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Exponential decay of a difference between a global solution to a reaction-diffusion system and its spatial average

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§1. Introduction.

This report is based on Hoshino [9].

We are concerned with asymptotic behavior of a unique nonnegative global solution $(u,v)(t,x)$ to the following system of reaction-diffusion equations with homogeneous Neumann boundary conditions:

$$\begin{cases} u_t = d_1\Delta u + f(u)v^n, & \text{in} \quad (0,\infty)\times\Omega, \\ v_t = d_2\Delta v - f(u)v^n, & \text{in} \quad (0,\infty)\times\Omega, \end{cases} \tag{1.1}$$

$$\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, \qquad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{1.2}$$

$$(u,v)(0,x) = (u_0,v_0)(x), \qquad \text{in} \quad \Omega. \tag{1.3}$$

Here Ω is a bounded domain in $\mathbf{R}^N$ $(N\geq 1)$ with smooth boundary $\partial\Omega$, and $\partial/\partial\nu$ stands for the outward normal derivative to $\partial\Omega$. We assume

Assumption 1. (i) d_1 and d_2 are positive constants.

(ii) u_0 and v_0 are bounded, $u_0\geq 0$, $v_0\geq 0$ and $\overline{u}_0>0$, $\overline{v}_0>0$, where $\overline{w}=|\Omega|^{-1}\int_\Omega w(x)dx$, and $|\Omega|$ is the volume of Ω.

(iii) f is smooth in $u\geq 0$ and $f(u)>0$ if $u>0$. Moreover, either

$$\lim_{u\to\infty} u^{-1}\log(1+f(u)) = 0$$

(cf. [6]) or

$$f(u)\leq e^{\alpha u} \quad \text{with } d_1\neq d_2 \quad \text{and } \sup_{x\in\Omega} v_0(x) < \frac{8d_1d_2}{\alpha N(d_1-d_2)^2}$$

(cf. [2]) holds.

Assumption 2. $n>1$.

Assumption 1 with $n\geq 1$ assures the existence of a unique nonnegative global solution $(u,v)(t,x)$ to (1.1) – (1.3). In fact, we have Alikakos [1], Masuda [11], Haraux and Kirane [5], Haraux and Youkana [6], Hollis, Martin and Pierre [7], Pao [12], Barabanova [2], Hoshino [8], and so on. Especially in [8], under Assumptions 1 and 2, Hoshino has shown a uniform convergence property of $(u,v)(t,x)$ to $(u_\infty,0)$ with a polynomial rate, that is to say,

$$\|(u-u_\infty,v)(t)\|_\infty \leq Kt^{-1/(n-1)} \qquad \text{as} \quad t\to\infty,$$

where

$$u_\infty = \overline{u}_0 + \overline{v}_0$$

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and has also proved that

$$\|(u-\overline{u}, v-\overline{v})(t)\|_\infty \le Kt^\mu e^{-d_0\lambda t} \tag{1.4}$$

as $t \to \infty$. Here, $\overline{w}(t) = |\Omega|^{-1}\int_\Omega w(t,x)dx$, $\mu = (\sqrt{2}-1)n/(2(n-1)) > 0$, λ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$, and

$$d_0 = \min\{d_1, d_2\}.$$

Here and hereafter, we make use of the notations

$$\|w\|_p = \|w\|_{L^p(\Omega)}, \qquad \|(w_1, w_2)\|_p = (\|w_1\|_p^2 + \|w_2\|_p^2)^{1/2}.$$

For the details of the previous results, see Theorem 1 in Section 2.

In this report, we will obtain a sharper decay rate of the difference between $(u,v)(t,x)$ and $(\overline{u},\overline{v})(t)$ than (1.4). In fact, we can show

$$(u,v)(t,x) = (\overline{u},\overline{v})(t) + O(e^{-d_0\lambda t}) \tag{1.5}$$

uniformly in $x \in \Omega$ as $t \to \infty$, and furthermore in the case $d_1 > d_2$ or $d_1 < d_2$,

$$u(t,x) = \overline{u}(t) + O(t^{-1}e^{-d_0\lambda t}), \tag{1.6}$$

or

$$v(t,x) = \overline{v}(t) + O(t^{-\min\{n,2n-2\}/(n-1)}e^{-d_0\lambda t}) \tag{1.7}$$

uniformly in $x \in \Omega$ as $t \to \infty$, respectively. For the details of our results, see Theorems 2 – 4 in Section 2 below.

Our results are related to those obtained by Conway, Hoff and Smoller [3] or Hale [4]. However, if we restrict ourselves to the case where we have a balance law in a reaction-diffusion system under homogeneous Neumann boundary conditions, then in comparison with previous results we confirm that we can improve the description of the approximation of $(u,v)(t,x)$ by $(\overline{u},\overline{v})(t)$ in the sense that we can sharpen the estimate of $\|(u-\overline{u}, v-\overline{v})(t)\|_\infty$ such as (1.5), (1.6) and (1.7). Actually, we have

$$\int_\Omega u(t,x)dx + \int_\Omega v(t,x)dx = \int_\Omega u_0(x)dx + \int_\Omega v_0(x)dx, \qquad t \ge 0$$

in our system (1.1) – (1.3).

Our idea for the analysis to get the results is that we make use of $(\phi,\psi)(t,x)$ which is defined by

$$\begin{cases} u(t,x) - u_\infty = (U(t)-u_\infty)(1+\phi(t,x)) = -V(t)(1+\phi(t,x)), \\ v(t,x) = V(t)(1+\psi(t,x)), \end{cases} \tag{1.8}$$

where $(U,V)(t)$ is a unique global solution to

$$\begin{cases} U' = f(U)V^n, \\ V' = -f(U)V^n, \end{cases} \qquad t > 0, \tag{1.9}$$

$$(U,V)(0) = (\overline{u}_0, \overline{v}_0) \tag{1.10}$$

with $n > 1$, where $' = d/dt$. Obviously, $V(t)$ verifies

$$V' = -f(u_\infty - V)V^n, \qquad t > 0$$

and we see that there is a positive constant C_0 such that

$$C_0^{-1}(1+t)^{-1/(n-1)} \le u_\infty - U(t) = V(t) \le C_0(1+t)^{-1/(n-1)} \tag{1.11}$$

for $t \ge 0$.

§2. Results.

First, let us recall some preliminary results on our system (1.1) – (1.3).

Theorem 1. (i) *Under Assumptions* 1 *and* 2, (1.1) – (1.3) *has a unique global solution* $(u,v)(t,x)$. *It holds true that*

$$0 \le v(t,x) \le \|v_0\|_\infty, \qquad t>0, \quad x \in \overline{\Omega},$$

and there exists a constant $M>0$ *such that*

$$0 \le u(t,x) \le M, \qquad t>0, \quad x \in \overline{\Omega}.$$

(ii) *There are positive constants* T *and* K *such that*

$$\begin{cases} \|(u-u_\infty, v)(t)\|_\infty \le K(1+t-T)^{-1/(n-1)}, \\ \|(u-\overline{u}, v-\overline{v})(t)\|_\infty \le K(1+t-T)^K e^{-d_0\lambda t}, \end{cases} \qquad t \ge T,$$

where $u_\infty = \overline{u}_0 + \overline{v}_0$, $d_0 = \min\{d_1, d_2\}$, *and* λ *is the smallest positive eigenvalue of* $-\Delta$ *with the homogeneous Neumann boundary condition on* $\partial\Omega$.

(iii) *Let* $(U,V)(t)$ *be the solution to* (1.9), (1.10). *Then,* $(U,V)(t)$ *plays a role of an asymptotic solution to* (1.1) – (1.3) *and*

$$(u,v)(t,x) = (U,V)(t) + O(t^{-1-1/(n-1)})$$

uniformly in $x \in \Omega$ *as* $t \to \infty$.

(iv) *Moreover,* $(\overline{u}, \overline{v})(t)$ *approximates* $(u,v)(t,x)$ *as follows:*

$$(u,v)(t,x) = (\overline{u}, \overline{v})(t) + O(t^\mu e^{-d_0\lambda t})$$

uniformly in $x \in \Omega$ *as* $t \to \infty$, *where* $\mu = (\sqrt{2}-1)n/(2(n-1)) > 0$.

Next, we state our main results, that is to say, we can sharpen the approximation of the global solution $(u,v)(t,x)$ to (1.1) – (1.3) by its spatial average $(\overline{u}, \overline{v})(t)$ than (iv) of Theorem 1.

Theorem 2. *The following asymptotic approximation of* $(u,v)(t,x)$ *by its spatial average holds true:*

$$(u,v)(t,x) = (\overline{u}, \overline{v})(t) + O(e^{-d_0\lambda t})$$

uniformly in $x \in \Omega$ *as* $t \to \infty$.

In the case $d_1 \neq d_2$, we can obtain stronger asymptotic relations.

Theorem 3. *When* $d_1 > d_2$,

$$u(t,x) = \overline{u}(t) + O(t^{-1} e^{-d_0\lambda t})$$

uniformly in $x \in \Omega$ *as* $t \to \infty$.

Theorem 4. *When* $d_1 < d_2$,

$$v(t,x) = \overline{v}(t) + O(t^{-\min\{n,2n-2\}/(n-1)} e^{-d_0\lambda t})$$

uniformly in $x \in \Omega$ *as* $t \to \infty$.

§3. Deformation of the problem.

Substituting (1.8) into (1.1) – (1.3), easy calculations give

$$\begin{cases} \phi_t = d_1\Delta\phi - V^{n-1}\{-f(u_\infty - V)\phi - Vf_u(u_\infty - V)\phi + nf(u_\infty - V)\psi + h\}, \\ \psi_t = d_2\Delta\psi - V^{n-1}\{-Vf_u(u_\infty - V)\phi + (n-1)f(u_\infty - V)\psi + h\}, \end{cases} \quad \text{in} \quad (0,\infty)\times\Omega, \tag{3.1}$$

$$\frac{\partial\phi}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0, \qquad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{3.2}$$

$$\begin{cases} \phi(0,x) = \phi_0(x) = -\dfrac{u_0(x)-\overline{u}_0}{\overline{v}_0}, \\ \psi(0,x) = \psi_0(x) = \dfrac{v_0(x)-\overline{v}_0}{\overline{v}_0}, \end{cases} \qquad \text{in} \quad \Omega, \tag{3.3}$$

where $f_u = df/du$, and $h = h(\phi,\psi)$ satisfies

$$-f(u_\infty - V)(1+\phi) + f(u_\infty - V(1+\phi))(1+\psi)^n = -f(u_\infty - V)\phi - Vf_u(u_\infty - V)\phi + nf(u_\infty - V)\psi + h.$$

Note that we also have

$$-f(u_\infty - V)(1+\psi) + f(u_\infty - V(1+\phi))(1+\psi)^n = -Vf_u(u_\infty - V)\phi + (n-1)f(u_\infty - V)\psi + h$$

at the same time and that $h = O(|\phi|^2 + |\psi|^2)$ as $(\phi,\psi)\to(0,0)$.

We will investigate the decay rate of $(\phi,\psi)(t,x)$ in order that we show Theorems $2-4$ (cf. [10]). We use the following projection operators.

Definition 3.1. $P_0 w = \overline{w} = |\Omega|^{-1}\int_\Omega w(x)dx, \qquad P_+ w = w - P_0 w.$

The following lemma is important for us.

Lemma 3.1. *There exists a nondecreasing function $L(r)$ on $[0,\infty)$ such that if*

$$K(t) \equiv L\left(\sup_{0\le\tau\le t}\|(\phi,\psi)(\tau)\|_\infty\right),$$

then for every $p\in[1,\infty]$,

$$\|h(t)\|_p \le K(t)\|(\phi,\psi)(t)\|_{2p}^2,$$

$$\|(P_+h)(t)\|_p \le K(t)\|(\phi,\psi)(t)\|_\infty\|(V^{1/2}P_+\phi, P_+\psi)(t)\|_p,$$

$$\|(P_+h)(t)\|_p \le K(t)\|(\phi,\psi)(t)\|_\infty\|(P_+\phi, P_+\psi)(t)\|_p.$$

When C is a constant, we will identify $CK(t)$ with $K(t)$ in the following sections.

Finally, we give the equations and the boundary and initial conditions which $(\phi^+,\psi^+)(t,x)$ satisfies:

$$\begin{cases} \phi_t^+ = d_1\Delta\phi^+ - V^{n-1}\{-f(u_\infty - V)\phi^+ - Vf_u(u_\infty - V)\phi^+ + nf(u_\infty - V)\psi^+ + h^+\}, \\ \psi_t^+ = d_2\Delta\psi^+ - V^{n-1}\left\{-Vf_u(u_\infty - V)\phi^+ + (n-1)f(u_\infty - V)\psi^+ + h^+\right\}, \end{cases} \quad \text{in} \quad (0,\infty)\times\Omega, \tag{3.4}$$

$$\frac{\partial\phi^+}{\partial\nu} = \frac{\partial\psi^+}{\partial\nu} = 0, \qquad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{3.5}$$

$$(\phi^+,\psi^+)(0,x) = (\phi_0,\psi_0)(x), \qquad \text{in} \quad \Omega. \tag{3.6}$$

Note that $P_0\phi_0 = P_0\psi_0 = 0$, in other words, $(P_+\phi_0)(x) = \phi_0(x)$, $(P_+\psi_0)(x) = \psi_0(x)$. Here and hereafter, we use the notation

$$w^+ = P_+ w$$

for simplicity.

§4. The case of small initial perturbation.

We will restrict ourselves to the case where $\|(\phi_0,\psi_0)\|_\infty$ is small and we will obtain the following theorem in terms of $(\phi,\psi)(t,x)$ instead of Theorem 2. We can reduce the case where the size of $\|(\phi_0,\psi_0)\|_\infty$ is large to the small case by virtue of Theorem 1 (ii) (see [8]).

Theorem 4.1. *There exists a constant $\delta_0 > 0$ such that if $\|(\phi_0, \psi_0)\|_\infty \leq \delta_0$, then*

$$\|(\phi^+, \psi^+)(t)\|_\infty \leq C\|(\phi_0, \psi_0)\|_\infty V(t)^{-1}e^{-d_0\lambda t}$$

for $t \geq 0$, where C is a positive constant.

We introduce some quantities as follows:

Definition 4.1. For $1 \leq p \leq \infty$,

$$I_p = \|(\phi_0, \psi_0)\|_p,$$

$$M_p(t) = \sup_{0\leq\tau\leq t} V(\tau)^{-(n-1)}\|(\phi, \psi)(\tau)\|_p,$$

$$M_\infty^0(t) = \sup_{0\leq\tau\leq t} V(\tau)^{-(n-1)}|(P_0\phi, P_0\psi)(\tau)|,$$

$$M_{p,V}^+(t) = \sup_{0\leq\tau\leq t} V(\tau)e^{d_0\lambda\tau}\|(V^{1/2}\phi^+, \psi^+)(\tau)\|_p,$$

$$M_p^+(t) = \sup_{0\leq\tau\leq t} V(\tau)e^{d_0\lambda\tau}\|(\phi^+, \psi^+)(\tau)\|_p,$$

where $V(t)$ is the solution for (1.9) and (1.10) satisfying (1.11), $d_0 = \min\{d_1, d_2\}$, and λ is the smallest positive eigenvalue of $-\Delta$ with the homogeneous Neumann boundary conditions on $\partial\Omega$.

According to the following scheme, we can show Theorem 4.1.

1. $M_\infty^0(t) \leq K(t)M_\infty(t)^2$.
2. $M_{p,V}^+(t) \leq CI_\infty + K(t)M_\infty(t)M_{2,V}^+(t)$ for $p \in [1, 2]$.
3. $M_{\infty,V}^+(t) \leq CI_\infty + K(t)M_\infty(t)M_{\infty,V}^+(t)$.
4. $M_2^+(t) \leq CI_\infty + K(t)M_\infty(t)M_2^+(t)$ for $p \in [1, 2]$.
5. $M_\infty^+(t) \leq CI_\infty + K(t)M_\infty(t)M_\infty^+(t)$.

In the Steps 2 and 4, we investigate $L^2(\Omega)$-energy of $(\phi^+, \psi^+)(t, x)$ with use of (3.4) – (3.6). On the other hand, in Steps 3 and 5 we treat (3.7) and (3.8) by means of $L^p(\Omega) - L^q(\Omega)$ estimate of an analytic semigroup $\{e^{-tA}\}_{t\geq0}$, where A means $-\Delta$ with the homogeneous Neumann boundary condition on $\partial\Omega$.

The following Theorem 4.2 (resp. 4.3) corresponds to Theorem 3 (resp. 4) in the case I_∞ is small.

Theorem 4.2. *Suppose that $d_1 > d_2$. If $\|(\phi_0, \psi_0)\|_\infty \leq \delta_0$, then*

$$V(t)e^{d_0\lambda t}\|\phi^+(t)\|_\infty \leq C\|(\phi_0, \psi_0)\|_\infty V(t)^{n-1}$$

for $t \geq 0$, where C is a positive constant.

Theorem 4.3. *Suppose that $d_1 < d_2$. If $\|(\phi_0, \psi_0)\|_\infty \leq \delta_0$, then*

$$V(t)e^{d_0\lambda t}\|\psi^+(t)\|_\infty \leq C\|(\phi_0, \psi_0)\|_\infty V(t)^{\min\{n,2n-2\}}$$

for $t \geq 0$, where C is a positive constant.

For the details of the proofs of our results in this report, refer to [9].

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