

# On weakly $(\tilde{\rho}, \tilde{D})$ -separable polynomials in skew polynomial rings

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## Abstract

Separable polynomials in skew polynomial rings were studied extensively by Y. Miyashita, T. Nagahara, S. Ikehata, and G. S. Ito. In particular, Ikehata gave the characterization of  $(\tilde{\rho}, \tilde{D})$ -separable polynomials in skew polynomial rings. In this article, we shall introduce the notion of weakly  $(\tilde{\rho}, \tilde{D})$ -separable polynomials in skew polynomial rings, and we shall give a characterization of the  $(\tilde{\rho}, \tilde{D})$ -separability and that of the weak  $(\tilde{\rho}, \tilde{D})$ -separability.

## 1 Introduction and Preliminaries

Throughout this paper,  $A/B$  will represent a ring extension with common identity 1. Let  $M$  be an  $A$ - $A$ -bimodule, and  $x, y$  arbitrary elements in  $A$ . An additive map  $\delta : A \rightarrow M$  is called a  $B$ -derivation of  $A$  to  $M$  if  $\delta(xy) = \delta(x)y + x\delta(y)$  and  $\delta(\alpha) = 0$  for any  $\alpha \in B$ . Moreover,  $\delta$  is called *inner* if  $\delta(x) = mx - xm$  for some fixed element  $m \in M$ . We say that a ring extension  $A/B$  is *separable* if the  $A$ - $A$ -homomorphism of  $A \otimes_B A$  onto  $A$  defined by  $a \otimes b \mapsto ab$  splits. It is well known that  $A/B$  is separable if and only if for any  $A$ - $A$ -bimodule  $M$ , every  $B$ -derivation of  $A$  to  $M$  is inner (cf. [1, Satz 4.2]). A ring extension  $A/B$  is said to be *weakly separable* if every  $B$ -derivation of  $A$  to  $A$  is inner. The notion of a weakly separable extension was introduced by N. Hamaguchi and A. Nakajima (cf. [2]). Obviously, a separable extension is weakly separable.

Let  $B$  be a ring,  $\rho$  an automorphism of  $B$ ,  $D$  a  $\rho$ -derivation of  $B$ .  $B[X; \rho, D]$  will mean the skew polynomial ring in which the multiplication is given by  $\alpha X = X\rho(\alpha) + D(\alpha)$  for any  $\alpha \in B$ . We set  $B[X; \rho] := B[X; \rho, 0]$  and  $B[X; D] := B[X; 1_A, D]$ . By  $B[X; \rho, D]_{(0)}$  we denote the set of all monic polynomials  $g$  in  $B[X; \rho, D]$  such that  $gB[X; \rho, D] = B[X; \rho, D]g$ . For a polynomial  $f \in B[X; \rho, D]_{(0)}$ , the residue ring  $B[X; \rho, D]/fB[X; \rho, D]$  is a free ring extension of  $B$ . We say that a polynomial  $f \in B[X; \rho, D]_{(0)}$  is *separable* (resp. *weakly separable*) in  $B[X; \rho, D]$  if  $B[X; \rho, D]/fB[X; \rho, D]$  is separable (resp. weakly separable) over  $B$ .

Throughout this article, we assume that  $\rho D = D\rho$ , and let  $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 \in B[X; \rho, D]_{(0)} \cap B^\rho[X]$  and

$$f' := mX^{m-1} + (m-1)X^{m-2}a_{m-1} \cdots + Xa_2 + a_1 \text{ (the derivative of } f),$$

$$Y_0 := X^{m-1} + X^{m-2}a_{m-1} + \cdots + Xa_2 + a_1,$$

$$\dots\dots$$

$$Y_j := X^{m-j-1} + X^{m-j-2}a_{m-1} + \cdots + Xa_{j+2} + a_{j+1},$$

$$\dots\dots$$

$$Y_{m-2} := X + a_{m-1},$$

$$Y_{m-1} := 1.$$

We shall use the following conventions:

$$B^\rho := \{\alpha \in B \mid \rho(\alpha) = \alpha\}$$

$$B^D := \{\alpha \in B \mid D(\alpha) = 0\}$$

$$B^{\rho,D} := B^\rho \cap B^D$$

$$C(B^{\rho,D}) := \{\beta \in B^{\rho,D} \mid b\beta = \beta b \ (\forall b \in B^{\rho,D})\} \text{ (the center of } B^{\rho,D})$$

$$A := B[X; \rho, D]/fB[X; \rho, D]$$

$$x := X + fB[X; \rho, D] \in A$$

$$f' := f' + fB[X; \rho, D] \in A$$

$$y_j := Y_j + fB[X; \rho, D] \in A \quad (0 \leq j \leq m-1)$$

$$\rho : \text{an automorphism of } A \text{ defined by } \rho \left( \sum_{j=0}^{m-1} x^j c_j \right) = \sum_{j=0}^{m-1} x^j \rho(c_j) \quad (c_j \in B)$$

$$D : \text{a } \rho\text{-derivation of } A \text{ defined by } D \left( \sum_{j=0}^{m-1} x^j c_j \right) = \sum_{j=0}^{m-1} x^j D(c_j) \quad (c_j \in B)$$

For any subsets  $T \subset B$  and  $S \subset A$ , we set

$$J_{m-1}(T) := \{z \in A \mid \rho^{m-1}(\alpha)z = z\alpha \ (\forall \alpha \in T)\},$$

$$V(T) := \{z \in A \mid \alpha z = z\alpha \ (\forall \alpha \in T)\},$$

$$W(S) := \left\{ \sum_{j=0}^{m-1} y_j \omega \otimes x^j \mid \omega \in S \right\},$$

$$(A \otimes_B A)^S := \{\varepsilon \in A \otimes_B A \mid \varepsilon w = w\varepsilon \ (\forall w \in S)\},$$

$$S^{\bar{\rho}} := \{z \in S \mid \rho(z) = z\},$$

$$S^{\bar{D}} := \{z \in S \mid D(z) = 0\},$$

$$S^{\bar{\rho}, \bar{D}} := S^{\bar{\rho}} \cap S^{\bar{D}}.$$

Note that  $J_{m-1}(B') = V(B')$  for any subset  $B'$  of  $B^\rho$ .

We shall state some basic results which were already known.

**Lemma 1.1** ([7, Lemma 1.6]).  *$f$  is in  $B[X; \rho, D]_{(0)}$  if and only if*

- (1)  $a_i \rho^m(\alpha) = \sum_{j=i}^m \binom{j}{i} \rho^j D^{j-i}(\alpha) a_j \quad (\alpha \in B, 0 \leq i \leq m-1, a_m = 1)$
- (2)  $D(a_i) = a_{i-1} - \rho(a_{i-1}) - a_i(\rho(a_{m-1}) - a_{m-1}) \quad (1 \leq i \leq m-1)$
- (3)  $D(a_0) = a_0(\rho(a_{m-1}) - a_{m-1})$

**Lemma 1.2** ([7, Corollary 1.7]). *If  $f$  is in  $B[X; \rho, D]_{(0)} \cap B^\rho[X]$  then  $f$  is in  $C(B^{\rho, D})[X]$ . Moreover,*

$$\alpha a_i = \sum_{j=i}^m (-1)^{j-i} \binom{j}{i} a_j \rho^{m-j} D^{j-i}(\alpha) \quad (\alpha \in B, 0 \leq j \leq m, a_m = 1).$$

**Lemma 1.3** ([6, Theorem 2.2]). *Let  $B$  be a commutative ring, and  $f(X)$  a monic polynomial in  $B[X]$ . The following are equivalent.*

- (1)  *$f(X)$  is weakly separable in  $B[X]$ .*
- (2)  *$f'(X)$  is a non-zero-divisor in  $B[X]$  modulo  $(f(X))$ , where  $f'(X)$  is a derivative of  $f(X)$ .*
- (3)  *$\delta(f(X))$  is a non-zero-divisor in  $B$ , where  $\delta(f(X))$  is a discriminant of  $f(X)$ .*

Now we consider the following  $A$ - $A$ -homomorphisms:

$$\begin{aligned} \mu : {}_A A \otimes_B A_A &\rightarrow {}_A A_A, \quad \mu(z \otimes w) = zw \\ \xi : {}_A A \otimes_B A_A &\rightarrow {}_A A \otimes_B A_A, \quad \xi(z \otimes w) = D(z) \otimes \rho(w) + z \otimes D(w) \\ \eta : {}_A A \otimes_B A_A &\rightarrow {}_A A \otimes_B A_A, \quad \eta(z \otimes w) = \rho(z) \otimes \rho(w) - z \otimes w \end{aligned}$$

By making of the above mappings, S. Ikehata gave the following definition.

**Definiton 1.4** ([4, pp.119]).  *$f$  is called  $(\rho, D)$ -separable in  $B[X; \rho, D]$  if there exists an  $A$ - $A$ -homomorphism  $\nu : A \rightarrow A \otimes_B A$  such that*

$$\mu\nu = 1_A, \quad \xi\nu = \nu D, \quad \eta\nu = \nu(\rho - 1_A).$$

Obviously, a  $(\rho, D)$ -separable polynomial in  $B[X; \rho, D]$  is separable. In [4], S. Ikehata studied  $(\rho, D)$ -separable polynomials in  $B[X; \rho, D]$  and he gave the following.

**Lemma 1.5** ([4, Theorem 2.1]). *The following are equivalent.*

- (1)  $f$  is  $(\rho, D)$ -separable in  $B[X; \rho, D]$ .
- (2) There exists  $h \in J_{m-1}(B)^{\tilde{\rho}, \tilde{D}}$  such that  $f'h = hf' = 1$ .
- (3)  $f$  is separable in  $C(B^{\rho, D})[X]$ .

Noting that Lemma 1.5 (3), we shall give the following definition as a generalization of  $(\rho, D)$ -separable polynomials in  $B[X; \rho, D]$ .

**Definition 1.6.**  $f$  is called *weakly  $(\rho, D)$ -separable* in  $B[X; \rho, D]$  if  $f$  is weakly separable in  $C(B^{\rho, D})[X]$ .

The purpose of this article is to give characterizations of weakly  $(\rho, D)$ -separable in  $B[X; \rho, D]$ . Moreover, we shall characterize the difference between the  $(\rho, D)$ -separability and the weak  $(\rho, D)$ -separability in  $B[X; \rho, D]$ .

## 2 Main results

The conventions and notations employed in the preceding section will be used in this section. In particular, recall that  $\rho D = D\rho$  and let  $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 \in B[X; \rho, D]_{(0)} \cap B^\rho[X]$ . Note that  $f$  is in  $C(B^{\rho, D})[X]$  by Corollary 1.2. First we shall state the following.

**Lemma 2.1.** *The following are equivalent.*

- (1)  $f$  is weakly  $(\rho, D)$ -separable in  $B[X; \rho, D]$ .
- (2)  $f'$  is a non-zero-divisor in  $C(B^{\rho, D})[X]/fC(B^{\rho, D})[X]$  ( $\cong V(B^{\rho, D})^{\tilde{\rho}, \tilde{D}}$ ).
- (3)  $\delta(f)$  is a non-zero-divisor in  $C(B^{\rho, D})$ , where  $\delta(f)$  is a discriminant of  $f$ .

*Proof.* It is obvious by Lemma 1.3. □

We recall that  $A$ - $A$ -homomorphism  $\mu : A \otimes_B A \rightarrow A$  defined by  $z \otimes w \mapsto zw$ . Noting that  $\alpha f' = f' \rho^{m-1}(\alpha)$  for any  $\alpha \in B$ , we can see that  $\mu \left( W(J_{m-1}(B)^{\tilde{\rho}, \tilde{D}}) \right) \subset V(B)^{\tilde{\rho}, \tilde{D}}$ . In addition, it is easy to see that  $\mu \left( W(V(B^{\rho, D})^{\tilde{\rho}, \tilde{D}}) \right) \subset V(B^{\rho, D})^{\tilde{\rho}, \tilde{D}}$ . Then we shall state the following.

**Theorem 2.2.** (1)  $f$  is  $(\rho, D)$ -separable in  $B[X; \rho, D]$  if and only if the following  $A$ - $A$ -homomorphism is onto:

$$\mu|_{W(J_{m-1}(B)^{\tilde{\rho}, \tilde{D}})} : W(J_{m-1}(B)^{\tilde{\rho}, \tilde{D}}) \longrightarrow V(B)^{\tilde{\rho}, \tilde{D}}$$

- (2)  $f$  is weakly  $(\rho, D)$ -separable in  $B[X; \rho, D]$  if and only if the following  $A$ - $A$ -homomorphism is one-to-one:

$$\mu|_{W(V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}})} : W(V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}}) \longrightarrow V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}}$$

*Proof.* Note that  $\mu\left(\sum_{j=0}^{m-1} y_j h \otimes x^j\right) = f'h = hf'$  for any  $h \in A^{\tilde{\rho}, \tilde{D}}$ .

(1) Assume that  $f$  is  $(\rho, D)$ -separable in  $B[X; \rho, D]$ . Then there exists  $h \in J_{m-1}(B)^{\tilde{\rho}, \tilde{D}}$  such that  $f'h = hf' = 1$  by Lemma 1.5 (2). For any  $g \in V(B)^{\tilde{\rho}, \tilde{D}}$ , we see that  $hg = gh \in J_{m-1}(B)^{\tilde{\rho}, \tilde{D}}$  and  $\mu\left(\sum_{j=0}^{m-1} y_j hg \otimes x^j\right) = f'hg = g$ . Thus  $\mu|_{W(J_{m-1}(B)^{\tilde{\rho}, \tilde{D}})}$  is onto.

Conversely, assume that  $\mu|_{W(J_{m-1}(B)^{\tilde{\rho}, \tilde{D}})}$  is onto. Since  $1 \in V(B)^{\tilde{\rho}, \tilde{D}}$ , there exists  $h \in J_{m-1}(B)^{\tilde{\rho}, \tilde{D}}$  such that  $1 = \mu\left(\sum_{j=0}^{m-1} y_j h \otimes x^j\right) = f'h = hf'$ . Therefore  $f$  is  $(\rho, D)$ -separable by Lemma 1.5 (2).

(2) Assume that  $f$  is weakly  $(\rho, D)$ -separable in  $B[X; \rho, D]$ . Then  $f'$  is a non-zero-divisor in  $V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}}$  by Lemma 2.1 (2). Let  $\sum_{j=0}^{m-1} y_j h \otimes x^j$  be in  $\text{Ker}\left(\mu|_{W(V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}})}\right)$  with  $h \in V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}}$ . Then we have  $0 = f'h = hf'$ . Since  $f'$  is a non-zero-divisor in  $V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}}$ , we obtain  $h = 0$  and hence  $\text{Ker}\left(\mu|_{W(V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}})}\right) = \{0\}$ . Thus  $\mu|_{W(V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}})}$  is one-to-one.

Conversely, assume that  $\mu|_{W(V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}})}$  is one-to-one. Let  $hf' = 0$  for some  $h \in V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}}$ . This implies that  $\mu\left(\sum_{j=0}^{m-1} y_j h \otimes x^j\right) = 0$ . Since  $\mu|_{W(V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}})}$  is one-to-one, we have  $\sum_{j=0}^{m-1} y_j h \otimes x^j = 0$ , namely,  $h = 0$ . Therefore  $f'$  is a non-zero-divisor in  $V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}}$ , and hence  $f$  is weakly  $(\rho, D)$ -separable by Lemma 2.1 (2).  $\square$

**Corollary 2.3.**  $f$  is  $(\rho, D)$ -separable in  $B[X; \rho, D]$  if and only if  $W(J_{m-1}(B)^{\tilde{\rho}, \tilde{D}}) \cong V(B)^{\tilde{\rho}, \tilde{D}}$  as an  $A$ - $A$ -bimodule.

*Proof.* Note that  $W(J_{m-1}(B)^{\tilde{\rho}, \tilde{D}}) \subset W(V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}})$  and  $V(B)^{\tilde{\rho}, \tilde{D}} \subset V(B^{\rho,D})^{\tilde{\rho}, \tilde{D}}$ . If  $f$  is  $(\rho, D)$ -separable in  $B[X; \rho, D]$  then  $f$  is also weakly  $(\rho, D)$ -separable, and so  $\mu|_{W(J_{m-1}(B)^{\tilde{\rho}, \tilde{D}})}$  is one-to-one. Therefore  $\mu|_{W(J_{m-1}(B)^{\tilde{\rho}, \tilde{D}})}$  is an isomorphism if and only if  $f$  is  $(\rho, D)$ -separable in  $B[X; \rho, D]$ .  $\square$

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