

Quantum α -determinants and q -deformations of hypergeometric polynomials

By

Kazufumi KIMOTO*

Abstract

The quantum α -determinant is a parametric deformation of the ordinary quantum determinant. We study the cyclic $\mathcal{U}_q(\mathfrak{sl}_2)$ -submodules of the quantum matrix algebra $\mathcal{A}_q(\text{Mat}_2)$ generated by the powers of the quantum α -determinant. The irreducible decomposition of this cyclic module is explicitly described in terms of certain polynomials in the parameter α , which is a q -deformation of the Gaussian hypergeometric polynomials.

This note is a summary of the author's recent paper [Kimoto, K., “Quantum alpha-determinants and q -deformed hypergeometric polynomials,” *Int. Math. Res. Not.*].

§ 1. Background

As a parametric interpolation of the determinant and permanent, we define the α -determinant of a matrix $X = (x_{ij})_{1 \leq i, j \leq n}$ by

$$\det^{(\alpha)}(X) := \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\nu(\sigma)} x_{\sigma(1)1} x_{\sigma(2)2} \cdots x_{\sigma(n)n},$$

where α is a complex parameter and $\nu(\sigma) = n - (m_1 + m_2 + \cdots + m_n)$ if the cycle type of a permutation σ in the symmetric group $\mathfrak{S}_n$ of degree n is $1^{m_1} 2^{m_2} \cdots n^{m_n}$ [13]. In fact, we have $\det(X) = \det^{(-1)}(X)$ and $\text{per}(X) = \det^{(1)}(X)$ by definition.

We are interested in the representation-theoretical properties of the α -determinant. Let us set the stage to formulate the problem. Denote by $\mathcal{A}(\text{Mat}_n)$ the $\mathbb{C}$ -algebra of polynomials in the n^2 commuting variables $\{x_{ij}\}_{1 \leq i, j \leq n}$, and $\mathcal{U}(\mathfrak{gl}_n)$ the universal

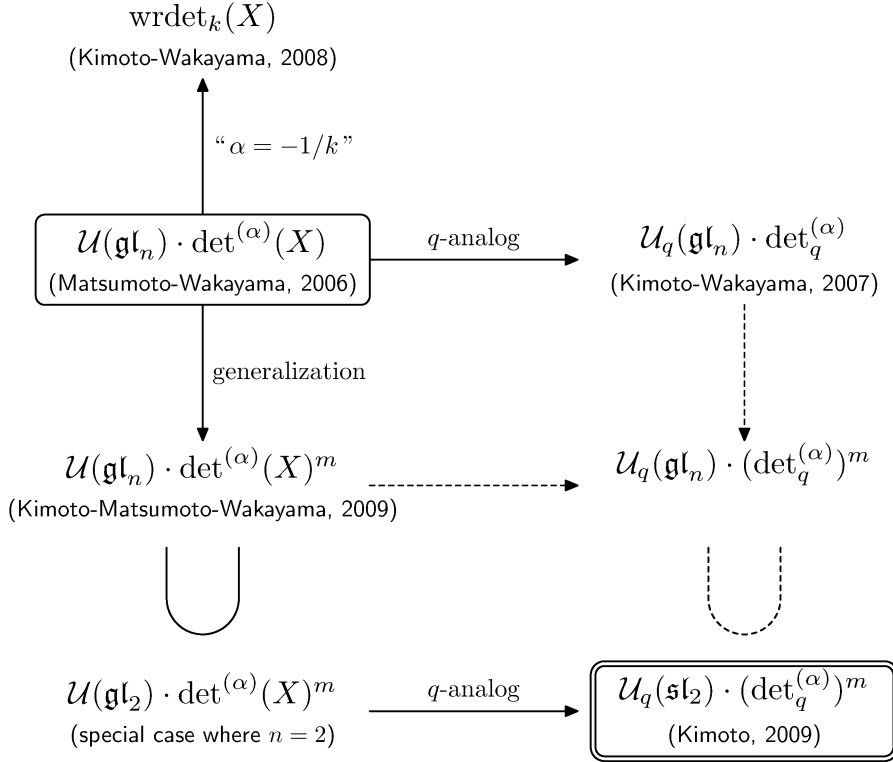
Received September 10, 2009. Accepted March 11, 2010.

2000 Mathematics Subject Classification(s): 20G42; 33C20

Key Words: Quantum groups, quantum alpha-determinant, cyclic modules, irreducible decomposition, hypergeometric polynomials, q -analog.

*Department of Mathematical Sciences, Faculty of Sciences, University of the Ryukyus, 1 Senbaru Nishihara-cho Okinawa 903-0213, Japan.

e-mail: kimoto@math.u-ryukyu.ac.jp

Figure 1. Representation-theoretical studies on α -determinants

enveloping algebra of the general linear Lie algebra $\mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C})$. We can introduce a $\mathcal{U}(\mathfrak{gl}_n)$ -module structure on $\mathcal{A}(\text{Mat}_n)$ by

$$E_{ij} \cdot f := \sum_{r=1}^n x_{ir} \frac{\partial f}{\partial x_{jr}} \quad (f \in \mathcal{A}(\text{Mat}_n)),$$

where $\{E_{ij}\}_{1 \leq i, j \leq n}$ is the standard basis of $\mathfrak{gl}_n$. Recall that the cyclic submodules $\mathcal{U}(\mathfrak{gl}_n) \cdot \det(X)$ and $\mathcal{U}(\mathfrak{gl}_n) \cdot \text{per}(X)$ are equivalent to the skew-symmetric tensor product $\bigwedge^n(\mathbb{C}^n)$ and symmetric tensor product $\text{Sym}^n(\mathbb{C}^n)$ of the natural representation $\mathbb{C}^n$ respectively, which are both irreducible. Thus the cyclic module $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$ can be regarded as an ‘interpolating’ family of $\mathcal{U}(\mathfrak{gl}_n)$ -submodules of the two irreducible submodules above, and it is natural and interesting to study the irreducible decomposition of $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$. This is the starting point of our study.

History of the representation-theoretical studies of α -determinants

Here we briefly explain the history of the studies of α -determinants from the viewpoint of representation theory to clarify the position of the matter which we deal with in this note (see also Figure 1 for a graphical summary).

The problem to determine the irreducible decomposition of $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$ was raised and settled by Matsumoto and Wakayama [5]. They described the irreducible decomposition of $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$ explicitly as follows:

$$(1.1) \quad \mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X) \cong \bigoplus_{\substack{\lambda \vdash n \\ f_\lambda(\alpha) \neq 0}} (\mathcal{M}_n^\lambda)^{\oplus f^\lambda}.$$

Here we identify the dominant integral weights and partitions, and denote by $\mathcal{M}_n^\lambda$ the irreducible highest weight $\mathcal{U}(\mathfrak{gl}_n)$ -module with highest weight λ . We also denote by $f_\lambda(x)$ the (modified) content polynomial of λ

$$(1.2) \quad f_\lambda(x) = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (1 + (j-i)x).$$

At this point, there are at least three direction to proceed with the studies of α -determinants.

- (A) The result (1.1) (together with (1.2)) by Matsumoto and Wakayama implies that the structure of the cyclic module $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)$ changes drastically when $\alpha = \pm 1/k$ for $k = 1, 2, \dots, n-1$. Study the α -determinant for such special values.
- (B) Study a q -analog of the problem above, that is, define a quantum version of the α -determinant in the quantum matrix algebra $\mathcal{A}_q(\text{Mat}_n)$ suitably, and consider the cyclic $\mathcal{U}_q(\mathfrak{gl}_n)$ -module generated by it.
- (C) Study the cyclic $\mathcal{U}(\mathfrak{gl}_n)$ -module generated by the powers $\det^{(\alpha)}(X)^m$ of the α -determinant in general.

(A) Wreath determinant

When $\alpha = -1/k$ for $k = 1, 2, \dots, n-1$, the α -determinant has a ‘ $-1/k$ -analog’ of the alternating property. Precisely, we have

$$\sum_{\sigma \in \mathfrak{S}_n(I)} \det^{(-1/k)}(\mathbf{a}_{\sigma(1)}, \mathbf{a}_{\sigma(2)}, \dots, \mathbf{a}_{\sigma(n)}) = 0$$

for any n by n matrix $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ and any subset $I \subset \{1, 2, \dots, n\}$ such that $\#I > k$. Here we put

$$\mathfrak{S}_n(I) = \{\sigma \in \mathfrak{S}_n; \sigma(x) = x, \forall x \in \{1, 2, \dots, n\} \setminus I\}.$$

This fact suggests that $\det^{(-1/k)}$ may have ‘determinant-like’ properties. Actually, if we define the k -wreath determinant $\text{wrdet}_k(\mathbf{a}_1, \dots, \mathbf{a}_n)$ of a kn by n matrix $(\mathbf{a}_1, \dots, \mathbf{a}_n)$

by

$$\mathrm{wrdet}_k(\mathbf{a}_1, \dots, \mathbf{a}_n) := \det^{(-1/k)}(\overbrace{\mathbf{a}_1, \dots, \mathbf{a}_1}^k, \dots, \overbrace{\mathbf{a}_n, \dots, \mathbf{a}_n}^k),$$

then it satisfies a relative GL_n -invariance

$$\mathrm{wrdet}_k(AP) = \mathrm{wrdet}_k(A) \det(P)^k, \quad P \in GL_n(\mathbb{C}).$$

We also have wreath-determinant analog for Vandermonde and Cauchy determinants. It would be interesting to seek various special wreath determinant formulas like these. See [3] for more details.

(B) Quantum α -determinant

Define the *quantum α -determinant* as an element in the quantum matrix algebra $\mathcal{A}_q(\mathrm{Mat}_n)$ by

$$\det_q^{(\alpha)} := \sum_{\sigma \in \mathfrak{S}_n} q^{\ell(\sigma)} \alpha^{\nu(\sigma)} x_{\sigma(1)1} x_{\sigma(2)2} \cdots x_{\sigma(n)n},$$

where $\ell(\sigma)$ denotes the inversion number of a permutation σ . This is nothing but the ordinary quantum determinant $\det_q$ when $\alpha = -1$. Since the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{gl}_n)$ acts on $\mathcal{A}_q(\mathrm{Mat}_n)$, we can consider the cyclic $\mathcal{U}_q(\mathfrak{gl}_n)$ -submodule $\mathcal{U}_q(\mathfrak{gl}_n) \cdot \det_q^{(\alpha)}$. Thus the irreducible decomposition of this cyclic submodule is regarded as a q -analog of the first problem studied by Matsumoto and Wakayama. The structure of $\mathcal{U}_q(\mathfrak{gl}_n) \cdot \det_q^{(\alpha)}$ is, however, quite complicated, so that we have only several less explicit results at present. See [4] for more details.

(C) Cyclic modules generated by powers of the α -determinant

Recently, Matsumoto, Wakayama and the author investigated the generalized case $\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^m$ and proved that

$$\mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^m \cong \bigoplus_{\substack{\lambda \vdash mn \\ \ell(\lambda) \leq n}} (\mathcal{M}_n^\lambda)^{\oplus \mathrm{rk} F_{n,m}^\lambda(\alpha)}$$

holds for certain square matrices $F_{n,m}^\lambda(\alpha)$ whose entries are polynomials in α . In this direct sum, λ runs over the partitions of mn whose length are at most n . Remark that the matrices $F_{n,m}^\lambda(\alpha)$ are determined up to conjugacy and non-zero scalar factor. In the particular case where $m = 1$, we explicitly have $F_{n,1}^\lambda(\alpha) = f_\lambda(\alpha)I$, where I is the identity matrix of size f^λ and $f_\lambda(\alpha)$ is given by (1.2). It seems quite difficult to describe $F_{n,m}^\lambda(\alpha)$ in an explicit manner in general. However, when $n = 2$, all the matrices $F_{2,m}^\lambda(\alpha)$ are *one by one*, and they are explicitly given by

$$(1.3) \quad F_{2,m}^{(m+s, m-s)}(\alpha) = (1 + \alpha)^s {}_2F_1 \left(\begin{matrix} s - m, s + 1 \\ -m \end{matrix}; -\alpha \right) \quad (s = 0, 1, \dots, m),$$

where ${}_2F_1(a, b; c; x)$ is the *Gaussian hypergeometric function* [2].

Goal of this note

The problem we study here is a q -analog of the study of $\mathcal{U}(\mathfrak{gl}_2) \cdot \det^{(\alpha)}(X)^m$. Namely, we investigate the cyclic $\mathcal{U}_q(\mathfrak{sl}_2)$ -submodule (instead of $\mathcal{U}_q(\mathfrak{gl}_2)$ -submodule just for simplicity of the description) of $\mathcal{A}_q(\text{Mat}_2)$ defined by

$$V_q^m(\alpha) = \mathcal{U}_q(\mathfrak{sl}_2) \cdot (\det_q^{(\alpha)})^m.$$

We prove that there exists a collection of polynomials $F_{m,j}(\alpha)$ ($j = 0, 1, \dots, m$) such that

$$V_q^m(\alpha) \cong \bigoplus_{\substack{0 \leq j \leq m \\ F_{m,j}(\alpha) \neq 0}} \mathcal{M}_q(2j+1),$$

where $\mathcal{M}_q(d)$ is the d -dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_2)$ given in the next section, and show that the polynomials $F_{m,j}(\alpha)$ are written in terms of a certain q -deformation of the hypergeometric polynomials (Theorem 3.3). Taking a limit $q \rightarrow 1$, we also obtain the formula (1.3) again (Corollary 3.6).

§ 2. Preliminaries

We first fix the convention on quantum groups (we basically follow to [7] and [11]). Assume that $q \in \mathbb{C}^\times$ is not a root of unity. The quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ is an associative algebra generated by k, k^{-1}, e, f with the fundamental relations

$$kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = q^2e, \quad kfk^{-1} = q^{-2}f, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

$\mathcal{U}_q(\mathfrak{sl}_2)$ has a (coassociative) coproduct

$$\begin{aligned} \Delta(k^{\pm 1}) &= k^{\pm 1} \otimes k^{\pm 1}, \\ \Delta(e) &= e \otimes 1 + k \otimes e, \\ \Delta(f) &= f \otimes k^{-1} + 1 \otimes f, \end{aligned}$$

which enables us to define tensor products of $\mathcal{U}_q(\mathfrak{sl}_2)$ -modules.

The quantum matrix algebra $\mathcal{A}_q(\text{Mat}_2)$ is an associative algebra generated by $x_{11}, x_{12}, x_{21}, x_{22}$ with the fundamental relations

$$\begin{aligned} (2.1) \quad x_{11}x_{12} &= qx_{12}x_{11}, & x_{21}x_{22} &= qx_{22}x_{21}, \\ x_{11}x_{21} &= qx_{21}x_{11}, & x_{12}x_{22} &= qx_{22}x_{12}, \\ x_{12}x_{21} &= x_{21}x_{12}, & x_{11}x_{22} - x_{22}x_{11} &= (q - q^{-1})x_{12}x_{21}. \end{aligned}$$

For convenience, we put

$$z_1 := x_{11}x_{22}, \quad z_2 := x_{12}x_{21}.$$

Notice that they *commute*:

$$z_1 z_2 = z_2 z_1.$$

The quantum α -determinant of size two is then a linear polynomial

$$\det_q^{(\alpha)} = x_{11}x_{22} + \alpha q x_{12}x_{21} = z_1 + \alpha q z_2$$

in commuting variables z_1, z_2 .

Remark. The quantum α -determinant of size two interpolates the quantum counterparts of the determinant and permanent. In fact, we have

$$\det_q = x_{11}x_{22} - q x_{12}x_{21} = \det_q^{(-1)}, \quad \text{per}_q = x_{11}x_{22} + q^{-1} x_{12}x_{21} = \det_q^{(q^{-2})}.$$

Here, in general, we define the *quantum permanent* of size n by

$$\text{per}_q := \sum_{\sigma \in \mathfrak{S}_n} q^{-\ell(\sigma)} x_{\sigma(1)1} x_{\sigma(2)2} \cdots x_{\sigma(n)n} \in \mathcal{A}_q(\text{Mat}_n),$$

which can be regarded as a q -analog of the usual permanent in the sense that the cyclic module $\mathcal{U}_q(\mathfrak{gl}_n) \cdot \text{per}_q$ is equivalent to the q -analog of n -th symmetric tensor product of the natural representation $\mathbb{C}^n$ of $\mathcal{U}_q(\mathfrak{gl}_n)$. However, the quantum α -determinant of size n does not coincide with the quantum permanent for any α if $n \geq 3$. This is because $\nu(\cdot)$ is a class function on $\mathfrak{S}_n$ in general, whereas the inversion number $\ell(\cdot)$ is *not* if $n \geq 3$.

We briefly recall necessary basic facts on representation theory of $\mathcal{U}_q(\mathfrak{sl}_2)$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis of the vector space $\mathbb{C}^2$. By defining

$$\begin{aligned} k^{\pm 1} \cdot \mathbf{e}_1 &:= q^{\pm 1} \mathbf{e}_1, & e \cdot \mathbf{e}_1 &:= \mathbf{0}, & f \cdot \mathbf{e}_1 &:= \mathbf{e}_2, \\ k^{\pm 1} \cdot \mathbf{e}_2 &:= q^{\mp 1} \mathbf{e}_2, & e \cdot \mathbf{e}_2 &:= \mathbf{e}_1, & f \cdot \mathbf{e}_2 &:= \mathbf{0}, \end{aligned}$$

$\mathbb{C}^2$ becomes a $\mathcal{U}_q(\mathfrak{sl}_2)$ -module. Put

$$\mathcal{M}_q(l+1) := \mathcal{U}_q(\mathfrak{sl}_2) \cdot \mathbf{v}_0^{(l)} \subset (\mathbb{C}^2)^{\otimes l}, \quad \mathbf{v}_0^{(l)} := \underbrace{\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \cdots \otimes \mathbf{e}_1}_l$$

for $l = 1, 2, \dots$. We also put $\mathcal{M}_q(1) := \mathbb{C} \cdot \mathbf{v}_0^{(0)}$ (with $\mathbf{v}_0^{(0)} := 1 \in \mathbb{C}$) and define $e \cdot \mathbf{v}_0^{(0)} = f \cdot \mathbf{v}_0^{(0)} := 0$ and $k^{\pm 1} \cdot \mathbf{v}_0^{(0)} := \mathbf{v}_0^{(0)}$. If we set $\mathbf{v}_r^{(l)} := f^r \cdot \mathbf{v}_0^{(l)}$ ($r = 0, 1, \dots, l$) and $\mathbf{v}_{-1}^{(l)} = \mathbf{v}_{l+1}^{(l)} = \mathbf{0}$, then it follows that $\mathcal{M}_q(l+1) = \bigoplus_{r=0}^l \mathbb{C} \cdot \mathbf{v}_r^{(l)}$ and

$$\begin{aligned} k^{\pm 1} \cdot \mathbf{v}_r^{(l)} &= q^{\pm(l-2r)} \mathbf{v}_r^{(l)}, \\ e \cdot \mathbf{v}_r^{(l)} &= [l-r+1]_q \mathbf{v}_{r-1}^{(l)}, \\ f \cdot \mathbf{v}_r^{(l)} &= [r+1]_q \mathbf{v}_{r+1}^{(l)} \end{aligned}$$

for $r = 0, 1, \dots, l$. This is an irreducible $(l+1)$ -dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$ -module.

Remark. There exists a convolution θ of $\mathcal{U}_q(\mathfrak{sl}_2)$ such that

$$\theta(e) = -e, \quad \theta(f) = f, \quad \theta(k^{\pm 1}) = k^{\mp 1}.$$

Using this, we can introduce inequivalent $\mathcal{U}_q(\mathfrak{sl}_2)$ -module structure on the vector space $\mathcal{M}_q(l+1)$, say $\mathcal{M}_q(l+1)^\theta$. However, such modules do not appear in the following discussion. Note that any finite dimensional irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$ -module is isomorphic to either $\mathcal{M}_q(l+1)$ or $\mathcal{M}_q(l+1)^\theta$ for some $l = 0, 1, 2, \dots$ (see, e.g. [7, 9]).

The algebra $\mathcal{A}_q(\text{Mat}_2)$ becomes a $\mathcal{U}_q(\mathfrak{sl}_2)$ -module by

$$\begin{aligned} k^{\pm 1} \cdot x_{i1} &:= q^{\pm 1} x_{i1}, & e \cdot x_{i1} &:= 0, & f \cdot x_{i1} &:= x_{i2}, \\ k^{\pm 1} \cdot x_{i2} &:= q^{\mp 1} x_{i2}, & e \cdot x_{i2} &:= x_{i1}, & f \cdot x_{i2} &:= 0 \end{aligned} \quad (i = 1, 2).$$

These are compatible with the fundamental relations (2.1) above. Notice that $e \cdot \det_q^r = f \cdot \det_q^r = 0$ and $k^{\pm 1} \cdot \det_q^r = \det_q^r$, (i.e. $\mathcal{U}_q(\mathfrak{sl}_2) \cdot \det_q^r \cong \mathcal{M}_q(1)$). It then follows that $X \cdot (v \det_q^r) = (X \cdot v) \det_q^r$ for $X \in \mathcal{U}_q(\mathfrak{sl}_2)$ and $v \in \mathcal{A}_q(\text{Mat}_2)$. We have

$$(2.2) \quad \mathcal{U}_q(\mathfrak{sl}_2) \cdot (x_{11}x_{21})^s \det_q^{m-s} \cong \mathcal{U}_q(\mathfrak{sl}_2) \cdot (x_{11}x_{21})^s \cong \mathcal{M}_q(2s+1) \quad (s = 0, 1, 2, \dots).$$

Actually, the linear map defined by $\mathbf{v}_r^{(l)} \mapsto f^r \cdot (x_{11}x_{21})^s$ gives a bijective intertwiner between $\mathcal{M}_q(2s+1)$ and $\mathcal{U}_q(\mathfrak{sl}_2) \cdot (x_{11}x_{21})^s$.

Define q -analogs of numbers, factorials and binomial coefficients by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! := \prod_{i=1}^n [i]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

§ 3. Cyclic modules generated by the quantum alpha-determinant

We fix a positive integer m and discuss the irreducible decomposition of the cyclic module $V_q^m(\alpha) := \mathcal{U}_q(\mathfrak{sl}_2) \cdot (\det_q^{(\alpha)})^m$. We refer to [1] for detailed discussion.

Put

$$v_{m,j} := (f^j \cdot (x_{11}x_{21})^j) \det_q^{m-j} \quad (j = 0, 1, \dots, m).$$

It is easy to see that the cyclic module $\mathcal{U}_q(\mathfrak{sl}_2) \cdot v_{m,j}$ is equivalent to $\mathcal{M}_q(2j+1)$. We show that $v_{m,j}$ is a homogeneous polynomial in z_1 and z_2 of degree m and give an explicit expression of it. For this purpose, we need the following two lemmas.

Lemma 3.1. *For each positive integer j , it follows that*

$$f^j \cdot (x_{11}x_{21})^j = q^{-j(j-1)/2} [j]_q! \sum_{r=0}^j q^{-r^2} \begin{bmatrix} j \\ r \end{bmatrix}_q^2 x_{11}^{j-r} x_{22}^{j-r} (x_{12}x_{21})^r.$$

Sketch of proof. For $1 \leq i \leq 2j$, put

$$f_j(i) := \overbrace{1 \otimes \cdots \otimes 1}^{i-1} \otimes f \otimes \overbrace{k^{-1} \otimes \cdots \otimes k^{-1}}^{2j-i} \in \mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2j}.$$

Then it follows that

$$\Delta^{2j-1}(f)^j = \sum_{1 \leq n_1, \dots, n_j \leq 2j} f_j(n_1) \cdots f_j(n_j).$$

Since $f_j(m)f_j(n) = q^{-2}f_j(n)f_j(m)$ if $m > n$ and $f^2 \cdot x_{11} = f^2 \cdot x_{21} = 0$, we have

$$\Delta^{2j-1}(f)^j = q^{-j(j-1)/2} [j]_q! \sum_{1 \leq n_1 < \cdots < n_j \leq 2j} f_j(n_1) \cdots f_j(n_j) + R,$$

where R is a certain element in $\mathcal{U}_q(\mathfrak{sl}_2)^{\otimes 2j}$ such that $R \cdot (x_{11}^j x_{21}^j) = 0$. Here we also use the well-known identity

$$\sum_{\sigma \in \mathfrak{S}_j} x^{\ell(\sigma)} = (1+x)(1+x+x^2) \cdots (1+x+\cdots+x^{j-1})$$

with $x = q^{-2}$. For given $n_1, \dots, n_j$ ($1 \leq n_1 < \cdots < n_j \leq 2j$), we get

$$f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{21}^j) = q^{-r^2+j(j-1)/2+2(n_1+\cdots+n_r)-2(m_1+\cdots+m_r)} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r$$

by a careful calculation (see [1] for detail). Using this, we have

$$\begin{aligned} & f^j \cdot (x_{11}^j x_{21}^j) \\ &= q^{-j(j-1)/2} [j]_q! \sum_{1 \leq n_1 < \cdots < n_j \leq 2j} f_j(n_1) \cdots f_j(n_j) \cdot (x_{11}^j x_{21}^j) \\ &= [j]_q! \sum_{r=0}^j q^{-r^2} \sum_{\substack{1 \leq n_1 < \cdots < n_r \leq j \\ 1 \leq m_1 < \cdots < m_r \leq j}} q^{2(n_1+\cdots+n_r)} q^{-2(m_1+\cdots+m_r)} x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r \\ &= [j]_q! \sum_{r=0}^j q^{-r^2} e_r(1, q^2, \dots, q^{2(j-1)}) e_r(1, q^{-2}, \dots, q^{-2(j-1)}) x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r, \end{aligned}$$

where $e_r(x_1, x_2, \dots, x_j)$ is the r th elementary symmetric polynomial in $x_1, x_2, \dots, x_j$. By the identity (see, e.g. [10])

$$e_r(1, q^2, \dots, q^{2j-2}) = q^{r(j-1)} \begin{bmatrix} j \\ r \end{bmatrix}_q$$

together with the symmetry $\begin{bmatrix} j \\ r \end{bmatrix}_q = \begin{bmatrix} j \\ r \end{bmatrix}_{q^{-1}}$, we obtain

$$f^j \cdot (x_{11}^j x_{21}^j) = [j]_q! \sum_{r=0}^j q^{-r^2} \begin{bmatrix} j \\ r \end{bmatrix}_q^2 x_{11}^{j-r} x_{22}^{j-r} (x_{12} x_{21})^r.$$

Since $(x_{11} x_{21})^j = q^{-j(j-1)/2} x_{11}^j x_{21}^j$, we have the desired conclusion. $\square$

Lemma 3.2. *For each positive integer l , it follows that*

$$x_{11}^l x_{22}^l = \prod_{s=1}^l (z_1 + (q^{2s-1} - q)z_2) = \sum_{j=0}^l q^{l(l-j)} \begin{bmatrix} l \\ j \end{bmatrix}_q (z_1 - qz_2)^j z_2^{l-j}.$$

Proof. The first equality is proved by induction on l by using the relation

$$(z_1 + (q^{2r-1} - q)z_2)x_{22} = x_{22}(z_1 + (q^{2r+1} - q)z_2).$$

The second equality is a specialization of the q -binomial theorem

$$\prod_{i=1}^n (x + q^{2i-n-1}y) = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix}_q x^r y^{n-r},$$

which is indeed applicable since z_1 and z_2 commute. $\square$

As a result, we get the explicit expression

$$(3.1) \quad v_{m,j} = \left\{ q^{-j(j-1)/2} [j]_q! \sum_{r=0}^j q^{-r^2} \begin{bmatrix} j \\ r \end{bmatrix}_q^2 z_2^r \prod_{s=1}^{j-r} (z_1 + (q^{2s-1} - q)z_2) \right\} (z_1 - qz_2)^{m-j},$$

which is indeed a homogeneous polynomial in z_1 and z_2 of degree m .

The vectors $\{v_{m,j}\}_{j=0}^m$ are linearly independent since they belong to inequivalent representations, and hence form a basis of the space consisting of homogeneous polynomials in z_1 and z_2 of degree m , whose dimension is $m+1$. Thus we conclude that $(\det_q^{(\alpha)})^m (\in \mathcal{A}_q(\text{Mat}_2) \otimes_{\mathbb{C}} \mathbb{C}[\alpha])$ can be expressed as a linear combination

$$(3.2) \quad \left(\det_q^{(\alpha)} \right)^m = \sum_{j=0}^m F_{m,j}(\alpha) v_{m,j}$$

of the vectors $v_{m,j}$, where $F_{m,j}(\alpha)$ are certain polynomial functions in α . This implies that the cyclic module $V_q^m(\alpha)$ contains an irreducible submodule equivalent to $\mathcal{M}_q(2j+1)$ (with multiplicity one) if $F_{m,j}(\alpha) \neq 0$. Consequently, it follows that

$$(3.3) \quad V_q^m(\alpha) \cong \bigoplus_{\substack{0 \leq j \leq m \\ F_{m,j}(\alpha) \neq 0}} \mathcal{M}_q(2j+1).$$

Let us determine the functions $F_{m,j}(\alpha)$ explicitly. The conditions (3.2) for the functions $F_{m,j}(\alpha)$ are given in terms of polynomials in commuting variables z_1, z_2 , so that it is meaningful to consider the specialization $z_1 = z, z_2 = 1$ in (3.2), where z is a

new variable. Put

$$g_j(z) := \prod_{i=1}^j (z + q^{2i-1} - q) = \sum_{i=0}^j q^{j(j-i)} \begin{bmatrix} j \\ i \end{bmatrix}_q (z - q)^i,$$

$$v_j(z) := q^{-j(j-1)/2} [j]_q! \sum_{r=0}^j q^{-r^2} \begin{bmatrix} j \\ r \end{bmatrix}_q^2 g_{j-r}(z).$$

Then (3.2) together with Lemmas 3.1 and 3.2 yields

$$(z + q\alpha)^m = \sum_{j=0}^m F_{m,j}(\alpha) v_j(z) (z - q)^{m-j}.$$

If we take the l th derivative of this formula with respect to z ($l = 0, 1, \dots, m$) and substitute $z = q$, then we get the relation

$$(3.4) \quad \binom{m}{l} q^{m-l} (1 + \alpha)^{m-l} = \sum_{j=m-l}^m F_{m,j}(\alpha) \frac{v_j^{(l-m+j)}(q)}{(l-m+j)!} = \sum_{s=0}^l F_{m,m-s}(\alpha) \frac{v_{m-s}^{(l-s)}(q)}{(l-s)!}.$$

Since

$$\begin{aligned} v_j(z) &= q^{j(j+1)/2} [j]_q! \sum_{i=0}^j q^{-ij} \left\{ \begin{bmatrix} j \\ i \end{bmatrix}_q \sum_{r=0}^{j-i} q^{r(i-2j)} \begin{bmatrix} j \\ j-r \end{bmatrix}_q \begin{bmatrix} j-i \\ r \end{bmatrix}_q \right\} (z - q)^i \\ &= q^{-j(j-1)/2} [j]_q! \sum_{i=0}^j \begin{bmatrix} j \\ i \end{bmatrix}_q \begin{bmatrix} 2j-i \\ j \end{bmatrix}_q (z - q)^i, \end{aligned}$$

we have

$$v_j(z) = q^{-j(j-1)/2} \sum_{i=0}^j \frac{[j]_q! [2j-i]_q!}{[i]_q! [j-i]_q!^2} (z - q)^i,$$

or

$$\frac{v_j^{(i)}(q)}{i!} = q^{-\binom{j}{2}} \frac{[j]_q! [2j-i]_q!}{[i]_q! [j-i]_q!^2}.$$

Here we use the q -Chu-Vandermonde formula

$$\sum_{r=0}^{j-i} q^{r(i-2j)} \begin{bmatrix} j \\ j-r \end{bmatrix}_q \begin{bmatrix} j-i \\ r \end{bmatrix}_q = q^{j(i-j)} \begin{bmatrix} 2j-i \\ j \end{bmatrix}_q.$$

Thus the formula (3.4) is rewritten more explicitly as

$$(3.5) \quad [m-l]_q!^2 \binom{m}{l} q^{m-l} (1 + \alpha)^{m-l} = \sum_{s=0}^l q^{-\binom{m-s}{2}} \frac{[m-s]_q! [2m-l-s]_q!}{[l-s]_q!} F_{m,m-s}(\alpha).$$

This also implies that the polynomial $F_{m,j}(\alpha)$ is divisible by $(1 + \alpha)^j$, that is

$$(3.6) \quad F_{m,j}(\alpha) = (1 + \alpha)^j Q_{m,j}(\alpha)$$

for some $Q_{m,j}(\alpha) \in \mathbb{C}[\alpha]$. By (3.6) and (3.5), we have

$$(3.7) \quad \begin{aligned} & \left[\begin{matrix} 2m - 2i \\ m - i \end{matrix} \right]_q^{-1} \binom{m}{i} q^{m-i} \\ &= \sum_{j=0}^i \left[\begin{matrix} 2i - 2m - 1 \\ i - j \end{matrix} \right]_q (-1)^{i-j} (1 + \alpha)^{i-j} \cdot q^{-\binom{m-j}{2}} [m - j]_q! Q_{m,m-j}(\alpha). \end{aligned}$$

Now we define the *mixed hypergeometric series* by

$$(3.8) \quad \Phi \left(\begin{matrix} a_1, \dots, a_k \\ b_1, \dots, b_l \end{matrix} ; \begin{matrix} c_1, \dots, c_m \\ d_1, \dots, d_n \end{matrix} ; q; x \right) = \sum_{i=0}^{\infty} \frac{(a_1; i) \cdots (a_k; i)}{(b_1; i) \cdots (b_l; i)} \frac{(c_1; i)_q \cdots (c_m; i)_q}{(d_1; i)_q \cdots (d_n; i)_q} \frac{x^i}{[i]_q!},$$

where $(a; i) = a(a+1) \cdots (a+i-1)$ and $(a; i)_q = [a]_q [a+1]_q \cdots [a+i-1]_q$ (cf. [8]).

Theorem 3.3. For $s = 0, 1, \dots, m$,

$$F_{m,s}(\alpha) = q^{\binom{s+1}{2}} \binom{m}{s} \frac{[s]_q!}{[2s]_q!} (1 + \alpha)^s \Phi \left(\begin{matrix} s - m \\ s + 1 \end{matrix} ; \begin{matrix} s + 1, s + 1 \\ 2s + 2 \end{matrix} ; q; q(1 + \alpha) \right)$$

holds.

Sketch of proof. We can prove the identity

$$\left(\left[\begin{matrix} 2i - 2m - 1 \\ i - j \end{matrix} \right]_q \right)_{0 \leq i, j \leq m}^{-1} = \left(\frac{[2m - 2i + 1]_q}{[2m - 2j + 1]_q} \left[\begin{matrix} 2m - 2j + 1 \\ i - j \end{matrix} \right]_q \right)_{0 \leq i, j \leq m}.$$

Using this, we solve (3.7) and find that

$$\begin{aligned} Q_{m,m-i}(\alpha) &= \frac{q^{\binom{m-i}{2}} [2m - 2i + 1]_q}{[m - i]_q!} \\ &\quad \times \sum_{j=0}^i \frac{(-1)^{i-j} q^{m-j}}{[2m - 2j + 1]_q} \left[\begin{matrix} 2m - 2j + 1 \\ i - j \end{matrix} \right]_q \left[\begin{matrix} 2m - 2j \\ m - j \end{matrix} \right]_q^{-1} \binom{m}{j} (1 + \alpha)^{i-j} \\ &= \frac{q^{\binom{m-i+1}{2}} m! [2m - 2i + 1]_q}{[m - i]_q!} \\ &\quad \times \sum_{r=0}^i \frac{(-q)^r [m - i + r]_q!^2}{(m - i + r)!(i - r)! [2m - 2i + r + 1]_q!} \frac{(1 + \alpha)^r}{[r]_q!}, \end{aligned}$$

where we set $r = i - j$. Since

$$(i - r)! = (-1)^r \frac{i!}{(-i; r)}, \quad (n + r)! = n! (n + 1; r), \quad [n + r]_q! = [n]_q! (n + 1; r)_q,$$

we have

$$\begin{aligned} & Q_{m, m-i}(\alpha) \\ (3.9) \quad &= \frac{q^{\binom{m-i+1}{2}} m! [m-i]_q!}{i! (m-i)! [2m-2i]_q!} \sum_{r=0}^i \frac{(-i; r) (m-i+1; r)_q^2}{(m-i+1; r) (2m-2i+2; r)_q} \frac{(q(1+\alpha))^r}{[r]_q!} \\ &= q^{\binom{m-i+1}{2}} \binom{m}{i} \frac{[m-i]_q!}{[2m-2i]_q!} \Phi \left(\begin{matrix} -i \\ m-i+1 \end{matrix}; \begin{matrix} m-i+1, m-i+1 \\ 2m-2i+2 \end{matrix}; q; q(1+\alpha) \right). \end{aligned}$$

If we substitute this into (3.6) and replace $m-i$ by s , then we have the conclusion. $\square$

Example 3.4 ($m = 1$). We have

$$\begin{aligned} F_{1,0}(\alpha) &= \Phi \left(\begin{matrix} -1 \\ 1 \end{matrix}; \begin{matrix} 1, 1 \\ 2 \end{matrix}; q; q(1+\alpha) \right) = \frac{1 - \alpha q^2}{1 + q^2}, \\ F_{1,1}(\alpha) &= q \frac{[1]_q!}{[2]_q!} (1 + \alpha) \Phi \left(\begin{matrix} 0 \\ 2 \end{matrix}; \begin{matrix} 2, 2 \\ 4 \end{matrix}; q; q(1+\alpha) \right) = \frac{q}{[2]_q} (1 + \alpha). \end{aligned}$$

Thus it follows that

$$\mathcal{U}_q(\mathfrak{sl}_2) \cdot \det_q^{(\alpha)} \cong \begin{cases} \mathcal{M}_q(3) & \alpha = q^{-2}, \\ \mathcal{M}_q(1) & \alpha = -1, \\ \mathcal{M}_q(1) \oplus \mathcal{M}_q(3) & \text{otherwise.} \end{cases}$$

Notice that $q^{-2} \neq -1$ since we assume that q is not a root of unity.

Example 3.5 ($m = 2$). We have

$$\begin{aligned} F_{2,0}(\alpha) &= \Phi \left(\begin{matrix} -2 \\ 1 \end{matrix}; \begin{matrix} 1, 1 \\ 2 \end{matrix}; q; q(1+\alpha) \right) \\ &= C_0(q) ((q^6 + q^4)\alpha^2 - 2q^2\alpha + q^4 + 1), \\ F_{2,1}(\alpha) &= 2q^2 \frac{[1]_q!}{[2]_q!} (1 + \alpha) \Phi \left(\begin{matrix} -1 \\ 2 \end{matrix}; \begin{matrix} 2, 2 \\ 4 \end{matrix}; q; q(1+\alpha) \right) \\ &= C_1(q)(1 + \alpha) ((q^4 + q^2)\alpha - q^4 + q^2 - 2), \\ F_{2,2}(\alpha) &= q^3 \frac{[2]_q!}{[4]_q!} (1 + \alpha)^2 \Phi \left(\begin{matrix} 0 \\ 3 \end{matrix}; \begin{matrix} 3, 3 \\ 6 \end{matrix}; q; q(1+\alpha) \right) \\ &= C_2(q)(1 + \alpha)^2, \end{aligned}$$

where $C_0(q), C_1(q), C_2(q)$ are certain rational functions in q . Hence, if we assume that q is *transcendental* (we have only to assume that $(q^4 - q^2 + 1)^2 + q^2(q^4 + q^2 + 1) \neq 0$ practically), then we see that

$$\mathcal{U}_q(\mathfrak{sl}_2) \cdot \left(\det_q^{(\alpha)}\right)^2 \cong \begin{cases} \mathcal{M}_q(1) & \alpha = -1, \\ \mathcal{M}_q(1) \oplus \mathcal{M}_q(5) & \alpha = \frac{q^4 - q^2 + 2}{q^4 + q^2}, \\ \mathcal{M}_q(3) \oplus \mathcal{M}_q(5) & \alpha = \frac{1 \pm q\sqrt{-q^4 - q^2 - 1}}{q^4 + q^2}, \\ \mathcal{M}_q(1) \oplus \mathcal{M}_q(3) \oplus \mathcal{M}_q(5) & \text{otherwise.} \end{cases}$$

When $(q^4 - q^2 + 1)^2 + q^2(q^4 + q^2 + 1) = 0$ (this does not implies that q is a root of unity), $\frac{q^4 - q^2 + 2}{q^4 + q^2}$ becomes a common root of $F_{2,0}(\alpha)$ and $F_{2,1}(\alpha)$, so that we have

$$\mathcal{U}_q(\mathfrak{sl}_2) \cdot \left(\det_q^{(\alpha)}\right)^2 \cong \mathcal{M}_q(5), \quad \alpha = \frac{q^4 - q^2 + 2}{q^4 + q^2}.$$

Remark. The mixed hypergeometric series (3.8) can be regarded as a common generalization of the generalized hypergeometric series and basic hypergeometric series as we see below:

$$\begin{aligned} \Phi\left(\begin{matrix} a_1, \dots, a_k \\ b_1, \dots, b_l, 1 \end{matrix}; 1; q; x\right) &= {}_kF_l\left(\begin{matrix} a_1, \dots, a_k \\ b_1, \dots, b_l \end{matrix}; x\right), \\ \Phi\left(\begin{matrix} c_1, \dots, c_m \\ d_1, \dots, d_n \end{matrix}; q; x\right) &= {}_m\phi_n\left(\begin{matrix} q^{2c_1}, \dots, q^{2c_m} \\ q^{2d_1}, \dots, q^{2d_n} \end{matrix}; q^2, (-1)^{1+n-m} q^{1+d-c} x\right), \end{aligned}$$

where $c = c_1 + \dots + c_m$ and $d = d_1 + \dots + d_n$.

Remark. The function Φ given by (3.8) satisfies the difference-differential equation

$$\begin{aligned} &\left\{ -(E + a_1) \cdots (E + a_k) [E + c_1]_q \cdots [E + c_m]_q \right. \\ &\quad \left. + \partial_q(E + b_1 - 1) \cdots (E + b_l - 1) [E + d_1 - 1]_q \cdots [E + d_n - 1]_q \right\} \Phi = 0, \end{aligned}$$

where we put

$$E = x \frac{d}{dx}, \quad [E + a]_q = \frac{q^{E+a} - q^{-E-a}}{q - q^{-1}}, \quad \partial_q f(x) = \frac{f(qx) - f(q^{-1}x)}{qx - q^{-1}x}.$$

If we take a limit $q \rightarrow 1$, then the equation above becomes a hypergeometric differential equation for ${}_{k+m}F_{l+n}(a_1, \dots, a_k, c_1, \dots, c_m; b_1, \dots, b_l, d_1, \dots, d_n; x)$.

All the discussion above also work in the classical case (i.e. the case where $q = 1$). Thus, by taking a limit $q \rightarrow 1$ in Theorem 3.3, we will obtain Theorem 4.1 in [2] (or

(1.3) up to constant) again. We abuse the same notations used in the discussion of quantum case above to indicate the classical counterparts. From (3.9), we have

$$\begin{aligned} Q_{m,s}(\alpha) &= \frac{m!}{(m-s)!(2s)!} {}_3F_2\left(\begin{matrix} s-m, s+1, s+1 \\ s+1, 2s+2 \end{matrix}; 1+\alpha\right) \\ &= \frac{m!}{(m-s)!(2s)!} {}_2F_1\left(\begin{matrix} s-m, s+1 \\ 2s+2 \end{matrix}; 1+\alpha\right). \end{aligned}$$

Notice that

$${}_2F_1\left(\begin{matrix} s-m, s+1 \\ 2s+2 \end{matrix}; 1-x\right) = \frac{m!(2s+1)!}{s!(m+s+1)!} {}_2F_1\left(\begin{matrix} s-m, s+1 \\ -m \end{matrix}; x\right).$$

Thus we also get

$$Q_{m,s}(\alpha) = \frac{m!^2(2s+1)}{(m-s)!s!(m+s+1)!} {}_2F_1\left(\begin{matrix} s-m, s+1 \\ -m \end{matrix}; -\alpha\right) \quad (s = 0, 1, \dots, m).$$

Summarizing these, we have the

Corollary 3.6 (Classical case). *It follows that*

$$\begin{aligned} F_{m,s}(\alpha) &= \frac{m!}{(m-s)!(2s)!} (1+\alpha)^s {}_2F_1\left(\begin{matrix} s-m, s+1 \\ 2s+2 \end{matrix}; 1+\alpha\right) \\ &= \frac{\binom{2m}{m-s} - \binom{2m}{m-s-1}}{\binom{2m}{m}s!} (1+\alpha)^s {}_2F_1\left(\begin{matrix} s-m, s+1 \\ -m \end{matrix}; -\alpha\right) \end{aligned}$$

for $s = 0, 1, \dots, m$.

References

- [1] Kimoto, K., Quantum alpha-determinants and q -deformed hypergeometric polynomials, *Int. Math. Res. Not.* **2009**, no. 22, 4168–4182. doi:10.1093/imrn/rnp083.
- [2] Kimoto, K., Matsumoto, S. and Wakayama, M., Alpha-determinant cyclic modules and Jacobi polynomials, *Trans. Amer. Math. Soc.*, **361** (2009), 6447–6473.
- [3] Kimoto, K. and Wakayama, M., Invariant theory for singular α -determinants, *J. Combin. Theory Ser. A*, **115** (2008), 1–31.
- [4] Kimoto, K. and Wakayama, M., Quantum α -determinant cyclic modules of $U_q(\mathfrak{gl}_n)$, *J. Algebra*, **313** (2007), 922–956.
- [5] Matsumoto, S. and Wakayama, M., Alpha-determinant cyclic modules of $\mathfrak{gl}_n(\mathbb{C})$, *J. Lie Theory*, **16** (2006), 393–405.
- [6] Jimbo, M., A q -analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation, *Lett. Math. Phys.*, **11** (1986), 247–252.
- [7] Jimbo, M., *Quantum groups and the Yang-Baxter equations*, Springer-Verlag, Tokyo, 1990.

- [8] Khan, M. A. and Khan, A. H., A note on mixed hypergeometric series, *Acta Math. Vietnam.*, **14**, no. 1 (1989): 95–98.
- [9] Klimyk, A. and Schmüdgen, K., *Quantum groups and their representations*, Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.
- [10] Macdonald, I. G., *Symmetric Functions and Hall Polynomials*, 2nd ed, Oxford University Press, 1995.
- [11] Noumi, M., Yamada, H. and Mimachi, K., Finite dimensional representations of the quantum group $GL_q(n; \mathbb{C})$ and the zonal spherical functions on $U_q(n-1) \backslash U_q(n)$, *Japan. J. Math.*, **19** (1993), 31–80.
- [12] Reshetikhin, N. Yu., Takhtadzhyan, L. A. and Faddeev, L. D., Quantization of Lie groups and Lie algebras, *Leningrad Math. J.*, **1**, no. 1 (1990), 193–225.
- [13] Vere-Jones, D., A generalization of permanents and determinants, *Linear Algebra Appl.*, **111** (1988), 119–124.