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EMBEDDABLE AW*-ALGEBRAS AND RELATED TOPICS

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One of the interesting problems in operator algebras is to find the essential condition for a C*-algebra to have a faithful representation as an embeddable AW*-algebra, where " embeddable " means that it is embedded as a double commutant in a type I AW*-algebra, with the same centre.

In [15], the final conclusion of the above problem was given by showing that

<u>AW*-algebras with a separating family of centre-valued states, which are completely additive on projectins, are embeddable</u>.

(See also [6], [18], [14], and [5]).

In this note, we would like to make a survey of the development of the problem of embeddability of AW*-algebras, with an outline of their proofs. We also would like to give an application of our result to the type determination problem of regular completions. (See [13] for details).

Let us recall that a C*-algebra A is an AW*-algebra if

(1) each maximal abelian *-subalgebra of A is generated by its projections,

(2) each family of orthogonal projections $\{e_\alpha\}$ in A has a supremum $\sum_A e_\alpha$ in Proj(A) (the set of all projections in A).

Kaplansy introduced AW*-algebras and obtained their theory.

In particular, by an elegant algebraic methods, he extended the Murray-von Neumann's type theory, classification of von Neumann algebras to these more general C*-algebras.

(*) Are all AW*-algebras embeddable ?

In 1970, Takenouchi and Dyer, independently, showed that there is an AW*-factor, which is not a von Neumann algebra and Saitô showed that their AW*-factors are of type III. Maitland Wright also gave an example of a non embeddable AW*-factor of type III. [19], [16] and [22].

So our next problem is this.

(**) Are all type II AW*-algebras embeddable ?

Since every embeddable type II_1 AW*-algebra has a centre-valued trace ([4]) and every semi-finite AW*-algebra A, with a faithful finite projection e in A such that eAe has a centre-valued trace, is embeddable ([3]), the question (**) is equivalent to the following, well-known, long standing open problm:

(***) Have all AW*-algebras of type II_1 got centre-valued traces ?

Several authors ([4], [7], [12], [13], [21], [23]) have considered this question. Maitland Wright, above all, proved that every II_1 AW*-factor with a strictly positive functional is a von Neumann algebra (and so has a trace) ([21]).

Recently, Ozawa established a transfer principle from von Neumann algebras in Boolean-valued set theory to embeddable AW*-algebras. This transfer principle will, then suggest the following:

Theorem 1. *Let* A *be an AW*-algebra with the centre* Z. *Suppose that* A *is finite and has a faithful* Z-*valued state. Then* A *is embeddable and has a centre-valued trace.*

Corollary 1. *Let* A *be a semi-finite (in particular, type* II_∞ *) AW*-algebra such that* A *has a faithful* Z-*valued state. Then* A *is embeddable.*

We can apply Corollary 1 to the following solution of the problem of type determination of the regular completions of separable C*-algebras.

Theorem 2. *Let* B *be a separable C*-algebra and let* $\hat{B}$ *be its regular completion. Then* $\hat{B}$ *has no type* II *direct summands.*

Theorem 3. *If* B *is* NGCR, *then* $\hat{B}$ *never be embeddable* ($\hat{B}$ *is of type* III *also*).

Definition 1. Let B be a C*-algebra. We say that B has a countable order dense subset if, there is a countable subset $\{ a_n \}$ in the hermitian part B_h of B, such that,

$$x = \mathrm{LUB}_{B_h}\{ a_n \mid a_n \leqq x \}$$

for every $x \in B_h$.

The lemma which links our Corollary 1 with Theorem 2 and Theorem 3 is the following:

Lemma 1. *Let A be an embeddable AW*-algebra. Suppose that A has a countable order dense subset. Then A is of type* I.

1. Outline of a proof of Theorem 1.

We can do this by using standard methods of set theory. Main idea is to use Widom's theory of embeddable AW*-algebras and generalized GNS-construction relative to centre-valued states (See [20]).

Let A be a unital C*-algebra and let Z be the centre of A. A centre-valued state ϕ on A is a positive Z-linear map from A into Z such taht $\|\phi(1)\| \leq 1$.

Recall that a Kaplansky-Hilbert module M, over an abelian AW*-algebra Z, has a Z-valued inner product $< , >$ such that

$$\xi \longrightarrow \|<\xi,\xi>\|^{1/2}$$

defines a Banach space norm on M. If M is faithful (in the sense that if $z \in Z$ satisfies $z\xi = 0$ for all $\xi \in M$, then $z = 0$), then the set $L_Z(M)$ of all bounded module endomorphisms of M is a type I AW*-algebra with centre Z. Conversely, given a type I AW*-algebra B, with the centre Z, one can construct a faithful Kaplansky-Hilbert module N, over Z, such that $B \cong L_Z(N)$. In this case, for every non-zero $a \in L_Z(N)$

there is $\xi \in \mathcal{N}$ such that

$$< a^*a\xi \ , \ \xi > \neq 0.$$

Just as a numerical state on a C*-algebra, if A is an AW*-algebra and ϕ is a centre-valued state, then one can construct a Kaplansky-Hilbert module $\mathcal{M}_\phi$, a unital *-homomorphism π_ϕ from A into $L_Z(\mathcal{M}_\phi)$ and a vector ξ_ϕ in $\mathcal{M}_\phi$ such that

$$\phi(x) = < \pi_\phi(x)\xi_\phi, \ \xi_\phi >$$

for all $x \in A$.

Let $S_Z(A)$ be the set of all Z-valued states on A. Let

$$\pi_A(x) = \oplus\{ \ \pi_\phi(x) \mid \phi \in S_Z(A) \ \},$$

and

$$\mathcal{M} = \oplus \ \{ \ \mathcal{M}_\phi \mid \phi \in S_Z(A) \ \}$$

as a Kaplansky-Hilbert module. Then $\mathcal{M}$ is a faithful Kaplansky-Hilbert module over Z and π_A is a faithful Z-representation of A in $L_Z(\mathcal{M})$. In fact, if A is an AW*-algebra, then one can easily check that there are sufficiently many elements in $S_Z(A)$.

Let $\mathcal{A} = \pi_A(A)$ in $L_Z(\mathcal{M})$ and let $\mathcal{A}''$ be the double commutant of $\mathcal{A}$ in $L_Z(\mathcal{M})$.

Definition 2. A projection p in $\mathcal{A}''$ is called <u>open</u> if there exists an increasing net $\{a_\alpha\}$ of non-negative elements in $\mathcal{A}$ such that

$$< a_\alpha\xi \ , \xi > \uparrow < p\xi \ , \xi > \quad \text{in} \quad Z_h,$$

for every $\xi \in \mathcal{M}$.

A projection q in $\mathcal{A}''$ is called <u>closed</u> if $1 - q$ is open.

Lemma 2. _Any projection_ p _in_ A'' _has the smallest closed projection in_ A'' _which dominates_ p.

Definition 3. Any given projection p in A'', the above _smallest closed_ projection which dominates p is called the _closure_ of p and denote it by $\overline{p}$.

Definition 4. A projection p in A'' is called _nowhere dense_ if there is a decreasing net L of projections in A such that $\pi_A(q) \geqq \overline{p}$ for all q in L and $\Pi_A\{ q \mid q \in L \} = 0$ in $\mathrm{Proj}(A)$.

Lemma 3. _Let_ A _be a finite AW*-algebra with the centre_ Z. _Let_ $\{p_n\}$ _be a sequence of nowhere dense closed projections in_ A''. _Then,_ $\overline{\sum_A'' p_n}$ (_in_ A'') _is also nowhere dense_.

Let, for each n, L_n be a decreasing net of projections in A such that $\pi_A(q) \geqq \overline{p_n}$ for all $q \in L_n$ and $\Pi_A\{q \mid q \in L_n\}=0$.

The key point of the proof is to construct a decreasing net $\{q_\alpha\}$ of projections in A such that $\pi_A(q_\alpha) \geqq \overline{\sum_A'' p_n}$ for each α and $\Pi_A\{q_\alpha \mid \alpha\} = 0$ via $\{L_n\}$. See the details for [13].

Next we shall sketch the proof of Theorem 1. Let ϕ be a _faithful_ centre-valued state. Then by our construction of (π_A, M) there is a ξ_ϕ in M such that

$$\phi(x) = \langle \pi_A(x)\xi_\phi , \xi_\phi \rangle$$

for every $x \in A$.

Let ψ be a <u>normal</u> Z-valued state on A'' defined by $\psi(x) = \langle x\xi_\phi , \xi_\phi \rangle$ for every $x \in A''$, then $(\Psi \mid A) \circ \pi_A = \phi$

Let P be the set of all closed nowhere dense projections in A'' and let

$$m = \text{LUB} \{ \psi(p) | p \in P \} \text{ in } Z$$

(Note that $\{ \psi(p) | p \in P\}$ is a bounded subset of Z). Then one can find an increasing net $\{ p_\alpha \}$ in P such that $\psi(p_\alpha) \uparrow m$ (in Z). In fact, if p_1 and $p_2 \in P$, then, by Lemma 3, $\overline{p_1 \vee p_2} \in M$, and so P is an increasing net. Take any non-zero projection z in Z and any positive number ε, one can choose an element $p(z) \in P$ and a non-zero subprojection z_1 in Z of z, such that

$$\|(m - \psi(p(z))z_1\| < \varepsilon .$$

So, we can choose an orthogonal family of central projections $\{z_\lambda\}$ such that $\sum_\lambda z_\lambda = 1$ and a family $\{p_\lambda\}$ in P such that

$$\|(m - \psi(p_\lambda))z_\lambda\| < \varepsilon$$

for all λ. Let $p = \sum_{A''} p_\lambda z_\lambda$, then

$$\|m - \psi(p)\| < \varepsilon.$$

Next, we shall show that $p \in P$. Let L_λ be a decreasing net of projections in A such that $\pi_A(q) \geqq \overline{p_\lambda}$ for all $q \in L_\lambda$ and $\Pi_A\{q \mid q \in L_\lambda\} = 0$ for each λ. Let Z be the set $\{\sum_A z_\lambda q_\lambda | q_\lambda \in L_\lambda$ for each $\lambda\}$. Then Z is a decreasing net of projections in A such that $\pi_A(\sum_A z_\lambda q_\lambda) \geqq \overline{p}$.. Since we can easily check that

$$\Pi_A\{ \sum_A q_\lambda z_\lambda \mid (q_\lambda) = q \in \Pi_\lambda L_\lambda \} = 0 ,$$

Z meets all the requirements for the nowhere density of p. Because p is closed, $p \in P$ follows.

Thus one can find a sequence $\{p_n\}$ in P such that

$$\|\psi(p_n) - m\| \to 0 \ (n \to \infty).$$

Let $d = \overline{\sum_{A''} p_n}$ $(\in A'')$, then, by Lemma 3, $d \in P$ and so $m \geqq \psi(d) \geqq \psi(p_n)$ $(n = 1,2,\cdots)$, which implies that $m = \psi(d)$. Put

$$\eta : A'' \to Z$$

by

$$\eta(x) = \psi((1 - d)x(1 - d)) \quad x \in A'',$$

then η is a normal Z-valued non-zero state on A''. Since $\eta(1) = \psi(1 - d) = \psi(1) - \psi(d) = \phi(1) - m$, if $\eta(1) = 0$, then $m = \phi(1)$. On the other hand, $1 - d$ is open, there is an increasing net $\{a_\alpha\}$ in A^+ such that

$$< \pi_A(a_\alpha)\xi , \xi > \uparrow <(1 - d)\xi , \xi> \text{ for all } \xi \in M$$

and so

$$\phi(a_\alpha) = <\pi_A(a_\alpha)\xi_\phi , \xi_\phi> \uparrow <(1 - d)\xi_\phi , \xi_\phi>,$$

thus, the faithfulness of ϕ tells us that $a_\alpha = 0$ for all α. Thus it follows that $d = 1$. This is a contradiction. Hence $\eta(1) \neq 0$.

Let $p \in P$ and let $r = \overline{p \vee d}$, then $r \in P$ by Lemma 3. Since $(1 - d)p(1 - d) \leqq r - d$, $\eta(p) \leqq \psi(r - d) = \psi(r) - \psi(d) = m - m = 0$.

Let $\{q_\lambda\}$ be any downward directed subset in $\mathrm{Proj}(A)$ with $\Pi_A\{q_\lambda \mid \lambda\} = 0$. Then, $\Pi_{A''}\{\pi_A(q_\lambda)|\lambda\} \in P$, which implies that

$$\mathrm{GLB}_\lambda\ \eta(\pi_A(q_\lambda)) = \eta(\Pi_{A''}\{\pi_A(q_\lambda)|\lambda\}) = 0$$

because η is normal on A''.

Hence, if we put $\phi_0 = \eta | \pi_A(A) \circ \pi_A$, then ϕ_0 is a non-zero Z-valued state of A and normal on Proj(A). Considering the complement of the support of ϕ_0 if necessary, one can construst sufficiently many Z-valued states $\{\phi_\alpha\}$, each of which is completely additive on Proj(A). Thus, by a theorem of Widom [20], A is embeddable and so A has a centre-valued trace by [4].

In [3], Elliott, Saitô and Wright showed the following:

Let A be a semi-finite AW*-algebra with the centre Z and let e be a finite projection in A whose central cover is 1. Then A is embeddable if, and only if, the finite corner eAe has a centre-valued trace.

So our Corollary 1 is a direct consequence of the above result.

2. Outline of proofs of Theorem 2 and Theorem 3.

Let $\hat{B}$ be the regular completion of a given separable C*-algebra B. Then $\hat{B}$ has a countable order dense subset and so $\hat{B}$ has a faithful centre-valued state. If $\hat{B}$ has semi-finite direct summand $\hat{B}e$, then $\hat{B}e$ has a faithful centre-valued state and so $\hat{B}e$ is embeddable. Thus by Lemma 1, $\hat{B}e$ is of type I and hence Theorem 2 follows.

If B is NGCR, then $\hat{B}$ has no type I direct summand ([17]). So the claim in Theorem 3 follows from Theorem 2.

An example. Let A be $F \otimes_{\alpha_0} C[0,1]$, the tensor product of the Fermion algebra by the C*-algebra of all complex-valued continuous functions on [0,1]; then one can easily show that A is a separable NGCR algebra such that $\hat{A}$ is not embeddable. Moreover, the centre Z of $\hat{A}$ is *-isomorphic to $D[0,1]$, the abelian AW*-algebra obtained from the regular completion of $C[0,1]$.

3. Appendix. Towards the additivity of traces in finite AW*-algebras.

In this section, we shall give a sufficient condition for the additivity of traces, which, we hope, will play a role in solving this problem.

Let D be the unique dimension function on a type II_1 AW*-factor A. Let $a \in A_h$ with $a = \int \lambda de_\lambda$. Put

$$Tr(a) = \int \lambda dD(e_\lambda),$$

then Tr satisfies the following properties:

(1) $Tr(u^*au) = Tr(a)$ for all $a \in A_h$, $u \in A_u$,

(2) $Tr(a) \geq 0$ for all $a \in A^+$

(3) $Tr(\lambda a) = \lambda Tr(a)$ for all $\lambda \in R$ and $a \in A_h$,

(4) $Tr(x + y) = Tr(x) + Tr(y)$ if $x, y \in A_h$ and $xy = yx$.

(****) Can we conclude that

$$Tr(x + y) = Tr(x) + Tr(y)$$

for any pair x and y in A_h ?

Proposition 1. <u>Let</u> A <u>be a type</u> II_1 <u>AW*-factor and let</u> Tr <u>be the above defined restricted trace on</u> A. <u>If</u> Tr <u>satisfies that</u>

(5) $Tr(e_1 + e_2 + e_3) = Tr(e_1) + Tr(e_2) + Tr(e_3)$
for every triplet $\{e_1, e_2, e_3\}$ *in* Proj(A), *then the answer to* (****) *is positive*.

Lemma 4 (T. Ono). *If* Tr *satisfies* (5), *then, for any projection* e *and any pair* $\{e_1, e_2\}$ *of orthogonal projections in* A. $Tr(e(e_1 + e_2)e) = Tr(ee_1e) + Tr(ee_2e)$.

Let $e_3 = 1 - e_1 - e_2$ and let ω be the primitive root of $x^3 = 1$. Take $v = e_1 + \omega e_2 + \omega^2 e_3$, then v is a unitary in A such that $e + v^*ev + v^{2*}ev^2 = 3(e_1ee_1 + e_2ee_2 + e_3ee_3)$ and so

$$\begin{aligned} Tr(e + v^*ev + v^{2*}ev^2) &= Tr(e) + Tr(v^*ev) + Tr(v^{2*}ev^2) \\ &= 3Tr(e). \end{aligned}$$

Hence it follows that $Tr(e) = Tr(e_1ee_1) + Tr(e_2ee_2) + Tr(e_3ee_3)$. On the other hand, the fact that $Tr(e_iee_i) + Tr(ee_ie)$ for each i, tells us that $Tr(e) = Tr(ee_1e) + Tr(ee_2e) + Tr(ee_3e)$. Observe that $Tr|eAe$ is a restricted trace, it follows that

$$\begin{aligned} Tr(ee_3e) &= Tr(e(1 - e_1 - e_2)e) \\ &= Tr(e - e(e_1 + e_2)e) \\ &= Tr(e) - Tr(e(e_1 + e_2)e). \end{aligned}$$

Thus it follows that $Tr(e(e_1 + e_2)e) = Tr(ee_1e) + Tr(ee_2e)$.

Since A is of type II, there are four orthogonal *equivalent* projections f_1, f_2, f_3 and f_4 such that $f_1 + f_2 + f_3 + f_4 = 1$. By using this system, one can canonically construct 4 by 4 system of matrix units $\{e_{ij}\}$ in A such that $e_{11} = f_1$.

We shall show, under the condition (5), that f_1Af_1 is a von Neumann algebra. Let a and b be in $(f_1Af_1)^+$ such that

$a \leqq (1/2)1$ and $b \leqq (1/2)1$. Let

$$p = \begin{pmatrix} a & a & (a - 2a^2)^{1/2} & 0 \\ a & a & (a - 2a^2)^{1/2} & 0 \\ (a - 2a^2)^{1/2} & (a - 2a^2)^{1/2} & 1 - 2a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$q = \begin{pmatrix} b & -b & 0 & -(b - 2b^2)^{1/2} \\ -b & b & 0 & (b - 2b^2)^{1/2} \\ 0 & 0 & 0 & 0 \\ -(b - 2b^2)^{1/2} & (b - 2b^2)^{1/2} & 0 & 1 - 2b \end{pmatrix}$$

via $\{e_{ij}\}$, then p and q are orthogonal projections in A such that $f_1pf_1 = a$ and $f_1qf_1 = b$. So, if, Tr satisfies (5), then

$$Tr(f_1(p + q)f_1) = Tr(f_1pf_1) + Tr(f_1qf_1),$$

by Lemma 4, which implies that $Tr(a + b) = Tr(a) + Tr(b)$, and so $Tr\ f_1Af_1$ is a trace on f_1Af_1. Hence f_1Af_1 is a von Neumann factor. Thus by the previous arguments before the proof of Corollary 1, we can get that A is a von Neumann factor, that is, Tr is additive. This completes the proof.

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