

Continued fractions and Dedekind sums for function fields

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1 Introduction

For coprime integers a and $c > 0$, the classical Dedekind sum $d(a, c)$ is defined by

$$d(a, c) = \frac{1}{4c} \sum_{k=1}^{c-1} \cot\left(\frac{\pi k}{c}\right) \cot\left(\frac{\pi ka}{c}\right). \quad (1)$$

For coprime positive integers a and c , it holds that

$$d(a, c) + d(c, a) = \frac{1}{12} \left(\frac{a}{c} + \frac{c}{a} + \frac{1}{ac} - 3 \right);$$

this is called the reciprocity law. The value of $d(a, c)$ has been investigated. Rewriting (1) in terms of the sawtooth function, we can easily see that $d(a, c)$ is a rational number. Rademacher [4] proved that $d(a, c)$ is not bounded above and below in the neighborhood of each a/c . Rademacher and Grosswald [5] posed the following two questions:

1. Is $\{(a/c, d(a, c)) \mid a/c \in \mathbb{Q}^*\}$ dense in $\mathbb{R}^2$?
2. Is $\{d(a, c) \mid a/c \in \mathbb{Q}^*\}$ dense in $\mathbb{R}$?

Hickerson [3] answered them using the theory of continued fractions.

As is well known, there is an analogy between algebraic number fields and function fields. For example, $A := \mathbb{F}_q[T]$, $K := \mathbb{F}_q(T)$, and $K_\infty := \mathbb{F}_q((1/T))$ are similar to $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively. Each A -lattice is an analog of a lattice in $\mathbb{C}$. In [1, 2], we introduced Dedekind sums and their higher-dimensional generalization for a given A -lattice in a function field, and we established the reciprocity law. The A -lattice L corresponding to the Carlitz module defines the Dedekind sum $s(a, c)$ (see Section 2), which is very similar to $d(a, c)$. In this report, we answer the analogous questions for $s(a, c)$.

2 Dedekind sums

2.1 A -lattices and Drinfeld modules

Let C_∞ be the completion of an algebraic closure of K_∞ ; it is an analog of $\mathbb{C}$. A rank r A -lattice is a finitely generated A -module of rank r such that it is discrete in

C_∞ . For such an A -lattice Λ , we define the infinite product $e_\Lambda(z)$ by

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right).$$

This product uniformly converges at a bounded set in C_∞ , and defines a map $e_\Lambda : C_\infty \rightarrow C_\infty$. The function $e_\Lambda(z)$ has the following properties:

- (E1) $e_\Lambda(z)$ is entire in the sense of rigid analysis;
- (E2) $e_\Lambda : C_\infty \rightarrow C_\infty$ is surjective $\mathbb{F}_q$ -linear, and Λ -periodic;
- (E3) e_Λ has a simple zero at each point in Λ , and no further zeros;
- (E4) $de_\Lambda(z)/dz = e'_\Lambda(z) = 1$.

For $a \in A$, there exists a unique polynomial $\phi_a(z) = \phi_a^\Lambda(z) = \sum l_i(\phi_a)z^{q^i}$ such that $\phi_a(e_\Lambda(z)) = e_\Lambda(az)$ holds. Let $\tau : z \mapsto z^q$ be the Frobenius map, and let $C_\infty\{\tau\}$ be a non-commutative ring in τ with the commutation rule $c^q\tau = \tau c$ ($c \in C_\infty$). There exists a unique positive integer r such that for any $a \in A \setminus \{0\}$,

$$\phi_a = \sum_{i=0}^{r \deg a} l_i(a)\tau^i \quad (l_0(a) = a).$$

Then, the map $\phi : A \rightarrow C_\infty\{\tau\}$, $a \mapsto \phi_a$ is called a rank r Drinfeld module over C_∞ . The map ϕ is an $\mathbb{F}_q$ -algebra homomorphism; hence, the values ϕ_a ($a \in A$) are determined by ϕ_T . The rank 1 Drinfeld module ρ with $\rho_T(z) = Tz + z^q$ is called the Carlitz module. The Carlitz module and a Drinfeld module of rank ≥ 2 are similar to the multiplicative group $\mathbb{G}_m$ and an elliptic curve, respectively. There exists a bijection between the set of rank r A -lattices and the set of rank r Drinfeld modules over C_∞ , defined by $\phi_a(e_\Lambda(z)) = e_\Lambda(az)$ ($a \in A$). The A -lattice L corresponding to ρ is similar to $2\pi i$, and each A -lattice of rank ≥ 2 is similar to a lattice in $\mathbb{C}$.

2.2 Dedekind sums

Let L be the A -lattice corresponding to the Carlitz module ρ . For coprime $a, c \in A \setminus \{0\}$, we define the inhomogeneous Dedekind sum $s(a, c)$ by

$$s(a, c) = \frac{1}{c} \sum_{0 \neq \ell \in L/cL} e_L\left(\frac{a\ell}{c}\right)^{-1} e_L\left(\frac{\ell}{c}\right)^{-1}.$$

When $L/cL = 0$, $s(a, c)$ is defined to be zero. Using the Galois theory, we see that $s(a, c) \in K$. By (E2), it holds that $s(a, c) = 0$ if $q > 3$. Thus, henceforth, we assume that $q = 3$ or 2 . The reciprocity law for $s(a, c)$ is as follows.

Theorem 2.1 (Reciprocity law) For coprime $a, c \in A$, we have

$$s(a, c) + s(c, a) = \begin{cases} \frac{1}{T^3 - T} \left(\frac{a}{c} + \frac{c}{a} + \frac{1}{ac} \right) & \text{if } q = 3, \\ \frac{1}{T^4 + T^2} \left(\frac{a}{c} + \frac{c}{a} + \frac{1}{a} + \frac{1}{c} + \frac{1}{ac} + 1 \right) & \text{if } q = 2. \end{cases}$$

This result follows from the fact that the sum of all residues of $1/(z\rho_a(z)\rho_c(z))$ is zero.

2.3 Continued fractions

Since the value $s(a, c)$ depends on a/c , we write $s(a/c) = s(a, c)$. Then $s(a/c+b) = s(a/c)$ is valid. For $x = a/c \in K$, we define the sequence $(x_n)_{n \geq 0}$ by $x_0 = x$, $x_{n+1} = 1/(x_n - a_n)$, where a_n is the polynomial part $\sum_{i=0}^k A_i T^i$ of the Laurent expansion $x_n = \sum_{i=-\infty}^k A_i T^i$. This sequence yields the continued fraction development of x :

$$x = [a_0, a_1, \dots, a_n] := a_0 + \frac{1}{a_1 + \frac{1}{\dots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}},$$

where a_i ($i \geq 1$) are non-constant. Note that if $x \in K_\infty \setminus K$, x is an infinite continued fraction. The following theorem gives us the value of $s(a/c)$.

Theorem 2.2 (i) If $q = 3$, then

$$s([a_0, \dots, a_r]) = \begin{cases} \frac{1}{T^3 - T} ([0, a_1, \dots, a_r] + (-1)^{r+1} [0, a_r, \dots, a_1] \\ \quad + a_1 - a_2 + \dots + (-1)^{r+1} a_r) & \text{if } r \geq 1, \\ 0 & \text{if } r = 0. \end{cases}$$

(ii) If $q = 2$, then

$$s([a_0, \dots, a_r]) = \begin{cases} \frac{1}{T^4 + T^2} ([0, a_1, \dots, a_r] + (-1)^{r+1} [0, a_r, \dots, a_1] \\ \quad + \prod_{i=1}^r [0, a_i, \dots, a_r] + a_1 - a_2 + \dots + (-1)^{r+1} a_r + r - 1) & \text{if } r \geq 1, \\ 0 & \text{if } r = 0. \end{cases}$$

We can prove this by induction on r by using Theorem 2.1.

Remark 2.3 Hickerson [3] proved the following result for $d(a/c) := d(a, c)$:

$$d([a_0, \dots, a_r]) = \begin{cases} \frac{-1+(-1)^r}{8} + \frac{1}{12} ([0, a_1, \dots, a_r] + (-1)^{r+1} [0, a_r, \dots, a_1] \\ \quad + a_1 - a_2 + \dots + (-1)^{r+1} a_r) & \text{if } r \geq 1, \\ 0 & \text{if } r = 0. \end{cases}$$

3 Density theorem

As an analog of Hickerson's result, the following two theorems are obtained.

Theorem 3.1 *If $q = 3$ or 2 , then $\{(a/c, s(a/c)) \mid a/c \in K^*\}$ is dense in K_∞^2 .*

Theorem 3.2 *If $q = 3$ or 2 , then $\{s(a/c) \mid a/c \in K^*\}$ is dense in K_∞ .*

Outline of proof of Theorems 3.1, 3.2. We consider the case $q = 3$. Since $(K_\infty \setminus K) \times K$ is dense in K_∞^2 , it suffices to prove that for any $(x, y) \in K_\infty \setminus K$ and for $\epsilon > 0$, there exists $a/c \in K^*$ such that $|x - a/c| < \epsilon$, $|y - s(a/c)| < 2\epsilon$. We write $x = [b_0, b_1, \dots]$. Take any element $\alpha \in K_\infty^*$. For any $\epsilon > 0$, taking fully large s , $|x - [b_0, \dots, b_{s-1}, \alpha]| < \epsilon$ holds. Similarly, we write $x - (T^3 - T)y = [d_0, d_1, \dots]$. Taking fully large t , $|x - (T^3 - T)y - [d_0, \dots, d_{t-1}, \alpha]| < \epsilon$ holds. Suppose that $s + t$ is even. There exists $m, n \in A \setminus \mathbb{F}_q$ such that

$$-b_0 + b_1 - b_2 + \dots + (-1)^s b_{s-1} + (-1)^{t-1} d_{t-1} + \dots - d_1 + d_0 = (-1)^s (m - n).$$

Putting

$$a/c = [b_0, \dots, b_{s-1}, m, n, d_{t-1}, \dots, d_1], \quad \alpha = [m, n, d_{t-1}, \dots, d_1],$$

we have $|x - a/c| < \epsilon$. By Theorem 2.2 (i), we obtain

$$\begin{aligned} s(a/c) = \frac{1}{T^3 - T} & ([0, b_1, \dots, b_{s-1}, m, n, d_{t-1}, \dots, d_1] \\ & - [0, d_1, \dots, d_{t-1}, n, m, b_{s-1}, \dots, b_1] \\ & + b_1 - b_2 + \dots + (-1)^s b_s + (-1)^{s+1} m + (-1)^{s+2} n \\ & + (-1)^{t-1} d_{t-1} + \dots + -d_1), \end{aligned}$$

which yields $|y - s(a/c)| < 2\epsilon$. Theorem 3.2 follows from Theorem 3.1. The case $q = 2$ can be proved in the same way.

References

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