press, Berlin, 2023. xvi+197. ISBN: 9783985470181 9783985475186. DOI: [10.4171/ETB/26](https://doi.org/10.4171/ETB/26).
- [Osi06] Denis V. Osin. “Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems”. In: *Memoirs of the American Mathematical Society* 179.843 (2006), pp. vi+100. ISSN: 0065-9266,1947-6221. DOI: [10.1090/memo/0843](https://doi.org/10.1090/memo/0843).
- [Osi16] D. Osin. “Acylindrically hyperbolic groups”. In: *Transactions of the American Mathematical Society* 368.2 (2016), pp. 851–888. ISSN: 0002-9947. DOI: [10.1090/tran/6343](https://doi.org/10.1090/tran/6343).
- [Pet21] Harry Petyt. “Mapping class groups are quasicubical”. In: (2021). arXiv:2112.10681 [math]. DOI: [10.48550/arXiv.2112.10681](https://doi.org/10.48550/arXiv.2112.10681).
- [PR14] Leonid Polterovich and Daniel Rosen. *Function theory on symplectic manifolds*. Vol. 34. CRM Monograph Series. American Mathematical Society, Providence, RI, 2014. xii+203. ISBN: 9781470416935. DOI: [10.1090/crmm/034](https://doi.org/10.1090/crmm/034).
- [PS17] Piotr Przytycki and Alessandro Sisto. “A note on acylindrical hyperbolicity of mapping class groups”. In: *Hyperbolic geometry and geometric group theory*. Vol. 73. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2017, pp. 255–264. DOI: [10.2969/aspm/07310255](https://doi.org/10.2969/aspm/07310255).
- [PS23] Harry Petyt and Davide Spriano. “Unbounded domains in hierarchically hyperbolic groups”. In: *Groups, Geometry, and Dynamics* (2023). ISSN: 1661-7207. DOI: [10.4171/ggd/706](https://doi.org/10.4171/ggd/706).
- [PSZ25] Harry Petyt, Davide Spriano, and Abdul Zalloum. “Stable cylinders and fine structures for hyperbolic groups and curve graphs”. In: arXiv:2501.13600 (2025). DOI: [10.48550/arXiv.2501.13600](https://doi.org/10.48550/arXiv.2501.13600).
- [Py06] Pierre Py. “Quasi-morphisms et invariant de Calabi”. In: *Annales Scientifiques de l’École Normale Supérieure. Quatrième Série* 39.1 (2006), pp. 177–195. ISSN: 0012-9593. DOI: [10.1016/j.ansens.2005.11.003](https://doi.org/10.1016/j.ansens.2005.11.003).
- [RS20] Bruno Robbio and Davide Spriano. “Hierarchical hyperbolicity of hyperbolic-2-decomposable groups”. In: (2020). arXiv:2007.13383 [math]. DOI: [10.48550/arXiv.2007.13383](https://doi.org/10.48550/arXiv.2007.13383).

- [RS94] E. Rips and Z. Sela. “Structure and rigidity in hyperbolic groups. I”. In: *Geometric and Functional Analysis* 4.3 (1994), pp. 337–371. ISSN: 1016-443X,1420-8970. DOI: [10.1007/BF01896245](https://doi.org/10.1007/BF01896245).
- [RS95] E. Rips and Z. Sela. “Canonical representatives and equations in hyperbolic groups”. In: *Inventiones Mathematicae* 120.3 (1995), pp. 489–512. ISSN: 0020-9910,1432-1297. DOI: [10.1007/BF01241140](https://doi.org/10.1007/BF01241140).
- [Sel97] Z. Sela. “Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II”. In: *Geometric and Functional Analysis* 7.3 (1997), pp. 561–593. ISSN: 1016-443X,1420-8970. DOI: [10.1007/s000390050019](https://doi.org/10.1007/s000390050019).
- [Ser77] Jean-Pierre Serre. *Arbres, amalgames, SL_2* . Vol. No. 46. Astérisque. Société Mathématique de France, Paris, 1977. 189 pp. (1 plate).
- [Szc25] Andrzej Szczepański. *Geometry of crystallographic groups*. Second. Vol. 5. Algebra and Discrete Mathematics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2025. xiii+257. ISBN: 9789811286599 9789811286605 9789811286612.
- [Tao24] Bingxue Tao. “Property (QT) of relatively hierarchically hyperbolic groups”. In: arXiv:2412.20065 (2024). DOI: [10.48550/arXiv.2412.20065](https://doi.org/10.48550/arXiv.2412.20065).
- [Tao25] Bingxue Tao. “An extension theorem for quasimorphisms”. In: arXiv:2511.21306 (2025). DOI: [10.48550/arXiv.2511.21306](https://doi.org/10.48550/arXiv.2511.21306).
- [TW25] Bingxue Tao and Renxing Wan. “Proper actions on finite products of hyperbolic spaces”. In: arXiv:2506.04856 (2025). DOI: [10.48550/arXiv.2506.04856](https://doi.org/10.48550/arXiv.2506.04856).
- [Ver24] Ignacio Vergara. “Quasi-trees, Lipschitz free spaces, and actions on ℓ^1 ”. In: arXiv:2409.14186 (2024). DOI: [10.48550/arXiv.2409.14186](https://doi.org/10.48550/arXiv.2409.14186).
- [Wan25] Renxing Wan. “Quasi-retracts of groups”. In: arXiv:2501.02238 (2025). DOI: [10.48550/arXiv.2501.02238](https://doi.org/10.48550/arXiv.2501.02238).
- [Wis00] Daniel T. Wise. “Subgroup separability of graphs of free groups with cyclic edge groups”. In: *The Quarterly Journal of Mathematics* 51.1 (2000), pp. 107–129. ISSN: 0033-5606,1464-3847. DOI: [10.1093/qmathj/50.1.107](https://doi.org/10.1093/qmathj/50.1.107).
- [WY25] Renxing Wan and Wenyuan Yang. “Uniform exponential growth for groups with proper product actions on hyperbolic spaces”. In: *Journal of Algebra* 676 (2025), pp. 189–235. ISSN: 0021-8693,1090-266X. DOI: [10.1016/j.jalgebra.2025.03.028](https://doi.org/10.1016/j.jalgebra.2025.03.028).
- [Yan19] Wen-yuan Yang. “Statistically convex-cocompact actions of groups with contracting elements”. In: *International Mathematics Research Notices. IMRN* 23 (2019), pp. 7259–7323. ISSN: 1073-7928. DOI: [10.1093/imrn/rny001](https://doi.org/10.1093/imrn/rny001).