

GIBBS MEASURES OF THE QUANTUM RABI MODEL

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1 Quantum Rabi model

This is the joint work with Tomoyuki Shirai and a review of [8]. The quantum Rabi model describes a two-level atom coupled to a single mode photon by the dipole interaction term. The single photon is represented by the 1D harmonic oscillator. Suppose that the eigenvalues of the two-level atom is $\{-\Delta, \Delta\}$. Here $\Delta > 0$ is a constant. Let σ_x, σ_y and σ_z be the 2×2 Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the Hamiltonian of the two-level atom is represented by $\Delta\sigma_z$. On the other hand let a and $a^\dagger$ be the annihilation operator and the creation operator in $L^2(\mathbb{R})$, respectively. They are given by

$$a = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + x \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + x \right).$$

They satisfy the canonical commutation relation $[a, a^\dagger] = \mathbb{1}$, and $a^* = a^\dagger$, where a^* denotes the adjoint of a . The harmonic oscillator is given by $a^\dagger a$, i.e.,

$$a^\dagger a = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2}.$$

The harmonic oscillator $a^\dagger a$ is self-adjoint on $D(\frac{d^2}{dx^2}) \cap D(x^2)$ and the spectrum of $a^\dagger a$ is $\text{spec}(a^\dagger a) = \mathbb{N} \cup \{0\}$. The quantum Rabi Hamiltonian is defined as a self-adjoint operator on the tensor product Hilbert space $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ by

$$K = \Delta\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes a^\dagger a + g\sigma_x \otimes (a + a^\dagger).$$

Here $g \in \mathbb{R}$ stands for a coupling constant. It can be seen that K has the parity symmetry:

$$[K, \sigma_z \otimes (-\mathbb{1})^{a^\dagger a}] = 0.$$

The parity symmetry is also referred to as $\mathbb{Z}_2$ -symmetry. We discuss measures associated with the ground state of the quantum Rabi Hamiltonian. The quantum Rabi model can be regarded as the one mode version of the spin-boson model in quantum field theory. In [5] the path measure associated with the ground state of the spin-boson model is discussed. In this note we also show the existence of the measure Π_∞ associated with the ground state Φ_g of the quantum Rabi Hamiltonian. Then under some condition we can see that

$$(\Phi_g, \mathcal{O}\Phi_g) = \mathbb{E}_{\Pi_\infty}[f_{\mathcal{O}}]$$

for some observable $\mathcal{O}$ with a function $f_{\mathcal{O}}$.

2 Probabilistic preparation

2.1 Unitary transformations

In this section we define a self-adjoint operators L . Let $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. The rotation group in $\mathbb{R}^3$ has an adjoint representation on $su(2)$. Let $n \in \mathbb{R}^3$ be a unit vector and $\theta \in [0, 2\pi)$. Thus $e^{(i/2)\theta n \cdot \sigma}(x \cdot \sigma)e^{-(i/2)\theta n \cdot \sigma} = Rx \cdot \sigma$, where R denotes the 3×3 matrix representing the rotation around n by an angle θ . In particular for $n = (0, 1, 0)$ and $\theta = \pi/2$, we have $U\sigma_x U^{-1} = \sigma_z$ and $U\sigma_z U^{-1} = -\sigma_x$, where

$$U = e^{i\frac{\pi}{4}\sigma_y}. \quad (2.1)$$

Then

$$UKU^{-1} = \begin{pmatrix} -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \sqrt{2}gx - \frac{1}{2} & -\Delta \\ -\Delta & -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + \sqrt{2}gx - \frac{1}{2} \end{pmatrix}.$$

Let us define the unitary operator $\mathcal{S}_g$. Let $p = -i\frac{d}{dx}$ and F denotes the Fourier transform on $L^2(\mathbb{R})$. Then $\mathcal{S}_g$ is defined by

$$\mathcal{S}_g = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} 0 & e^{i\sqrt{2}gp} \\ e^{-i\sqrt{2}gp} & 0 \end{pmatrix}. \quad (2.2)$$

Let φ_g be the normalized ground state of $a^\dagger a$, i.e., $a^\dagger a \varphi_g = 0$ and it is explicitly given by

$$\varphi_g(x) = \pi^{-1/4} e^{-|x|^2/2}.$$

Since φ_g is strictly positive, we can define the unitary operator $\mathcal{U}_{\varphi_g} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}, \varphi_g^2 dx)$ by

$$\mathcal{U}_{\varphi_g} f = \varphi_g^{-1} f. \quad (2.3)$$

We set the probability measure $\varphi_g^2(x)dx$ on $\mathbb{R}$ by $d\mu$, i.e.,

$$d\mu(x) = \frac{1}{\sqrt{\pi}} e^{-|x|^2} dx.$$

Define

$$\mathcal{H} = \mathbb{C}^2 \otimes L^2(\mathbb{R}, d\mu).$$

Let $\mathcal{U} = \mathcal{U}_{\varphi_g} U$. We define the self-adjoint operator L by

$$\begin{aligned} L &= \mathcal{U} K \mathcal{U}^{-1} = -\Delta \sigma_x \otimes \mathbb{1} + g \sigma_z \otimes (b^\dagger + b) + \mathbb{1} \otimes b^\dagger b \\ &= \begin{pmatrix} -\frac{1}{2} \frac{d^2}{dx^2} + x \frac{d}{dx} & 0 \\ 0 & -\frac{1}{2} \frac{d^2}{dx^2} + x \frac{d}{dx} \end{pmatrix} - \begin{pmatrix} -\sqrt{2} g x & \Delta \\ \Delta & \sqrt{2} g x \end{pmatrix}. \end{aligned} \quad (2.4)$$

Here b and $b^\dagger$ are the annihilation operator and the creation operator in $L^2(\mathbb{R}, d\mu)$, which are defined by $\varphi_g^{-1} a^\sharp \varphi_g = b^\sharp$. It is actually given by

$$b = a + \frac{x}{\sqrt{2}}, \quad b^\dagger = a^\dagger - \frac{x}{\sqrt{2}}.$$

2.2 Ornstein-Uhlenbeck process

Let $(X_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck process on a probability space

$$(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, P^x).$$

We see that $P^x(X_0 = x) = 1$ and

$$\int_{\mathbb{R}} \mathbb{E}_P^x [X_t] d\mu(x) = 0, \quad \int_{\mathbb{R}} \mathbb{E}_P^x [X_t X_s] d\mu(x) = \frac{1}{2} e^{-|t-s|}.$$

Here $\mathbb{E}_P^x [\dots]$ denotes the expectation with respect to the probability measure P^x . Let $h = b^\dagger b$. The generator of X_t is given by $-h$ and

$$(\phi, e^{-th} \psi)_{L^2(\mathbb{R}, d\mu)} = \int_{\mathbb{R}} \mathbb{E}_P^x \left[\overline{\phi(X_0)} \psi(X_t) \right] d\mu(x). \quad (2.5)$$

It is well known that the Ornstein-Uhlenbeck process can be represented by 1D-Brownian motion. Let $(B_t)_{t \geq 0}$ be 1D-Brownian motion starting from x at $t = 0$ on a probability space $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mathcal{W}^0)$. The distributions of X_s under P^x and $e^{-s} \left(x + \frac{1}{\sqrt{2}} B_{e^{2s}-1} \right)$ under $\mathcal{W}^0$ are identical. We denote this as

$$X_s \stackrel{d}{=} e^{-s} \left(x + \frac{1}{\sqrt{2}} B_{e^{2s}-1} \right) \quad s \geq 0. \quad (2.6)$$

We can compute the density function κ_t of X_t as

$$\mathbb{E}_P^x [f(X_t)] = \int_{\mathbb{R}} f(y) \kappa_t(y, x) dy,$$

where

$$\kappa_t(y, x) = \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \exp \left(-\frac{|y - e^{-t}x|^2}{1 - e^{-2t}} \right). \quad (2.7)$$

The Mehler kernel M_t is defined by

$$M_t(x, y) = \frac{\varphi_g(x)}{\varphi_g(y)} \kappa_t(y, x) = \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \exp \left(-\frac{1}{2} \frac{(1 + e^{-2t})(x^2 + y^2) - 4xye^{-t}}{1 - e^{-2t}} \right).$$

For the later use we extend the Ornstein-Uhlenbeck process $(X_t)_{t \geq 0}$ to the Ornstein-Uhlenbeck process $(\hat{X}_t)_{t \in \mathbb{R}}$ on the whole real line on the probability space $(\bar{\mathcal{X}}, \mathcal{B}_{\bar{\mathcal{X}}}, \bar{\mathbb{P}}^x)$. Here $\bar{\mathcal{X}} = \mathcal{X} \times \mathcal{X}$, $\mathcal{B}_{\bar{\mathcal{X}}} = \mathcal{B}_{\mathcal{X}} \times \mathcal{B}_{\mathcal{X}}$ and $\bar{\mathbb{P}}^x = \mathbb{P}^x \otimes \mathbb{P}^x$. Define for $w = (w_1, w_2) \in \mathcal{X} \times \mathcal{X}$

$$\hat{X}_t(w) = \begin{cases} X_t(w_1), & t \geq 0, \\ X_{-t}(w_2), & t < 0. \end{cases} \quad (2.8)$$

Then $\hat{X}_t$ and $\hat{X}_{-s}$ for any $s, t > 0$ are independent. We also see that

$$(\phi, e^{-th}\psi)_{L^2(\mathbb{R}, d\mu)} = \int_{\mathbb{R}} \mathbb{E}_{\bar{\mathbb{P}}^x} \left[\overline{\phi(\hat{X}_0)} \psi(\hat{X}_t) \right] d\mu(x) = \int_{\mathbb{R}} \mathbb{E}_{\bar{\mathbb{P}}^x} \left[\overline{\phi(\hat{X}_{-s})} \psi(\hat{X}_{t-s}) \right] d\mu(x) \quad (2.9)$$

for any $0 \leq s \leq t$.

2.3 Spin process

In order to show the spin part by a path measure we introduce a Poisson process. Let $(N_t)_{t \geq 0}$ be a Poisson process on a probability space

$$(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}}, \Pi)$$

with the unit intensity, i.e.,

$$\mathbb{E}_{\Pi} [\mathbb{1}_{\{N_t=n\}}] = \frac{t^n}{n!} e^{-t}, \quad n \geq 0.$$

Note that N_t is a nonnegative integer-valued random process, $N_0 = 0$ and $t \mapsto N_t$ is not decreasing. Furthermore $t \mapsto N_t$ is right continuous and its left limit exists (càdlàg). Let

$$\mathbb{Z}_2 = \{-1, +1\}.$$

Then for $u \in L^2(\mathbb{Z}_2)$,

$$\|u\|_{L^2(\mathbb{Z}_2)}^2 = \sum_{\alpha \in \mathbb{Z}_2} |u(\alpha)|^2.$$

Introducing the norm on $\mathbb{C}^2$ by $(u, v)_{\mathbb{C}^2} = \sum_{i=1}^2 \bar{u}_i v_i$, we identify $\mathbb{C}^2 \cong L^2(\mathbb{Z}_2)$ by $\mathbb{C}^2 \ni u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cong u(\alpha)$ with $u(+1) = u_1$ and $u(-1) = u_2$. Note that

$$(u, v)_{\mathbb{C}^2} = (u, v)_{L^2(\mathbb{Z}_2)}.$$

Under this identification σ_x, σ_y and σ_z are represented as the operators U_x, U_y and U_z , respectively on $L^2(\mathbb{Z}_2)$ by

$$U_x u(\alpha) = u(-\alpha), \quad U_y u(\alpha) = -i\alpha u(-\alpha), \quad U_z u(\alpha) = \alpha u(\alpha), \quad u \in L^2(\mathbb{Z}_2). \quad (2.10)$$

We define

$$S_t = (-1)^{N_t} \alpha, \quad \alpha \in \mathbb{Z}_2.$$

Here $(S_t)_{t \geq 0}$ is a dichotomous process which is referred to as a spin process in this note. Let $\sigma_F = \frac{1}{2}(\sigma_z + i\sigma_y)(\sigma_z - i\sigma_y) = -\sigma_x + \mathbb{1}$ be the fermionic harmonic oscillator. Then it is known that for $u, v \in \mathbb{C}^2$, $(u, e^{-t\sigma_F} v)_{\mathbb{C}^2} = \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\Pi}[\overline{u(S_0)} v(S_t)]$. Hence

$$(u, e^{t\sigma_x} v)_{\mathbb{C}^2} = e^t \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\Pi}[\overline{u(S_0)} v(S_t)]. \quad (2.11)$$

We also extend the Poisson process $(N_t)_{t \geq 0}$ to the Poisson process $(\hat{N}_t)_{t \in \mathbb{R}}$ on the whole real line on a probability space $(\tilde{\mathcal{Y}}, \mathcal{B}_{\tilde{\mathcal{Y}}}, \tilde{\Pi})$, where $\tilde{\mathcal{Y}} = \mathcal{Y} \times \mathcal{Y}$, $\mathcal{B}_{\tilde{\mathcal{Y}}} = \mathcal{B}_{\mathcal{Y}} \times \mathcal{B}_{\mathcal{Y}}$ and $\tilde{\Pi} = \Pi \otimes \Pi$. Let $(\tilde{N}_t)_{t \geq 0}$ be a Poisson process on $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}}, \Pi)$ such that $t \mapsto \tilde{N}_t$ is left continuous and its right limit exists (càglàd). Define for $w = (w_1, w_2) \in \mathcal{Y} \times \mathcal{Y}$,

$$\hat{N}_t(w) = \begin{cases} N_t(w_1), & t \geq 0, \\ \tilde{N}_{-t}(w_2), & t < 0. \end{cases}$$

Then $\mathbb{R} \ni t \mapsto \hat{N}_t$ is a càdlàg path. Note that $\hat{N}_t$ is independent of $\hat{N}_{-s}$ for any $s, t > 0$. We define

$$\hat{S}_t = (-1)^{\hat{N}_t} \alpha, \quad \alpha \in \mathbb{Z}_2.$$

By the shift invariance of $\hat{S}_s$ [9, Proposition 3.44] we can see that for $u, v \in \mathbb{C}^2$,

$$(u, e^{t\sigma_x} v)_{\mathbb{C}^2} = e^t \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\tilde{\Pi}}[\overline{u(\hat{S}_0)} v(\hat{S}_t)] = e^t \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\tilde{\Pi}}[\overline{u(\hat{S}_{-s})} v(\hat{S}_{t-s})]$$

for any $0 \leq s \leq t$.

3 Path measure associated with the ground state

In this section we construct the path measure associated with the ground state of the quantum Rabi model. We recall that $L = -\Delta_{\sigma_x} \otimes \mathbb{1} + \mathbb{1} \otimes b^\dagger b + g\sigma_z \otimes (b + b^\dagger)$. Let Φ_g be the ground state of L such that

$$L\Phi_g = E\Phi_g$$

with $E = \inf \text{spec}(L)$. It is shown that $\Phi_g > 0$ in [4] under the identification (3.1). Hence $(\mathbb{1}, \Phi_g)_{\mathcal{H}} \neq 0$. Then

$$\Phi_g = \lim_{t \rightarrow \infty} \frac{e^{-tL} \mathbb{1}}{\|e^{-tL} \mathbb{1}\|_{\mathcal{H}}}.$$

Let us set

$$\langle \mathcal{O} \rangle = (\Phi_g, \mathcal{O}\Phi_g)_{\mathcal{H}}$$

for a bounded operator $\mathcal{O}$. Then we have

$$\langle \mathcal{O} \rangle = \lim_{t \rightarrow \infty} \frac{(e^{-tL} \mathbb{1}, \mathcal{O}e^{-tL} \mathbb{1})_{\mathcal{H}}}{\|e^{-tL} \mathbb{1}\|_{\mathcal{H}}^2}.$$

The right-hand side can be represented in terms of Feynman-Kac formula, and under some condition we can also see that

$$\langle \mathcal{O} \rangle = \mathbb{E}_{\Pi_\infty}[f_{\mathcal{O}}]$$

with some probability measure Π_∞ and a function $f_{\mathcal{O}}$. The probability measure Π_∞ is referred to as the path measure associated with the ground state Φ_g . The similar results are investigated in models in quantum field theory [10, 1, 5, 6, 7], but as far as we know there is no example in quantum mechanics.

3.1 Feynman-Kac formula

Combining (2.5) and (2.11) we can represent $(\phi, e^{-tH}\psi)$ by a path measure. Let

$$q_s = (S_s, X_s) \quad s \geq 0$$

be the $(\mathbb{Z}_2 \times \mathbb{R})$ -valued random process on the probability space $(\mathcal{X} \otimes \mathcal{Y}, \mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{Y}}, \mathbb{P}^x \otimes \Pi)$. We introduce the identification:

$$\mathcal{H} \cong L^2(\mathbb{Z}_2 \times \mathbb{R}) \quad (3.1)$$

by

$$\begin{pmatrix} \phi_+(x) \\ \phi_-(x) \end{pmatrix} \cong \phi(\alpha, x) = \delta_{+1\alpha}\phi_+(x) + \delta_{-1\alpha}\phi_-(x), \quad (\alpha, x) \in \mathbb{Z}_2 \times \mathbb{R}. \quad (3.2)$$

Here $\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$. We use identification (3.1) without notices unless no confusion arises. Let $W : \mathbb{Z}_2 \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$W(\alpha, x) = \sqrt{2}\alpha x.$$

Thus $W(q_{s-}) = \sqrt{2}S_{s-}X_s$. The Poisson integral $\int_0^{t+} W(q_{s-})dN_s$ is a random process on the probability space $(\mathcal{X} \otimes \mathcal{Y}, \mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{Y}}, \mathbb{P}^x \otimes \Pi)$, which is defined by

$$\left(\int_0^{t+} W(q_{s-})dN_s \right) (w_1, w_2) = \sum_{j=1}^n W(q_{s_j}(w_1, w_2)) = \sqrt{2} \sum_{j=1}^n S_{s_j-}(w_1)X_{s_j}(w_2).$$

Here $\{s_j\}$ is the set of jump points such that $N_{s_j-}(w_1) \neq N_{s_j+}(w_1)$ for $0 \leq s_j \leq t$. Let

$$\mathbf{E}[\dots] = \frac{1}{2} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}^x}^{\alpha} \mathbb{E}_{\Pi}[\dots] d\mu(x).$$

Lemma 3.1 *Let $\phi, \psi \in \mathcal{H}$. Then under the identification (3.2), it follows that*

$$(\phi, e^{-tL}\psi) = 2e^t \mathbf{E} \left[\overline{\phi(q_0)} \psi(q_t) \Delta^{N_t} e^{-g \int_0^t W(q_s) ds} \right]. \quad (3.3)$$

Proof: We refer the reader to [8]. ■

Lemma 3.1 can be extended to the path integral representations of Euclidean Green functions. Let $h = -\Delta/2$ and $(B_t)_{t \geq 0}$ be 1D Brownian motion on $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mathcal{W}^x)$. Suppose that $0 < t_0 < t_1 < \dots < t_n$. Let $C^{\{t_0, t_1, \dots, t_n\}}(A_0 \times \dots \times A_n) = \{\omega \in \mathcal{X} \mid \omega(t_j) \in A_j, j = 0, 1, \dots, n\}$ be a cylinder set. Then it is known that

$$\mathcal{W}^x(C^{\{t_0, t_1, \dots, t_n\}}(A_0 \times \dots \times A_n)) = \mathbb{E}^x \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(B_{t_j}) \right) \right].$$

We know furthermore that for $f, g \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} \mathbb{E}^x \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(B_{t_j}) \right) \bar{f}(B_0) g(B_t) \right] dx = (f, e^{-t_0 h} \mathbb{1}_{A_0} e^{-(t_1 - t_0)h} \dots e^{-(t_n - t_{n-1})h} \mathbb{1}_{A_n} e^{-(t - t_n)h} g).$$

Lemma 3.2 *Let $f_j = f_j(\alpha, x)$ be bounded function on $\mathbb{Z}_2 \times \mathbb{R}$ for $j = 0, 1, \dots, n$. Suppose that $0 < t_0 < t_1 < \dots < t_n$. Then*

$$\begin{aligned} & (\phi, e^{-t_0 L} f_0 e^{-(t_1 - t_0)L} f_1 e^{-(t_2 - t_1)L} \dots e^{-(t_n - t_{n-1})L} f_n e^{-(t - t_n)L} \psi) \\ &= 2e^t \mathbf{E} \left[\bar{\phi}(q_0) \psi(q_t) \left(\prod_{j=0}^n f_j(q_{t_j}) \right) e^{-g \int_0^t W(q_s) ds} \right]. \end{aligned}$$

Proof: Denote the natural filtrations of $(N_t)_{t \geq 0}$ and $(X_t)_{t \geq 0}$ by $\mathcal{N}_s = \sigma(N_r, 0 \leq r \leq s)$ and $\mathcal{M}_s = \sigma(X_r, 0 \leq r \leq s)$, respectively. The Markov properties of $(N_t)_{t \geq 0}$ and $(X_t)_{t \geq 0}$ yield that

$$\begin{aligned} & (e^{-sL} f e^{-tL} \phi)(\alpha, x) \\ &= e^{s+t} \mathbb{E}_{\Pi} \mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^s W(q_r) dr} f(q_s) \mathbb{E}_{\Pi}^{S_s} \mathbb{E}_{\mathbb{P}}^{X_s} \left[e^{-g \int_0^t W(q_r) dr} \phi(q_t) \right] \right] \\ &= e^{s+t} \mathbb{E}_{\Pi} \mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^s W(q_r) dr} f(q_s) \mathbb{E}_{\Pi} \mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^t W(q_{r+s}) dr} \phi(q_{t+s}) \mid \mathcal{N}_s \times \mathcal{M}_s \right] \right] \\ &= e^{s+t} \mathbb{E}_{\Pi} \mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^s W(q_r) dr} f(q_s) e^{-g \int_0^t W(q_{r+s}) dr} \phi(q_{t+s}) \right] \\ &= e^{s+t} \mathbb{E}_{\Pi} \mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^{s+t} W(q_r) dr} f(q_s) \phi(q_{t+s}) \right]. \end{aligned}$$

Repeating these procedures we have the lemma. ■

3.2 Probability measure Π_{∞} associated with the ground state

We set $T_s = S_{\Delta s}$ and $q_s^{\Delta} = (T_s, X_s)$. We assume that $\Delta > 0$ in what follows.

Lemma 3.3 *Let $\phi, \psi \in \mathcal{H}$. Then*

$$(\phi, e^{-tL} \psi) = e^{\Delta t} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\Pi} \mathbb{E}_{\mathbb{P}}^x \left[\overline{\phi(q_0^{\Delta})} \psi(q_t^{\Delta}) e^{-g \int_0^t W(q_s^{\Delta}) ds} \right] d\mu(x). \quad (3.4)$$

Proof: Since

$$\frac{1}{\Delta}L = -\sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes \frac{1}{\Delta}b^\dagger b + \frac{g}{\Delta}\sigma_z \otimes (b^\dagger + b),$$

the Feynman-Kac formula (3.3) yields that

$$(\phi, e^{-tL}\phi) = (\phi, e^{-\Delta t \frac{1}{\Delta}L}\phi) = e^{\Delta t} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\Pi} \mathbb{E}_{\mathbb{P}}^x \left[\overline{\phi(S_0, X_0)} \psi(S_{\Delta t}, X_t) e^{-\frac{g}{\Delta} \int_0^{\Delta t} \sqrt{2} S_s X_{s/\Delta} ds} \right] d\mu(x).$$

By the change of variable s to Δs in $\frac{g}{\Delta} \int_0^{\Delta t} \sqrt{2} S_s X_{s/\Delta} ds$, we see (3.4). ■

For the later use we have a technical lemma below.

Lemma 3.4 *We have*

$$\mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^t W(\hat{q}_s^\Delta) ds} \right] = e^{-g \left(\int_0^t e^{-s} (-1)^{N_{\Delta s}} ds \right) x} e^{\frac{g^2}{4} \int_0^{(1-e^{-2t})/2} \left| \int_y^t (-1)^{N_{\Delta s}} ds \right|^2 dy}.$$

In particular

$$\mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^t W(\hat{q}_s^\Delta) ds} \right] \leq e^{|g|(1-e^{-t})x} e^{\frac{g^2}{4} \int_0^{(1-e^{-2t})/2} |t-y|^2 dy}.$$

Proof: We have

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^t W(\hat{q}_s^\Delta) ds} \right] &= \mathbb{E}_{\mathcal{W}}^0 \left[e^{-g \int_0^t e^{-s} (x + \frac{1}{\sqrt{2}} B_{e^{2s}-1}) (-1)^{N_{\Delta s}} ds} \right] \\ &= e^{-g \left(\int_0^t e^{-s} (-1)^{N_{\Delta s}} ds \right) x} \mathbb{E}_{\mathcal{W}}^0 \left[e^{-g \int_0^t B_{(1-e^{-2s})/2} (-1)^{N_{\Delta s}} ds} \right] \\ &= e^{-g \left(\int_0^t e^{-s} (-1)^{N_{\Delta s}} ds \right) x} e^{\frac{g^2}{4} \left\| \int_0^t \mathbb{1}_{(1-e^{-2s})/2}(\cdot) (-1)^{N_{\Delta s}} ds \right\|_{L^2(\mathbb{R})}^2}. \end{aligned}$$

Then the lemma is proven. ■

Now we extend $(T_t)_{t \geq 0}$ to the process on the whole real line. Let

$$\hat{T}_t = (-1)^{\hat{N}_{\Delta t}} \alpha \quad t \in \mathbb{R}.$$

We can realize $(\hat{T}_t)_{t \in \mathbb{R}}$ as a coordinate process as usual. Let $\mathcal{D} = D(\mathbb{R})$ be the space of càdlàg paths on $\mathbb{R}$. There exists a topology d° on $\mathcal{D}$ such that $(\mathcal{D}, d^\circ)$ is a separable and complete metric space (e.g. [3, Section 3.5] and [2, Section 16]). Let $\mathcal{B}_{\mathcal{D}}$ be the Borel sigma-field of $\mathcal{D}$. Thus

$$\hat{T}_\bullet : (\tilde{\mathcal{Y}}, \mathcal{B}_{\tilde{\mathcal{Y}}}, \bar{\Pi}) \rightarrow (\mathcal{D}, \mathcal{B}_{\mathcal{D}})$$

is an $\mathcal{D}$ -valued random variable. We denote its image measure on $(\mathcal{D}, \mathcal{B}_{\mathcal{D}})$ by Q^α , i.e., $Q^\alpha(A) = \bar{\Pi}(\hat{T}_\bullet^{-1}(A))$ for $A \in \mathcal{B}_{\mathcal{D}}$, and the coordinate process on $(\mathcal{D}, \mathcal{B}_{\mathcal{D}})$ by the same symbol $(\hat{T}_t)_{t \geq 0}$, i.e., $\hat{T}_t(\omega) = \omega(t)$ for $\omega \in \mathcal{D}$. Let $\pi_\Lambda : \mathcal{D} \rightarrow \mathbb{R}^\Lambda$ be the projection defined by $\pi_\Lambda(\omega) = (\omega(t_0), \dots, \omega(t_n))$ for $\omega \in \mathcal{D}$ and $\Lambda = \{t_0, \dots, t_n\}$. Then

$$\mathcal{A} = \{\pi_\Lambda^{-1}(E) \mid \Lambda \subset \mathbb{R}, \#\Lambda < \infty, E \in \mathcal{B}(\mathbb{R}^\Lambda)\}$$

is the family of cylinder sets. It is known that the sigma-field generated by cylinder sets coincides with $\mathcal{B}_{\mathcal{D}}$. Moreover let $\mathcal{D}_T = D([-T, T])$ be the space of càdlàg paths on $[-T, T]$

and $\pi_T : \mathcal{D} \rightarrow \mathcal{D}_T$ be the projection defined by $\pi_T \omega = \omega|_{[-T, T]}$. Let $\mathcal{B}_T$ be the Borel sigma-field of $\mathcal{D}_T$. Let $\pi_\Lambda : \mathcal{D}_T \rightarrow \mathbb{R}^\Lambda$ be the projection defined by $\pi_\Lambda(\omega) = (\omega(t_0), \dots, \omega(t_n))$ for $\omega \in \mathcal{D}_T$ and $\Lambda = \{t_0, \dots, t_n\}$. Note that we use the same notation π as the projection from $\mathcal{D}$ to $\mathbb{R}^\Lambda$. Then

$$\mathcal{A}_T = \{\pi_\Lambda^{-1}(E) \mid \Lambda \subset [-T, T], \# \Lambda < \infty, E \in \mathcal{B}(\mathbb{R}^\Lambda)\}$$

is the family of cylinder sets. We set

$$\overset{\circ}{\mathcal{B}} = \bigcup_{s \geq 0} \pi_s^{-1}(\mathcal{B}_s), \quad \overset{\circ}{\mathcal{B}}_T = \bigcup_{0 \leq s \leq T} \pi_s^{-1}(\mathcal{B}_s).$$

It is also seen that the sigma-field generated by $\overset{\circ}{\mathcal{B}}$ (resp. $\overset{\circ}{\mathcal{B}}_T$) coincides with $\mathcal{B}_{\mathcal{D}}$ (resp. $\mathcal{B}_T$). Together with them we have

$$\mathcal{B}_{\mathcal{D}} = \sigma(\mathcal{A}) = \sigma(\overset{\circ}{\mathcal{B}}), \quad \mathcal{B}_T = \sigma(\mathcal{A}_T) = \sigma(\overset{\circ}{\mathcal{B}}_T). \quad (3.5)$$

Hence (3.3) can be reformulated in terms of the coordinate process $(\hat{T}_t)_{t \geq 0}$ on $(\mathcal{D}, \mathcal{B}_{\mathcal{D}}, \mathbb{Q}^\alpha)$ instead of $(\bar{\mathcal{Y}}, \mathcal{B}_{\bar{\mathcal{Y}}}, \bar{\Pi})$ as

$$(\phi, e^{-tL}\psi) = e^{\Delta t} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}}^\alpha \mathbb{E}_{\mathbb{P}}^x \left[\overline{\phi(\hat{q}_0^\Delta)} e^{-g \int_0^t W(\hat{q}_s^\Delta) ds} \psi(\hat{q}_t^\Delta) \right] d\mu(x). \quad (3.6)$$

Here

$$\hat{q}_s^\Delta = (\hat{T}_s, \hat{X}_s) \quad s \in \mathbb{R},$$

where $\hat{X}_t$ is the Ornstein-Uhlenbeck process on the whole real line. The advantage of (3.4) is that Δ^{N_t} disappears. Δ^{N_t} is not shift invariant but $\hat{T}_s$ in (3.4) is shift invariant. Then

$$\begin{aligned} & \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}}^\alpha \mathbb{E}_{\mathbb{P}}^x \left[\overline{\phi(\hat{q}_0^\Delta)} e^{-g \int_0^t W(\hat{q}_s^\Delta) ds} \psi(\hat{q}_t^\Delta) \right] d\mu(x) \\ &= \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}}^\alpha \mathbb{E}_{\mathbb{P}}^x \left[\overline{\phi(\hat{q}_{-r}^\Delta)} e^{-g \int_0^t W(\hat{q}_{s-r}^\Delta) ds} \psi(\hat{q}_{t-r}^\Delta) \right] d\mu(x) \end{aligned}$$

for any $0 \leq r \leq t$. Let

$$W_\Delta(t, s) = \hat{T}_t \hat{T}_s e^{-|t-s|}. \quad (3.7)$$

Lemma 3.5 *We have*

$$(\mathbb{1}, e^{-tL}\mathbb{1}) = 2e^{\Delta t} \mathbb{E}_{\mathbb{Q}}^\alpha \left[\exp \left(\frac{g^2}{2} \int_0^t ds \int_0^t dr W_\Delta(s, r) \right) \right].$$

Proof: By the Feynman-Kac formula given by (3.4) and inserting (2.6), we can see that

$$(\mathbb{1}, e^{-tL}\mathbb{1}) = e^{\Delta t} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathbb{Q}}^\alpha \left[\mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^t \hat{T}_s e^{-s} B_{e^{2s-1}} ds} \right] \int_{\mathbb{R}} e^{-(\sqrt{2}g \int_0^t \hat{T}_s e^{-s} ds)x} d\mu(x) \right]$$

Since

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^t \hat{T}_s e^{-s} B_{e^{2s-1}} ds} \right] &= \exp \left(\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_s \hat{T}_r e^{-(s+r)} (e^{2(s \wedge r)} - 1) \right) \\ \int_{\mathbb{R}} e^{-(\sqrt{2}g \int_0^t \hat{T}_s e^{-s} ds)x} d\mu(x) &= \exp \left\{ \frac{g^2}{2} \left(\int_0^t \hat{T}_s e^{-s} ds \right)^2 \right\},\end{aligned}$$

we obtain that

$$(\mathbb{1}, e^{-tL} \mathbb{1}) = e^{\Delta t} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\exp \left(\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_s \hat{T}_r e^{-|s-r|} \right) \right].$$

Hence the lemma follows. ■

Remark 3.6 (1) Since $W_{\Delta}(s, r)$ is independent of α , $\mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\exp \left(\frac{g^2}{2} \int_0^t ds \int_0^t dr W_{\Delta}(s, r) \right) \right]$ is also independent of σ .

(2) By the shift invariance of $\hat{T}_s$ we can also see that

$$\mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\exp \left(\frac{g^2}{2} \int_0^t ds \int_0^t dr W_{\Delta}(s, r) \right) \right] = \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\exp \left(\frac{g^2}{2} \int_{-u}^{t-u} ds \int_{-u}^{t-u} dr W_{\Delta}(s, r) \right) \right]$$

for any $0 \leq u \leq t$. Thus we see that

$$\begin{aligned}(e^{-tL} \mathbb{1}, e^{-tL} \mathbb{1}) &= 2e^{2\Delta t} \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\exp \left(\frac{g^2}{2} \int_0^{2t} ds \int_0^{2t} dr W_{\Delta}(s, r) \right) \right] \\ &= 2e^{2\Delta t} \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\exp \left(\frac{g^2}{2} \int_{-t}^t ds \int_{-t}^t dr W_{\Delta}(s, r) \right) \right].\end{aligned}\tag{3.8}$$

We can also compute $(e^{-tL} \mathbb{1}, e^{-\beta b^\dagger b} e^{-tL} \mathbb{1})$ for $\beta > 0$.

Lemma 3.7 Let $\beta > 0$. Then

$$\begin{aligned}(e^{-tL} \mathbb{1}, e^{-\beta b^\dagger b} e^{-tL} \mathbb{1}) &= 2e^{2\Delta t} \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\exp \left(\frac{g^2}{2} \int_{-t}^t \int_{-t}^t W_{\Delta}(s, r) ds dr - g^2(1 - e^{-\beta}) \int_{-t}^0 \int_0^t W_{\Delta}(s, r) ds dr \right) \right].\end{aligned}$$

Proof: Since

$$(\phi, e^{-\beta b^\dagger b} \psi) = \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \bar{\phi}(\alpha, X_0) \mathbb{E}_{\mathbb{P}}^x[\psi(\alpha, X_{\beta})] d\mu(x),$$

we see that

$$(e^{-tL} \mathbb{1}, e^{-\beta b^\dagger b} e^{-tL} \mathbb{1}) = \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} (e^{-tL} \mathbb{1})(\alpha, X_0) \mathbb{E}_{\mathbb{P}}^x[(e^{-tL} \mathbb{1})(\alpha, X_{\beta})] d\mu(x).$$

It is straightforward to compute $(e^{-tL}\mathbb{1})(\alpha, X_0)$ and $(e^{-tL}\mathbb{1})(\alpha, X_\beta)$. We have

$$\begin{aligned} (e^{-tL}\mathbb{1})(\alpha, X_0) &= e^{\Delta t} \mathbb{E}_{\mathbb{Q}}^\alpha \mathbb{E}_{\mathbb{P}}^x \left[e^{-\sqrt{2}g \int_0^t \hat{T}_s X_s^x ds} \right] \\ &= e^{\Delta t} \mathbb{E}_{\mathbb{Q}}^\alpha \left[e^{-\sqrt{2}g \int_0^t \hat{T}_s e^{-s} ds} \mathbb{E}_{\mathcal{W}}^0 \left[e^{-g \int_0^t \hat{T}_s e^{-s} B_{e^{2s-1}} ds} \right] \right] \\ &= e^{\Delta t} \mathbb{E}_{\mathbb{Q}}^\alpha \left[e^{-\sqrt{2}g \int_0^t \hat{T}_s e^{-s} ds} e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_s \hat{T}_r e^{-(s+r)} (e^{2(s \wedge r)} - 1)} \right]. \end{aligned}$$

The computation of $\mathbb{E}_{\mathbb{P}}^x [(e^{-tL}\mathbb{1})(\alpha, X_\beta)]$ is more complicated than that of $(e^{-tL}\mathbb{1})(\alpha, X_0)$. We have

$$\mathbb{E}_{\mathbb{P}}^x [(e^{-tL}\mathbb{1})(\alpha, X_\beta)] = e^{\Delta t} \mathbb{E}_{\mathbb{P}}^x \left[\mathbb{E}_{\mathbb{Q}}^\alpha \left[e^{-\sqrt{2}g \int_0^t \hat{T}_s e^{-s} ds} X_\beta e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_s \hat{T}_r e^{-(s+r)} (e^{2(s \wedge r)} - 1)} \right] \right].$$

Inserting (2.6) to X_β above again, we obtain that

$$\begin{aligned} &= e^{\Delta t} \mathbb{E}_{\mathcal{W}}^0 \left[\mathbb{E}_{\mathbb{Q}}^\alpha \left[e^{-\sqrt{2}g \int_0^t \hat{T}_s e^{-s} ds} e^{-\beta \left(x + \frac{1}{\sqrt{2}} B_{e^{2\beta-1}} \right)} e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_s \hat{T}_r e^{-(s+r)} (e^{2(s \wedge r)} - 1)} \right] \right] \\ &= e^{\Delta t} \mathbb{E}_{\mathbb{Q}}^\alpha \left[e^{-(\sqrt{2}g \int_0^t \hat{T}_s e^{-s} ds)} e^{-\beta x} e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_s \hat{T}_r e^{-(s+r)} (1 - e^{-2\beta})} e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_s \hat{T}_r e^{-(s+r)} (e^{2(s \wedge r)} - 1)} \right] \\ &= e^{\Delta t} \mathbb{E}_{\mathbb{Q}}^\alpha \left[e^{-(\sqrt{2}g \int_0^t \hat{T}_{s-t} e^{-s} ds)} e^{-\beta x} e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_{s-t} \hat{T}_{r-t} e^{-(s+r)} (1 - e^{-2\beta})} e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_{s-t} \hat{T}_{r-t} e^{-(s+r)} (e^{2(s \wedge r)} - 1)} \right]. \end{aligned}$$

In the last line above we shift $\hat{T}_s$ by t . Since $\hat{T}_u$ for $0 \leq u \leq t$ and $\hat{T}_{s-t}$ for $0 \leq s \leq t$ are independent, combining above computations, we have

$$\begin{aligned} &(e^{-tL}\mathbb{1}, e^{-\beta b^\dagger b} e^{-tL}\mathbb{1}) \\ &= \sum_{\alpha \in \mathbb{Z}_2} e^{2\Delta t} \int_{\mathbb{R}} \frac{e^{-x^2}}{\sqrt{\pi}} \mathbb{E}_{\mathbb{Q}}^\alpha \left[e^{-(\sqrt{2}g \int_0^t \hat{T}_s e^{-s} ds)} x e^{-(\sqrt{2}g \int_0^t \hat{T}_{s-t} e^{-s} ds)} e^{-\beta x} e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_s \hat{T}_r e^{-(s+r)} (e^{2(s \wedge r)} - 1)} \right. \\ &\quad \left. \times e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_{s-t} \hat{T}_{r-t} e^{-(s+r)} (1 - e^{-2\beta})} e^{\frac{g^2}{2} \int_0^t ds \int_0^t dr \hat{T}_{s-t} \hat{T}_{r-t} e^{-(s+r)} (e^{2(s \wedge r)} - 1)} \right] dx. \quad (3.9) \end{aligned}$$

Terms dependent on x on the exponent above can be computed as

$$\begin{aligned} &-x^2 - \sqrt{2}g \left(e^{-\beta} \int_0^t \hat{T}_{s-t} e^{-s} ds + \int_0^t \hat{T}_s e^{-s} ds \right) x \\ &= - \left(x + \frac{g}{\sqrt{2}} \int_0^t \hat{T}_s e^{-s} ds + \frac{g}{\sqrt{2}} e^{-\beta} \int_0^t \hat{T}_{s-t} e^{-s} ds \right)^2 + \frac{g^2}{2} \left(\int_0^t \hat{T}_s e^{-s} ds + e^{-\beta} \int_0^t \hat{T}_{s-t} e^{-s} ds \right)^2. \end{aligned}$$

The first term on the right-hand side can be integrated with respect to dx as

$$\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\left(x + \frac{g}{\sqrt{2}} \int_0^t \hat{T}_s e^{-s} ds + \frac{g}{\sqrt{2}} e^{-\beta} \int_0^t \hat{T}_{s-t} e^{-s} ds \right)^2} dx = 1.$$

The second term on the right-hand side can be computed as

$$\begin{aligned} & \left(\int_0^t \hat{T}_s e^{-s} ds + e^{-\beta} \int_0^t \hat{T}_{s-t} e^{-s} ds \right)^2 \\ &= \int_0^t \int_0^t \hat{T}_s \hat{T}_r e^{-(s+r)} ds dr + 2e^{-\beta} \int_0^t \int_0^t \hat{T}_{s-t} \hat{T}_r e^{-(s+r)} ds dr + e^{-2\beta} \int_0^t \int_0^t \hat{T}_{s-t} \hat{T}_{r-t} e^{-(s+r)} ds dr. \end{aligned} \quad (3.10)$$

Terms independent of x on (3.9) are

$$\begin{aligned} & \int_0^t ds \int_0^t \hat{T}_s \hat{T}_r e^{-(s+r)} (e^{2(s \wedge r)} - 1) dr + \int_0^t ds \int_0^t \hat{T}_{s-t} \hat{T}_{r-t} e^{-(s+r)} (1 - e^{-2\beta}) dr \\ & \quad + \int_0^t ds \int_0^t \hat{T}_{s-t} \hat{T}_{r-t} e^{-(s+r)} (e^{2(s \wedge r)} - 1) dr. \end{aligned} \quad (3.11)$$

Then the sum of (3.10) and (3.11) is

$$\begin{aligned} & (3.10) + (3.11) \\ &= \int_0^t ds \int_0^t \hat{T}_{s-t} \hat{T}_{r-t} e^{-|s-r|} dr + \int_0^t ds \int_0^t \hat{T}_s \hat{T}_r e^{-|s-r|} dr + 2e^{-\beta} \int_0^t \int_0^t \hat{T}_{s-t} \hat{T}_r e^{-(s+r)} ds dr \\ &= \int_{-t}^0 ds \int_{-t}^0 \hat{T}_s \hat{T}_r e^{-|s-r|} dr + \int_0^t ds \int_0^t \hat{T}_s \hat{T}_r e^{-|s-r|} dr + 2e^{-\beta} \int_{-t}^0 \int_0^t \hat{T}_s \hat{T}_r e^{-|s-r|} ds dr. \end{aligned}$$

By the trick $\int_{-t}^t \int_{-t}^t = \int_{-t}^0 \int_{-t}^0 + \int_0^t \int_0^t + 2 \int_{-t}^0 \int_0^t$, we see that

$$(3.10) + (3.11) = \int_{-t}^t ds \int_{-t}^t \hat{T}_s \hat{T}_r e^{-|s-r|} dr - 2(1 - e^{-\beta}) \int_{-t}^0 \int_0^t \hat{T}_s \hat{T}_r e^{-|s-r|} ds dr.$$

Then the lemma follows. ■

Define the probability measure Π_T on $(\mathcal{D}, \mathcal{B}_{\mathcal{D}})$ by

$$\Pi_T(A) = \frac{1}{Z_T} \frac{1}{2} e^{2T\Delta} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\mathbb{1}_A e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_{\Delta}(t, s)} \right], \quad A \in \mathcal{B}_{\mathcal{D}}, \quad (3.12)$$

where $Z_T = \frac{1}{2} e^{2T\Delta} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_{\Delta}(t, s)} \right]$ is the normalizing constant. Note that pair interaction $W_{\Delta}(t, s)$ is independent of σ and hence one can replace $\sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathbb{Q}}^{\alpha}$ with $2\mathbb{E}_{\mathbb{Q}}^{\alpha}$ in (3.12). We also notice that $1 = \|\Phi_g\|_{\mathcal{H}}^2 = \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} |\Phi_g(\alpha, x)|^2 d\mu(x)$, $2 = \|\mathbb{1}\|_{\mathcal{H}}^2 = \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} d\mu(x)$ and $2Z_T = \|e^{-TL}\mathbb{1}\|^2$.

Let $A_j \in \mathcal{B}(\mathbb{R})$ for $j = 0, 1, \dots, n$ and $\Lambda = \{t_0, t_1, \dots, t_n\} \subset [-T, T]$. The cylinder set is defined by

$$C_T^{\Lambda}(A_0 \times \dots \times A_n) = \{\omega \in \mathcal{D}_T \mid \omega(t_j) \in A_j, j = 0, 1, \dots, n\}.$$

Recall that the family of cylinder sets is denoted by $\mathcal{A}_T$. We also note that $\sigma(\mathcal{A}_T) = \mathcal{B}_T$. Let

$$m_t(A) = e^{2Et} e^{2\Delta t} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}}^{\alpha} \mathbb{E}_{\mathbb{P}}^x \left[\mathbb{1}_A \Phi_g(\hat{q}_{-t}^{\Delta}) \Phi_g(\hat{q}_t^{\Delta}) e^{-g \int_{-t}^t W(\hat{q}_s^{\Delta}) ds} \right] d\mu(x). \quad (3.13)$$

Since $\overset{\circ}{\mathcal{B}}$ is a finitely additive family of sets, we define the finitely additive set function ν on $(\mathcal{D}, \overset{\circ}{\mathcal{B}})$ by $\nu(A) = m_t(A)$ for $A \in \pi_t^{-1}(\mathcal{B}_t)$.

Lemma 3.8 ν is well defined, i.e., $m_t(A) = m_s(A)$ for $A \in \pi_t^{-1}(\mathcal{B}_t) \subset \pi_s^{-1}(\mathcal{B}_s)$.

Proof: Notice that $m_t \circ \pi_t^{-1}$ and $m_s \circ \pi_t^{-1}$ are probability measures on $(\mathcal{D}_t, \mathcal{B}_t)$. We compute finite dimensional distributions of $m_t \circ \pi_t^{-1}$ and $m_s \circ \pi_t^{-1}$. Let $\Lambda = \{t_0, t_1, \dots, t_n\} \subset [-t, t] \subset [-s, s]$. Since $e^{-r\bar{L}}\Phi_g = \Phi_g$ for any $r \geq 0$, we have by (4.4),

$$\begin{aligned} & m_t \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)) \\ &= e^{2Et} e^{2\Delta t} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_Q^\alpha \mathbb{E}_P^x \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(\hat{T}_{t_j}) \right) \Phi_g(\hat{q}_{-t}^\Delta) \Phi_g(\hat{q}_t^\Delta) e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x) \\ &= (e^{-(t_0+t)\bar{L}} \Phi_g, \mathbb{1}_{A_0} e^{-(t_1-t_0)\bar{L}} \mathbb{1}_{A_1} \dots e^{-(t_n-t_{n-1})\bar{L}} \mathbb{1}_{A_n} e^{-(t-t_n)\bar{L}} \Phi_g) \\ &= (\Phi_g, \mathbb{1}_{A_0} e^{-(t_1-t_0)\bar{L}} \mathbb{1}_{A_1} \dots e^{-(t_n-t_{n-1})\bar{L}} \mathbb{1}_{A_n} \Phi_g) \\ &= (e^{-(t_0+s)\bar{L}} \Phi_g, \mathbb{1}_{A_0} e^{-(t_1-t_0)\bar{L}} \mathbb{1}_{A_1} \dots e^{-(t_n-t_{n-1})\bar{L}} \mathbb{1}_{A_n} e^{-(s-t_n)\bar{L}} \Phi_g) \\ &= m_s \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)). \end{aligned}$$

It is straightforward to see that the Kolmogorov consistency condition also holds true:

$$m_t \circ \pi_t^{-1} \left(C_t^{\{\Lambda, s_1, \dots, s_m\}} \left(A_0 \times \dots \times A_n \times \prod_{i=1}^m \mathbb{R} \right) \right) = m_t \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)).$$

Let $\pi_\Lambda : [-t, t]^\mathbb{R} \rightarrow \mathbb{R}^\Lambda$ be the projection such that for $\omega \in [-t, t]^\mathbb{R}$, $\pi_\Lambda \omega = (\omega(t_0), \dots, \omega(t_n))$. Thus by the Kolmogorov extension theorem there exists a unique probability measure $\tilde{m}_t$ on $([-t, t]^\mathbb{R}, \sigma(\mathcal{A}_t))$ such that

$$\tilde{m}_t(\pi_\Lambda^{-1}(A_0 \times \dots \times A_n)) = m_t \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)) \quad (3.14)$$

for all $\Lambda \subset [-t, t]$ with $\#\Lambda < \infty$ and $A_j \in \mathcal{B}(\mathbb{R})$. Since the extension is unique, $m_t \circ \pi_t^{-1} = \tilde{m}_t$. Similarly there exists a unique probability measure $\tilde{m}_s$ on $([-t, t]^\mathbb{R}, \sigma(\mathcal{A}_t))$ such that $m_s \circ \pi_t^{-1} = \tilde{m}_s$. Then $m_s \circ \pi_t^{-1} = m_t \circ \pi_t^{-1}$ on $\mathcal{B}_t$, which implies the lemma. $\blacksquare$

The first task is to extend ν to a probability measure by the Hopf extension theorem.

Lemma 3.9 ν can be uniquely extended to a probability measure Π_∞ on $(\mathcal{D}, \mathcal{B}_\mathcal{D})$.

Proof: Suppose that $E_n \in \overset{\circ}{\mathcal{B}}$ such that $E_n \supset E_{n+1} \supset \dots$ and $\lim_{n \rightarrow \infty} \nu(E_n) = \alpha > 0$. It is enough to show that $\bigcap_n E_n \neq \emptyset$ by the Hopf extension theorem. Let $E_n = \pi_{T_n}^{-1}(E'_n)$ with $E'_n \in \mathcal{B}_{T_n}$. We can assume that $T_n < T_{n+1} < \dots \rightarrow \infty$. Let $\mu_T = \nu \circ \pi_T^{-1}$ be a probability measure on $\mathcal{D}_T$. Since $\mathcal{D}_T$ is a Polish space, μ_T is regular, i.e., for $A \in \mathcal{B}_T$ and $\epsilon > 0$ there exist a compact set K and an open set O in $\mathcal{D}_T$ such that $K \subset A \subset O$ and $\mu_T(O \setminus K) < \epsilon$. There exists a compact set $K'_n \subset \mathcal{D}_{T_n}$ such that $\mu_{T_n}(E'_n \setminus K'_n) < \alpha/2^n$. Let $K_n = \pi_{T_n}^{-1}(K'_n)$, $D_n = \bigcap_{j=1}^n K_j$ and $D = \bigcap_{n=1}^\infty D_n$. Since $D \subset \bigcap_n E_n$, it is enough to show that $D \neq \emptyset$. We

see that

$$\begin{aligned}
\alpha - \nu(D_n) &\leq \nu(E_n) - \nu(D_n) \leq \nu(E_n \setminus D_n) \\
&= \nu(\cup_{j=1}^n E_n \setminus K_j) = \nu(\pi_{T_n}^{-1} \cup_{j=1}^n E'_n \setminus K'_j) = \mu_{T_n}(\cup_{j=1}^n E'_n \setminus K'_j) \\
&= \sum_{j=1}^n \mu_{T_n}(E'_n \setminus K'_j) \leq \sum_{j=1}^n \mu_{T_n}(E'_j \setminus K'_j) \leq \sum_{j=1}^n \alpha/2^j.
\end{aligned}$$

Then $0 < \nu(D_n)$ and we see that $D_n \neq \emptyset$. Let $f_n \in D_n$, i.e., $f_n \in \bigcap_{j=1}^n K_j$. Thus

$$f_n \in K_\ell \text{ for any } n \geq \ell.$$

Let $\ell = 1$. Then $\pi_{T_1}(f_n) \in K'_1$ for any $n \geq 1$. Taking a subsequence n' , we see that $\lim_{n' \rightarrow \infty} \pi_{T_1}(f_{n'}) = h_1 \in K'_1$ exists. Let $\ell = 2$. Then $\pi_{T_2}(f_{n'}) \in K'_2$ for any $n' \geq 2$. Take a subsequence n'' of n' again, then $\lim_{n'' \rightarrow \infty} \pi_{T_2}(f_{n''}) = h_2 \in K'_2$ exists. Proceeding this procedure, we can obtain a subsequence $\{m\}$ that $\lim_{m \rightarrow \infty} \pi_{T_\ell}(f_m) = h_\ell \in K'_\ell$ exists for any ℓ . Let $g_\ell = \pi_{T_\ell}^{-1}(h_\ell) \in L_\ell$. Define $g \in \mathcal{D}$ by $g(x) = g_\ell(x)$ for $x \in [-T_\ell, T_\ell]$. By the construction this is well defined, i.e., $g_\ell(x) = g_{\ell+1}(x)$ for $x \in [-T_\ell, T_\ell]$. We see that $g \in D$ and $D \neq \emptyset$. ■

For probability measures Π_T and Π_∞ on $(\mathcal{D}, \mathcal{B}_D)$ in order to show that $\Pi_T(A) \rightarrow \Pi_\infty(A)$ for every $A \in \mathring{\mathcal{B}}$, we define the finitely additive set function ρ_T on $(\mathcal{D}_T, \mathring{\mathcal{B}}_T)$. Let $\mathbb{1}_T = e^{-T\bar{L}} \mathbb{1}$ for $t \geq 0$. Then $s\text{-}\lim_{T \rightarrow \infty} \mathbb{1}_T = \Phi_g$ and $\|\mathbb{1}_T\|^2 = 2e^{2TE} Z_T$. The finitely additive set function ρ_T on $(\mathcal{D}_T, \mathring{\mathcal{B}}_T)$ is defined by

$$\rho_T(A) = e^{2Et} e^{2t\Delta} \frac{1}{\|\mathbb{1}_T\|^2} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}}^\alpha \mathbb{E}_{\mathbb{P}}^x \left[\mathbb{1}_A \mathbb{1}_{T-t}(\hat{q}_{-t}^\Delta) \mathbb{1}_{T-t}(\hat{q}_t^\Delta) e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x) \quad (3.15)$$

for $A \in \pi_t^{-1}(\mathcal{B}_t)$ but $t \leq T$. The right-hand side of (3.15) is denoted by $M_{T,t}(A)$.

Lemma 3.10 ρ_T is well defined, i.e., $M_{T,t}(A) = M_{T,s}(A)$ for $A \in \pi_t^{-1}(\mathcal{B}_t) \subset \pi_s^{-1}(\mathcal{B}_s)$.

Proof: This is shown in a similar manner to Lemma 3.8. Let

$$M_{T,t}(A) = e^{2Et} e^{2t\Delta} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}}^\alpha \mathbb{E}_{\mathbb{P}}^x \left[\mathbb{1}_A \mathbb{1}_{T-t}(\hat{q}_{-t}^\Delta) \mathbb{1}_{T-t}(\hat{q}_t^\Delta) e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x).$$

Then $M_{T,t} \circ \pi_t^{-1}$ and $M_{T,s} \circ \pi_t^{-1}$ are probability measures on $(\mathcal{D}_t, \mathcal{B}_t)$. Let $\Lambda = \{t_0, t_1, \dots, t_n\} \subset [-t, t] \subset [-s, s]$. We have by (4.4),

$$\begin{aligned}
&M_{T,t} \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)) \\
&= e^{2Et} e^{2t\Delta} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}}^\alpha \mathbb{E}_{\mathbb{P}}^x \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(\hat{T}_{t_j}) \right) \mathbb{1}_{T-t}(\hat{q}_{-t}^\Delta) \mathbb{1}_{T-t}(\hat{q}_t^\Delta) e^{-g \int_{-t}^t W(\hat{q}_r^\Delta) dr} \right] d\mu(x) \\
&= (e^{-(t_0+t)\bar{L}} \mathbb{1}_{T-t}, \mathbb{1}_{A_0} e^{-(t_1-t_0)\bar{L}} \mathbb{1}_{A_1} \dots e^{-(t_n-t_{n-1})\bar{L}} \mathbb{1}_{A_n} e^{-(t-t_n)\bar{L}} \mathbb{1}_{T-t}) \\
&= (e^{-(t_0+s)\bar{L}} \mathbb{1}_{T-s}, \mathbb{1}_{A_0} e^{-(t_1-t_0)\bar{L}} \mathbb{1}_{A_1} \dots e^{-(t_n-t_{n-1})\bar{L}} \mathbb{1}_{A_n} e^{-(s-t_n)\bar{L}} \mathbb{1}_{T-s}) \\
&= M_{T,s} \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)).
\end{aligned}$$

It is straightforward to see that the Kolmogorov consistency condition also holds true:

$$M_{T,t} \circ \pi_t^{-1} \left(C_t^{\{\Lambda, s_1, \dots, s_m\}} \left(A_0 \times \dots \times A_n \times \prod_{i=1}^m \mathbb{R} \right) \right) = M_{T,t} \circ \pi_t^{-1} (C_t^\Lambda (A_0 \times \dots \times A_n)).$$

Thus by the Kolmogorov extension theorem there exists a unique probability measure $\bar{M}_{T,t}$ on $([-t, t]^\mathbb{R}, \sigma(\mathcal{A}_t))$ such that

$$\bar{M}_{T,t}(\pi_\Lambda^{-1}(A_0 \times \dots \times A_n)) = M_{T,t} \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)) \quad (3.16)$$

for all $\Lambda \subset [-T, T]$ with $\#\Lambda < \infty$ and $A_j \in \mathcal{B}(\mathbb{R})$. Since the extension is unique, $M_{T,t} \circ \pi_t^{-1} = \bar{M}_{T,t}$. Similarly there exists a unique probability measure $\bar{M}_{T,s}$ on $([-t, t]^\mathbb{R}, \sigma(\mathcal{A}_t))$ such that $M_{T,s} \circ \pi_t^{-1} = \bar{M}_{T,s}$. Then $M_{T,s} \circ \pi_t^{-1} = M_{T,t} \circ \pi_t^{-1}$ on $\mathcal{B}_t$, which implies the lemma. ■

We shall show that $\rho_T = \Pi_T$ on $\overset{\circ}{\mathcal{B}}_T$ for any $T > 0$.

Lemma 3.11 *We have $\rho_T = \Pi_T$ on $\overset{\circ}{\mathcal{B}}_T$.*

Proof: Let $t \leq T$. It is enough to show that $\Pi_T(A) = \rho_T(A)$ for $A \in \pi_t^{-1}(\mathcal{B}_t)$. Let $\Lambda = \{t_0, t_1, \dots, t_n\} \subset [-t, t] \subset [-T, T]$ and $A_0 \times \dots \times A_n \in \mathcal{B}(\mathbb{R}^\Lambda)$. We have

$$\Pi_T \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)) = \frac{1}{Z_T} e^{2T\Delta} \frac{1}{2} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_Q^\alpha \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(\hat{T}_{t_j}) \right) e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_\Delta(t,s)} \right], \quad (3.17)$$

$$\begin{aligned} \rho_T \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)) \\ = e^{2Et} e^{2\Delta t} \frac{1}{\|\mathbb{1}_T\|^2} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_Q^\alpha \mathbb{E}_{\mathbb{P}}^x \left[\left(\prod_{j=0}^n \mathbb{1}_{A_j}(\hat{T}_{t_j}) \right) \mathbb{1}_{T-t}(\hat{q}_{-t}^\Delta) \mathbb{1}_{T-t}(\hat{q}_t^\Delta) e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x). \end{aligned} \quad (3.18)$$

By (4.4) we see that

$$\begin{aligned} (3.17) &= \frac{1}{\|\mathbb{1}_T\|^2} (\mathbb{1}, e^{-(t_0+T)L} \mathbb{1}_{A_0} e^{-(t_1-t_0)L} \mathbb{1}_{A_1} \dots \mathbb{1}_{A_n} e^{-(T-t_n)L} \mathbb{1}) \\ &= \frac{e^{2Et}}{\|\mathbb{1}_T\|^2} (\mathbb{1}_{T-t}, e^{-(t_0+t)L} \mathbb{1}_{A_0} e^{-(t_1-t_0)L} \mathbb{1}_{A_1} \dots \mathbb{1}_{A_n} e^{-(t-t_n)L} \mathbb{1}_{T-t}) = (3.18). \end{aligned}$$

Then we have

$$\Pi_T \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)) = \rho_T \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)). \quad (3.19)$$

Since both sides of (3.19) satisfy the Kolmogorov consistency condition, there exists a unique probability measure μ on $(\mathcal{D}_T, \mathcal{B}_t)$ such that

$$\mu(\pi_\Lambda^{-1}(A_0 \times \dots \times A_n)) = \Pi_T \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)) = \rho_T \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \dots \times A_n)).$$

$\Pi_T \circ \pi_t^{-1}$ and $\rho_T \circ \pi_t^{-1}$ are probability measures on $(\mathcal{D}_t, \mathcal{B}_t)$, and $\Pi_T \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \cdots \times A_n)) = \Pi_T \circ \pi_t^{-1}(\pi_\Lambda^{-1}(A_0 \times \cdots \times A_n)) = \rho_T \circ \pi_t^{-1}(C_t^\Lambda(A_0 \times \cdots \times A_n)) = \rho_T \circ \pi_t^{-1}(\pi_\Lambda^{-1}(A_0 \times \cdots \times A_n))$. Since the extension is unique, $\Pi_T \circ \pi_t^{-1} = \mu = \rho_T \circ \pi_t^{-1}$ on $(\mathcal{D}_t, \mathcal{B}_t)$ follows. $\blacksquare$

The following proposition is shown for spin boson model in [5, Theorem 3.8] and for relativistic Pauli-Fierz model in [6, Lemma 7.6], and the proof for the quantum Rabi Hamiltonian is a minor modification of [5, 6].

Proposition 3.12 *There exists a probability measure Π_∞ on $(\mathcal{D}, \mathcal{B}_D)$ such that*

$$\lim_{T \rightarrow \infty} \Pi_T(A) = \Pi_\infty(A) \quad A \in \mathring{\mathcal{B}}.$$

Proof: By $s\text{-}\lim_{T \rightarrow \infty} \mathbb{1}_T = \Phi_g$ we obtain that $s\text{-}\lim_{T \rightarrow \infty} \mathbb{1}_{T-t} = \Phi_g$ and $\lim_{T \rightarrow \infty} \|\mathbb{1}_T\| = 1$. Then for each $\alpha \in \mathbb{Z}_2$, $(\mathbb{1}_{T-t}/\|\mathbb{1}_T\|)(\cdot, \sigma) \rightarrow \varphi_g(\cdot, \sigma)$ as $T \rightarrow \infty$ in $L^2(\mathbb{R}, d\mu)$. Let $\Phi_g^T = \frac{\mathbb{1}_{T-t}}{\|\mathbb{1}_T\|}$. Note that $\Phi_g, \Phi_g^T \in L^\infty(\mathbb{Z}_2 \times \mathbb{R})$. Let $A \in \pi_t^{-1}(\mathcal{B}_t)$. Then $\Pi_T(A) = \rho_T(A)$ by Lemma 3.11 and $\nu(A) = \Pi_\infty(A)$ by Lemma 3.8. We have

$$\begin{aligned} \Pi_T(A) - \Pi_\infty(A) &= \rho_T(A) - \nu(A) \\ &= e^{2Et} e^{2\Delta t} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathbb{Q}}^\alpha \left[\mathbb{1}_A \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left(\Phi_g(\hat{q}_{-t}^\Delta) \Phi_g(\hat{q}_t^\Delta) - \Phi_g^T(\hat{q}_{-t}^\Delta) \Phi_g^T(\hat{q}_t^\Delta) \right) e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x) \right]. \end{aligned}$$

Then

$$\begin{aligned} & \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left| \Phi_g(\hat{q}_{-t}^\Delta) \Phi_g(\hat{q}_t^\Delta) - \Phi_g^T(\hat{q}_{-t}^\Delta) \Phi_g^T(\hat{q}_t^\Delta) \right| e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x) \\ & \leq \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left| \Phi_g(\hat{q}_{-t}^\Delta) - \Phi_g^T(\hat{q}_{-t}^\Delta) \right| \left| \Phi_g(\hat{q}_t^\Delta) \right| e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x) \\ & \quad + \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left| \Phi_g^T(\hat{q}_{-t}^\Delta) \right| \left| \Phi_g(\hat{q}_t^\Delta) - \Phi_g^T(\hat{q}_t^\Delta) \right| e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x). \end{aligned}$$

We estimate $\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left| \Phi_g(\hat{q}_{-t}^\Delta) - \Phi_g^T(\hat{q}_{-t}^\Delta) \right| \left| \Phi_g(\hat{q}_t^\Delta) \right| e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x)$. By the shift invariance we have

$$\begin{aligned} & \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left| \Phi_g(\hat{q}_{-t}^\Delta) - \Phi_g^T(\hat{q}_{-t}^\Delta) \right| \left| \Phi_g(\hat{q}_t^\Delta) \right| e^{-g \int_{-t}^t W(\hat{q}_s^\Delta) ds} \right] d\mu(x) \\ & = \int_{\mathbb{R}} \left| \Phi_g(\hat{q}_0^\Delta) - \Phi_g^T(\hat{q}_0^\Delta) \right| \mathbb{E}_{\mathbb{P}}^x \left[\left| \Phi_g(\hat{q}_{2t}^\Delta) \right| e^{-g \int_0^{2t} W(\hat{q}_s^\Delta) ds} \right] d\mu(x). \end{aligned}$$

By the Schwarz inequality we also have

$$\leq \left(\int_{\mathbb{R}} \left| \Phi_g(\hat{q}_0^\Delta) - \Phi_g^T(\hat{q}_0^\Delta) \right|^2 d\mu(x) \right)^{1/2} \left(\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left| \Phi_g(\hat{q}_{2t}^\Delta) \right|^2 \right] d\mu(x) \right)^{1/2} \left(\mathbb{E}_{\mathbb{P}}^x \left[e^{-2g \int_0^{2t} W(\hat{q}_s^\Delta) ds} \right] \right)^{1/2}.$$

Since by Lemma 3.4,

$$\mathbb{E}_{\mathbb{P}}^x \left[e^{-2g \int_0^{2t} W(\hat{q}_s^\Delta) ds} \right] \leq e^{|g|(1-e^{-2t})|x|} e^{g^2 \int_0^{(1-e^{-4t})/2} |2t-y|^2 dy},$$

we have

$$\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left| (\Phi_g(\hat{q}_{-t}^{\Delta}) - \Phi_g^T(\hat{q}_{-t}^{\Delta})) \Phi_g(\hat{q}_t^{\Delta}) \right| e^{-g \int_{-t}^t W(\hat{q}_s^{\Delta}) ds} \right] d\mu(x) \\ \leq C \left(\int_{\mathbb{R}} |\Phi_g(\hat{q}_0^{\Delta}) - \Phi_g^T(\hat{q}_0^{\Delta})|^2 d\mu(x) \right)^{1/2} \left(\int_{\mathbb{R}} e^{|g|(1-e^{-2t})|x|} d\mu(x) \right)^{1/2}.$$

Here we employed that $\Phi_g \in L^\infty(\mathbb{Z}_2 \times \mathbb{R})$. Since $\int_{\mathbb{R}} |\Phi_g(\hat{q}_0^{\Delta}) - \Phi_g^T(\hat{q}_0^{\Delta})|^2 d\mu(x) \rightarrow 0$ as $T \rightarrow \infty$,

$$\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left| \Phi_g(\hat{q}_{-t}^{\Delta}) - \Phi_g^T(\hat{q}_{-t}^{\Delta}) \right| \Phi_g(\hat{q}_t^{\Delta}) \right| e^{-g \int_{-t}^t W(\hat{q}_s^{\Delta}) ds} \right] d\mu(x) \rightarrow 0$$

as $T \rightarrow \infty$. Similarly we can also show that

$$\int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[\left| \Phi_g^T(\hat{q}_{-t}^{\Delta}) \right| \left| \Phi_g(\hat{q}_t^{\Delta}) - \Phi_g^T(\hat{q}_t^{\Delta}) \right| e^{-g \int_{-t}^t W(\hat{q}_s^{\Delta}) ds} \right] d\mu(x) \rightarrow 0$$

as $T \rightarrow \infty$. Then the proof is complete. $\blacksquare$

The sequence of probability measures $(\Pi_T)_{T>0}$ is said to locally converge to the probability measure Π_∞ whenever $\lim_{T \rightarrow \infty} \Pi_T(A) = \Pi_\infty(A)$ for all $A \in \pi_t^{-1}(\mathcal{B}_t)$ and for all $t \geq 0$.

Corollary 3.13 *Let f be a $\mathcal{B}_t$ -measurable and bounded function. Then*

$$\lim_{T \rightarrow \infty} \mathbb{E}_{\Pi_T}[f] = \mathbb{E}_{\Pi_\infty}[f].$$

Proof: It is enough to show the corollary for a nonnegative function f . Since f is bounded and $\mathcal{B}_t$ -measurable, there exists a sequence $\{f_n\}$ such that $\lim_{n \rightarrow \infty} \sup_{x \in \mathcal{D}} |f_n(x) - f(x)| = 0$. Here f_n is of the form $f_n = \sum_{j=1}^{m_n} a_j \mathbb{1}_{A_j}$ with $A_j \in \mathcal{B}_t$ and $a_j > 0$. Let $\epsilon > 0$ be arbitrary. We assume that $\sup_{x \in \mathcal{D}} |f_n(x) - f(x)| \leq \epsilon$. Then we see that

$$|\mathbb{E}_{\Pi_T}[f] - \mathbb{E}_{\Pi_\infty}[f]| \leq \mathbb{E}_{\Pi_T}[|f - f_n|] + |\mathbb{E}_{\Pi_T}[f_n] - \mathbb{E}_{\Pi_\infty}[f_n]| + \mathbb{E}_{\Pi_\infty}[|f_n - f|] \\ \leq 2\epsilon + |\mathbb{E}_{\Pi_T}[f_n] - \mathbb{E}_{\Pi_\infty}[f_n]|$$

and from Proposition 3.12 it follows that $\lim_{T \rightarrow \infty} |\mathbb{E}_{\Pi_T}[f] - \mathbb{E}_{\Pi_\infty}[f]| \leq 2\epsilon$. Then the corollary follows. $\blacksquare$

4 Expectations by Π_∞

In this section we give some examples of application of Π_∞ . These examples are one mode versions of the spin boson model [10, 5]. Then we show only outlines of proofs.

4.1 Number operator $b^\dagger b$

Theorem 4.1 *Let $\beta \in \mathbb{C}$. Then*

$$\langle e^{\beta b^\dagger b} \rangle = \mathbb{E}_{\Pi_\infty} \left[e^{-g^2(1-e^\beta) \int_{-\infty}^0 \int_0^\infty W_\Delta(s,r) ds dr} \right], \quad (4.1)$$

$$\langle (b^\dagger b)^m \rangle = \sum_{l=1}^m a_l(m) g^{2l} \mathbb{E}_{\Pi_\infty} \left[\left(\int_{-\infty}^0 \int_0^\infty W_\Delta(s,r) ds dr \right)^l \right]. \quad (4.2)$$

Here $a_l(m) = \frac{(-1)^l}{l!} \sum_{s=1}^l (-1)^s \binom{l}{s}$ are the Stirling numbers. In particular $\langle (b^\dagger b)^m \rangle \leq e^{2g^2} - 1$ for any $m \geq 0$.

Simple but non trivial application is as follows. We know that $\langle \sigma_x \otimes (-\mathbb{1})^{b^\dagger b} \rangle < 0$ since the parity of Φ_g is -1 . As a corollary of Theorem 4.1 we can show that $\langle (-\mathbb{1})^{b^\dagger b} \rangle > 0$.

Corollary 4.2 *We have*

$$\langle (-\mathbb{1})^{b^\dagger b} \rangle = \mathbb{E}_{\Pi_\infty} \left[e^{-2g^2 \int_{-\infty}^0 \int_0^\infty W_\Delta(s,r) ds dr} \right] > 0.$$

Proof: Put $\beta = i\pi$ in Theorem 4.1. Then the corollary follows. ■

4.2 Gaussian functions

We construct a path integral representation of $\langle e^{i\beta x} \rangle$.

Theorem 4.3 *We have*

$$\langle e^{i\beta x} \rangle = e^{-\beta^2/4} \mathbb{E}_{\Pi_\infty} [e^{i\beta K}],$$

where

$$K = -\frac{g}{\sqrt{2}} \int_{-\infty}^{\infty} \hat{T}_s e^{-|s|} ds.$$

Corollary 4.4 *Let $\beta \in \mathbb{C}$ such that $|\beta| < 1$. Then*

$$\langle e^{\beta x^2} \rangle = \frac{1}{\sqrt{1-\beta}} \mathbb{E}_{\Pi_\infty} \left[e^{\frac{\beta K^2}{1-\beta}} \right]. \quad (4.3)$$

In particular $\lim_{\beta \uparrow 1} \|e^{\beta x^2/2} \Phi_g\|^2 = \infty$.

Proof: By Theorem 4.2 we see that

$$\begin{aligned} \langle e^{-\beta^2 x^2/2} \rangle &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\Phi_g, e^{ik\beta x} \Phi_g) e^{-k^2/2} dk = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-k^2\beta^2/4} \mathbb{E}_{\Pi_\infty} [e^{ik\beta K}] e^{-k^2/2} dk \\ &= \mathbb{E}_{\Pi_\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-k^2\beta^2/4} e^{ik\beta K} e^{-k^2/2} dk \right] = \frac{1}{\sqrt{1+\beta^2/2}} \mathbb{E}_{\Pi_\infty} \left[e^{-\frac{\beta^2 K^2}{\beta^2+2}} \right]. \end{aligned}$$

By an analytic continuation we obtain (4.3) for $\beta \in \mathbb{C}$ such that $|\beta| < 1$. Then the corollary follows. ■

4.3 Spin σ_z

Let $\bar{L} = L - E$. Path integral representations of Euclidean Green functions by Lemma 3.2 can be rewritten as follows.

Corollary 4.5 (1) Suppose that $\phi, \psi \in \mathcal{H}$ and $f_j = f_j(\alpha, x) \in L^\infty(\mathbb{Z}_2 \times \mathbb{R})$ for $j = 0, 1, \dots, n$, and $0 < t_0 < t_1 < \dots < t_n < t$. Then

$$\begin{aligned} & (\phi, e^{-t_0 \bar{L}} f_0 e^{-(t_1-t_0) \bar{L}} f_1 e^{-(t_2-t_1) \bar{L}} \dots e^{-(t_n-t_{n-1}) \bar{L}} f_n e^{-(t-t_n) \bar{L}} \psi) \\ &= e^{\Delta t} e^{Et} \sum_{\alpha \in \mathbb{Z}_2} \int_{\mathbb{R}} \mathbb{E}_{\mathbb{Q}}^{\alpha} \mathbb{E}_{\mathbb{P}}^x \left[\bar{\phi}(\hat{q}_0^{\Delta}) \psi(\hat{q}_t^{\Delta}) \left(\prod_{j=0}^n f_j(\hat{q}_{t_j}^{\Delta}) \right) e^{-g \int_0^t W(\hat{q}_s^{\Delta}) ds} \right]. \end{aligned} \quad (4.4)$$

(2) Suppose that $g_j = g_j(\alpha) \in L^\infty(\mathbb{Z}_2)$ for $j = 0, 1, \dots, n$ and $0 < t_0 < t_1 < \dots < t_n < t$. Then

$$\begin{aligned} & (\mathbb{1}, e^{-t_0 \bar{L}} g_0(\sigma_z) e^{-(t_1-t_0) \bar{L}} g_1(\sigma_z) e^{-(t_2-t_1) \bar{L}} \dots e^{-(t_n-t_{n-1}) \bar{L}} g_n(\sigma_z) e^{-(t-t_n) \bar{L}} \mathbb{1}) \\ &= e^{\Delta t} e^{Et} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\left(\prod_{j=0}^n g_j(\hat{T}_{t_j}) \right) \int_{\mathbb{R}} \mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^t W(\hat{q}_s^{\Delta}) ds} \right] d\mu(x) \right]. \end{aligned} \quad (4.5)$$

Proof: (1) is a simple reworking of Lemma 3.2 and (2) is a special case of (1). ■

One can see that the integrand in (4.5) is

$$\mathbb{E}_{\mathbb{P}}^x \left[e^{-g \int_0^t W(\hat{q}_s^{\Delta}) ds} \right] = e^{-g \left(\int_0^t e^{-s} (-1)^{N_{\Delta s}} ds \right) x} e^{\frac{g^2}{4} \int_0^{(1-e^{-2t})/2} \left| \int_y^t (-1)^{N_{\Delta s}} ds \right|^2 dy}.$$

by Lemma 3.4.

Theorem 4.6 We have $\langle \sigma_z e^{-|t-s| \bar{L}} \sigma_z \rangle = \mathbb{E}_{\Pi_\infty} [\hat{T}_t \hat{T}_s]$ for any $t, s \in \mathbb{R}$.

Proof: By Lemma 4.5 and a limiting argument, we see that

$$\begin{aligned} (\sigma_z \Phi_g, e^{-t \bar{L}} \sigma_z \Phi_g) &= \lim_{T \rightarrow \infty} \frac{1}{\|\mathbb{1}_{T-t/2}\|^2} (\sigma_z \mathbb{1}_{T-t/2}, e^{-t \bar{L}} \sigma_z \mathbb{1}_{T-t/2}) \\ &= \lim_{T \rightarrow \infty} \frac{e^{2ET} e^{2T\Delta}}{\|\mathbb{1}_{T-t/2}\|^2} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\hat{T}_{-t/2} \hat{T}_{t/2} e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_{\Delta}(t,s)} \right]. \end{aligned}$$

Then we have

$$\begin{aligned} (\sigma_z \Phi_g, e^{-t \bar{L}} \sigma_z \Phi_g) &= \lim_{T \rightarrow \infty} \frac{\|\mathbb{1}_T\|^2}{\|\mathbb{1}_{T-t/2}\|^2} \frac{e^{2ET} e^{2T\Delta}}{\|\mathbb{1}_T\|^2} \sum_{\alpha \in \mathbb{Z}_2} \mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\hat{T}_{-t/2} \hat{T}_{t/2} e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_{\Delta}(t,s)} \right] \\ &= \lim_{T \rightarrow \infty} \frac{\|\mathbb{1}_T\|^2}{\|\mathbb{1}_{T-t/2}\|^2} \frac{\mathbb{E}_{\mathbb{Q}}^{\alpha} \left[\hat{T}_{-t/2} \hat{T}_{t/2} e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_{\Delta}(t,s)} \right]}{\mathbb{E}_{\mathbb{Q}}^{\alpha} \left[e^{\frac{g^2}{2} \int_{-T}^T dt \int_{-T}^T ds W_{\Delta}(t,s)} \right]} = \mathbb{E}_{\Pi_\infty} [\hat{T}_{-t/2} \hat{T}_{t/2}]. \end{aligned}$$

Hence for $t > s$,

$$(\sigma_z \Phi_g, e^{-(t-s) \bar{L}} \sigma_z \Phi_g) = \mathbb{E}_{\Pi_\infty} [\hat{T}_{-(t-s)/2} \hat{T}_{(t-s)/2}] = \mathbb{E}_{\Pi_\infty} [\hat{T}_t \hat{T}_s]$$

by the shift invariance. ■

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