

Toward a topological description of Legendrian contact homology
of unit conormal bundles

Yukihiro Okamoto

Abstract

For a smooth compact submanifold K of a Riemannian manifold Q , its unit conormal bundle Λ_K is a Legendrian submanifold of the unit cotangent bundle of Q with a canonical contact structure. Using pseudo-holomorphic curve techniques, the Legendrian contact homology of Λ_K is defined when, for instance, $Q = \mathbb{R}^n$. In this paper, aiming at giving another description of this homology, we define a graded \mathbb{R} -algebra for any pair (Q, K) with orientations from a perspective of string topology and prove its invariance under smooth isotopies of K . The author conjectures that it is isomorphic to the Legendrian contact homology of Λ_K with coefficients in \mathbb{R} in all degrees. This is a reformulation of a homology group, called string homology, introduced by Cieliebak, Ekholm, Latschev and Ng when the codimension of K is 2, though the coefficient is reduced from original $\mathbb{Z}[\pi_1(\Lambda_K)]$ to \mathbb{R} . We compute our invariant (i) in all degrees for specific examples, and (ii) in the 0-th degree when the normal bundle of K is a trivial 2-plane bundle. We also outline a strategy for the proof of the conjecture that our invariant is isomorphic to the Legendrian contact homology of Λ_K with coefficients in \mathbb{R} in all degrees.

Acknowledgements

I would like to show my deepest gratitude to my supervisor, Kei Irie. He made lots of time for discussion and gave me many valuable insights, and also continuously encouraged me during my master and doctoral course. I respect his attitude toward mathematical sciences. I believe that when facing problems (maybe non-mathematical problems) in future, I will be helped by the knowledge and lessons which I learned from him.

After I moved to Kyoto, I could experience a research environment which is quite stimulating for me to study symplectic geometry. I owe this to Kaoru Ono and deeply appreciate his kindness. I would also like to thank the members of the seminar in Kyoto for valuable discussions: Mayuko Yamashita, Suguru Ishikawa, Toru Yoshiyasu, Tomohiro Asano and Taisuke Shibata.

What I studied in my undergraduate period has become the foundation of my research when studying pseudo-holomorphic curves. It was fortunate for me that I could study gauge theory for one year under the advice of Mikio Furuta. I would also like to thank Nobuo Iida, Jin Miyazawa and Masaki Taniguchi, who taught me analysis of linear and non-linear elliptic partial differential equations on manifolds.

I appreciate the hospitality of Kai Cieliebak, Urs Frauenfelder and Marián Poppr during my research stay in Universität Augsburg, Germany. I would also like to thank Kazushi Ueda for being my supervisor for one year and giving some advice during my master course.

The mental counseling office in Kyoto University helped me while I was suffering from depression. I would like to thank the counselor, Itsuki Yamamoto.

Lastly, my family supported me from many aspects. I would like to show my deep gratitude to my grandparents, father, mother, brother and sister.

Contents

1	Introduction	1
2	Differentiable space and de Rham chains	5
2.1	Notations and conventions	5
2.2	de Rham chain complex	6
3	Differentiable space of paths and an operation from string topology	9
3.1	Preparation	9
3.2	Differentiable space of paths	10
3.3	Homology groups	11
3.4	Splitting and concatenating paths	18
3.5	Operation on de Rham chains	20
3.6	$[-1, 1]$ and $[-1, 1]^2$ -modeled de Rham chains	24
4	Construction of $H_*^{\text{string}}(Q, K)$	30
4.1	Definition of chain complex	30
4.2	Variants from $[-1, 1]$ and $[-1, 1]^2$ -modeled de Rham chains	33
4.3	The limit of $\varepsilon \rightarrow 0$	35
4.4	Definition of $H_*^{\text{string}}(Q, K)$	40
4.5	Invariance	43
5	Examples	46
5.1	Computation of $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_1)$	46
5.2	Computation of $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_2)$	54
5.3	A corollary and its potential application	55
6	Cord algebra and $H_0^{\text{string}}(Q, K)$	56
6.1	Cord algebra and string homology	56
6.2	Connecting string homology and $H_*^{\text{string}}(Q, K)$	60
6.3	Proof of isomorphism	66
7	Connection to Legendrian contact homology	68
7.1	Legendrian contact homology	68
7.2	Conjecture	71
7.3	Idea toward the proof of Conjecture 7.3	72
	References	82

1 Introduction

Convention. Throughout this paper, all manifolds are of class C^∞ without boundary and second countable, and all submanifolds are of class C^∞ without boundary, unless otherwise specified.

Backgrounds. Let Q be a manifold with a Riemannian metric, and K be a compact submanifold of Q . For any pair (Q, K) , one can associate the unit cotangent bundle UT^*Q of Q and the unit conormal bundle Λ_K of K . It is well-known that UT^*Q has a canonical contact structure and Λ_K is a Legendrian submanifold of UT^*Q .

As an isotopy invariant for Legendrian submanifolds, the *Legendrian contact homology* has been studied for pairs (M, Λ) of a contact manifold M and its compact Legendrian submanifold Λ . It is the homology of a differential graded algebra generated by the set of Reeb chords of Λ , and was introduced by Chekanov [6] and Eliashberg [17]. The differential is defined by using pseudo-holomorphic curves in the symplectization of M . A rigorous definition was given by Ekholm, Etnyre, and Sullivan in [11, 14] when there is a diffeomorphism from M to the contactization of a Liouville manifold which preserves contact forms. As is mentioned in [14, Section 5.1], this included the case of $M = UT^*\mathbb{R}^n$. (We remark that the definition of [14] is given by pseudo-holomorphic curves in the Liouville manifold. These curves can be lifted to pseudo-holomorphic curves in the symplectization of M . See [10].)

Suppose conceptually that we have an algebraic invariant in symplectic or contact topology defined by using pseudo-holomorphic curves, and apply it to an object related to the cotangent bundle T^*Q . For instance, we consider the symplectic homology of T^*Q or the wrapped Floer homology of the conormal bundle L_K of K in T^*Q . In such case, it is known by the following results that these invariants have another view from the topology of the loop or path space of Q without using pseudo-holomorphic curves (Here we assume that Q is a closed spin manifold and all homology groups have the coefficients in \mathbb{Z}):

- The symplectic homology $SH_*(T^*Q)$ of T^*Q is isomorphic to the singular homology of the free loop space of Q [2, 3, 30].
- The wrapped Floer homology $WF_*(L_K, L_K)$ of L_K is isomorphic to the singular homology of the space of paths in Q with end points in K [1].

These results lead us to an expectation that if the Legendrian contact homology of a pair (UT^*Q, Λ_K) is defined, it has another description in terms of the topology of the path space of Q . This expectation has already been confirmed in particular cases. When the codimension of K is 2, Cieliebak, Ekholm, Latschev, and Ng defined in [9] a graded $\mathbb{Z}[\pi_1(\Lambda_K)]$ -algebra, called *string homology*, which is inspired by string topology of the path space of Q . They showed that when Q is \mathbb{R}^3 with the standard metric and K is a knot, the 0-th degree part of the string homology is isomorphic to the 0-th degree part of the *fully non-commutative* Legendrian contact homology of $(UT^*\mathbb{R}^3, \Lambda_K)$ with coefficients in $\mathbb{Z}[\pi_1(\Lambda_K)]$. However, such topological descriptions have not yet been found in higher degrees or for K which is not a knot in \mathbb{R}^3 .

Main results. Let Q be an oriented manifold and K be its compact oriented submanifold of codimension $d \geq 1$. The main purpose of this paper is to define a graded \mathbb{R} -algebra $H_*^{\text{string}}(Q, K)$ and observe its basic properties. This graded \mathbb{R} -algebra can be regarded as a reformulation of the string homology of [9], whose coefficient is reduced from $\mathbb{Z}[\pi_1(\Lambda_K)]$ to \mathbb{R} . The feature of our formulation is that $H_*^{\text{string}}(Q, K)$ is defined for K of an arbitrary codimension and in all degrees, compared to the string homology defined for K of codimension 2 and generated by singular chains of degree less than or equal to 2. The two main differences from the string homology in its construction are the reduction of the coefficient and the substitution of singular chains by *de Rham chains* explained below.

The construction of $H_*^{\text{string}}(Q, K)$ can be summarized as follows: We first choose auxiliary data including a complete Riemannian metric on Q . As a graded \mathbb{R} -vector space, it is defined to be

$$H_*^{\text{string}}(Q, K) := \varinjlim_{a \rightarrow \infty} \varprojlim_{(\varepsilon, \delta) \in \mathcal{T}_a : \varepsilon \rightarrow 0} H_*^{<a}(\varepsilon, \delta), \quad (1)$$

where $H_*^{<a}(\varepsilon, \delta)$ for $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ and $(\varepsilon, \delta) \in \mathcal{T}_a$ is the homology of a chain complex

$$\left(C_*^{<a}(\varepsilon) := \bigoplus_{m=0}^{\infty} C_{*-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0), D_\delta \right). \quad (2)$$

Let us give an explanation for each piece of the above definition:

1. $C_*^{\text{dR}}(X, A)$ is the \mathbb{R} -vector space of de Rham chains defined for a pair of differentiable spaces (X, A) . Together with the boundary operator

$$\partial: C_*^{\text{dR}}(X, A) \rightarrow C_{*+1}^{\text{dR}}(X, A),$$

$(C_*^{\text{dR}}(X, A), \partial)$ becomes a chain complex. de Rham chains can be used as substitutions of singular chains over \mathbb{R} . Their basic properties are summarized in Section 2. The advantage is that the fiber product of de Rham chains can be defined in a natural way. The main references are [23, 24].

2. For $a \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 1}$, Σ_m^a is a differentiable space of sequences $(\gamma_1, \dots, \gamma_m)$ of paths $\gamma_k: [0, T_k] \rightarrow Q$ ($k = 1, \dots, m$) with end points in K . It contains all $(\gamma_k)_{k=1, \dots, m}$ whose total length is less than a . For the precise definition, see Section 3.2. Exceptionally, Σ_0^a is the one point set for $a > 0$ and Σ_0^0 is the empty set.

3. D_δ is defined by

$$D_\delta(x) := \partial x + \sum_{k=1}^m (-1)^{p+kd+1} f_{k,\delta}(x)$$

for $x \in C_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$. Here

$$f_{k,\delta}: C_*^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) \rightarrow C_{*+1-d}^{\text{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon}, \Sigma_{m+1}^0) \quad (k = 1, \dots, m)$$

are linear maps which play the key role in our construction. The idea comes from an operation of string topology explained by three steps:

- (i) Fix a pair of short paths $(\sigma_i: [0, \varepsilon_i] \rightarrow N_\varepsilon)_{i=1,2}$ in a tubular neighborhood N_ε of K such that $\sigma_1(\varepsilon_1), \sigma_2(0) \in K$ and $\sigma_1(0) = \sigma_2(\varepsilon_2)$.
- (ii) For any sequence of m -paths $(\gamma_k)_{k=1, \dots, m}$, we split the k -th path $\gamma_k: [0, T_k] \rightarrow Q$ at a time, say τ , if the image $\gamma_k(\tau)$ coincides with $\sigma_1(0)$. We then concatenate $\gamma_k|_{[0, \tau]}$ (resp. $\gamma_k|_{[\tau, T_k]}$) with σ_1 (resp. σ_2) to get a new sequence of $(m+1)$ -paths

$$(\gamma_1, \dots, \gamma_{k-1}, (\gamma_k|_{[0, \tau]} \cdot \sigma_1), (\sigma_2 \cdot \gamma_k|_{[\tau, T_k]}), \gamma_{k+1}, \dots, \gamma_m).$$

- (iii) We extend the procedures (i), (ii) for families (or de Rham chains) of paths parameterized over manifolds.

For the precise definition, we need to take fiber products of chains. See Section 3.4 and 3.5. $f_{k,\delta}$ depends on a chain $\delta \in C_{n-d}^{\text{dR}}(S_\varepsilon)$, where S_ε for $\varepsilon > 0$ is a differentiable space of pairs of short paths in N_ε introduced in Section 3.2. For $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$, where $\mathcal{L}(K)$ is a closed subset of Lebesgue measure 0, a set \mathcal{T}_a consisting of pairs (ε, δ) is defined in Definition 4.4. It is necessary to prove $D_\delta \circ D_\delta = 0$ for any $(\varepsilon, \delta) \in \mathcal{T}_a$ in Proposition 4.1 to define the chain complex (2).

4. The inverse limit in (1) is defined from an inverse system

$$(\{H_*^{<a}(\varepsilon, \delta)\}_{(\varepsilon, \delta) \in \mathcal{T}_a}, \{k_{(\varepsilon', \delta'), (\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon}).$$

Its construction is given in Section 4.3. To define the linear map $k_{(\varepsilon', \delta'), (\varepsilon, \delta)} : H_*^{<a}(\varepsilon', \delta') \rightarrow H_*^{<a}(\varepsilon, \delta)$, we need to factor through another homology group constructed from “[−1, 1]-modeled de Rham chains”. Furthermore, to check its well-definedness and a claim about composition, we need one more homology group constructed from “[−1, 1]²-modeled de Rham chains”. These variants of de Rham chains are introduced in Section 3.6.

5. The inverse limit is denoted by $H_*^{<a}(Q, K) := \varprojlim_{\varepsilon \rightarrow 0} H_*^{<a}(\varepsilon, \delta)$. The direct limit in (1) is defined from $(\{H_*^{<a}(Q, K)\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}, \{I^{a,b}\}_{a \leq b})$, where $I^{a,b} : H_*^{<a}(Q, K) \rightarrow H_*^{<b}(Q, K)$ is induced by the inclusion maps $\Sigma_m^{a+m\varepsilon} \rightarrow \Sigma_m^{b+m\varepsilon}$ for all $m \in \mathbb{Z}_{\geq 0}$. See Section 4.4.
6. A graded associative product structure on $H_*^{\text{string}}(Q, K)$ is induced by natural maps $\Sigma_m^a \times \Sigma_{m'}^{a'} \rightarrow \Sigma_{m+m'}^{a+a'}$ for all $m, m' \in \mathbb{Z}_{\geq 0}$. The unit comes from $1 \in \mathbb{R} = C_0^{\text{dR}}(\Sigma_0^a, \Sigma_0^0)$ for $a > 0$. See Subsection 4.4.2.

A fundamental property of $H_*^{\text{string}}(Q, K)$ is the invariance by C^∞ isotopies of K .

Theorem 1.1. *The isomorphism class of the unital graded \mathbb{R} -algebra $H_*^{\text{string}}(Q, K)$ is independent on auxiliary data and invariant by changing the orientation of K . Moreover, it is invariant under C^∞ isotopies of K . (See Proposition 4.20.)*

We also give non-trivial computations when $Q = \mathbb{R}^{2d-1}$ for $d \geq 2$. For two specific submanifolds in \mathbb{R}^{2d-1} , both of which are diffeomorphic to $S^{d-1} \sqcup S^{d-1}$, we prove that our invariant is isomorphic to the homology of a finitely generated differential graded algebra. Using this computation, we obtain the next result.

Theorem 1.2. *For every $d \geq 2$, there are two non-isotopic oriented submanifolds K, K' in \mathbb{R}^{2d-1} of codimension d such that Λ_K is isotopic to $\Lambda_{K'}$ in $UT^*\mathbb{R}^{2d-1}$ as a C^∞ submanifold with a spin structure, while $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K) \not\cong H_*^{\text{string}}(\mathbb{R}^{2d-1}, K')$. (See Corollary 5.10.)*

The spin structure on Λ_K for any submanifold K in a spin manifold Q is explained in Proposition 5.11.

Another purpose of this paper is to observe the relation to Legendrian contact homology. The following result is non-trivial from the construction of $H_*^{\text{string}}(Q, K)$.

Theorem 1.3. *When the codimension of K is 2 and the normal bundle of K is trivial, $H_0^{\text{string}}(Q, K)$ is isomorphic to the cord algebra of (Q, K) over \mathbb{R} . (See Corollary 6.11.)*

If K is connected, the cord algebra over \mathbb{R} we consider in this paper is a reduction of the cord algebra over $\mathbb{Z}[H_1(\Lambda_K)]$ defined by Ng in [27]. Combining with the result by Ekholm, Etnyre, Ng, and Sullivan in [15], the cord algebra for a knot K in \mathbb{R}^3 was proved to be isomorphic to the 0-th degree part of the Legendrian contact homology of $(UT^*\mathbb{R}^3, \Lambda_K)$. Later, another direct proof was given in [9].

The author makes the following conjecture when $Q = \mathbb{R}^n$. (The author expects that this conjecture holds not only for $Q = \mathbb{R}^n$ but for more general Q .)

Conjecture 1.4. *For any compact oriented submanifold K in \mathbb{R}^n , $H_*^{\text{string}}(\mathbb{R}^n, K)$ is isomorphic to the Legendrian contact homology of $(UT^*\mathbb{R}^n, \Lambda_K)$ with coefficients in \mathbb{R} . (See Conjecture 7.3.)*

The Legendrian contact homology with coefficients in \mathbb{R} is an isotopy invariant of Legendrian submanifolds with a spin structure [13, 14]. If Conjecture 1.4 is true, then our invariant can be applied to study the contact topology of $UT^*\mathbb{R}^n$. For instance, assuming this conjecture,

Theorem 1.2 would imply that Λ_K is not isotopic to $\Lambda_{K'}$ as a Legendrian submanifold with a spin structure.

The author is working toward the proof of Conjecture 1.4. The idea toward the proof is explained in Section 7. The exposition is intuitive and not rigorous at many points, but the author think that it convinces that the conjecture is reasonable. The outline is similar to [9]. To give the isomorphism of the conjecture, we observe moduli spaces of pseudo-holomorphic curves with boundaries in L_K and \mathbb{R}^n . If $n = 3$ and K is a knot, these moduli spaces were used in the proof of [9, Theorem 1.2]. The difference from [9] is that we will consider *Kuranishi structures* on these moduli spaces in order to solve a problem about the transversality of the fiber product of moduli spaces. In Section 7, we sketch how the moduli spaces with Kuranishi structure are related to de Rham chains which are used to define $H_*^{\text{string}}(\mathbb{R}^n, K)$. The reference of this idea is [24].

Future applications.

Let us mention to two possible applications of Conjecture 1.4.

- Distinguishing unit conormal bundles modulo Legendrian isotopy: Let K_0 and K_1 be compact submanifolds of \mathbb{R}^n . If K_0 and K_1 are C^∞ isotopic in \mathbb{R}^n , it is obvious that Λ_{K_0} is Legendrian isotopic to Λ_{K_1} in $UT^*\mathbb{R}^n$. It is natural to ask whether the converse is true. There are several preceding researches on this subject. Shende [29] and Ekholm-Ng-Shende [18] observed the case where $n = 3$ and K is a knot or a link in \mathbb{R}^3 . [29] used the derived category of sheaves with microsupport contained in $L_K \cup \mathbb{R}^3$ and [16] used an enhanced version of the Legendrian contact homology of Λ_K , in order to show the following: For any knots K_0 and K_1 in \mathbb{R}^3 , if Λ_{K_0} is Legendrian isotopic to Λ_{K_1} in $U^*\mathbb{R}^3$, then K_0 is isotopic to either K_1 or the mirror of K_1 .

In higher dimensions, such a complete answer has not been discovered. Asplund [4] observed higher dimensional cases by using wrapped Floer cohomology stopped by a unit conormal bundle. From the example in [4] of submanifolds of S^n for $n = 5$ or $n \geq 7$ and a point $x_0 \in S^n$, we obtain examples of pairs of codimension 2 spheres in $\mathbb{R}^n \cong S^n \setminus \{x_0\}$ for $n = 5$ or $n \geq 7$ whose unit conormal bundles are not Legendrian isotopic. If we find a systematic way to compute $H_*^{\text{string}}(\mathbb{R}^n, K)$ for every K under certain conditions, this would give a partial answer to the question.

- Application to clean Lagrangian intersections: Consider $T^*\mathbb{R}^3$ and identify the image of the zero section with \mathbb{R}^3 . In [28], the author considered clean intersections in $T^*\mathbb{R}^3$ of \mathbb{R}^3 and an exact Lagrangian submanifold L along a knot K in \mathbb{R}^3 . By using some computations of the Legendrian contact homology of Λ_K via the cord algebra of K , we can obtain constraints on the knot types of K under a specific condition on L . The computations of $H_*^{\text{string}}(\mathbb{R}^n, K)$ will be used to extend this result to clean Lagrangian intersections in $T^*\mathbb{R}^n$ for $n \geq 3$.

Organization of paper.

In Section 2, general notions of a differentiable space and its de Rham chain complex are introduced. In Section 3.2, the differentiable spaces Σ_m^α and S_ε are defined. Their de Rham chain complexes are studied in Section 3.3. Through Section 3.4 and 3.5, the operator $f_{k,\delta}$ is defined. In Section 3.6, $[-1, 1]$ -modeled and $[-1, 1]^2$ -modeled de Rham chains for path spaces are introduced. In Section 4.1, we define the chain complexes (2) and give a couple of computations. In Section 4.2, we consider their variants using those chains in Section 3.6. They are necessary to define the map $k_{(\varepsilon', \delta'), (\varepsilon, \delta)}$ in Section 4.3. The definition of $H_*^{\text{string}}(Q, K)$ is given in Section 4.4. The independence on auxiliary data is checked in Section 4.5, from which the isotopy invariance follows. In Section 5, we examine the algebraic structure of $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K)$ when K is a

higher-dimensional generalization of the Hopf link or the unlink in \mathbb{R}^3 . In Section 6.1, referring to [9], we define the cord algebra over \mathbb{R} and its another description as the 0-th degree part of the string homology. In Section 6.2, we construct a graded map from the string homology to $H_*^{\text{string}}(Q, K)$. In Section 6.3, this map is proved to be an isomorphism on the 0-th degree part. In Section 7.1, we review the definition of Legendrian contact homology. In Section 7.2, we state Conjecture 1.4 and Section 7.3 is devoted to outline a strategy for proving the conjecture.

Acknowledgements. The author would like to express his deep gratitude to his supervisor Kei Irie for spending hours of discussion and carefully reading the draft. He also would like to thank the referee of this paper for carefully reading it and making valuable comments. This work was supported by JST, the establishment of university fellowships towards the creation of science technology innovation, Grant Number JPMJFS2123 and JSPS KAKENHI Grant Number JP23KJ1238. Part of this work was supported by the WINGS-FMSP program at the Graduate School of Mathematical Science, the University of Tokyo.

2 Differentiable space and de Rham chains

In this section, the notions of differentiable spaces and de Rham chains are introduced. We also summarize results applied in the latter sections

Remark 2.1. The notion of differentiable space goes back to [7] by K.-T. Chen. The notion of de Rham chains was proposed by Fukaya in [19], and later, Irie gave a definition in [23, 24]. We mainly refer, especially about sign conventions, to [24]. As is mentioned in [23, Remark 4.1], the definition of plots (elements of a differentiable structure) in this paper is different from that of [7].

2.1 Notations and conventions

For $m, N \in \mathbb{Z}_{\geq 0}$, let $\mathcal{U}_{m,N}$ be the set of oriented m -dimensional submanifolds of \mathbb{R}^N . We then define $\mathcal{U} := \bigcup_{m,N \in \mathbb{Z}_{\geq 0}} \mathcal{U}_{m,N}$. Let us fix a few conventions about orientations. If we write \mathbb{R}^n for $n \in \mathbb{Z}_{\geq 1}$, this means the manifold $\mathbb{R}^n \in \mathcal{U}_{n,n}$ whose orientation is given so that $dx_1 \wedge \cdots \wedge dx_n$ is a positive volume form when (x_1, \dots, x_n) is the standard coordinate of \mathbb{R}^n . If we write $\{0\}$, this means $\{0\} \in \mathcal{U}_{0,0}$ with a positive sign assigned.

Let us think about the orientation of fiber products of oriented manifolds. For $U, V, M \in \mathcal{U}$, suppose that there are two C^∞ maps $f: U \rightarrow M, g: V \rightarrow M$. We also assume that g is a submersion. (Hereafter, all submersions are of class C^∞ .) Then, the fiber product

$$U \times_f V := \{(u, v) \in U \times V \mid f(u) = g(v)\}$$

is a C^∞ submanifold of $U \times V$. In order to determine the orientation at $(u, v) \in U \times_f V$, we take a right inverse $s: T_{g(v)}M \rightarrow T_vV$ of $(dg)_v$ (i.e. $(dg)_v \circ s = \text{id}_{T_{g(v)}M}$). Then, there are two isomorphisms

$$\begin{aligned} T_{g(v)}M \times \text{Ker}(dg)_v &\rightarrow T_vV: (z, y) \mapsto s(z) + y, \\ T_uU \times \text{Ker}(dg)_v &\rightarrow T_{(u,v)}(U \times_f V): (x, y) \mapsto (x, y + s \circ (df)_u(x)). \end{aligned}$$

The orientations of $\text{Ker}(dg)_v$ and $T_{(u,v)}(U \times_f V)$ are determined so that the above isomorphisms preserve orientations. Of course, when X and Y are oriented \mathbb{R} -vector spaces, we assign the product orientation on $X \times Y$. In particular, when $M = \{0\}$, this gives the orientation of the product manifold $U \times V$.

For $U \in \mathcal{U}$, $\Omega_c^p(U)$ is the vector space of compactly supported C^∞ differential p -forms on U . When $p < 0$ or $p > \dim U$, we define $\Omega_c^p(U) := 0$. For $U, U' \in \mathcal{U}$ and a submersion $\pi: U' \rightarrow U$, we have an \mathbb{R} -linear map

$$\pi_!: \Omega_c^p(U') \rightarrow \Omega_c^{p-(\dim U' - \dim U)}(U),$$

called the *integration along fibers*. When $U' = \mathbb{R}^d \times \mathbb{R}^k$, $U = \mathbb{R}^k$ and $\pi(t, x) = x$ for every $(t, x) \in U'$, this map is characterized by the following: For $f \in \Omega_c^0(U')$, $1 \leq i_1 < \dots < i_a \leq d$ and $1 \leq j_1 < \dots < j_b \leq k$, if we take $\omega := f(dt_{i_1} \wedge \dots \wedge dt_{i_a} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_b})$, then for every $x \in U$,

$$(\pi_!(\omega))_x = \begin{cases} 0 & \text{if } a < d, \\ \left(\int_{\mathbb{R}^d} f(\cdot, x) dt_1 \wedge \dots \wedge dt_d\right) (dx_{i_1} \wedge \dots \wedge dx_{i_a})_x & \text{if } a = d. \end{cases}$$

For an arbitrary submersion $\pi: U' \rightarrow U$, $\pi_!$ is defined by taking local charts and a partition of unity on U .

2.2 de Rham chain complex

2.2.1 Differentiable space

We give the definition of a differentiable space.

Definition 2.2. Let X be a set and P_X be a set of pairs (U, φ) of $U \in \mathcal{U}$ and a map $\varphi: U \rightarrow X$. We say P_X is a *differentiable structure* on X if it satisfies the following condition:

- For any $(U, \varphi) \in P_X$, $U' \in \mathcal{U}$ and a submersion $\pi: U' \rightarrow U$, the pair $(U', \varphi \circ \pi)$ is also an element of P_X .

We call such pair (X, P_X) a *differentiable space*. An element of P_X is called a *plot* of (X, P_X) .

Example 2.3. Let M be a manifold. There are two types of canonical differentiable structures on M :

$$\begin{cases} P_M & := \{(U, \varphi) \mid \varphi: U \rightarrow M \text{ is a } C^\infty \text{ map}\}, \\ P_M^{\text{reg}} & := \{(U, \varphi) \mid \varphi: U \rightarrow M \text{ is a submersion}\}. \end{cases}$$

Clearly, (M, P_M) and (M, P_M^{reg}) are differentiable spaces. The latter is denoted by M^{reg} . We consider the differentiable structure P_M for any manifold M , unless we declare to use M^{reg} .

Definition 2.4. Let (X, P_X) and (Y, P_Y) be differentiable spaces and Z be a subset of X . Denote the projection map from $X \times Y$ to X (resp. Y) by pr_X (resp. pr_Y) and the inclusion map from Z to X by ι_Z .

1. We define differentiable structures on $X \times Y$ and Z by

$$\begin{aligned} P_{X \times Y} &:= \{(U, \varphi) \mid (U, \text{pr}_X \circ \varphi) \in P_X \text{ and } (U, \text{pr}_Y \circ \varphi) \in P_Y\}, \\ P_Z &:= \{(U, \varphi) \mid (U, \iota_Z \circ \varphi) \in P_X\}. \end{aligned}$$

2. Let $f: X \rightarrow Y$ be a map. We say f is a *smooth map* if $(U, f \circ \varphi) \in P_Y$ for any $(U, \varphi) \in P_X$.

In the case of the above definition, we simply call (Z, P_Z) a *subspace* of (X, P_X) . Note that given a set W and two maps $f: X \rightarrow W$ and $g: Y \rightarrow W$, the fiber product $X \times_f Y$ becomes a differentiable space as a subspace of $(X \times Y, P_{X \times Y})$.

2.2.2 de Rham chains

Next, we introduce the notion of de Rham chain complex. Hereafter, if we say that X is a differentiable space, this means that X is equipped with a differentiable structure denoted by P_X .

Let X be a differentiable space. We consider a graded \mathbb{R} -vector space

$$A_*(X) := \bigoplus_{(U, \varphi) \in P_X} \Omega_c^{\dim U - *} (U).$$

For any $(U, \varphi) \in P_X$ and $\omega \in \Omega_c^{\dim U - *}(U)$, let (U, φ, ω) denote the element of $A_*(X)$ such that its component for $(V, \psi) \in P_X$ is

$$(U, \varphi, \omega)_{(V, \psi)} = \begin{cases} \omega & \text{if } (V, \psi) = (U, \varphi), \\ 0 & \text{if } (V, \psi) \neq (U, \varphi). \end{cases}$$

We take a linear subspace $Z_*(X)$ of $A_*(X)$ generated by

$$\{(U', \varphi \circ \pi, \omega) - (U, \varphi, \pi_! \omega) \mid (U, \varphi) \in P_X \text{ and } \pi: U' \rightarrow U \text{ is a submersion}\}.$$

Then we define a quotient vector space

$$C_*^{\text{dR}}(X) := A_*(X)/Z_*(X).$$

The equivalence class of $(U, \varphi, \omega) \in A_*(X)$ in $C_*^{\text{dR}}(X)$ is denoted by $[U, \varphi, \omega]$. We also define an \mathbb{R} -linear map $\partial: C_*^{\text{dR}}(X) \rightarrow C_{* - 1}^{\text{dR}}(X)$ of degree (-1) by

$$\partial[U, \varphi, \omega] := (-1)^{|\omega| + 1} [U, \varphi, d\omega].$$

This map is well-defined and $\partial \circ \partial = 0$ holds. $(C_*^{\text{dR}}(X), \partial)$ is called the *de Rham chain complex* of a differentiable space X , and an element of $C_*^{\text{dR}}(X)$ is called a *de Rham chain* of X . By taking its homology, we obtain

$$H_*^{\text{dR}}(X) := H_*(C_*^{\text{dR}}(X), \partial).$$

In addition, a functoriality holds. Namely, any smooth map $f: X \rightarrow Y$ induces a chain map

$$f_*: C_*^{\text{dR}}(X) \rightarrow C_*^{\text{dR}}(Y): [U, \varphi, \omega] \mapsto [U, f \circ \varphi, \omega].$$

Remark 2.5. The following are fundamental techniques to compute de Rham chains:

1. For $[U, \varphi, \omega] \in C_*^{\text{dR}}(X)$, suppose that $V \subset U$ is an open subset containing $\text{supp } \omega$. Then $[U, \varphi, \omega] = [V, \varphi|_V, \omega|_V] \in C_*^{\text{dR}}(X)$.
2. If $(\mathbb{R} \times U, \varphi) \in P_X$ satisfies $\varphi(s, \cdot) = \begin{cases} \varphi_0 & \text{if } s \leq 0, \\ \varphi_1 & \text{if } s \geq 1, \end{cases}$ for some $(U, \varphi_0), (U, \varphi_1) \in P_X$, then

$$\partial[\mathbb{R} \times U, \varphi, (-1)^{|\omega|} \chi \times \omega] = [U, \varphi_1, \omega] - [U, \varphi_0, \omega] \in C_*^{\text{dR}}(X)$$

for any closed form $\omega \in \Omega_c^{\dim U - *}(U)$ and $\chi: \mathbb{R} \rightarrow [0, 1]$ such that $\text{supp } \chi$ is compact and $\chi(s) = 1$ for every $s \in [0, 1]$.

Example 2.6. Let M be an oriented manifold. The de Rham chain complex of M^{reg} is naturally isomorphic to $(\Omega_c^{\dim M - *}(M), d)$ through the map

$$C_p^{\text{dR}}(M^{\text{reg}}) \rightarrow \Omega_c^{\dim M - p}(M): [U, \varphi, \omega] \mapsto (-1)^{s(p)} \varphi_! \omega.$$

Here $s(p) := (p - \dim M)(p - \dim M - 1)/2$. Hence $H_*^{\text{dR}}(M^{\text{reg}})$ is isomorphic to the compactly supported de Rham cohomology $H_{c, \text{dR}}^{\dim M - *}(M)$.

Let us define the de Rham chain complex for a pair of differentiable spaces. A smooth map $f: X \rightarrow \mathbb{R}$ is said to be *approximately smooth* if there exists a decreasing sequence $(f_j)_{j \in \mathbb{Z}_{\geq 1}}$ of smooth maps from X to \mathbb{R} such that $\lim_{j \rightarrow \infty} f_j(x) = f(x)$ for every $x \in X$. The following lemma is proved in [23, Lemma 4.11].

Lemma 2.7. For an approximately smooth function $f: X \rightarrow \mathbb{R}$, let $X^a := f^{-1}((-\infty, a))$ for every $a \in \mathbb{R} \cup \{\infty\}$. Then for $a, b \in \mathbb{R} \cup \{\infty\}$ with $a \leq b$, the linear map $i_*: C_*^{\text{dR}}(X^a) \rightarrow C_*^{\text{dR}}(X^b)$, which is induced by the inclusion map $i: X^a \rightarrow X^b$, is injective.

In the case of the above lemma, we define a quotient complex

$$C_*^{\text{dR}}(X^b, X^a) := C_*^{\text{dR}}(X^b) / i_*(C_*^{\text{dR}}(X^a)).$$

Its homology is denoted by $H_*^{\text{dR}}(X^b, X^a)$.

Next, we define a fiber product of de Rham chains.

Definition 2.8. Let (X, P_X) and (Y, P_Y) be differentiable spaces. Suppose that we have an oriented manifold M of dimension n and two smooth maps

$$f: (X, P_X) \rightarrow (M, P_M), \quad g: (Y, P_Y) \rightarrow (M, P_M^{\text{reg}}) = M^{\text{reg}}$$

Then, we define a linear map

$$C_{p+n}^{\text{dR}}(X) \otimes C_{q+n}^{\text{dR}}(Y) \rightarrow C_{p+q+n}^{\text{dR}}(X \times_f Y): x \otimes y \rightarrow x \times_f y$$

by

$$x \times_f y := (-1)^{p|\eta|} [W, (\varphi \times \psi)|_W, (\omega \times \eta)|_W]$$

for $x = [U, \varphi, \omega] \in C_{p+n}^{\text{dR}}(X)$ and $y = [V, \psi, \eta] \in C_{q+n}^{\text{dR}}(Y)$. Here, $W := U \times_{f \circ \varphi} \times_{g \circ \psi} V$ is a fiber product over M .

It is straightforward to check the well-definedness of $x \times_f y$. It can also be checked that

$$\partial(x \times_f y) = (\partial x) \times_f y + (-1)^p x \times_f (\partial y)$$

holds for any $x \in C_{p+n}^{\text{dR}}(X)$ and $y \in C_{q+n}^{\text{dR}}(Y)$. When $M = \{0\}$, we simply write $x \times_f y$ by $x \times y$.

2.2.3 Collection of results about de Rham chain complex

In the rest of this section, let us summarize a couple of basic results about de Rham chain complex. The first result can be compared with the computation for M^{reg} in Example 2.6. Hereafter, $H_*^{\text{sing}}(\cdot)$ denotes the singular homology with coefficients in \mathbb{R} .

Proposition 2.9. For every oriented manifold M , there exists a canonical isomorphism

$$\Psi_M: H_*^{\text{sing}}(M) \rightarrow H_*^{\text{dR}}(M)$$

such that for any C^∞ map $f: M \rightarrow N$ between oriented manifolds, $\Psi_N \circ f_* = f_* \circ \Psi_M$ holds.

For the details of the construction of Ψ_M , see [23, Section 4.7]. It is the composition of a natural isomorphism between $H_*^{\text{sing}}(M)$ and $H_*^{\text{sm}}(M)$ (the homology of smooth singular chains in M) and a canonical map from $H_*^{\text{sm}}(M)$ to $H_*^{\text{dR}}(M)$. For the proof that Ψ_M is an isomorphism, see [23, Section 5]. This result can be extended to relative homology groups for (M, N) , where N is an open submanifold of M such that $N = f^{-1}((-\infty, a))$ for some approximately smooth map $f: M \rightarrow \mathbb{R}$.

Next, let $f, g: X \rightarrow Y$ be smooth maps between differentiable spaces. We say f is *homotopic* to g if there exists a smooth map $H: \mathbb{R} \times X \rightarrow Y$ such that $H(t, x) = f(x)$ for $t \leq 0$ and $H(t, x) = g(x)$ for $t \geq 1$. Then we have the following result. For the proof, see [23, Proposition 4.7].

Proposition 2.10. *For two smooth maps $f, g: X \rightarrow Y$, if f is homotopic to g , then there exists a chain homotopy $K: C_*^{\text{dR}}(X) \rightarrow C_{*+1}^{\text{dR}}(Y)$ such that $\partial K + K\partial = f_* - g_*$. In particular, $f_* = g_*: H_*^{\text{dR}}(X) \rightarrow H_*^{\text{dR}}(Y)$ holds.*

Remark 2.11. For three smooth maps $f, g, h: X \rightarrow Y$ such that f is homotopic to g and g is homotopic to h , we can ask whether f is homotopic to h . In fact, if the differentiable structure P_Y of Y satisfies the following condition, such transitivity holds (The proof is straightforward.):

- For any $U \in \mathcal{U}$ and $(U_1, \varphi_1), (U_2, \varphi_2) \in P_Y$ such that $(U_i)_{i=1,2}$ is an open cover of U and $\varphi_1|_{U_1 \cap U_2} = \varphi_2|_{U_1 \cap U_2}$, we have $(U, \varphi) \in P_Y$ for $\varphi: U \rightarrow Y$ which maps $u \in U_i$ to $\varphi_i(u)$ ($i = 1, 2$).

All differentiable spaces appearing after Section 3 satisfy this condition. However, as mentioned in [23, Remark 4.4], it seems difficult in general case to prove such transitivity.

The last one is a result about excisions.

Proposition 2.12. *Let X be a differentiable space and $Y = f^{-1}((-\infty, a)) \subset X$ for some approximately smooth function $f: X \rightarrow \mathbb{R}$ and $a \in \mathbb{R}$. Suppose there is another approximately smooth function $g: X \rightarrow \mathbb{R}$ and $b_0 \in \mathbb{R}$ such that $g^{-1}((b_0, \infty)) \subset Y$. For every $b > b_0$, let $X^b := g^{-1}((-\infty, b))$ and $Y^b := (g|_Y)^{-1}((-\infty, b))$. Then, the inclusion map of pairs $i: (X^b, Y^b) \rightarrow (X, Y)$ induces an isomorphism $i_*: C_*^{\text{dR}}(X^b, Y^b) \rightarrow C_*^{\text{dR}}(X, Y)$.*

Proof. We first prove the assertion when $g: X \rightarrow \mathbb{R}$ is a smooth map. For $b > b_0$, choose $\delta > 0$ and a smooth function $\kappa: \mathbb{R} \rightarrow [0, 1]$ such that $2\delta < b - b_0$ and $\kappa(b') = \begin{cases} 1 & \text{if } b' \leq b_0 + \delta, \\ 0 & \text{if } b' \geq b - \delta. \end{cases}$ Then we define a linear map

$$r: C_*^{\text{dR}}(X) \rightarrow C_*^{\text{dR}}(X^b): [U, \varphi, \omega] \mapsto [U^b, \varphi|_{U^b}, (\kappa \circ g \circ \varphi) \cdot \omega|_{U^b}],$$

where $U^b := (g \circ \varphi)^{-1}((-\infty, b))$. This reduces to a map $\bar{r}: C_*^{\text{dR}}(X, Y) \rightarrow C_*^{\text{dR}}(X^b, Y^b)$. We claim that \bar{r} is the inverse map of i_* . Indeed, for any $x = [U, \varphi, \omega] \in C_*^{\text{dR}}(X)$, we have

$$\begin{aligned} x - i_* \circ \bar{r}(x) &= [U, \varphi, \omega] - [U, \varphi, (\kappa \circ g \circ \varphi) \cdot \omega] \\ &= [U_0, \varphi|_{U_0}, ((1 - \kappa) \circ g \circ \varphi) \cdot \omega|_{U_0}] \in C_*^{\text{dR}}(Y) \text{ for } U_0 := (g \circ \varphi)^{-1}((b_0, \infty)). \end{aligned}$$

Similarly, we can show that $x - \bar{r} \circ i_*(x) \in C_*^{\text{dR}}(Y^b)$ for $x \in C_*^{\text{dR}}(X^b)$.

In a general case, there exists a decreasing sequence $(g_j)_{j \geq 1}$ of smooth maps $g_j: X \rightarrow \mathbb{R}$ such that $g_j(x) \rightarrow g(x)$ ($j \rightarrow \infty$) for every $x \in X$. For $b > b_0$, let $X_j^b := g_j^{-1}((-\infty, b))$ and $Y_j^b := (g_j|_Y)^{-1}((-\infty, b))$. From [23, Corollary 4.12 (i)], $\varinjlim_{j \rightarrow \infty} C_*^{\text{dR}}(X_j^b, Y_j^b) \rightarrow C_*^{\text{dR}}(X^b, Y^b)$, induced by inclusion maps, is an isomorphism. We have shown that $(i|_{(X_j^b, Y_j^b)})_*: C_*^{\text{dR}}(X_j^b, Y_j^b) \rightarrow C_*^{\text{dR}}(X, Y)$ is an isomorphism for every $j \geq 1$, so i_* is also an isomorphism. \square

3 Differentiable space of paths and an operation from string topology

3.1 Preparation

Throughout this paper, Q is a manifold of dimension n and K is a compact submanifold of Q of codimension $d \geq 1$. In addition, both Q and K are required to have fixed orientations. The construction of $H_*^{\text{string}}(Q, K)$ depends on the following auxiliary data:

1. a complete Riemannian metric g on Q . (We write $g(v, w) = \langle v, w \rangle_g$ and $\sqrt{g(v, v)} = |v|_g$.)

2. a constant $C_0 \geq 1$.
3. a positive real number ε_0 for which the map

$$\{(x, v) \in (TK)^\perp \mid |v|_g < \varepsilon_0\} \rightarrow Q: (x, v) \mapsto \exp_x(v) \quad (3)$$

is an open embedding. Here, $\exp_x: T_x Q \rightarrow Q$ is the exponential map with respect to g

4. a C^∞ function $\mu: [0, \frac{3}{2}] \rightarrow [0, 1]$ such that $\mu(t) = \begin{cases} t & \text{near } t = 0, \\ 1 & \text{near } t = \frac{3}{2}, \end{cases}$ and $0 \leq \mu'(t) \leq 1$ for every $t \in [0, \frac{3}{2}]$.

The independence of $H_*^{\text{string}}(Q, K)$ on these data up to isomorphism is proved in Section 4.5. Until then, these data are fixed, so $\langle v, w \rangle_g$ and $|v|_g$ are denoted by $\langle v, w \rangle$ and $|v|$ respectively.

We define $\mathcal{C}(K)$ to be the set of geodesics $\gamma: [0, T] \rightarrow Q$ with unit speed such that $\gamma(0), \gamma(T) \in K$ and $\gamma'(0) \in (T_{\gamma(0)}K)^\perp$, $\gamma'(T) \in (T_{\gamma(T)}K)^\perp$. Such geodesics are called *binormal chords* of K . We also define for $m \in \mathbb{Z}_{\geq 1}$

$$\begin{aligned} \mathcal{L}_m(K) &:= \{\sum_{k=1}^m \text{length} \gamma_k \mid \gamma_1, \dots, \gamma_m \in \mathcal{C}(K)\}, \\ \mathcal{L}(K) &:= \bigcup_{m=1}^\infty \mathcal{L}_m(K). \end{aligned}$$

These are closed subsets of $\{a \in \mathbb{R} \mid a \geq 2\varepsilon_0\}$ since K is compact. Moreover, they are null sets with respects to the Lebesgue measure. For the proof, see Lemma 3.10.

3.2 Differentiable space of paths

In this section, we introduce two differentiable spaces of paths, Σ_m^a and S_ε . Let $\Omega_K(Q)$ be the set of C^∞ paths $\gamma: [0, T] \rightarrow Q$ with $T > 0$ such that $\gamma(0), \gamma(T) \in K$ and $|\gamma'(t)| \leq C_0$ for any $t \in [0, T]$. For any C^∞ path $\gamma: [0, T] \rightarrow Q$, its length is denoted by

$$\text{length} \gamma := \int_0^T |\gamma'(t)| dt.$$

For $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $m \in \mathbb{Z}_{\geq 1}$, we define Σ_m^a to be a subset of $\Omega_K(Q)^{\times m}$ which consists of $(\gamma_k: [0, T_k] \rightarrow Q)_{k=1, \dots, m}$ satisfying *either* of the following two conditions:

- $\sum_{k=1}^m \text{length} \gamma_k < a$.
- $\min_{1 \leq k \leq m} \text{length} \gamma_k < \varepsilon_0$.

The differentiable structure on Σ_m^a is defined by

$$P_{\Sigma_m^a} := \{(U, \varphi) \mid U \in \mathcal{U} \text{ and } \varphi: U \rightarrow \Sigma_m^a \text{ is smooth}\}.$$

Here, we say φ is smooth in the following sense: If we write $\varphi(u) = (\gamma_k^u: [0, T_k^u] \rightarrow Q)_{k=1, \dots, m}$ for $u \in U$, then for each $k \in \{1, \dots, m\}$, the function $U \rightarrow \mathbb{R}_{>0}: u \mapsto T_k^u$ is of class C^∞ and

$$\{(u, t) \in U \times \mathbb{R} \mid 0 \leq t \leq T_k^u\} \rightarrow Q: (u, t) \mapsto \gamma_k^u(t)$$

is a C^∞ map. As an exception, let us define $\Sigma_0^a := \begin{cases} \{*\} & \text{if } a > 0, \\ \emptyset & \text{if } a = 0, \end{cases}$ together with the differentiable structure $P_{\Sigma_0^a} := \{(U, \varphi) \mid U \in \mathcal{U}, \varphi: U \rightarrow \Sigma_0^a\}$.

We consider the de Rham chain complex $(C_*^{\text{dR}}(\Sigma_m^a), \partial)$ for $a \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 0}$. Lemma 2.7 implies that we may think of $C_*^{\text{dR}}(\Sigma_m^a)$ as a linear subspace of $C_*^{\text{dR}}(\Sigma_m^b)$ when $a \leq b$, since

$$\begin{cases} \Sigma_m^b \rightarrow \mathbb{R}: (\gamma_l)_{l=1, \dots, m} \mapsto \text{length} \gamma_k \quad (k = 1, \dots, m), \\ \mathbb{R}^m \rightarrow \mathbb{R}: (a_k)_{k=1, \dots, m} \mapsto \min_{1 \leq k \leq m} a_k, \end{cases}$$

are approximately smooth functions. Thus the quotient complex $(C_*^{\text{dR}}(\Sigma_m^b, \Sigma_m^a), \partial)$ is defined.

Remark 3.1. When $a \leq m\varepsilon_0$, the condition that $\sum_{k=1}^m \text{length}\gamma_k < a$ implies that one of γ_k ($k = 1, \dots, m$) has length less than ε_0 . Thus, $\Sigma_m^a = \Sigma_m^0$ if $a \leq m\varepsilon_0$. When $a = \infty$, $\Sigma_m^\infty = \Omega_K(Q)^{\times m}$, which will be used only in Section 6.

Next, we define another differentiable space of paths. For every $\varepsilon \in (0, \varepsilon_0]$, the open subset in Q

$$N_\varepsilon := \{\exp_x(v) \mid x \in K, v \in (T_x K)^\perp \text{ and } |v| < \varepsilon/2\}$$

is a tubular neighborhood of K in Q . Then we define a set S_ε which consists of pairs of C^∞ paths $(\sigma_i: [0, \varepsilon_i] \rightarrow N_\varepsilon)_{i=1,2}$ satisfying:

- $0 < \varepsilon_i \leq \varepsilon/2$ for $i = 1, 2$.
- $\sigma_1(\varepsilon_1), \sigma_2(0) \in K$ and $\sigma_1(0) = \sigma_2(\varepsilon_2)$.
- $|\sigma'_i(t)| \leq 1$ for $i = 1, 2$ and any $t \in [0, \varepsilon_i]$.

On this set, the evaluation map ev_0 is defined by

$$\text{ev}_0: S_\varepsilon \rightarrow N_\varepsilon: (\sigma_1, \sigma_2) \mapsto \sigma_1(0).$$

The differentiable structure on S_ε is defined by

$$P_{S_\varepsilon} := \{(V, \psi) \mid V \in \mathcal{U}, \psi \text{ is a smooth map such that } \text{ev}_0 \circ \psi: V \rightarrow N_\varepsilon \text{ is a submersion}\}$$

Here we say ψ is smooth in the following sense: If we write $\psi(v) = (\sigma_i^v: [0, \varepsilon_i^v] \rightarrow N_\varepsilon)_{i=1,2}$ for $v \in V$, then for $i \in \{1, 2\}$, the function $V \rightarrow \mathbb{R}_{>0}: v \mapsto \varepsilon_i^v$ is of class C^∞ and

$$\{(v, t) \in V \times \mathbb{R} \mid 0 \leq t \leq \varepsilon_i^v\} \rightarrow N_\varepsilon: (v, t) \mapsto \sigma_i^v(t)$$

is a C^∞ map. Note that ev_0 is a smooth map from $(S_\varepsilon, P_{S_\varepsilon})$ to $(N_\varepsilon, P_{N_\varepsilon}^{\text{reg}}) = N_\varepsilon^{\text{reg}}$ defined in Example 2.3.

3.3 Homology groups

In this section, we examine the homology groups $H_*^{\text{dR}}(\Sigma_m^b, \Sigma_m^a)$ and $H_*^{\text{dR}}(S_\varepsilon)$. The main results are Proposition 3.7 and Proposition 3.9. At the end, several additional results are proved.

3.3.1 Finite dimensional approximation of Σ_m^a

Let us fix $b_0 \in \mathbb{R}_{>0}$ and prepare several notations related to the Riemannian metric g . We note that there is a compact subset of Q which contains the images of all paths $\gamma \in \Omega_K(Q)$ with $\text{length}\gamma \leq b_0$, since K is compact and g is complete. For any two points $q, q' \in Q$, let $d(q, q')$ be the distance between q and q' . Let us also fix $\varepsilon_g > 0$ such that if q and q' in this compact set satisfy $d(q, q') < \varepsilon_g$, then there exists a unique geodesic path on $[0, 1]$ of length $d(q, q')$ from q to q' . We write this geodesic by $\overline{qq'}: [0, 1] \rightarrow Q$.

For every $a \in [0, b_0)$ and $m \in \mathbb{Z}_{\geq 1}$, let $\bar{\Sigma}_m^a$ be the subspace of Σ_m^a which consists of $(\gamma_k)_{k=1, \dots, m}$ satisfying $\sum_{k=1}^m \text{length}\gamma_k < b_0$. From Proposition 2.12 about excision, the inclusion map $\iota: \bar{\Sigma}_m^b \rightarrow \Sigma_m^b$ induces an isomorphism

$$\iota_*: H_*^{\text{dR}}(\bar{\Sigma}_m^b, \bar{\Sigma}_m^a) \rightarrow H_*^{\text{dR}}(\Sigma_m^b, \Sigma_m^a) \quad (4)$$

for $a, b \in [0, b_0)$ with $a \leq b$. This means that we cut out a subset

$$\{(\gamma_k)_{k=1, \dots, m} \mid \min_{1 \leq k \leq m} \text{length}\gamma_k < \varepsilon_0 \text{ and } \sum_{k=1}^m \text{length}\gamma_k \geq b_0\}$$

to compute the homology group.

First, we approximate $\bar{\Sigma}_m^a$ for $a \in [0, b_0)$ by finite dimensional manifolds. For every $\nu \in \mathbb{Z}_{\geq 1}$, let us define

$$\bar{\Sigma}_m^a(\nu) := \{(\gamma_k: [0, T_k] \rightarrow Q)_{k=1, \dots, m} \in \bar{\Sigma}_m^a \mid \max_{1 \leq k \leq m} T_k < C_0^{-1} \varepsilon_g \nu\} \quad (5)$$

so that $\bigcup_{\nu=1}^{\infty} \bar{\Sigma}_m^a(\nu) = \bar{\Sigma}_m^a$. Let us also define $B_m(\nu)$ to be the submanifold of $(Q^{\times(\nu+1)})^{\times m}$ which consists of $(q_k^l)_{k=1, \dots, m}^{l=0, \dots, \nu}$ satisfying:

- $q_k^0, q_k^\nu \in K$ for every $k = 1, \dots, m$.
- $\sum_{k=1}^m \sum_{l=0}^{\nu-1} d(q_k^l, q_k^{l+1}) < b_0$.
- $d(q_k^l, q_k^{l+1}) < \varepsilon_g$ for every $k = 1, \dots, m$ and $l = 0, \dots, \nu - 1$.

We then define $B_m^a(\nu)$ for $a < b_0$ to be an open submanifold of $B_m(\nu)$ which consists of $(q_k^l)_{k=1, \dots, m}^{l=0, \dots, \nu}$ satisfying *either* of the following two conditions:

- $\sum_{k=1}^m \sum_{l=0}^{\nu-1} d(q_k^l, q_k^{l+1}) < a$.
- $\min_{1 \leq k \leq m} \left(\sum_{l=0}^{\nu-1} d(q_k^l, q_k^{l+1}) \right) < \varepsilon_0$.

The differentiable structures on this manifold is $P_{B_m^a(\nu)}$ defined in Example 2.3. For every $\nu \in \mathbb{Z}_{\geq 1}$, there are two maps: $\iota_{\Sigma, \nu}: \bar{\Sigma}_m^a(\nu) \rightarrow \bar{\Sigma}_m^a(2\nu)$ is just the inclusion map, and $\iota_{B, \nu}: B_m^a(\nu) \rightarrow B_m^a(2\nu)$ is an embedding of a manifold which maps $(q_k^l)_{k=1, \dots, m}^{l=0, \dots, \nu} \in B_m^a(\nu)$ to $(\bar{q}_k^{l'})_{k=1, \dots, m}^{l'=0, \dots, 2\nu} \in B_m^a(2\nu)$, where

$$\bar{q}_k^{l'} = \begin{cases} q_k^l & \text{if } l' \text{ is even and } l' = 2l, \\ q_k^l q_k^{l+1} \left(\frac{1}{2}\right) & \text{if } l' \text{ is odd and } l' = 2l + 1. \end{cases}$$

In addition, we define two maps

$$f_\nu: \bar{\Sigma}_m^a(\nu) \rightarrow B_m^a(\nu), \quad g_\nu: B_m^a(\nu) \rightarrow \bar{\Sigma}_m^a(2\nu),$$

as follows: f_ν maps $(\gamma_k: [0, T_k] \rightarrow Q)_{k=1, \dots, m} \in \bar{\Sigma}_m^a(\nu)$ to $(\gamma_k \left(\frac{l}{\nu} T_k\right))_{k=1, \dots, m}^{l=0, \dots, \nu} \in B_m^a(\nu)$. Note that for $l = 0, \dots, \nu - 1$ and $k = 1, \dots, m$,

$$d\left(\gamma_k \left(\frac{l}{\nu} T_k\right), \gamma_k \left(\frac{l+1}{\nu} T_k\right)\right) \leq \text{length } \gamma_k|_{[\frac{l}{\nu} T_k, \frac{l+1}{\nu} T_k]} \leq \frac{T_k}{\nu} \cdot \sup_{t \in [0, T_k]} |\gamma'(t)| < \varepsilon_g.$$

On the other hand, g_ν maps $(q_k^l)_{k=1, \dots, m}^{l=0, \dots, \nu} \in B_m^a(\nu)$ to

$$(\gamma_k: [0, \frac{3}{2} C_0^{-1} \varepsilon_g \nu] \rightarrow Q)_{k=1, \dots, m} \in \bar{\Sigma}_m^a(2\nu),$$

where, for $l = 0, \dots, \nu - 1$,

$$\gamma_k(t) := \overline{q_k^l q_k^{l+1}} \circ \chi \left(\frac{C_0}{\varepsilon_g} \cdot t - \frac{3}{2} l \right) \text{ if } \frac{3}{2} C_0^{-1} \varepsilon_g l \leq t \leq \frac{3}{2} C_0^{-1} \varepsilon_g (l + 1).$$

Here $\chi: [0, \frac{3}{2}] \rightarrow [0, 1]$ is a C^∞ function such that $\chi(t) = \begin{cases} 0 & \text{near } t = 0, \\ \frac{1}{2} & \text{near } t = \frac{3}{4}, \text{ and } 0 \leq \chi'(t) \leq 1 \\ 1 & \text{near } t = \frac{3}{2}, \end{cases}$

for any $t \in [0, \frac{3}{2}]$. Note that $|(\gamma_k)'(t)| \leq d(q_k^l, q_k^{l+1}) \cdot \sup_{t \in [0, 3/2]} |\chi'(t)| \cdot \frac{C_0}{\varepsilon_g} \leq C_0$. The next lemma shows that $B_m^a(\nu)$ approximates $\bar{\Sigma}_m^a(\nu)$ as $\nu \rightarrow \infty$. We refer to [23, Lemma 6.3].

Lemma 3.2. *The following diagram commutes up to homotopy:*

$$\begin{array}{ccc} \bar{\Sigma}_m^a(\nu) & \xrightarrow{\iota_{\Sigma, \nu}} & \bar{\Sigma}_m^a(2\nu) \\ f_\nu \downarrow & \nearrow g_\nu & \downarrow f_{2\nu} \\ B_m^a(\nu) & \xrightarrow{\iota_{B, \nu}} & B_m^a(2\nu) \end{array}$$

Proof. The lower right triangle commutes in the strict sense. For the upper left triangle, we need to show that $\iota_{\Sigma, \nu}$ is homotopic to $g_\nu \circ f_\nu$.

We abbreviate $C_0^{-1} \cdot \varepsilon_g$ by c_0 . For $(\gamma_k: [0, T_k] \rightarrow Q)_{k=1, \dots, m} \in \bar{\Sigma}_m^a(\nu)$, let us define a path $\gamma_k^s: [0, c_0\nu] \rightarrow Q$ for $k = 1, \dots, m$ and $s \in [0, 1]$ by

$$\gamma_k^s(t) := \begin{cases} \overline{\gamma_k(\frac{l}{\nu}T_k)\gamma_k(\frac{l+s}{\nu}T_k)}(t - c_0l) & \text{if } c_0l \leq t \leq c_0(l+s), \\ \gamma_k(\frac{T_k}{c_0\nu}(t - c_0l) + \frac{l}{\nu}T_k) & \text{if } c_0(l+s) \leq t \leq c_0(l+1), \end{cases} \quad (l = 0, \dots, \nu - 1).$$

Then, γ_k^0 is equal to $\gamma_k(\frac{T_k}{c_0\nu} \cdot)$ and γ_k^1 is a broken geodesic connecting $(\gamma_k(\frac{l}{\nu}T_k))^{l=0, \dots, \nu}$. We modify γ_k^s to a C^∞ path. For instance, we take a C^∞ function $\tilde{\chi}: [0, 1] \times [0, \frac{3}{2}c_0\nu] \rightarrow [0, c_0\nu]$ satisfying $0 \leq \frac{\partial}{\partial t}\tilde{\chi}(s, t) \leq 1$ and

$$\tilde{\chi}(s, t) = \begin{cases} c_0l & \text{on a neighborhood of } \{t = \frac{3}{2}c_0l\}, \\ \frac{2}{3}(t - \frac{1}{4}c_0) & \text{on a neighborhood of } \{t = s + \frac{3}{2}c_0l + \frac{1}{4}c_0\}. \end{cases}$$

Then, we define $\tilde{\gamma}_k^s := \gamma_k^s \circ \tilde{\chi}(s, \cdot): [0, \frac{3}{2}c_0\nu] \rightarrow Q$. If we take a C^∞ function $\kappa: \mathbb{R} \rightarrow [0, 1]$ such that $\kappa(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t \geq 1, \end{cases}$ then we get a smooth map

$$H: \mathbb{R} \times \bar{\Sigma}_m^a(\nu) \rightarrow \bar{\Sigma}_m^a(2\nu): (s, (\gamma_k)_{k=1, \dots, m}) \mapsto (\tilde{\gamma}_k^{\kappa(s)})_{k=1, \dots, m}.$$

This gives a homotopy from $H(0, \cdot)$ to $H(1, \cdot)$. Moreover, $\iota_{\Sigma, \nu}$ is homotopic to $H(0, \cdot)$ since the paths of $\iota_{\Sigma, \nu}((\gamma_k)_{k=1, \dots, m})$ and the paths of $H(0, (\gamma_k)_{k=1, \dots, m})$ differ only by parameterizations, so the homotopy can be constructed by interpolating these parameterizations. For the same reason, $g_\nu \circ f_\nu$ is homotopic to $H(1, \cdot)$. Therefore, $\iota_{\Sigma, \nu}$ is homotopic to $g_\nu \circ f_\nu$. \square

From Lemma 3.2, it follows that for any $a, b \in \mathbb{R}_{\geq 0}$ with $a \leq b < b_0$,

$$\varinjlim_{j \rightarrow \infty} (f_{2^j})_*: H_*^{\text{dR}}(\bar{\Sigma}_m^b, \bar{\Sigma}_m^a) = \varinjlim_{j \rightarrow \infty} H_*^{\text{dR}}(\bar{\Sigma}_m^b(2^j), \bar{\Sigma}_m^a(2^j)) \rightarrow \varinjlim_{j \rightarrow \infty} H_*^{\text{dR}}(B_m^b(2^j), B_m^a(2^j))$$

is an isomorphism. Combining with (4), we get an isomorphism

$$\left(\varinjlim_{j \rightarrow \infty} (f_{2^j})_* \right) \circ (\iota_*)^{-1}: H_*^{\text{dR}}(\Sigma_m^b, \Sigma_m^a) \rightarrow \varinjlim_{\nu \rightarrow \infty} H_*^{\text{dR}}(B_m^b(\nu), B_m^a(\nu)). \quad (6)$$

Furthermore, from Proposition 2.9, $H_*^{\text{sing}}(B_m^b(\nu), B_m^a(\nu)) \cong H_*^{\text{dR}}(B_m^b(\nu), B_m^a(\nu))$.

3.3.2 Computation of homology by Morse theory

Next, we examine the homology group $H_*^{\text{sing}}(B_m^b(\nu), B_m^a(\nu))$ in terms of Morse theory. Fix $m \in \mathbb{Z}_{\geq 1}$ and $\nu \in \mathbb{Z}_{\geq 1}$. For $k \in \{1, \dots, m\}$ and $l \in \{0, \dots, \nu - 1\}$, we set

$$h_k^l: B_m(\nu) \rightarrow \mathbb{R}: (q_k^l)_{k=1, \dots, m}^{l=0, \dots, \nu} \mapsto d(q_k^l, q_k^{l+1})^2.$$

For every $r > 0$, let us introduce the following:

- a C^∞ function $\sigma_r: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}: z \mapsto \sqrt{z+r}$.
- a C^∞ function

$$L_r: B_m(\nu) \rightarrow \mathbb{R}: \mathbf{q} \mapsto \sum_{k=1}^m \sum_{l=0}^{\nu-1} \sigma_r \circ h_k^l(\mathbf{q}).$$

- compact subsets of $B_m(\nu)$

$$Z_r := \{\mathbf{q} \in L_r^{-1}([0, b_0]) \mid \sigma_r \circ h_k^l(\mathbf{q}) \leq \varepsilon_g \text{ for every } k = 1, \dots, m \text{ and } l = 0, \dots, \nu - 1\},$$

$$Z_r^0 := \{\mathbf{q} \in Z_r \mid \min_{1 \leq k \leq m} \sum_{l=1}^{\nu-1} \sigma_r \circ h_k^l(\mathbf{q}) \leq \varepsilon_0\}.$$

The role of $\{\sigma_r\}_{r>0}$ is to approximate \sqrt{z} by C^∞ functions. We define for every $a \in [0, b_0)$ and $r > 0$

$$Z_r^a := (L_r|_{Z_r})^{-1}([0, a]) \cup Z_r^0.$$

Then, $Z_r^a \subset Z_{r'}^a$ holds if $0 < r' < r$. Furthermore, $\bigcup_{r>0} Z_r^a = B_m^a(\nu)$ holds. Therefore, we have an isomorphism induced by the inclusion maps

$$\lim_{r \rightarrow 0} H_*^{\text{sing}}(Z_r^b, Z_r^a) \rightarrow H_*^{\text{sing}}(B_m^b(\nu), B_m^a(\nu))$$

for $a, b \in \mathbb{R}_{\geq 0}$ with $a \leq b < b_0$. In order to apply Morse theory to L_r , we need to determine its critical points. The next lemma is fundamental. We omit the proof, but a similar result is proved, for instance in [25], when there is no boundary condition.

Lemma 3.3. *For $k_0 \in \{1, \dots, m\}$ and $l_0 \in \{0, \dots, \nu - 1\}$, let $X_{k_0}^{l_0}$ be the gradient vector field of $h_{k_0}^{l_0}$. We define π^{l_0} to be the orthogonal projection from $TQ|_K$ to TK if $l_0 = 0$ or ν , and otherwise $\pi^{l_0} := \text{id}_{TQ}$. Then, each component of $X_{k_0}^{l_0} = (v_k^l)_{k=1, \dots, m}^{l=0, \dots, \nu}$ is determined by the following: If $(k, l) \neq (k_0, l_0), (k_0, l_0 + 1)$, then $v_k^l = 0$. Otherwise,*

$$v_{k_0}^{l_0} = \pi^{l_0} \left(- \frac{d}{dt} \Big|_{t=0} \overline{q_{k_0}^{l_0} q_{k_0}^{l_0+1}} \right), \quad v_{k_0}^{l_0+1} = \pi^{l_0+1} \left(\frac{d}{dt} \Big|_{t=1} \overline{q_{k_0}^{l_0} q_{k_0}^{l_0+1}} \right).$$

Proposition 3.4. *Suppose that $a \in [0, b_0) \setminus \mathcal{L}_m(K)$ and $r > 0$ is sufficiently small. Then $\mathbf{q} \in Z_r \setminus Z_r^0$ is a critical point of L_r with its value in $[0, a]$ if and only if there exist $\gamma_1, \dots, \gamma_m \in \mathcal{C}(K)$ such that $(\gamma_k)_{k=1, \dots, m} \in \bar{\Sigma}_m^a(\nu)$ and $\mathbf{q} = f_\nu((\gamma_k)_{k=1, \dots, m})$.*

Proof. From Lemma 3.3, $\mathbf{q} = (q_k^l)_{k=1, \dots, m}^{l=0, \dots, \nu} \in B_m(\nu)$ is a critical point of L_r if and only if the following conditions hold for every $k \in \{1, \dots, m\}$:

$$\begin{aligned} (\sigma_r)'(h_k^0(\mathbf{q})) \cdot \frac{d}{dt} \Big|_{t=0} \overline{q_k^0 q_k^1} &\in (T_{q_k^0} K)^\perp \text{ and } (\sigma_r)'(h_k^{\nu-1}(\mathbf{q})) \cdot \frac{d}{dt} \Big|_{t=1} \overline{q_k^{\nu-1} q_k^\nu} \in (T_{q_k^\nu} K)^\perp, \\ (\sigma_r)'(h_k^{l-1}(\mathbf{q})) \cdot \frac{d}{dt} \Big|_{t=1} \overline{q_k^{l-1} q_k^l} &= (\sigma_r)'(h_k^l(\mathbf{q})) \cdot \frac{d}{dt} \Big|_{t=0} \overline{q_k^l q_k^{l+1}} \text{ for every } l \in \{1, \dots, \nu - 1\}. \end{aligned} \tag{7}$$

Comparing the norms of both sides in the second line of (7), we have for $l \in \{1, \dots, \nu - 1\}$

$$(\sigma_r)'(h_k^{l-1}(\mathbf{q})) \cdot \sqrt{h_k^{l-1}(\mathbf{q})} = (\sigma_r)'(h_k^l(\mathbf{q})) \cdot \sqrt{h_k^l(\mathbf{q})}.$$

Since $(\sigma_r)'(z)\sqrt{z}$ is a strictly increasing function of z , this equation means that $d(q_k^{l-1}, q_k^l) = d(q_k^l, q_k^{l+1})$ for $l \in \{1, \dots, \nu - 1\}$. Since $(\sigma_r)' > 0$, the conditions of (7) are equivalent to the

following:

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \overline{q_k^0 q_k^1} &\in (T_{q_k^0} K)^\perp \text{ and } \frac{d}{dt} \Big|_{t=1} \overline{q_k^{\nu-1} q_k^\nu} \in (T_{q_k^\nu} K)^\perp, \\ \frac{d}{dt} \Big|_{t=1} \overline{q_k^{l-1} q_k^l} &= \frac{d}{dt} \Big|_{t=0} \overline{q_k^l q_k^{l+1}} \text{ for every } l \in \{1, \dots, \nu-1\}. \end{aligned} \quad (8)$$

For every $\mathbf{q} \in B_m(\nu)$, let us define $T_{k,\mathbf{q}} := \sum_{l=0}^{\nu-1} d(q_k^l, q_k^{l+1})$ and a path $\gamma_{k,\mathbf{q}}: [0, T_{k,\mathbf{q}}] \rightarrow Q$ determined by

$$\gamma_{k,\mathbf{q}}(T_{k,\mathbf{q}}t) := \overline{q_k^l q_k^{l+1}}(\nu t - l) \text{ if } \frac{l}{\nu} \leq t \leq \frac{l+1}{\nu} \text{ for } l \in \{0, \dots, \nu-1\}.$$

Note that if $(\gamma_{k,\mathbf{q}})_{k=1,\dots,m} \in \bar{\Sigma}_m^a(\nu)$, then $\mathbf{q} = f_\nu((\gamma_{k,\mathbf{q}})_{k=1,\dots,m})$ holds. We take $r > 0$ so small that $T_{k,\mathbf{q}} > 0$ holds for every $k = 1, \dots, m$ and $\mathbf{q} \in Z_r \setminus Z_r^0$. Then, for $\mathbf{q} \in Z_r \setminus Z_r^0$, the condition (8) is equivalent to that $\gamma_{k,\mathbf{q}}$ is a binormal chord of K in Q , that is, $\gamma_{k,\mathbf{q}} \in \mathcal{C}(K)$. In addition,

$$\lim_{r \rightarrow 0} L_r(\mathbf{q}) = \sum_{k=1}^m \text{length} \gamma_{k,\mathbf{q}}$$

for every $\mathbf{q} \in Z_r$ uniformly. Recall that $\mathcal{L}_m(K)$ is a closed subset of $\mathbb{R}_{>0}$. Assuming that $a \notin \mathcal{L}_m(K)$ and $r > 0$ is sufficiently small, it follows that a critical point $\mathbf{q} \in Z_r \setminus Z_r^0$ of L_r satisfies $L_r(\mathbf{q}) \leq a$ if and only if the binormal chords $(\gamma_{k,\mathbf{q}})_{k=1,\dots,m}$ satisfy $\sum_{k=1}^m \text{length} \gamma_{k,\mathbf{q}} < a$. This proves the proposition. \square

Let Y_r be the gradient vector field of L_r . To prove the next lemma, which is rather technical, let us prepare a few computations. $X_{k,r}^l := ((\sigma_r)' \circ h_k^l) \cdot X_k^l$ is the gradient vector field of $\sigma_r \circ h_k^l$. For every $k \in \{1, \dots, m\}$, $l \in \{1, \dots, \nu-2\}$ and $\mathbf{q} \in B_m(\nu)$, we have

$$\begin{aligned} \langle Y_r, X_k^l \rangle(\mathbf{q}) &= \left\langle (\sigma_r)'(h_k^{l-1}(\mathbf{q})) \cdot \frac{d}{dt} \Big|_{t=1} \overline{q_k^{l-1} q_k^l}, - \frac{d}{dt} \Big|_{t=0} \overline{q_k^l q_k^{l+1}} \right\rangle \\ &\quad + (\sigma_r)'(h_k^l(\mathbf{q})) \cdot \left| \frac{d}{dt} \Big|_{t=0} \overline{q_k^l q_k^{l+1}} \right|^2 + (\sigma_r)'(h_k^l(\mathbf{q})) \cdot \left| \frac{d}{dt} \Big|_{t=1} \overline{q_k^l q_k^{l+1}} \right|^2 \\ &\quad + \left\langle -(\sigma_r)'(h_k^{l+1}(\mathbf{q})) \cdot \frac{d}{dt} \Big|_{t=0} \overline{q_k^{l+1} q_k^{l+2}}, \frac{d}{dt} \Big|_{t=1} \overline{q_k^l q_k^{l+1}} \right\rangle. \end{aligned}$$

We abbreviate the increasing function $(\sigma_r)' \sqrt{z}$ by τ_r . Then, by Cauchy-Schwarz inequality,

$$\begin{aligned} \langle Y_r, X_{k,r}^l \rangle(\mathbf{q}) &\geq -\tau_r(h_k^{l-1}(\mathbf{q})) \cdot \tau_r(h_k^l(\mathbf{q})) + 2(\tau_r(h_k^l(\mathbf{q})))^2 - \tau_r(h_k^{l+1}(\mathbf{q})) \cdot \tau_r(h_k^l(\mathbf{q})) \\ &= (2\tau_r(h_k^l(\mathbf{q})) - \tau_r(h_k^{l-1}(\mathbf{q})) - \tau_r(h_k^{l+1}(\mathbf{q}))) \cdot \tau_r(h_k^l(\mathbf{q})). \end{aligned}$$

Since σ_r and τ_r are increasing functions, we have for $k_0 \in \{1, \dots, m\}$ and $l_0 \in \{1, \dots, \nu-2\}$,

$$\sigma_r \circ h_{k_0}^{l_0}(\mathbf{q}) = \max_{k,l} \sigma_r \circ h_k^l(\mathbf{q}) \Rightarrow \langle -Y_r, X_{k_0,r}^{l_0} \rangle(\mathbf{q}) \leq 0. \quad (9)$$

The same result holds when $l_0 = 0$ or $\nu-1$. We also note that for every $k_0 \in \{1, \dots, m\}$,

$$\langle -Y_r, \sum_{l=1}^{\nu-1} X_{k_0,r}^l \rangle(\mathbf{q}) \leq 0. \quad (10)$$

Lemma 3.5. *The trajectory of any point in Z_r (resp. Z_r^0) along $-Y_r$ never goes outside Z_r (resp. Z_r^0) at positive time.*

Proof. Suppose that $\Gamma: [0, T] \rightarrow B_m(\nu)$ is a trajectory along $-Y_r$. Let us consider two continuous functions $f, g: [0, T] \rightarrow \mathbb{R}$ defined by

$$f(t) := \max_{k,l} \sigma_r \circ h_k^l(\Gamma(t)), \quad g(t) := \min_k \sum_{l=1}^{\nu-1} \sigma_r \circ h_k^l(\Gamma(t)).$$

To prove this lemma, it suffices to show that they are decreasing functions. Indeed, there exists a discrete subset $A \subset [0, T]$ such that f and g are differentiable at every $t \in [0, T] \setminus A$. (9) and (10) imply that $f'(t) \leq 0$ and $g'(t) \leq 0$ for every $t \in [0, T] \setminus A$. Hence, f and g are decreasing on $[0, T]$. \square

We apply a general result from Morse theory.

Lemma 3.6. *Let B be a manifold and $L: B \rightarrow \mathbb{R}$ be a C^∞ function. For $a, b \in \mathbb{R}$ with $a \leq b$ and two compact subsets $Z, Z^0 \subset B$, suppose that there is no critical point of L in $(L|_Z)^{-1}([a, b]) \setminus Z^0$ and that the trajectory of any point in Z (resp. Z^0) along the negative gradient vector field of L never goes outside Z (resp. Z^0) at positive times. Let us define $Z^{a'} := (L|_Z)^{-1}((-\infty, a')) \cup Z^0$ for $a' \in \{a, b\}$. Then $H_*^{\text{sing}}(Z^b, Z^a) = 0$ holds.*

Proof. The conditions on a, b, Z and Z^0 show that Z^b can be deformed into Z^a along the negative gradient flow of L . Therefore, we get a map from (Z^b, Z^a) to (Z^a, Z^a) which gives the inverse map of the inclusion map up to homotopy. \square

Combining the above results, we prove the first main proposition in this section.

Proposition 3.7. *If $\mathcal{L}_m(K) \cap [a, b] = \emptyset$, then $H_*^{\text{dR}}(\Sigma_m^b, \Sigma_m^a) = 0$.*

Proof. From Proposition 3.4 and Lemma 3.5, we can apply Lemma 3.6 to show that if $\mathcal{L}_m(K) \cap [a, b] = \emptyset$ and $r > 0$ is sufficiently small, then $H_*^{\text{sing}}(Z_r^b, Z_r^a) = 0$ holds, and thus

$$H_*^{\text{sing}}(B_m^b(\nu), B_m^a(\nu)) \cong \varinjlim_{r \rightarrow 0} H_*^{\text{sing}}(Z_r^b, Z_r^a) = 0.$$

From (6), it follows that $H_*^{\text{dR}}(\Sigma_m^b, \Sigma_m^a) \cong \varinjlim_{j \rightarrow \infty} H_*^{\text{sing}}(B_m^b(2^j), B_m^a(2^j)) = 0$. \square

3.3.3 $H_*^{\text{dR}}(S_\varepsilon)$ and the evaluation map

Next, we examine $H_*^{\text{dR}}(S_\varepsilon)$ for $\varepsilon \in (0, \varepsilon_0]$. Choose a Riemannian metric g' on N_{ε_0} for which K is a totally geodesic submanifold of N_{ε_0} . Then, we can take a constant $C_1 \geq 1$ such that for any $q, q' \in N_{\varepsilon_0/C_1}$ with $d(q, q') < \varepsilon_0/C_1$, there exists a unique shortest geodesic in N_{ε_0} with respect to g' from q to q' . Let us write this geodesic by $\widetilde{qq'}: [0, 1] \rightarrow N_{\varepsilon_0}$. In this subsection, \exp denotes the exponential map with respect to g .

There is a smooth map between differentiable spaces $s_\varepsilon: N_\varepsilon^{\text{reg}} \rightarrow S_\varepsilon$ which maps $\exp_x v \in N_\varepsilon$ ($x \in K, v \in (T_x K)^\perp$ with $|v| < \varepsilon/2$) to

$$s_\varepsilon(\exp_x v) := (\sigma_i^v: [0, \varepsilon/2] \rightarrow N_\varepsilon)_{i=1,2},$$

where $\sigma_1^v(t) = \exp_x(\frac{\varepsilon-2t}{\varepsilon}v)$ and $\sigma_2^v(t) = \exp_x(\frac{2t}{\varepsilon}v)$ for $t \in [0, \varepsilon/2]$. This satisfies $\text{ev}_0 \circ s_\varepsilon = \text{id}_{N_\varepsilon}$. For $\varepsilon, \bar{\varepsilon} \in (0, \varepsilon_0]$ with $\varepsilon \leq \bar{\varepsilon}$, let $i_{\varepsilon, \bar{\varepsilon}}: S_\varepsilon \rightarrow S_{\bar{\varepsilon}}$ denote the inclusion map.

Lemma 3.8. *There exists a constant $C \geq C_1$ such that for any $\varepsilon \in (0, \varepsilon_0/C]$, the inclusion map $i_{\varepsilon, C\varepsilon}: S_\varepsilon \rightarrow S_{C\varepsilon}$ is homotopic to $i_{\varepsilon, C\varepsilon} \circ s_\varepsilon \circ \text{ev}_0$.*

Proof. We define a C^∞ map

$$G: \{(q, q') \in N_{\varepsilon_0/C_1} \times N_{\varepsilon_0/C_1} \mid d(q, q') < \varepsilon_0/C_1\} \times [0, 1] \rightarrow N_{\varepsilon_0}: ((q, q'), s) \mapsto \widetilde{qq'}(s).$$

Then there is a constant $C \geq C_1$ so that

$$|d(G(\cdot, s))_{(q, q')}(v, v')|_g \leq \frac{C}{2} \cdot (|v|_g + |v'|_g)$$

for every $s \in [0, 1]$ and $(v, v') \in T_q N_{\varepsilon_0/C} \times T_{q'} N_{\varepsilon_0/C}$ with $d(q, q') < \varepsilon_0/C_1$. For any $\varepsilon \in (0, \varepsilon_0/C]$ and $(\sigma_i: [0, \varepsilon_i] \rightarrow N_\varepsilon)_{i=1,2} \in S_\varepsilon$, we set $\bar{\varepsilon} := C\varepsilon$ and define $(\sigma_i^{(s)}: [0, \varepsilon_i^s] \rightarrow N_{\bar{\varepsilon}})_{i=1,2} \in S_{\bar{\varepsilon}}$ for $s \in \mathbb{R}$ as follows: Take $x \in K$ and $v \in T_x K$ such that $\sigma_1(0) = \exp_x v$ and $|v| < \varepsilon/2$. Then we define

$$\varepsilon_i^s := \begin{cases} (1 - \kappa(s))\varepsilon_i + \kappa(s)\bar{\varepsilon} & \text{if } s \leq \frac{1}{3}, \\ \bar{\varepsilon} & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3}, \\ (1 - \kappa(s - \frac{2}{3}))\bar{\varepsilon} + \kappa(s - \frac{2}{3})\varepsilon & \text{if } s \geq \frac{2}{3}, \end{cases}$$

$$\sigma_i^{(s)}(t) := \begin{cases} \sigma_i(\varepsilon_i t / \varepsilon_i^s) & \text{if } s \leq \frac{1}{3}, \\ G(\sigma_i(\varepsilon_i t / \bar{\varepsilon}), \sigma_i^v(t/C), \kappa(s - \frac{1}{3})) & \text{if } \frac{1}{3} \leq s \leq \frac{2}{3}, \\ \sigma_i^v(\varepsilon t / \varepsilon_i^s) & \text{if } \frac{2}{3} \leq s. \end{cases}$$

Here, $\kappa: \mathbb{R} \rightarrow [0, 1]$ is a C^∞ function such that $\kappa(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ 1 & \text{if } s \geq \frac{1}{3}. \end{cases}$ Note that when $\frac{1}{3} \leq s \leq \frac{2}{3}$,

$$|(\sigma_i^{(s)})'(t)|_g \leq \frac{C}{2} \cdot \left(\frac{\varepsilon_i}{\bar{\varepsilon}} \sup |\sigma_i'|_g + \frac{1}{C} \sup |(\sigma_i^v)'|_g \right) \leq \frac{C}{2} \cdot \left(\frac{\varepsilon_i}{\bar{\varepsilon}} + \frac{1}{C} \right) \leq 1.$$

Now the homotopy from $i_{\varepsilon, \bar{\varepsilon}}$ to $i_{\varepsilon, \bar{\varepsilon}} \circ s_\varepsilon \circ \text{ev}_0$ is given by the map

$$\mathbb{R} \times S_\varepsilon \rightarrow S_{\bar{\varepsilon}}: (s, (\sigma_i)_{i=1,2}) \rightarrow (\sigma_i^{(s)}: [0, \varepsilon_i^s] \rightarrow N_{\bar{\varepsilon}})_{i=1,2}.$$

□

Proposition 3.9. *Let C be the constant of Lemma 3.8. For any $\varepsilon \in (0, \varepsilon_0/C]$ and $x \in H_*^{\text{dR}}(S_\varepsilon)$,*

$$(\text{ev}_0)_*(x) = 0 \in H_*^{\text{dR}}(N_\varepsilon) \Rightarrow (i_{\varepsilon, C\varepsilon})_*(x) = 0 \in H_*^{\text{dR}}(S_{C\varepsilon}).$$

Proof. By Lemma 3.8, $(i_{\varepsilon, C\varepsilon})_*(x) = (i_{\varepsilon, C\varepsilon} \circ s_\varepsilon)_*((\text{ev}_0)_*(x)) = 0$ if $(\text{ev}_0)_*(x) = 0 \in H_*^{\text{dR}}(N_\varepsilon)$. □

3.3.4 Additional results

Using the computations obtained in the former subsections, we prove several additional results. b_0 is a fixed real number as before. We also fix $a \in [0, b_0)$ which may belong to $\mathcal{L}(K)$.

Lemma 3.10. *There exists a manifold Z and a C^∞ function $f: Z \rightarrow \mathbb{R}$ such that $\mathcal{L}_1(K) \cap [0, a)$ is contained in the set of critical values of f .*

Proof. We use the notations in the proof of Proposition 3.4 for $m = 1$. For every critical point $\mathbf{q} \in Z_r \setminus Z_r^0$ of L_r , $\text{length} \gamma_{1, \mathbf{q}} = f_r(L_r(\mathbf{q})) \in (0, f_r(b_0))$, where f_r is determined by $f_r(\nu \sigma_r(l)) = \nu \sqrt{l}$ for every $l \in \mathbb{R}_{>0}$. We choose $r > 0$ so that $a < f_r(b_0)$. Then, from the correspondence between binormal chords and critical points of L_r , $\mathcal{L}_m(K) \cap [0, a)$ is contained in the critical value of $f := f_r \circ L_r$ on $Z := Z_r \setminus Z_r^0$. □

This proves that $\mathcal{L}(K) = \bigcup_{m=1}^\infty \mathcal{L}_m(K)$ is a null set. Indeed, for any $m \in \mathbb{Z}$, $\mathcal{L}_m(K) \cap [0, a)$ is contained in the set of critical values of the C^∞ function $Z^{\times m} \rightarrow \mathbb{R}: (x_1, \dots, x_m) \mapsto f(x_1) + \dots + f(x_m)$. By Sard's theorem, it is a null set in \mathbb{R} . Since b_0 was chosen arbitrarily, it follows that $\mathcal{L}_m(K)$ is a null set.

Next, recall that the definition of Σ_m^a depends on auxiliary data C_0 and ε_0 .

Lemma 3.11. *$H_*^{\text{dR}}(\Sigma_m^a)$ does not depend on the choice of C_0 and ε_0 . More precisely, the following hold:*

- If we write Σ_m^a by Σ_{m,C_0}^a to clarify the dependence on C_0 , the inclusion map $\Sigma_{m,C_0}^a \rightarrow \Sigma_{m,C'_0}^a$ for $C_0 \leq C'_0$ induces an isomorphism on homology.
- If we write Σ_m^a by $\Sigma_{m,\varepsilon_0}^a$ to clarify the dependence on ε_0 , the inclusion map $\Sigma_{m,\varepsilon'_0}^a \rightarrow \Sigma_{m,\varepsilon_0}^a$ for $\varepsilon'_0 < \varepsilon_0$ induces an isomorphism on homology.

Proof. We define a smooth map $\Sigma_{m,C'_0}^a \rightarrow \Sigma_{m,C_0}^a$ which maps $(\gamma_k: [0, T_k] \rightarrow Q)_{k=1,\dots,m}$ to

$$([0, C'_0 T_k / C_0] \rightarrow Q: t \mapsto \gamma_k(C_0 t / C'_0))_{k=1,\dots,m}.$$

This gives the inverse map of the inclusion map up to homotopy. This proves the assertion for C_0 .

To prove the assertion for ε_0 , let us write Z_r^a by Z_{r,ε_0}^a . Then, there is no critical point of $\sum_{l=0}^{\nu-1} \sigma_r \circ h_k^l$ in $Z_{r,\varepsilon_0}^a \setminus Z_{r,\varepsilon'_0}^a$ for every $k \in \{1, \dots, m\}$. By deforming along the negative gradient vector field $-\sum_{l=0}^{\nu-1} X_{k,r}^l$ inductively on $k = 1, 2, \dots, m$, we can see that Z_{r,ε'_0}^a is a deformation retract of Z_{r,ε_0}^a . This implies that $H_*^{\text{sing}}(Z_{r,\varepsilon_0}^a, Z_{r,\varepsilon'_0}^a) = 0$ and thus $H_*^{\text{dR}}(\Sigma_{m,\varepsilon_0}^a, \Sigma_{m,\varepsilon'_0}^a) = 0$. \square

For the sake of discussions in Section 6, let us fix a topology on the set Σ_m^a for $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ as follows: $\Omega_K(Q)$ becomes a topological space so that the injection

$$\Omega_K(Q) \rightarrow C^\infty([0, 1], Q) \times \mathbb{R}_{>0}: (\gamma: [0, T] \rightarrow Q) \mapsto (\gamma(T^{-1}\cdot), T)$$

is a homeomorphism onto its image when $C^\infty([0, 1], Q)$ is equipped with C^∞ -topology. We give Σ_m^a the restricted topology from $\Omega_K(Q)^{\times m}$. Then, we can consider singular homology groups, such as $H_*^{\text{sing}}(\Sigma_m^b, \Sigma_m^a)$ for $a \leq b$.

Suppose that $a, b \in \mathbb{R}_{\geq 0}$ and $a \leq b < b_0$. By excision theorem, $\iota_*: H_*^{\text{sing}}(\bar{\Sigma}_m^b, \bar{\Sigma}_m^a) \rightarrow H_*^{\text{sing}}(\Sigma_m^b, \Sigma_m^a)$ is an isomorphism. All maps in Lemma 3.2 are continuous and the diagram commutes up to continuous homotopy. Therefore, we have an isomorphism

$$\left(\varinjlim_{j \rightarrow \infty} (f_{2^j})_*\right) \circ (\iota_*)^{-1}: H_*^{\text{sing}}(\Sigma_m^b, \Sigma_m^a) \rightarrow \varinjlim_{j \rightarrow \infty} H_*^{\text{sing}}(B_m^b(2^j), B_m^a(2^j)). \quad (11)$$

3.4 Splitting and concatenating paths

For $\varepsilon \in (0, \varepsilon_0 / (5C_0)]$, we define an open subset of \mathbb{R}^2

$$A_\varepsilon := \{(T, \tau) \mid T > 4\varepsilon \text{ and } 2\varepsilon < \tau < T - 2\varepsilon\}.$$

This becomes a differentiable space as a subspace of $(\mathbb{R}^2)^{\text{reg}}$. For $a \in \mathbb{R}_{\geq 0}$, $m \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \dots, m\}$, define smooth maps

$$\begin{aligned} \text{tl}_k: \Sigma_m^a &\rightarrow \mathbb{R}: (\gamma_l: [0, T_l] \rightarrow Q)_{l=1,\dots,m} \mapsto T_k, \\ \text{pr}_T: A_\varepsilon &\rightarrow \mathbb{R}^{\text{reg}}: (T, \tau) \mapsto T. \end{aligned}$$

(tl stands for the time length.) These maps define a fiber product $\Sigma_m^a \times_{\text{pr}_T} A_\varepsilon$ over \mathbb{R} , and the k -th evaluation map ev_k is defined on it by

$$\text{ev}_k: \Sigma_m^a \times_{\text{pr}_T} A_\varepsilon \rightarrow Q: ((\gamma_l)_{l=1,\dots,m}, (T, \tau)) \mapsto \gamma_k(\tau).$$

From ev_k and $\text{ev}_0: S_\varepsilon \rightarrow Q^{\text{reg}}$, we obtain a fiber product over Q . We define a map on this fiber product

$$\text{con}_k: (\Sigma_m^a \times_{\text{pr}_T} A_\varepsilon) \times_{\text{ev}_k \times \text{ev}_0} S_\varepsilon \rightarrow \Sigma_{m+1}^{a+\varepsilon},$$

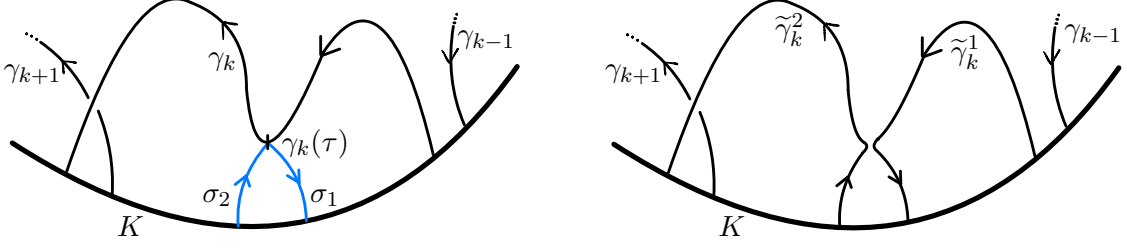


Figure. 1: The process to define $\tilde{\gamma}_k^1$ and $\tilde{\gamma}_k^2$.

which maps $((\gamma_l)_{l=1,\dots,m}, (T, \tau), (\sigma_i: [0, \varepsilon_i] \rightarrow N_\varepsilon)_{i=1,2})$ to $(\gamma_1, \dots, \gamma_{k-1}, \tilde{\gamma}_k^1, \tilde{\gamma}_k^2, \gamma_{k+1}, \dots, \gamma_m)$, where $\tilde{\gamma}_k^i$ ($i = 1, 2$) are the following paths:

$$\tilde{\gamma}_k^1: [0, \tau + 2\varepsilon_1] \rightarrow Q: t \mapsto \begin{cases} \gamma_k(t) & \text{if } 0 \leq t \leq \tau - \varepsilon_1, \\ \gamma_k(\tau - \varepsilon_1 + \varepsilon_1 \mu(\frac{t - \tau + \varepsilon_1}{\varepsilon_1})) & \text{if } \tau - \varepsilon_1 \leq t \leq \tau + \frac{\varepsilon_1}{2}, \\ \sigma_1(\varepsilon_1 - \varepsilon_1 \mu(\frac{\tau + 2\varepsilon_1 - t}{\varepsilon_1})) & \text{if } \tau + \frac{\varepsilon_1}{2} \leq t \leq \tau + 2\varepsilon_1, \end{cases} \quad (12)$$

$$\tilde{\gamma}_k^2: [0, T - \tau + 2\varepsilon_2] \rightarrow Q: t \mapsto \begin{cases} \sigma_2(\varepsilon_2 \mu(\frac{t}{\varepsilon_2})) & \text{if } 0 \leq t \leq \frac{3}{2}\varepsilon_2, \\ \gamma_k(\tau + \varepsilon_2 - \varepsilon_2 \mu(\frac{3\varepsilon_2 - t}{\varepsilon_2})) & \text{if } \frac{3}{2}\varepsilon_2 \leq t \leq 3\varepsilon_2, \\ \gamma_k(t + \tau - 2\varepsilon_2) & \text{if } 3\varepsilon_2 \leq t \leq T - \tau + 2\varepsilon_2. \end{cases}$$

Here, $\mu: [0, \frac{3}{2}] \rightarrow [0, 1]$ is one of fixed data we have chosen in Subection 3.1.

This definition can be explained as follows (See Figure 1.): We split the k -th path $\gamma_k: [0, T] \rightarrow Q$ at $\tau \in (2\varepsilon, T - 2\varepsilon)$ where $\gamma_k(\tau) = \sigma_1(0) (= \sigma_2(\varepsilon_2)) \in N_\varepsilon$, and then concatenate $\gamma_k|_{[0, \tau]}$ (resp. $\gamma_k|_{[\tau, T]}$) with σ_1 (resp. σ_2). The reparameterizations via μ is necessary in order to modify them to C^∞ paths. Note that

$$\text{length}\tilde{\gamma}_k^1 + \text{length}\tilde{\gamma}_k^2 = \text{length}\gamma_k + \text{length}\sigma_1 + \text{length}\sigma_2 < \text{length}\gamma_k + \varepsilon.$$

The following lemma shows the cases where an element in the fiber product is mapped by con_k into Σ_{m+1}^0 .

Lemma 3.12. *For $((\gamma_l)_{l=1,\dots,m}, (T, \tau), (\sigma_i)_{i=1,2}) \in (\Sigma_m^a \text{tl}_k \times_{\text{pr}_T} A_\varepsilon)_{\text{ev}_k \times \text{ev}_0} S_\varepsilon$, we have*

$$\text{con}_k((\gamma_l)_{l=1,\dots,m}, (T, \tau), (\sigma_i)_{i=1,2}) \in \Sigma_{m+1}^0,$$

if either of the following three condition holds:

- (i) $(\gamma_l)_{l=1,\dots,m} \in \Sigma_m^0$.
- (ii) $\tau < 4\varepsilon_0/(5C_0)$ or $T - 4\varepsilon_0/(5C_0) < \tau$.
- (iii) γ_k satisfies for every $\tau' \in (\gamma_k)^{-1}(N_\varepsilon)$ that either $\gamma_k|_{[0, \tau']}$ or $\gamma_k|_{[\tau', T]}$ has length less than $4\varepsilon_0/5$.

Proof. As in the definition of con_k , let us write

$$\text{con}_k((\gamma_l)_{l=1,\dots,m}, (T, \tau), (\sigma_i)_{i=1,2}) = (\gamma_1, \dots, \tilde{\gamma}_k^1, \tilde{\gamma}_k^2, \dots, \gamma_m).$$

Under the condition (i), $\text{length}\gamma_l < \varepsilon_0$ for some $l \in \{1, \dots, m\}$. If $l \neq k$, the assertion is trivial. If $l = k$, either $\gamma_k|_{[0, \tau]}$ or $\gamma_k|_{[\tau, T]}$ has length less than $\varepsilon_0/2$, and thus either $\tilde{\gamma}_k^1$ or $\tilde{\gamma}_k^2$ has length less than

$$\varepsilon_0/2 + \max\{\text{length}\sigma_i \mid i = 1, 2\},$$

which is smaller than ε_0 . Under the condition (ii), if $\tau < 4\varepsilon_0/(5C_0)$ (resp. $T - 4\varepsilon_0/(5C_0) < \tau$) holds, then $\text{length}\gamma_k|_{[0, \tau]} < 4\varepsilon_0/5$ (resp. $\text{length}\gamma_k|_{[\tau, T]} < 4\varepsilon_0/5$). Therefore, either $\tilde{\gamma}_k^1$ or $\tilde{\gamma}_k^2$ has length smaller than ε_0 . Under the condition (iii), we can also show that either $\tilde{\gamma}_k^1$ or $\tilde{\gamma}_k^2$ has length smaller than ε_0 . \square

3.5 Operation on de Rham chains

For $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, let us choose a C^∞ cutoff function $\rho_\varepsilon: A_\varepsilon \rightarrow [0, 1]$ such that

$$\rho_\varepsilon(T, \tau) = \begin{cases} 0 & \text{if } t \leq \frac{10}{3}\varepsilon \text{ or } T - \frac{10}{3}\varepsilon \leq \tau, \\ 1 & \text{if } \frac{11}{3}\varepsilon \leq \tau \leq T - \frac{11}{3}\varepsilon. \end{cases}$$

In particular, $\rho_\varepsilon(T, \tau) = 0$ if $T \leq 5\varepsilon$. We also choose a C^∞ function $\chi_\nu: A_\varepsilon \rightarrow [0, 1]$ for every $\nu \in \mathbb{Z}_{\geq 1}$ such that $\chi_\nu(T, \tau) = \begin{cases} 1 & \text{if } T \leq \nu, \\ 0 & \text{if } T \geq \nu + 1. \end{cases}$ The support of $\chi_\nu \rho_\varepsilon$ is compact, so we obtain a de Rham chain

$$\alpha_{\varepsilon, \nu} := [A_\varepsilon, \text{id}_{A_\varepsilon}, \chi_\nu \rho_\varepsilon] \in C_2^{\text{dR}}(A_\varepsilon).$$

In addition, we define $\Sigma_m^a(\nu) := \bigcap_{k=1}^m (\text{tl}_k)^{-1}([0, \nu])$, which is a subspace of Σ_m^a .

For $m \in \mathbb{Z}_{\geq 1}$, $k \in \{1, \dots, m\}$ and $\xi \in C_q^{\text{dR}}(S_\varepsilon)$, we define a linear map

$$f_{k, \xi}: C_*^{\text{dR}}(\Sigma_m^a) \rightarrow C_{*+1+q-n}^{\text{dR}}(\Sigma_{m+1}^{a+\varepsilon})$$

so that for every $\nu \in \mathbb{Z}_{\geq 1}$ and $x \in C_*^{\text{dR}}(\Sigma_m^a(\nu)) \subset C_*^{\text{dR}}(\Sigma_m^a)$,

$$f_{k, \xi}(x) = (\text{con}_k)_* ((x \text{tl}_k \times_{\text{pr}_T} \alpha_{\varepsilon, \nu}) \text{ev}_k \times_{\text{ev}_0} \xi).$$

This map is well-defined since $x \text{tl}_k \times_{\text{pr}_T} (\alpha_{\varepsilon, \nu} - \alpha_{\varepsilon, \nu'}) = 0$ when $x \in C_*^{\text{dR}}(\Sigma_m^a(\nu))$ and $\nu' \geq \nu$.

Returning to Definition 2.8, we can describe the de Rham chain $f_{k, \xi}(x)$ explicitly. For $x = [U, \varphi, \omega] \in C_p^{\text{dR}}(\Sigma_m^a)$, we write $\varphi(u) = (\gamma_l^u: [0, T_l^u] \rightarrow Q)_{l=1, \dots, m}$ for $u \in U$. If we take $\nu > \sup_{u \in \text{supp } \omega} T_k^u$, then $x \in C_p^{\text{dR}}(\Sigma_m^a(\nu))$. First, we have $x \text{tl}_k \times_{\text{pr}_T} \alpha_{\varepsilon, \nu} = [\tilde{U}_k, \tilde{\varphi}_k, \tilde{\omega}_k]$, where

$$\begin{aligned} \tilde{U}_k &:= \{(u, \tau) \in U \times \mathbb{R} \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon\}, \\ \tilde{\varphi}_k: \tilde{U}_k &\rightarrow \Sigma_m^a \text{tl}_k \times_{\text{pr}_T} A_\varepsilon: (u, \tau) \mapsto (\varphi(u), (T_k^u, \tau)), \\ \tilde{\omega}_k \in \Omega_c^*(\tilde{U}_k): (\tilde{\omega}_k)_{(u, \tau)} &:= \rho_\varepsilon(T_k^u, \tau) \cdot \chi_\nu(T_k^u, \tau) \cdot \omega_u = \rho_\varepsilon(T_k^u, \tau) \cdot \omega_u. \end{aligned}$$

Here, \tilde{U}_k is oriented as an open submanifold of $U \times \mathbb{R}$. The last equality holds since $\chi_\nu(T_k^u, \tau) = 1$ for $u \in \text{supp } \omega$. This shows the independence of $f_{k, \xi}(x)$ on the choice of χ_ν . For $\xi = [V, \psi, \eta] \in C_q^{\text{dR}}(S_\varepsilon)$, we write $\psi(v) = (\sigma_i^v)_{i=1, 2}$ for $v \in V$. Then we have $f_{k, \xi}(x) = (-1)^r [W_k, \Phi_k, \zeta_k]$, where

$$\begin{aligned} W_k &:= \{(u, \tau, v) \in U \times \mathbb{R} \times V \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon, \gamma_k^u(\tau) = \sigma_1^v(0)\}, \\ \Phi_k: W_k &\rightarrow \Sigma_{m+1}^{a+\varepsilon}: (u, \tau, v) \mapsto \text{con}_k(\varphi(u), (T_k^u, \tau), \psi(v)), \\ \zeta_k \in \Omega_c^*(W_k): (\zeta_k)_{(u, \tau, v)} &:= \rho_\varepsilon(T_k^u, \tau) \cdot (\omega_u \times \eta_v), \\ r &:= (p+1-n)|\eta|. \end{aligned} \tag{13}$$

Here W_k is oriented as a fiber product over Q of $\tilde{U}_k \rightarrow Q: (u, \tau) \mapsto \gamma_k^u(\tau)$ and $\text{ev}_0 \circ \psi: V \rightarrow Q$.

Lemma 3.13. For $x \in C_p^{\text{dR}}(\Sigma_m^a)$ and $\xi \in C_q^{\text{dR}}(S_\varepsilon)$,

$$\partial \circ f_{k, \xi}(x) - f_{k, \xi} \circ \partial(x) - (-1)^{p+1-n} f_{k, \partial\xi}(x) \in C_{p+q-n}^{\text{dR}}(\Sigma_m^0).$$

Proof. Using the notation of (13) for $x = [U, \varphi, \omega] \in C_p^{\text{dR}}(\Sigma_m^a)$ and $\xi = [V, \psi, \eta] \in C_q^{\text{dR}}(S_\varepsilon)$, we have

$$\begin{aligned} \partial \circ f_{k, \xi}(x) - f_{k, \xi} \circ \partial(x) - (-1)^{p+1-n} f_{k, \partial\xi}(x) &= (-1)^{p-1} (\text{con}_k)_* ((x \text{tl}_k \times_{\text{pr}_T} (\partial\alpha_{\varepsilon, \nu})) \text{ev}_k \times_{\text{ev}_0} \xi) \\ &= (-1)^{(p-n)|\eta|+1} [W_k, \Phi_k, \theta_k], \end{aligned}$$

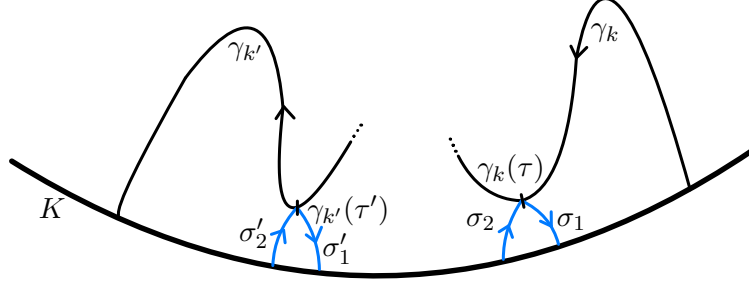


Figure. 2: The case where $(\gamma_l)_{l=1,\dots,m}$ intersects both $(\sigma_i)_{i=1,2}$ and $(\sigma'_i)_{i=1,2}$.

where $\theta_k \in \Omega_c^*(W_k)$ is defined by $(\theta_k)_{(u,\tau,v)} = \frac{\partial \rho_\varepsilon}{\partial \tau}(T_k^u, \tau) \cdot (\omega_u \times d\tau \times \eta_v)$. From the condition on ρ_ε , the support of θ lies in an open subset

$$\bar{W}_k := \{(u, \tau, v) \in W_k \mid \tau < 4\varepsilon \text{ or } T_k^u - 4\varepsilon < \tau\}.$$

Since $(\varphi(u), (T_k^u, \tau), \psi(v)) \in (\Sigma_m^a \text{tl}_k \times_{\text{pr}_T} A_\varepsilon) \text{ev}_k \times_{\text{ev}_0} S_\varepsilon$ for $(u, \tau, v) \in \bar{W}_k$ satisfies the condition (ii) of Lemma 3.12, it follows that $\Phi_k(\bar{W}_k) \subset \Sigma_m^0$. Therefore,

$$[W_k, \Phi_k, \theta] = [\bar{W}_k, \Phi_k|_{\bar{W}_k}, \theta|_{\bar{W}_k}] \in C_{p+q-n}^{\text{DR}}(\Sigma_m^0).$$

This proves the lemma. \square

The next lemma is crucial to define chain complexes in Section 4.1. Before stating it, let us give an observation. Suppose that we have

$$\begin{cases} (\gamma_l)_{l=1,\dots,m} \in \Sigma_m^a, \\ (\sigma_i: [0, \varepsilon_i] \rightarrow N_\varepsilon)_{i=1,2}, (\sigma'_i: [0, \varepsilon'_i] \rightarrow N_\varepsilon)_{i=1,2} \in S_\varepsilon, \\ k, k' \in \{1, \dots, m\} \text{ with } k \leq k', \end{cases}$$

and there exist two points $\tau \in (2\varepsilon, T_k - 2\varepsilon)$ and $\tau' \in (2\varepsilon, T_{k'} - 2\varepsilon)$ such that

$$\gamma_k(\tau) = \sigma_1(0), \quad \gamma_{k'}(\tau') = \sigma'_1(0).$$

When $k = k'$, we additionally assume that $\tau + 2\varepsilon < \tau'$. (Figure 2 describes the situation we consider.) Then, we can split γ_k at $t = \tau$ and $\gamma_{k'}$ at $t = \tau'$, and concatenate them with $(\sigma_i)_{i=1,2}$ and $(\sigma'_i)_{i=1,2}$ respectively. Depending on which point we use first, there are two elements

$$\begin{aligned} \Phi &:= \text{con}_k((\gamma_l)_{l=1,\dots,m}, (T_k, \tau), (\sigma_i)_{i=1,2}) \in \Sigma_{m+1}^{a+\varepsilon}, \\ \Phi' &:= \text{con}_{k'}((\gamma_l)_{l=1,\dots,m}, (T_{k'}, \tau'), (\sigma'_i)_{i=1,2}) \in \Sigma_{m+1}^{a+\varepsilon}. \end{aligned}$$

In either case, there remains another point which we have not yet used. When $k < k'$, we can split the $(k' + 1)$ -th path of Φ at $t = \tau'$ and the k -th path of Φ' at $t = \tau$, and concatenate them with $(\sigma'_i)_{i=1,2}$ and $(\sigma_i)_{i=1,2}$ respectively. When $k = k'$, we can split the $(k + 1)$ -th path of Φ at $t = \tau' - \tau + 2\varepsilon_2$ and the k -th path of Φ' at $t = \tau$, and concatenate them with $(\sigma'_i)_{i=1,2}$ and $(\sigma_i)_{i=1,2}$ respectively. After these two steps, we get the following equations:

$$\begin{cases} \text{con}_{k'+1}(\Phi, (T_{k'}, \tau'), (\sigma'_i)_{i=1,2}) = \text{con}_k(\Phi', (T_k, \tau), (\sigma_i)_{i=1,2}) & \text{if } k < k', \\ \text{con}_{k+1}(\Phi, (\tilde{T}_k^2, \tau' - \tau + 2\varepsilon_2), (\sigma'_i)_{i=1,2}) = \text{con}_k(\Phi', (\tilde{T}_k^1, \tau), (\sigma_i)_{i=1,2}) & \text{if } k = k'. \end{cases} \quad (14)$$

Here, $\tilde{T}_k^2 := T_k - \tau + 2\varepsilon_2$ and $\tilde{T}_k^1 := \tau' + 2\varepsilon_1$. This observation leads us to the following lemma about de Rham chains.

Lemma 3.14. For $x \in C_p^{\text{dR}}(\Sigma_m^a)$, $\xi \in C_q^{\text{dR}}(S_\varepsilon)$ and $k, k' \in \{1, \dots, m\}$ with $k \leq k'$, the following hold:

$$\begin{cases} f_{k'+1, \xi} \circ f_{k, \xi}(x) + (-1)^{q-n} f_{k, \xi} \circ f_{k', \xi}(x) = 0 & \text{if } k < k', \\ f_{k'+1, \xi} \circ f_{k, \xi}(x) + (-1)^{q-n} f_{k, \xi} \circ f_{k', \xi}(x) \in C_{p+2+2q-2n}^{\text{dR}}(\Sigma_{m+2}^0) & \text{if } k = k'. \end{cases}$$

Proof. We use the notations of (13) for $x = [U, \varphi, \omega]$ and $\xi = [V, \psi, \eta]$. For short, let us abbreviate for $(T, \tau, T', \tau') \in A_\varepsilon \times A_\varepsilon$

$$\rho(T, \tau, T', \tau') := \rho_\varepsilon(T, \tau) \cdot \rho_\varepsilon(T', \tau').$$

Case 1. We consider the case $k < k'$. We have $f_{k'+1, \xi} \circ f_{k, \xi}(x) = (-1)^s [W_{k, k'}, \Phi_{k, k'}, \zeta_{k, k'}]$ for

$$\begin{aligned} W_{k, k'} &:= \{(u, \tau, v, \tau', v') \mid (u, \tau, v) \in W_k, 2\varepsilon < \tau' < T_k^u - 2\varepsilon, \gamma_{k'}^u(\tau') = \sigma_1^{v'}(0)\}, \\ \Phi_{k, k'} &: W_{k, k'} \rightarrow \Sigma_{m+2}^{a+2\varepsilon}: (u, \tau, v, \tau', v') \mapsto \text{con}_{k'+1}(\Phi_k(u, \tau, v), (T_{k'}^u, \tau'), \psi(v')), \\ \zeta_{k, k'} &\in \Omega_c^*(W_{k, k'}): (\zeta_{k, k'})_{(u, \tau, v, \tau', v')} := \rho(T_k^u, \tau, T_{k'}^u, \tau') \cdot (\omega_u \times \eta_v \times \eta_{v'}), \\ s &:= (q+1-n)|\eta|, \end{aligned}$$

by substituting k and $[U, \varphi, \omega]$ in (13) with k' and $[W_k, \Phi_k, (-1)^r \zeta_k]$. Similarly, $f_k \circ f_{k'}(x) = (-1)^s [W'_{k, k'}, \Phi'_{k, k'}, \zeta'_{k, k'}]$ for

$$\begin{aligned} W'_{k, k'} &:= \{(u, \tau', v', \tau, v) \mid (u, \tau', v') \in W_{k'}, 2\varepsilon < \tau < T_k^u - 2\varepsilon, \gamma_k^u(\tau) = \sigma_1^v(0)\}, \\ \Phi'_{k, k'} &: W'_{k, k'} \rightarrow \Sigma_{m+2}^{a+2\varepsilon}: (u, \tau', v', \tau, v) \mapsto \text{con}_k(\Phi_{k'}(u, \tau', v'), (T_k^u, \tau), \psi(v)), \\ \zeta'_{k, k'} &\in \Omega_c^*(W'_{k, k'}): (\zeta'_{k, k'})_{(u, \tau', v', \tau, v)} = \rho(T_{k'}^u, \tau', T_k^u, \tau) \cdot (\omega_u \times \eta_{v'} \times \eta_v). \end{aligned}$$

We define a diffeomorphism

$$h: W_{k, k'} \rightarrow W'_{k, k'}: (u, \tau, v, \tau', v') \mapsto (u, \tau', v', \tau, v),$$

which changes the sign of orientation by $(-1)^{(1+\dim V-n)^2}$. From (14), it follows that $\Phi'_{k, k'} \circ h = \Phi_{k, k'}$. Moreover, $h_*(\zeta_{k, k'}) = (-1)^{|\eta|^2} \zeta'_{k, k'}$ holds. Combining these computations,

$$f_{k'+1, \xi} \circ f_{k, \xi}(x) = (-1)^{s+(1+\dim V-n+|\eta|)} [W'_{k, k'}, \Phi'_{k, k'}, \zeta'_{k, k'}] = (-1)^{q-n+1} f_{k, \xi} \circ f_{k', \xi}(x).$$

Case 2. We consider the case $k = k'$. We have $f_{k+1} \circ f_k(x) = (-1)^s [W_{k, k}, \Phi_{k, k}, \zeta_{k, k}]$ for

$$\begin{aligned} W_{k, k} &:= \{(u, \tau, v, \tau', v') \mid (u, \tau, v) \in W_k, 2\varepsilon < \tau' < \tilde{T}_k^2(u, \tau, v) - 2\varepsilon, \gamma_k^u(\tau' + \tau - 2\varepsilon_2^v) = \sigma_2^{v'}(0)\}, \\ \Phi_{k, k} &: W_{k, k} \rightarrow \Sigma_{m+2}^{a+2\varepsilon}: (u, \tau, v, \tau', v') \mapsto \text{con}_{k+1}(\Phi_k(u, \tau, v), (\tilde{T}_k^2(u, \tau, v), \tau'), \psi(v')), \\ \zeta_{k, k} &\in \Omega_c^*(W_{k, k}): (\zeta_{k, k})_{(u, \tau, v, \tau', v')} = \rho(T_k^u, \tau, \tilde{T}_k^2(u, \tau, v), \tau') \cdot (\omega_u \times \eta_v \times \eta_{v'}), \end{aligned}$$

where $\tilde{T}_k^2(u, \tau, v) := T_k^u - \tau + 2\varepsilon_2^v$. Similarly, $f_{k, \xi} \circ f_{k, \xi}(x) = (-1)^s [W'_{k, k}, \Phi'_{k, k}, \zeta'_{k, k}]$ for

$$\begin{aligned} W'_{k, k} &:= \{(u, \tau', v', \tau, v) \mid (u, \tau', v') \in W_k, 2\varepsilon < \tau < \tilde{T}_k^1(\tau', v') - 2\varepsilon, \gamma_k^u(\tau) = \sigma_1^v(0)\}, \\ \Phi'_{k, k} &: W'_{k, k} \rightarrow \Sigma_{m+2}^{a+2\varepsilon}: (u, \tau', v', \tau, v) \mapsto \text{con}_k(\Phi_k(u, \tau', v'), (\tilde{T}_k^1(\tau', v'), \tau), \psi(v)), \\ \zeta'_{k, k} &\in \Omega_c^*(W'_{k, k}): (\zeta'_{k, k})_{(u, \tau', v', \tau, v)} = \rho(T_k^u, \tau', \tilde{T}_k^1(\tau', v'), \tau) \cdot (\omega_u \times \eta_{v'} \times \eta_v), \end{aligned}$$

where $\tilde{T}_k^1(\tau', v') := \tau' + 2\varepsilon_1^v$. Since $\rho(T_k, \tau, \tilde{T}_k^2(u, \tau, v), \tau') = 0$ for $\tau' \leq \frac{10}{3}\varepsilon$, we have $f_{k+1} \circ f_k(x) = (-1)^s [\bar{W}_{k, k}, \bar{\Phi}_{k, k}, \bar{\zeta}_{k, k}]$ for $\bar{W}_{k, k} := W_{k, k} \cap \{\tau' > 3\varepsilon\}$.

This time, we define a map

$$h: \bar{W}_{k, k} \rightarrow W'_{k, k}: (u, \tau, v, \tau', v') \mapsto (u, \tau' + \tau - 2\varepsilon_2^v, v', \tau, v),$$

which is an open embedding and changes the sign of orientation by $(-1)^{(1+\dim V-n)^2}$. From (14), it follows that $\Phi'_{k,k} \circ h = \Phi_{k,k}$. Indeed, if we set $\tau'_* := \tau' + \tau - 2\varepsilon_2^v$, then

$$\begin{aligned}\Phi'_{k,k} \circ h(u, \tau, v, \tau', v') &= \text{con}_k(\Phi_k(u, \tau'_*, v'), (\tilde{T}_k^1(\tau'_*, v'), \tau), \psi(v)) \\ &= \text{con}_{k+1}(\Phi_k(u, \tau, v), (\tilde{T}_k^1(u, \tau, v), \tau'_* - \tau + 2\varepsilon_2^v), \psi(v')) \\ &= \Phi_{k,k}(u, \tau, v, \tau', v')\end{aligned}$$

for every $(u, \tau, v, \tau', v') \in \bar{W}_{k,k}$. Therefore, we have

$$(-1)^{q-n+1} f_{k+1,\xi} \circ f_{k,\xi}(x) - f_{k,\xi} \circ f_{k,\xi}(x) = (-1)^s [W'_{k,k}, \Phi'_{k,k}, (-1)^{|\eta|^2} h_*(\zeta_{k,k}) - \zeta'_{k,k}].$$

For $(u, \tau', v', \tau, v) \in W'_{k,k}$,

$$\begin{aligned}&(-1)^{|\eta|^2} (h_*(\zeta_{k,k}))_{(u, \tau', v', \tau, v)} - (\zeta'_{k,k})_{(u, \tau', v', \tau, v)} \\ &= \left(\rho(T_k^u, \tau, \tilde{T}_k^2(u, \tau, v), \tau' - \tau + 2\varepsilon_2^v) - \rho(T_k^u, \tau', \tilde{T}_k^1(\tau', v'), \tau) \right) \cdot (\omega_u \times \eta_{v'} \times \eta_v).\end{aligned}$$

We can check that if $\frac{11}{3}\varepsilon \leq \tau' \leq T_k^u - \frac{11}{3}\varepsilon$ and $\frac{11}{3}\varepsilon \leq \tau \leq \tilde{T}_k^1(\tau', v') - \frac{11}{3}\varepsilon$, then $\frac{11}{3}\varepsilon \leq \tau' - \tau + 2\varepsilon_2^v \leq \tilde{T}_k^2(u, \tau, v) - \frac{11}{3}\varepsilon$ and $\frac{11}{3}\varepsilon \leq \tau \leq T_k^u - \frac{11}{3}\varepsilon$ hold, and thus

$$\rho(T_k^u, \tau, \tilde{T}_k^2(u, \tau, v), \tau' - \tau + 2\varepsilon_2^v) = 1 = \rho(T_k^u, \tau', \tilde{T}_k^1(\tau', v'), \tau).$$

Therefore, $\text{supp}((-1)^{|\eta|^2} h_*(\zeta_{k,k}) - \zeta'_{k,k})$ lies in $W_{k,k}^1 \cup W_{k,k}^2$, where

$$\begin{aligned}W_{k,k}^1 &:= \{(u, \tau', v', \tau, v) \in W'_{k,k} \mid \tau' < 4\varepsilon \text{ or } T_k^u - 4\varepsilon < \tau'\}, \\ W_{k,k}^2 &:= \{(u, \tau', v', \tau, v) \in W'_{k,k} \mid \tau < 4\varepsilon \text{ or } \tilde{T}_k^1(\tau', v') - 4\varepsilon < \tau\}.\end{aligned}$$

From Lemma 3.12, we have $\Phi_k(u, \tau', v') = \text{con}_k(\varphi(u), (T_k^u, \tau'), \psi(v')) \in \Sigma_{m+1}^0$ for all $(u, \tau', v', \tau, v) \in W_{k,k}^1$. Then, Lemma 3.12 is applied again to show that

$$\Phi'_{k,k}(u, \tau', v', \tau, v) = \text{con}_k(\Phi_k(u, \tau', v'), (\tilde{T}_k^1(\tau', v'), \tau), \psi(v))$$

is an element of Σ_{m+2}^0 for every $(u, \tau', v', \tau, v) \in W_{k,k}^1 \cup W_{k,k}^2$. Indeed, we can apply the case (i) of Lemma 3.12 for $(u, \tau', v', \tau, v) \in W_{k,k}^1$, and the case (ii) for $(u, \tau', v', \tau, v) \in W_{k,k}^2$. As a consequence,

$$\begin{aligned}&(-1)^{q-n+1} f_{k+1,\xi} \circ f_{k,\xi}(x) - f_{k,\xi} \circ f_{k,\xi}(x) \\ &= \left[W_{k,k}^1 \cup W_{k,k}^2, \Phi'_{k,k} \Big|_{W_{k,k}^1 \cup W_{k,k}^2}, \left((-1)^{|\eta|^2} h_*(\zeta_{k,k}) - \zeta'_{k,k} \right) \Big|_{W_{k,k}^1 \cup W_{k,k}^2} \right] \in C_{p+2+2q-2n}^{\text{dR}}(\Sigma_{m+2}^0).\end{aligned}$$

□

In the definition of $f_{k,\xi}$ for $\xi \in C_q^{\text{dR}}(S_\varepsilon)$, there is an ambiguity about the choice of $\rho_\varepsilon: A_\varepsilon \rightarrow [0, 1]$. Suppose that we choose another ρ'_ε , and define $\alpha'_{\varepsilon,\nu} f'_{k,\xi}$ in the same way as $\alpha_{\varepsilon,\nu}$ and $f_{k,\xi}$. Take an arbitrary chain $x \in C_p^{\text{dR}}(\Sigma_m^a)$. Since $\text{supp}(\rho'_\varepsilon - \rho_\varepsilon)$ lies in $\{(T, \tau) \in A_\varepsilon \mid \tau < 4\varepsilon \text{ or } T - 4\varepsilon < \tau\}$, $x \text{ tl}_k \times_{\text{pr}_T} (\alpha_{\varepsilon,\nu} - \alpha'_{\varepsilon,\nu})$ is a chain in the subspace

$$\{((\gamma_k)_{k=1,\dots,m}, (T, \tau)) \in \Sigma_m^a \text{ tl}_k \times_{\text{pr}_T} A_\varepsilon \mid \tau < 4\varepsilon \text{ or } T - 4\varepsilon < \tau\}.$$

From Lemma 3.12, we have

$$f_{k,\xi}(x) - f'_{k,\xi}(x) = (\text{con}_k)_*((x \text{ tl}_k \times_{\text{pr}_T} (\alpha_{\varepsilon,\nu} - \alpha'_{\varepsilon,\nu})) \text{ ev}_k \times_{\text{ev}_0} \xi) \in C_{p+1+q-n}^{\text{dR}}(\Sigma_{m+1}^0).$$

Therefore, for $a \in \mathbb{R}_{\geq 0}$, the induced map on the quotient spaces, which is denoted by the same symbol,

$$f_{k,\xi}: C_*^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \rightarrow C_{*+1+q-n}^{\text{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0)$$

is independent on the choice of ρ_ε . For this map, the equation

$$\partial \circ f_{k,\xi} - f_{k,\xi} \circ \partial = (-1)^{p+1-n} f_{k,\partial\xi}: C_p^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \rightarrow C_{p+1+q-n}^{\text{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0) \quad (15)$$

follows from Lemma 3.13, and the equation for $k' \geq k$

$$f_{k'+1,\xi} \circ f_{k,\xi} + (-1)^{q-n} f_{k,\xi} \circ f_{k',\xi} = 0: C_p^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \rightarrow C_{p+2+2q-2n}^{\text{dR}}(\Sigma_{m+2}^{a+2\varepsilon}, \Sigma_{m+2}^0) \quad (16)$$

follows from Lemma 3.14. If $\partial\xi = 0$, (15) implies that $f_{k,\xi}$ is a chain map shifting the degree by $(1+q-n)$.

3.6 $[-1, 1]$ and $[-1, 1]^2$ -modeled de Rham chains

In this section, we introduce two types of variants of de Rham chains. In this paper, they are necessary for only four kinds of differentiable spaces: Σ_m^a , $\Sigma_m^a \times \Sigma_{m'}^{a'}$, S_ε and (M, P_M) for a manifold M . Throughout this section, X represents these differentiable spaces.

3.6.1 $[-1, 1]$ -modeled de Rham chains

We introduce chains in $\mathbb{R} \times X$. We define \bar{P}_X as the set of tuples $(U, \varphi, (\tau_+, \tau_-))$ such that:

- $(U, \varphi) \in P_{\mathbb{R}^{\text{reg}} \times X}$. If $X = S_\varepsilon$, we additionally require that $(\text{id}_{\mathbb{R}} \times \text{ev}_0) \circ \varphi: U \rightarrow \mathbb{R} \times Q$ is a submersion. Let us write $\varphi = (\varphi_{\mathbb{R}}, \varphi_X): U \rightarrow \mathbb{R} \times X$ and $U_I := \varphi_{\mathbb{R}}^{-1}(I)$ for any subset $I \subset \mathbb{R}$.
- $\tau_+: U_{\mathbb{R}_{\geq 1}} \rightarrow \mathbb{R}_{\geq 1} \times U_{\{1\}}$ and $\tau_-: U_{\mathbb{R}_{\leq -1}} \rightarrow \mathbb{R}_{\leq -1} \times U_{\{-1\}}$ are diffeomorphisms such that

$$\varphi \circ \tau_+^{-1} = i_{\mathbb{R}_{\geq 1}} \times \varphi_X|_{U_{\{1\}}}, \quad \varphi \circ \tau_-^{-1} = i_{\mathbb{R}_{\leq -1}} \times \varphi_X|_{U_{\{-1\}}}.$$

Here, $i_{\mathbb{R}_{\geq 1}}$ (resp. $i_{\mathbb{R}_{\leq -1}}$) is the inclusion map from $\mathbb{R}_{\geq 1}$ (resp. $\mathbb{R}_{\leq -1}$) to \mathbb{R} .

Remark 3.15. When $X = S_\varepsilon$, the condition that $(U, \varphi) \in P_{\mathbb{R}^{\text{reg}} \times S_\varepsilon}$ implies only that the composition of $(\text{id}_{\mathbb{R}} \times \text{ev}_0) \circ \varphi: U \rightarrow \mathbb{R} \times Q$ with $\text{pr}_{\mathbb{R}}$ (resp. pr_Q) is a submersion to \mathbb{R} (resp. Q). The condition that $(\text{id}_{\mathbb{R}} \times \text{ev}_0) \circ \varphi$ itself is a submersion is necessary to define a fiber product of $[-1, 1]$ -modeled de Rham chains in the later subsection.

For $(U, \varphi, (\tau_+, \tau_-)) \in \bar{P}_X$, we define a linear subspace $\Omega_c^p(U, \varphi, (\tau_+, \tau_-))$ of $\Omega^p(U)$ which consists of p -forms ω on U such that $\text{supp } \omega \cap U_{[-1, 1]}$ is compact, $(\tau_+^{-1})^* \omega = 1 \times \omega|_{U_{\{1\}}}$ and $(\tau_-^{-1})^* \omega = 1 \times \omega|_{U_{\{-1\}}}$. We consider a graded \mathbb{R} -vector space

$$\bar{A}_*(X) := \bigoplus_{(U, \varphi, (\tau_+, \tau_-)) \in \bar{P}_X} \Omega_c^{\dim U - 1 - *} (U, \varphi, (\tau_+, \tau_-)).$$

For $\mathbf{U} = (U, \varphi, (\tau_+, \tau_-)) \in \bar{P}_X$ and $\omega \in \Omega_c^p(U, \varphi, (\tau_+, \tau_-))$, let $(U, \varphi, (\tau_+, \tau_-), \omega)$ denote the element of $\bar{A}_*(X)$ such that its component for $\mathbf{V} \in \bar{P}_X$ is

$$(U, \varphi, (\tau_+, \tau_-), \omega)_{\mathbf{V}} = \begin{cases} \omega & \text{if } \mathbf{V} = \mathbf{U}, \\ 0 & \text{if } \mathbf{V} \neq \mathbf{U}. \end{cases}$$

We take a linear subspace $\bar{Z}_*(X)$ of $\bar{A}_*(X)$ generated by vectors

$$(V, \varphi \circ \pi, (\sigma_+, \sigma_-), \omega) - (U, \varphi, (\tau_+, \tau_-), \pi_! \omega)$$

for any submersion $\pi: V \rightarrow U$ such that

$$(\text{id}_{\mathbb{R}_{\geq 1}} \times \pi|_{V_{\{1\}}}) \circ \sigma_+ = \tau_+ \circ \pi, \quad (\text{id}_{\mathbb{R}_{\leq -1}} \times \pi|_{V_{\{-1\}}}) \circ \sigma_- = \tau_- \circ \pi.$$

We define a quotient vector space

$$\bar{C}_*^{\text{dR}}(X) := \bar{A}_*(X) / \bar{Z}_*(X),$$

whose elements we call $[-1, 1]$ -modeled de Rham chains. $[U, \varphi, (\tau_+, \tau_-), \omega]$ denotes the equivalence class of $(U, \varphi, (\tau_+, \tau_-), \omega)$. We define a degree (-1) linear map $\partial: \bar{C}_*^{\text{dR}}(X) \rightarrow \bar{C}_{*+1}^{\text{dR}}(X)$ by

$$\partial[U, \varphi, (\tau_+, \tau_-), \omega] := (-1)^{|\omega|+1} [U, \varphi, (\tau_+, \tau_-), d\omega].$$

Obviously $\partial \circ \partial = 0$ holds and we obtain a chain complex $(\bar{C}_*^{\text{dR}}(X), \partial)$. Its homology is denoted by $\bar{H}_*^{\text{dR}}(X)$.

Naturally, there are three chain maps:

$$\bar{i}: C_*^{\text{dR}}(X) \rightarrow \bar{C}_*^{\text{dR}}(X): [V, \psi, \omega] \mapsto (-1)^{\dim V} [\mathbb{R} \times V, \text{id}_{\mathbb{R}} \times \psi, (\text{id}_{\mathbb{R}_{\geq 1}} \times V, \text{id}_{\mathbb{R}_{\leq -1}} \times V), 1 \times \omega] \quad (17)$$

and

$$\begin{aligned} e_+ &: \bar{C}_*^{\text{dR}}(X) \rightarrow C_*^{\text{dR}}(X): [U, \varphi, (\tau_+, \tau_-), \omega] \mapsto (-1)^{\dim U - 1} [U_{\{1\}}, \varphi_X|_{U_{\{1\}}}, \omega|_{U_{\{1\}}}], \\ e_- &: \bar{C}_*^{\text{dR}}(X) \rightarrow C_*^{\text{dR}}(X): [U, \varphi, (\tau_+, \tau_-), \omega] \mapsto (-1)^{\dim U - 1} [U_{\{-1\}}, \varphi_X|_{U_{\{-1\}}}, \omega|_{U_{\{-1\}}}], \end{aligned} \quad (18)$$

Here, $U_{\{1\}}$ and $U_{\{-1\}}$ are oriented so that τ_+ and τ_- preserve orientations. Clearly, $e_+ \circ \bar{i} = e_- \circ \bar{i} = \text{id}_{C_*^{\text{dR}}(X)}$. For $\bar{i} \circ e_+$ and $\bar{i} \circ e_-$, the next result holds.

Lemma 3.16. $\bar{i} \circ e_+$ and $\bar{i} \circ e_-$ are chain homotopic to the identity map $\text{id}_{\bar{C}_*^{\text{dR}}(X)}$.

Proof. This assertion is essentially proved in [24, Lemma 4.8]. We should notice that the result in the reference is proved for a specific differentiable space $\mathcal{L}_{k+1}(a)$ (a differentiable space of marked Moore loops in a manifold). However, even for a differentiable space X considered in this section, we can extend the definition of a chain

$$K([U, \varphi, (\tau_+, \tau_-), \omega]) := (-1)^{|\omega|+1} [\mathbb{R} \times U, \bar{\varphi}, (\bar{\tau}_+, \bar{\tau}_-), \bar{\omega}] \in \bar{C}_{*+1}^{\text{dR}}(X)$$

for any $[U, \varphi, (\tau_+, \tau_-), \omega] \in \bar{C}_*^{\text{dR}}(X)$, which appears in the proof of [24, Lemma 4.8]. Then $K: \bar{C}_*^{\text{dR}}(X) \rightarrow \bar{C}_{*+1}^{\text{dR}}(X)$ gives a chain homotopy from $\text{id}_{\bar{C}_*^{\text{dR}}(X)}$ to $\bar{i} \circ e_+$. The proof for $\bar{i} \circ e_-$ is completely parallel. \square

3.6.2 $[-1, 1]^2$ -modeled de Rham chains

Let us define a smooth map

$$\iota: (\mathbb{R}^2)^{\text{reg}} \times X \rightarrow (\mathbb{R}^2)^{\text{reg}} \times X: ((r_1, r_2), x) \mapsto ((r_2, r_1), x).$$

We often use the coordinate (r_1, r_2) of \mathbb{R}^2 to denote its subsets, for instance $\mathbb{R}_{\geq 1} \times \mathbb{R} = \{r_1 \geq 1\}$.

We introduce chains in $\mathbb{R}^2 \times X$. We define \hat{P}_X as the set of tuples $(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2))$ such that:

- $(U, \varphi) \in P_{(\mathbb{R}^2)^{\text{reg}} \times X}$. If $X = S_\varepsilon$, we additionally require that $(\text{id}_{\mathbb{R}^2} \times \text{ev}_0) \circ \varphi: U \rightarrow \mathbb{R}^2 \times Q$ is a submersion. Let us write $\varphi = ((\varphi_{\mathbb{R}}^1, \varphi_{\mathbb{R}}^2), \varphi_X): U \rightarrow \mathbb{R}^2 \times X$ and $U_D := \{u \in U \mid (\varphi_{\mathbb{R}}^1(u), \varphi_{\mathbb{R}}^2(u)) \in D\}$ for any subset $D \subset \mathbb{R}^2$.
- τ_+^j, τ_-^j ($j = 1, 2$) are diffeomorphisms such that

$$\begin{aligned} \varphi \circ (\tau_+^1)^{-1} &= i_{\mathbb{R}_{\geq 1}} \times (\varphi_{\mathbb{R}}^2 \times \varphi_X)|_{U_{\{r_1=1\}}}, \quad \varphi \circ (\tau_-^1)^{-1} = i_{\mathbb{R}_{\leq -1}} \times (\varphi_{\mathbb{R}}^2 \times \varphi_X)|_{U_{\{r_1=-1\}}}, \\ \iota \circ \varphi \circ (\tau_+^2)^{-1} &= i_{\mathbb{R}_{\geq 1}} \times (\varphi_{\mathbb{R}}^1 \times \varphi_X)|_{U_{\{r_2=1\}}}, \quad \iota \circ \varphi \circ (\tau_-^2)^{-1} = i_{\mathbb{R}_{\leq -1}} \times (\varphi_{\mathbb{R}}^1 \times \varphi_X)|_{U_{\{r_2=-1\}}}. \end{aligned}$$

For $(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2)) \in \hat{P}_X$, we define a linear subspace $\Omega_c^p(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2))$ of $\Omega^p(U)$ which consists of p -forms ω on U such that $\text{supp } \omega \cap U_{[-1,1] \times [-1,1]}$ is compact and

$$((\tau_+^j)^{-1})^* \omega = 1 \times \omega|_{U_{\{r_j=1\}}}, \quad ((\tau_-^j)^{-1})^* \omega = 1 \times \omega|_{U_{\{r_j=-1\}}} \quad (j = 1, 2).$$

We consider the graded \mathbb{R} -vector space

$$\hat{A}_*(X) := \bigoplus_{(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2)) \in \hat{P}_X} \Omega_c^{\dim U - 2 - *}(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2)).$$

For $\mathbf{U} = (U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2)) \in \hat{P}_X$ and $\omega \in \Omega_c^p(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2))$, let

$$(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega)$$

denote the element of $\hat{A}_*(X)$ such that its component for $\mathbf{V} \in \hat{P}_X$ is

$$(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega)_{\mathbf{V}} = \begin{cases} \omega & \text{if } \mathbf{V} = \mathbf{U}, \\ 0 & \text{if } \mathbf{V} \neq \mathbf{U}. \end{cases}$$

We take a linear subspace $\hat{Z}_*(X)$ of $\hat{A}_*(X)$ generated by vectors

$$(V, \varphi \circ \pi, (\sigma_+^1, \sigma_-^1), (\sigma_+^2, \sigma_-^2), \omega) - (U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \pi_! \omega)$$

for any submersion $\pi: V \rightarrow U$ such that for $j = 1, 2$

$$(\text{id}_{\mathbb{R}_{\geq 1}} \times \pi|_{V_{\{r_j=1\}}}) \circ \sigma_+^j = \tau_+^j \circ \pi, \quad (\text{id}_{\mathbb{R}_{\leq -1}} \times \pi|_{V_{\{r_j=-1\}}}) \circ \sigma_-^j = \tau_-^j \circ \pi.$$

Now we define a quotient vector space

$$\hat{C}_*^{\text{dR}}(X) := \hat{A}_*(X) / \hat{Z}_*(X),$$

whose elements we call $[-1, 1]^2$ -*modeled de Rham chains*. $[U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega]$ denotes the equivalence class of $(U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega)$. We define a degree (-1) linear map $\partial: \hat{C}_*^{\text{dR}}(X) \rightarrow \hat{C}_{*-1}^{\text{dR}}(X)$ by

$$\partial[U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] := (-1)^{|\omega|+1} [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), d\omega].$$

Obviously $\partial \circ \partial = 0$ holds and we obtain a chain complex $(\hat{C}_*^{\text{dR}}(X), \partial)$. Its homology is denoted by $\hat{H}_*^{\text{dR}}(X)$.

Naturally, there are six chain maps

$$\begin{aligned} \hat{i}^1, \hat{i}^2: \bar{C}_*^{\text{dR}}(X) &\rightarrow \hat{C}_*^{\text{dR}}(X), \\ e_+^1, e_+^2, e_-^1, e_-^2: \hat{C}_*^{\text{dR}}(X) &\rightarrow \bar{C}_*^{\text{dR}}(X), \end{aligned} \tag{19}$$

define as follows: \hat{i}^1 and \hat{i}^2 map $x = [V, \psi, (\tau_+, \tau_-), \omega] \in \bar{C}_*^{\text{dR}}(X)$ to

$$\begin{aligned} \hat{i}^1 x &:= (-1)^{\dim V - 1} [\mathbb{R} \times V, \text{id}_{\mathbb{R}} \times \psi, (\text{id}_{\mathbb{R}_{\geq 1}} \times V, \text{id}_{\mathbb{R}_{\leq -1}} \times V), (\hat{\tau}_+, \hat{\tau}_-), 1 \times \omega], \\ \hat{i}^2 x &:= (-1)^{\dim V} [\mathbb{R} \times V, \iota \circ (\text{id}_{\mathbb{R}} \times \psi), (\hat{\tau}_+, \hat{\tau}_-), (\text{id}_{\mathbb{R}_{\geq 1}} \times V, \text{id}_{\mathbb{R}_{\leq -1}} \times V), 1 \times \omega], \end{aligned}$$

where $\hat{\tau}_+: \mathbb{R} \times V_{\mathbb{R}_{\geq 1}} \rightarrow \mathbb{R}_{\geq 1} \times (\mathbb{R} \times V_{\{1\}})$ and $\hat{\tau}_-: \mathbb{R} \times V_{\mathbb{R}_{\leq -1}} \rightarrow \mathbb{R}_{\leq -1} \times (\mathbb{R} \times V_{\{-1\}})$ are determined by

$$\begin{aligned} \hat{\tau}_+(r', \tau_+^{-1}(r, u_+)) &= (r, (r', u_+)) \text{ for } r' \in \mathbb{R} \text{ and } (r, u_+) \in \mathbb{R}_{\geq 1} \times V_{\{1\}}, \\ \hat{\tau}_-(r', \tau_-^{-1}(r, u_-)) &= (r, (r', u_-)) \text{ for } r' \in \mathbb{R} \text{ and } (r, u_-) \in \mathbb{R}_{\leq -1} \times V_{\{-1\}}. \end{aligned}$$

In addition, e_+^j and e_-^j ($j = 1, 2$) map $y = [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] \in \hat{C}_*^{\text{dR}}(X)$ to

$$\begin{aligned} e_+^1 y &:= (-1)^{\dim U} [U_{\{1\} \times \mathbb{R}}, (\varphi_{\mathbb{R}}^2, \varphi_X) \Big|_{U_{\{1\} \times \mathbb{R}}}, (\tau_+^2 \Big|_{U_{\{1\} \times \mathbb{R}_{\geq 1}}}, \tau_-^2 \Big|_{U_{\{1\} \times \mathbb{R}_{\leq -1}}}), \omega \Big|_{U_{\{1\} \times \mathbb{R}}}], \\ e_+^2 y &:= (-1)^{\dim U - 1} [U_{\mathbb{R} \times \{1\}}, (\varphi_{\mathbb{R}}^1, \varphi_X) \Big|_{U_{\mathbb{R} \times \{1\}}}, (\tau_+^1 \Big|_{U_{\mathbb{R}_{\geq 1} \times \{1\}}}, \tau_-^1 \Big|_{U_{\mathbb{R}_{\leq -1} \times \{1\}}}), \omega \Big|_{U_{\mathbb{R} \times \{1\}}}], \\ e_-^1 y &:= (-1)^{\dim U} [U_{\{-1\} \times \mathbb{R}}, (\varphi_{\mathbb{R}}^2, \varphi_X) \Big|_{U_{\{-1\} \times \mathbb{R}}}, (\tau_+^2 \Big|_{U_{\{-1\} \times \mathbb{R}_{\geq 1}}}, \tau_-^2 \Big|_{U_{\{-1\} \times \mathbb{R}_{\leq -1}}}), \omega \Big|_{U_{\{-1\} \times \mathbb{R}}}], \\ e_-^2 y &:= (-1)^{\dim U - 1} [U_{\mathbb{R} \times \{-1\}}, (\varphi_{\mathbb{R}}^1, \varphi_X) \Big|_{U_{\mathbb{R} \times \{-1\}}}, (\tau_+^1 \Big|_{U_{\mathbb{R}_{\geq 1} \times \{-1\}}}, \tau_-^1 \Big|_{U_{\mathbb{R}_{\leq -1} \times \{-1\}}}), \omega \Big|_{U_{\mathbb{R} \times \{-1\}}}]. \end{aligned}$$

Here, the orientations of U_D for $D = \{r_j = 1\}, \{r_j = -1\}$ ($j = 1, 2$) are determined so that τ_+^j and τ_-^j preserve orientations. The signs are chosen so that

$$e_+ \circ e_+^1 = e_+ \circ e_+^2, \quad e_- \circ e_+^1 = e_+ \circ e_-^2, \quad e_+ \circ e_-^1 = e_- \circ e_+^2, \quad e_- \circ e_-^1 = e_- \circ e_-^2. \quad (20)$$

hold. Clearly, $e_+^j \circ \hat{i}^j = e_-^j \circ \hat{i}^j = \text{id}_{\hat{C}_*^{\text{dR}}(X)}$ for $j = 1, 2$. For $\hat{i}^j \circ e_+^j$ and $\hat{i}^j \circ e_-^j$ ($j = 1, 2$), the next result holds

Lemma 3.17. $\hat{i}^j \circ e_+^j$ and $\hat{i}^j \circ e_-^j$ ($j = 1, 2$) are chain homotopic to $\text{id}_{\hat{C}_*^{\text{dR}}(X)}$.

Proof. We omit the detailed proof as Lemma 3.16. For any $x = [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] \in \hat{C}_*^{\text{dR}}(X)$, let us define diffeomorphisms for $j = 1, 2$

$$\tilde{\tau}_+^j: \mathbb{R} \times U_{\{r_j \geq 1\}} \rightarrow \mathbb{R}_{\geq 1} \times (\mathbb{R} \times U_{\{r_j = 1\}}), \quad \tilde{\tau}_-^j: \mathbb{R} \times U_{\{r_j \leq -1\}} \rightarrow \mathbb{R}_{\leq -1} \times (\mathbb{R} \times U_{\{r_j = -1\}}),$$

so that $(\tilde{\tau}_+^j)^{-1}(r_j, (r, u)) = (r, (\tau_+^j)^{-1}(r_j, u))$ and $(\tilde{\tau}_-^j)^{-1}(r_j, (r, u)) = (r, (\tau_-^j)^{-1}(r_j, u))$. Referring to the proof of [24, Lemma 4.8], we can find $\tilde{\varphi}^j, \tilde{\tau}_\pm^j, \tilde{\omega}^j$ ($j = 1, 2$) to define

$$\begin{aligned} K^1(x) &:= (-1)^{|\omega|+1} [\mathbb{R} \times U, \tilde{\varphi}^1, (\tilde{\tau}_+^1, \tilde{\tau}_-^1), (\tilde{\tau}_+^2, \tilde{\tau}_-^2), \tilde{\omega}^1], \\ K^2(x) &:= (-1)^{|\omega|+1} [\mathbb{R} \times U, \tilde{\varphi}^2, (\tilde{\tau}_+^1, \tilde{\tau}_-^1), (\tilde{\tau}_+^2, \tilde{\tau}_-^2), \tilde{\omega}^2], \end{aligned}$$

so that $K^j: \hat{C}_*^{\text{dR}}(X) \rightarrow \hat{C}_{*+1}^{\text{dR}}(X)$ is a chain homotopy from $\text{id}_{\hat{C}_*^{\text{dR}}(X)}$ to $\hat{i}^j \circ e_+^j$ for $j = 1, 2$. The proof for $\hat{i}^j \circ e_-^j$ is completely parallel. \square

3.6.3 Collection of analogies with ordinary de Rham chains

As above, X and Y are chosen from one of the following differentiable space: $\Sigma_m^a, \Sigma_m^a \times \Sigma_{m'}^{a'}$, S_ε and (M, P_M) for a manifold M . Let $f: X \rightarrow Y$ be a smooth map. If $Y = S_\varepsilon$, we require that $X = S_{\varepsilon'}$ and $\text{ev}_0 \circ f = \text{ev}_0$. Then, f induces chain maps

$$\begin{aligned} f_*: \bar{C}_*^{\text{dR}}(X) &\rightarrow \bar{C}_*^{\text{dR}}(Y): [U, \varphi, (\tau_+, \tau_-), \omega] \mapsto [U, f \circ \varphi, (\tau_+, \tau_-), \omega], \\ f_*: \hat{C}_*^{\text{dR}}(X) &\rightarrow \hat{C}_*^{\text{dR}}(Y): [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] \mapsto [U, f \circ \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega]. \end{aligned}$$

If $X = \Sigma_m^a$ and $Y = \Sigma_m^b$ for $a \leq b$ and f is the inclusion map, then we claim that the above maps are injective. This can be proved as Lemma 2.7, so we omit the proof. As a consequence, we can define $\bar{C}_*^{\text{dR}}(\Sigma_m^b, \Sigma_m^a)$ and $\hat{C}_*^{\text{dR}}(\Sigma_m^b, \Sigma_m^a)$ as quotient complexes.

Next, let $(X, Y) = (\Sigma_m^a, \Sigma_{m'}^{a'})$ or $(\{0\}, S_\varepsilon)$. We identify $\{0\} \times S_\varepsilon$ with S_ε . Then, a cross product $x \times y \in \bar{C}_{p+q}^{\text{dR}}(X \times Y)$ is defined for $x = [U, \varphi, (\tau_+, \tau_-), \omega] \in \bar{C}_p^{\text{dR}}(X)$ and $y = [V, \psi, (\sigma_+, \sigma_-), \eta] \in \bar{C}_q^{\text{dR}}(Y)$ by

$$x \times y := (-1)^{p|\eta|} [W, \tilde{\varphi}, (\tilde{\tau}_+, \tilde{\tau}_-), \omega \times \eta].$$

Here, $W := U \times_{\varphi_{\mathbb{R}} \times \psi_{\mathbb{R}}} V$ is a fiber product over \mathbb{R} and $\tilde{\varphi}, \tilde{\tau}_+, \tilde{\tau}_-$ are determined by

$$\begin{aligned}\tilde{\varphi}: W &\rightarrow \mathbb{R} \times (X \times Y): (u, v) \mapsto (\varphi_{\mathbb{R}}(u), \varphi_X(u), \psi_Y(v)), \\ \tilde{\tau}_+(u, v) &= (r, (u_+, v_+)) \text{ for } (u, v) = ((\tau_+)^{-1}(r, u_+), (\sigma_+)^{-1}(r, v_+)) \in W_{\mathbb{R}_{\geq 1}}, \\ \tilde{\tau}_-(u, v) &= (r, (u_-, v_-)) \text{ for } (u, v) = ((\tau_-)^{-1}(r, u_-), (\sigma_-)^{-1}(r, v_-)) \in W_{\mathbb{R}_{\leq -1}}.\end{aligned}$$

Similarly, a cross product $x \times y \in \hat{C}_{p+q}^{\text{dR}}(X \times Y)$ is defined for $x = [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] \in \hat{C}_p^{\text{dR}}(X)$ and $y = [V, \psi, (\sigma_+^1, \sigma_-^1), (\sigma_+^2, \sigma_-^2), \eta] \in \hat{C}_q^{\text{dR}}(Y)$ by

$$x \times y := (-1)^{p|\eta|} [W, \tilde{\varphi}, (\tilde{\tau}_+^1, \tilde{\tau}_-^1), (\tilde{\tau}_+^2, \tilde{\tau}_-^2), \omega \times \eta].$$

Here, $W := U \times_{(\varphi_{\mathbb{R}}^1, \varphi_{\mathbb{R}}^2) \times (\psi_{\mathbb{R}}^1, \psi_{\mathbb{R}}^2)} V$ is a fiber product over \mathbb{R}^2 and $\tilde{\varphi}, \tilde{\tau}_+^j, \tilde{\tau}_-^j$ ($j = 1, 2$) are determined by

$$\begin{aligned}\tilde{\varphi}: W &\rightarrow \mathbb{R} \times (X \times Y): (u, v) \mapsto (\varphi_{\mathbb{R}}(u), \varphi_X(u), \psi_Y(v)), \\ \tilde{\tau}_+^j(u, v) &= (r, (u_+, v_+)) \text{ for } (u, v) = ((\tau_+^j)^{-1}(r, u_+), (\sigma_+^j)^{-1}(r, v_+)) \in W_{\{r_j=1\}}, \\ \tilde{\tau}_-^j(u, v) &= (r, (u_-, v_-)) \text{ for } (u, v) = ((\tau_-^j)^{-1}(r, u_-), (\sigma_-^j)^{-1}(r, v_-)) \in W_{\{r_j=-1\}}.\end{aligned}$$

The next result is analogous to Proposition 3.7 and Proposition 3.9. It follows immediately from the fact that $(e_+)_*: \bar{H}_*^{\text{dR}}(X) \rightarrow H_*^{\text{dR}}(X)$ is an isomorphism (see Lemma 3.16.).

Proposition 3.18. *Let $a, b \in \mathbb{R}_{>0}$ with $a \leq b$ and $\varepsilon \in (0, \varepsilon_0/C]$ for the constant C of Lemma 3.8. Then, the following hold:*

- If $\mathcal{L}_m(K) \cap [a, b] = \emptyset$, then $\bar{H}_*^{\text{dR}}(\Sigma_m^b, \Sigma_m^a) = 0$.
- For any $x \in \bar{H}_*^{\text{dR}}(S_\varepsilon)$,

$$(\text{ev}_0)_* \circ (e_+)_*(x) = 0 \in H_*^{\text{dR}}(N_\varepsilon) \Rightarrow (i_{\varepsilon, C\varepsilon})_*(x) = 0 \in \bar{H}_*^{\text{dR}}(S_{C\varepsilon})$$

3.6.4 Operations on $[-1, 1]$ and $[-1, 1]^2$ -modeled de Rham chains

In the rest of this section, let us define linear maps corresponding to $f_{k, \xi}$. We rather refer to the explicit description (13) of $f_{k, \xi}(x)$ than its original definition using fiber products of chains. Let $\varepsilon \in (0, \varepsilon_0/(5C_0))$ and $\rho_\varepsilon: A_\varepsilon \rightarrow [0, 1]$ be the C^∞ function we have chosen in Section 3.5

For $k = 1, \dots, m$ and $\bar{\xi} \in \bar{C}_q^{\text{dR}}(S_\varepsilon)$, we define a linear map

$$\bar{f}_{k, \bar{\xi}}: \bar{C}_*^{\text{dR}}(\Sigma_m^a) \rightarrow \bar{C}_{*+1+q-n}^{\text{dR}}(\Sigma_{m+1}^{a+\varepsilon})$$

as follows: Let

$$x = [U, \varphi, (\tau_+, \tau_-), \omega] \in \bar{C}_p^{\text{dR}}(\Sigma_m^a), \quad \bar{\xi} = [V, \psi, (\sigma_+, \sigma_-), \eta] \in \bar{C}_q^{\text{dR}}(S_\varepsilon),$$

and denote

$$\begin{aligned}\varphi(u) &= (\varphi_{\mathbb{R}}(u), \varphi_\Sigma(u)) = (\varphi_{\mathbb{R}}(u), (\gamma_l^u: [0, T_l^u] \rightarrow Q)_{l=1, \dots, m}) \in \mathbb{R} \times \Sigma_m^a, \\ \psi(v) &= (\psi_{\mathbb{R}}(v), \psi_S(v)) = (\psi_{\mathbb{R}}(v), (\sigma_i^v)_{i=1, 2}) \in \mathbb{R} \times S_\varepsilon,\end{aligned}$$

for every $u \in U$ and $v \in V$. Then we define $\bar{f}_{k, \bar{\xi}}(x) := (-1)^s [W_k, \Phi_k, (\tilde{\tau}_+, \tilde{\tau}_-), \zeta_k]$, where

$$\begin{aligned}W_k &:= \{(u, \tau, v) \in U \times \mathbb{R} \times V \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon, (\varphi_{\mathbb{R}}(u), \gamma_k^u(\tau)) = (\psi_{\mathbb{R}}(u), \sigma_1^v(0))\}, \\ \Phi_k: W_k &\rightarrow \mathbb{R} \times \Sigma_{m+1}^{a+\varepsilon}: (u, \tau, v) \mapsto (\varphi_{\mathbb{R}}(u), \text{con}_k(\varphi_\Sigma(u), (T_k^u, \tau), \psi_S(v))), \\ \zeta_k &\in \Omega_c^*(W_k): (\zeta_k)_{(u, \tau, v)} := \rho_\varepsilon(T_k^u, \tau) \cdot (\omega_u \times \eta_v), \\ s &:= (p+1-n)|\eta| + n + 1,\end{aligned} \tag{21}$$

and

$$\tilde{\tau}_+(u, \tau, v) := (r, (u_+, \tau, v_+)) \in \mathbb{R} \times (W_k)_{\{r=1\}}$$

for $(u, \tau, v) = (\tau_+^{-1}(r, u_+), \tau, \sigma_+^{-1}(r, v_+)) \in (W_k)_{\{r \geq 1\}}$, and

$$\tilde{\tau}_-(u, \tau, v) := (r, (u_-, \tau, v_-)) \in \mathbb{R} \times (W_k)_{\{r=-1\}}$$

for $(u, \tau, v) = (\tau_-^{-1}(r, u_-), \tau, \sigma_-^{-1}(r, v_-)) \in (W_k)_{\{r \leq -1\}}$. Here, W_k is oriented as a fiber product over $\mathbb{R} \times Q$ of the map

$$\{(u, \tau) \in U \times \mathbb{R} \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon\} \rightarrow \mathbb{R} \times Q: (u, \tau) \rightarrow (\varphi_{\mathbb{R}}(u), \gamma_k^u(\tau))$$

and the submersion $(\text{id}_{\mathbb{R}} \times \text{ev}_0) \circ \psi: V \rightarrow \mathbb{R} \times Q$. It can be checked that \bar{i} and e_+, e_- intertwine this operator and $f_{k, \xi}$ for $\xi \in C_q^{\text{dR}}(S_\varepsilon)$. Namely,

$$\bar{i} \circ f_{k, \xi} = f_{k, \bar{i}\xi} \circ \bar{i}, \quad e_+ \circ \bar{f}_{k, \bar{\xi}} = f_{k, e_+\bar{\xi}} \circ e_+, \quad e_- \circ \bar{f}_{k, \bar{\xi}} = f_{k, e_-\bar{\xi}} \circ e_-.$$

Similar results as $f_{k, \xi}$ are the following: $\bar{f}_{k, \bar{\xi}}$ induces a linear map

$$\bar{f}_{k, \bar{\xi}}: \bar{C}_*^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \rightarrow \bar{C}_{*+1+q-n}^{\text{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0),$$

which is independent on the choice of ρ_ε . The next equations are variants of (15) and (16), and they follow from similar computations as Lemma 3.13 and Lemma 3.14, so we omit the proof.

Proposition 3.19. *For $k' \geq k$, the following equations hold:*

$$\begin{aligned} \partial \circ \bar{f}_{k, \bar{\xi}} - \bar{f}_{k, \bar{\xi}} \circ \partial &= (-1)^{p+1-n} \bar{f}_{k, \partial \bar{\xi}}: \bar{C}_p^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \rightarrow \bar{C}_{p+1+q-n}^{\text{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0), \\ \bar{f}_{k'+1, \bar{\xi}} \circ \bar{f}_{k, \bar{\xi}} + (-1)^{q-n} \bar{f}_{k, \bar{\xi}} \circ \bar{f}_{k', \bar{\xi}} &= 0: \bar{C}_p^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \rightarrow \bar{C}_{p+2+2q-2n}^{\text{dR}}(\Sigma_{m+2}^{a+2\varepsilon}, \Sigma_{m+2}^0). \end{aligned}$$

Next, for $k = 1, \dots, m$ and $\hat{\xi} \in \hat{C}_q^{\text{dR}}(S_\varepsilon)$, we define a linear map

$$\hat{f}_{k, \hat{\xi}}: \bar{C}_*^{\text{dR}}(\Sigma_m^a) \rightarrow \bar{C}_{*+1+q-n}^{\text{dR}}(\Sigma_{m+1}^{a+\varepsilon})$$

as follows: Let

$$\begin{aligned} x &= [U, \varphi, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \omega] \in \hat{C}_p^{\text{dR}}(\Sigma_m^a), \\ \hat{\xi} &= [V, \psi, (\sigma_+^1, \sigma_-^1), (\sigma_+^2, \sigma_-^2), \eta] \in \hat{C}_q^{\text{dR}}(S_\varepsilon), \end{aligned}$$

and denote

$$\begin{aligned} \varphi(u) &= (\varphi_{\mathbb{R}^2}(u), \varphi_\Sigma(u)) = ((\varphi_{\mathbb{R}}^1(u), \varphi_{\mathbb{R}}^2(u)), (\gamma_l^u: [0, T_l^u] \rightarrow Q)_{l=1, \dots, m}) \in \mathbb{R}^2 \times \Sigma_m^a, \\ \psi(v) &= (\psi_{\mathbb{R}^2}(v), \psi_S(v)) = ((\psi_{\mathbb{R}}^1(v), \psi_{\mathbb{R}}^2(v)), (\sigma_i^v)_{i=1, 2}) \in \mathbb{R}^2 \times S_\varepsilon, \end{aligned}$$

for every $u \in U$ and $v \in V$. Then we define $\hat{f}_{k, \hat{\xi}}(x) := (-1)^s [W_k, \Phi_k, (\tilde{\tau}_+^1, \tilde{\tau}_-^1), (\tilde{\tau}_+^2, \tilde{\tau}_-^2), \zeta_k]$, where

$$\begin{aligned} W_k &:= \{(u, \tau, v) \in U \times \mathbb{R} \times V \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon, (\varphi_{\mathbb{R}^2}(u), \gamma_k^u(\tau)) = (\psi_{\mathbb{R}^2}(u), \sigma_1^v(0))\}, \\ \Phi_k &: W_k \rightarrow \mathbb{R}^2 \times \Sigma_{m+1}^{a+\varepsilon}: (u, \tau, v) \mapsto (\varphi_{\mathbb{R}^2}(u), \text{con}_k(\varphi_\Sigma(u), (T_k^u, \tau), \psi_S(v))), \\ \zeta_k &\in \Omega_c^*(W_k): (\zeta_k)_{(u, \tau, v)} := \rho_\varepsilon(T_k^u, \tau) \cdot (\omega_u \times \eta_v), \\ s &:= (p+1-n)|\eta|, \end{aligned}$$

and for $j = 1, 2$,

$$\tilde{\tau}_+^j(u, \tau, v) := (r_j, (u_+^j, \tau, v_+^j)) \in \mathbb{R}_{\geq 1} \times (W_k)_{\{r_j=1\}}$$

for $(u, \tau, v) = ((\tau_+^j)^{-1}(r_j, u_+^j), \tau, (\sigma^j)_+^{-1}(r_j, v_+^j)) \in (W_k)_{\{r_j \geq 1\}}$ and

$$\tilde{\tau}_-^j(u, \tau, v) := (r_j, (u_-^j, \tau, v_-^j)) \in \mathbb{R}_{\leq -1} \times (W_k)_{\{r_j = -1\}}$$

for $(u, \tau, v) = ((\tau_-^j)^{-1}(r_j, u_-^j), \tau, (\sigma^j)_-^{-1}(r_j, v_-^j)) \in (W_k)_{\{r_j \leq -1\}}$. Here, W_k is oriented as a fiber product over $\mathbb{R}^2 \times Q$ of the map

$$\{(u, \tau) \in U \times \mathbb{R} \mid 2\varepsilon < \tau < T_k^u - 2\varepsilon\} \rightarrow \mathbb{R}^2 \times Q: (u, \tau) \rightarrow (\varphi_{\mathbb{R}^2}(u), \gamma_k^u(\tau))$$

and the submersion $(\text{id}_{\mathbb{R}^2} \times \text{ev}_0) \circ \psi: V \rightarrow \mathbb{R}^2 \times Q$. It can be checked that \hat{v}^j and e_+^j, e_-^j ($j = 1, 2$) intertwine $\hat{f}_{k, \hat{\xi}}$ and $\bar{f}_{k, \bar{\xi}}$ for $\bar{\xi} \in \bar{C}_q^{\text{dR}}(S_\varepsilon)$.

Similar results as $f_{k, \xi}$ are the following: $\hat{f}_{k, \hat{\xi}}$ induces a linear map

$$\hat{f}_{k, \hat{\xi}}: \hat{C}_*^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \rightarrow \hat{C}_{*+1+q-n}^{\text{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0),$$

which is independent on ρ_ε . The next equations are variants of (15) and (16), and they follow from similar computations as Lemma 3.13 and Lemma 3.14, so we omit the proof.

Proposition 3.20. *For $k' \geq k$, the following equations hold:*

$$\begin{aligned} \partial \circ \hat{f}_{k, \hat{\xi}} - \hat{f}_{k, \hat{\xi}} \circ \partial &= (-1)^{p+1-n} \hat{f}_{k, \partial \hat{\xi}}: \hat{C}_p^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \rightarrow \hat{C}_{p+1+q-n}^{\text{dR}}(\Sigma_{m+1}^{a+\varepsilon}, \Sigma_{m+1}^0), \\ \hat{f}_{k'+1, \hat{\xi}} \circ \hat{f}_{k, \hat{\xi}} + (-1)^{q-n} \hat{f}_{k, \hat{\xi}} \circ \hat{f}_{k', \hat{\xi}} &= 0: \hat{C}_p^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \rightarrow \hat{C}_{p+2+2q-2n}^{\text{dR}}(\Sigma_{m+2}^{a+2\varepsilon}, \Sigma_{m+2}^0). \end{aligned}$$

4 Construction of $H_*^{\text{string}}(Q, K)$

4.1 Definition of chain complex

For $a \in \mathbb{R}_{>0}$ and $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, we define a graded \mathbb{R} -vector space

$$C_*^{<a}(\varepsilon) := \bigoplus_{m=0}^{\infty} C_{*-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

If $m \geq \frac{2a}{\varepsilon_0}$, then $a + m\varepsilon \leq m(\frac{1}{2}\varepsilon_0 + \varepsilon) \leq m\varepsilon_0$. In such case, $\Sigma_m^{a+m\varepsilon} = \Sigma_m^0$ by Remark 3.1. Therefore, the components for $m \in \mathbb{Z}_{\geq 0}$ vanishes if $m \geq \frac{2a}{\varepsilon_0}$.

For each $m \in \mathbb{Z}_{\geq 0}$, we think of $C_{*-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$ as a linear subspace of $C_*^{<a}(\varepsilon)$ in a natural way. For $\delta \in C_{n-d}^{\text{dR}}(S_\varepsilon)$, we define a degree (-1) linear map

$$D_\delta: C_*^{<a}(\varepsilon) \rightarrow C_{*-1}^{<a}(\varepsilon)$$

so that for $x \in C_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$,

$$D_\delta(x) = \partial x + \sum_{k=1}^m (-1)^{p+1+kd} f_{k, \delta}(x) \in C_{p-1}^{<a}(\varepsilon).$$

When $m = 0$, the RHS is just equal to ∂x .

Proposition 4.1. *If $\delta \in C_{n-d}^{\text{dR}}(S_\varepsilon)$ satisfies $\partial \delta = 0$, then $D_\delta \circ D_\delta = 0$ holds.*

Proof. Take an arbitrary $x \in C_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$. Since $\partial \circ \partial = 0$,

$$\begin{aligned} D_\delta \circ D_\delta(x) &= \sum_{k=1}^m \left((-1)^{p+1+kd} \partial \circ f_{k, \delta}(x) + (-1)^{p+kd} f_{k, \delta} \circ \partial(x) \right) \\ &\quad + \sum_{k'=1}^{m+1} \sum_{k=1}^m (-1)^{(k+k')d-1} f_{k', \delta} \circ f_{k, \delta}(x). \end{aligned}$$

Applying (15) for $q = n - d$ and $\xi = \delta$ for which $\partial\delta = 0$ holds, we can see that the first summand is equal to 0. For the second summand, we apply (16) for $q = n - d$ and $\xi = \delta$. Then

$$\begin{aligned} & \sum_{k'=1}^{m+1} \sum_{k=1}^m (-1)^{(k+k')d-1} f_{k',\delta} \circ f_{k,\delta}(x) \\ &= \sum_{1 \leq k \leq k' \leq m} \left((-1)^{(k+k'+1)d-1} f_{k'+1,\delta} \circ f_{k,\delta}(x) + (-1)^{(k+k')d-1} f_{k,\delta} \circ f_{k',\delta}(x) \right) \\ &= 0. \end{aligned}$$

This shows that $D_\delta \circ D_\delta(x) = 0$. \square

In summary, for $a \in \mathbb{R}_{>0}$, $\varepsilon \in (0, \varepsilon_0/(5C_0)]$ and $\delta \in C_{n-d}^{\text{dR}}(S_\varepsilon)$ with $\partial\delta = 0$, a chain complex $(C_*^{<a}(\varepsilon), D_\delta)$ is defined. Let $H_*^{<a}(\varepsilon, \delta)$ denote its homology.

The chain complex $(C_*^{<a}(\varepsilon), D_\delta)$ is filtered by subcomplexes $\{\mathcal{F}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ defined by

$$\mathcal{F}_{\varepsilon,p}^{<a} := \bigoplus_{m \geq -p}^{\infty} C_{*-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0). \quad (22)$$

Let $E_{(\varepsilon,\delta)}^{<a} := (\{(E_{(\varepsilon,\delta)}^{<a})_{p,q}^r\}, \{(d_{(\varepsilon,\delta)}^{<a})_{p,q}^r\})$ be the spectral sequence determined by $\{\mathcal{F}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$. Note that $\mathcal{F}_{\varepsilon,p}^{<a} = 0$ for $p \leq -\frac{2a}{\varepsilon_0}$ and $\mathcal{F}_{\varepsilon,p}^{<a} = \mathcal{F}_{\varepsilon,0}^{<a}$ for $p \geq 0$, and thus this spectral sequence converges to $H_*^{<a}(\varepsilon, \delta)$ in the sense of [31, Bounded Convergence 5.2.5]. The first page is given by

$$(E_{(\varepsilon,\delta)}^{<a})_{-m,q}^1 = \begin{cases} H_{(q-m)-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) = H_{q-m(d-1)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0. \end{cases}$$

Let us state an abstract lemma about morphisms in the category of spectral sequences. This result, which is a refinement of [31, Comparison Theorem 5.2.12], will be repeatedly used in the rest of this paper.

Lemma 4.2. *Let $E = (\{E_{p,q}^r\}, \{d_{p,q}^r\})$ and $E' = (\{E'_{p,q}{}^r\}, \{d'_{p,q}{}^r\})$ be bounded spectral sequences which converge to H_* and H'_* respectively in the sense of [31, Bounded Convergence 5.2.5]. Let $f = \{f_{p,q}^r\}$ be a morphism from E to E' which is compatible with $\{h_n: H_n \rightarrow H'_n\}_{n \in \mathbb{Z}}$. Then, the following assertion hold:*

- *Suppose that for some $r_0 \geq 1$ and $n_0 \in \mathbb{Z}$, $f_{p,q}^{r_0}$ is an isomorphism if $p+q < n_0$, and a surjection if $p+q = n_0$. Then, h_n is an isomorphism if $n < n_0$ and a surjection if $n = n_0$. In particular, if $f_{p,q}^{r_0}$ is an isomorphism for every $p, q \in \mathbb{Z}$ for some $r_0 \geq 1$, then h_n is an isomorphism for every $n \in \mathbb{Z}$.*

Proof. Suppose that $\{f_{p,q}^r\}$ satisfy the condition of the assertion for $r_0 \geq 1$ and $n_0 \in \mathbb{Z}$. Note that for any $r \geq 1$ and $p, q \in \mathbb{Z}$,

$$f_{p,q}^{r+1}: E_{p,q}^{r+1} \cong \text{Ker } d_{p,q}^r / \text{Im } d_{p+r,q-r+1}^r \rightarrow E'_{p,q}{}^{r+1} \cong \text{Ker } d'_{p,q}{}^r / \text{Im } d'_{p+r,q-r+1}{}^r$$

is induced by $\{f_{p,q}^r\}_{p,q}$. Therefore, by inductive arguments about $\{f_{p,q}^r\}$ on $r = r_0, r_0+1, \dots$, we can prove that $f_{p,q}^\infty: E_{p,q}^\infty \rightarrow E'_{p,q}{}^\infty$ is an isomorphism if $p+q < n_0$, and a surjection if $p+q = n_0$. We omit the concluding argument about h_n , since it is parallel to [31, Comparison Theorem 5.2.12]. \square

For $\varepsilon, \bar{\varepsilon} \in (0, \varepsilon_0/(5C_0)]$ with $\varepsilon \leq \bar{\varepsilon}$, let $j_{\varepsilon,\bar{\varepsilon}}: \Sigma_m^{a+m\varepsilon} \rightarrow \Sigma_m^{a+m\bar{\varepsilon}}$ ($m \in \mathbb{Z}_{\geq 0}$) be the inclusion maps. These maps induce a linear map

$$(j_{\varepsilon,\bar{\varepsilon}})_*: C_*^{<a}(\varepsilon) \rightarrow C_*^{<a}(\bar{\varepsilon}).$$

Moreover, for any $\delta \in C_{n-d}^{\text{dR}}(S_\varepsilon)$ with $\partial\delta = 0$, $(j_{\varepsilon,\bar{\varepsilon}})_*$ is a chain map from $(C_*^{<a}(\varepsilon), D_\delta)$ to $(C_*^{<a}(\bar{\varepsilon}), D_{(i_{\varepsilon,\bar{\varepsilon}})_*\delta})$ and preserves the filtrations $\{\mathcal{F}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathcal{F}_{\bar{\varepsilon},p}^{<a}\}_{p \in \mathbb{Z}}$.

Lemma 4.3. *Suppose that $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$. Then the induced map on homology*

$$(j_{\varepsilon, \bar{\varepsilon}})_* : H_*^{<a}(\varepsilon, \delta) \rightarrow H_*^{<a}(\bar{\varepsilon}, (i_{\varepsilon, \bar{\varepsilon}})_* \delta)$$

is an isomorphism if $\bar{\varepsilon}$ satisfies $[a, a + \frac{2a}{\varepsilon_0} \bar{\varepsilon}] \cap \mathcal{L}(K) = \emptyset$.

Proof. If $m \geq \frac{2a}{\varepsilon_0}$, $\Sigma_m^{a+m\bar{\varepsilon}} = \Sigma_m^{a+m\varepsilon} = \Sigma_m^0$. If $0 \leq m \leq \frac{2a}{\varepsilon_0}$, $[a + m\varepsilon, a + m\bar{\varepsilon}] \cap \mathcal{L}(K) = \emptyset$ from the condition on $\bar{\varepsilon}$. Thus, Proposition 3.7 is applied to show that $H_*^{\text{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}, \Sigma_m^{a+m\varepsilon}) = 0$ for all $m \in \mathbb{Z}_{\geq 0}$. Therefore, the induced map on the $(-m, q)$ -term ($m \geq 0$) of the first page

$$(j_{\varepsilon, \bar{\varepsilon}})_* : (E_{(\varepsilon, \delta)}^{<a})_{-m, q}^1 = H_{q-m(d-1)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) \rightarrow (E_{(\bar{\varepsilon}, (i_{\varepsilon, \bar{\varepsilon}})_* \delta)}^{<a})_{-m, q}^1 = H_{q-m(d-1)}^{\text{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}, \Sigma_m^0)$$

is an isomorphism. Now the assertion follows from Lemma 4.2. \square

As we have seen in Example 2.6, $H_*^{\text{dR}}(N_\varepsilon^{\text{reg}}) \cong H_c^d(N_\varepsilon; \mathbb{R})$. Therefore we can determine a unique homology class $\text{Th}_\varepsilon \in H_{n-d}^{\text{dR}}(N_\varepsilon^{\text{reg}})$ which corresponds to the Thom class of $(TK)^\perp$ through the diffeomorphism $\{(x, v) \in (TK)^\perp \mid v < \varepsilon\} \rightarrow N_\varepsilon : (x, v) \mapsto \exp_x v$.

The above lemma leads us to define a set of data (ε, δ) as follows.

Definition 4.4. Let $C \geq 1$ be the constant of Lemma 3.8. We define \mathcal{T}_a for every $a \in \mathbb{R}_{\geq 0} \setminus \mathcal{L}(K)$ to be the set of pairs (ε, δ) of $\varepsilon \in (0, \varepsilon_0/(5C^4)]$ and $\delta \in C_{n-d}^{\text{dR}}(S_\varepsilon)$ such that:

1. $[a, a + \frac{2a}{\varepsilon_0} \hat{\varepsilon}] \cap \mathcal{L}(K) = \emptyset$ for $\hat{\varepsilon} := C^3 \varepsilon$.
2. $\partial \delta = 0$ and $(\text{ev}_0)_*[\delta] = \text{Th}_\varepsilon \in H_{n-d}^{\text{dR}}(N_\varepsilon^{\text{reg}})$.

Let $a, b \in \mathbb{R}_{>0}$ with $a < b$. For $\varepsilon \in (0, \varepsilon_0/(5C_0)]$ and $\delta \in C_{n-d}^{\text{dR}}(S_\varepsilon)$ with $\partial \delta = 0$, there exists a chain map $(I_\varepsilon^{a,b})_*$ from $(C_*^{<a}(\varepsilon), D_\delta)$ to $(C_*^{<b}(\varepsilon), D_\delta)$ induced by inclusion maps $I_\varepsilon^{a,b} : \Sigma_m^{a+m\varepsilon} \rightarrow \Sigma_m^{b+m\varepsilon}$ for all $m \in \mathbb{Z}_{\geq 0}$. We define a quotient complex

$$(C_*^{[a,b]}(\varepsilon) := C_*^{<b}(\varepsilon)/C_*^{<a}(\varepsilon), D_\delta).$$

Let $H_*^{[a,b]}(\varepsilon, \delta)$ denote its homology. Obviously, there exists a long exact sequence

$$\dots \longrightarrow H_*^{<a}(\varepsilon, \delta) \xrightarrow{(I_\varepsilon^{a,b})_*} H_*^{<b}(\varepsilon, \delta) \longrightarrow H_*^{[a,b]}(\varepsilon, \delta) \longrightarrow H_{*-1}^{<a}(\varepsilon, \delta) \xrightarrow{(I_\varepsilon^{a,b})_*} \dots \quad (23)$$

The next result is computations from the spectral sequence.

Proposition 4.5. *For $a, b \in \mathbb{R} \setminus \mathcal{L}(K)$ with $a < b$ and $(\varepsilon, \delta) \in \mathcal{T}_a \cap \mathcal{T}_b$, the following hold:*

- *If $[a, b] \cap \mathcal{L}(K) = \emptyset$, then $H_*^{[a,b]}(\varepsilon, \delta) = 0$.*
- *If there exist $c \in \mathcal{L}(K)$ and $m_0 \in \mathbb{Z}_{\geq 1}$ such that $[a, b] \cap \mathcal{L}_m(K) = \begin{cases} \{c\} & \text{if } m = m_0, \\ \emptyset & \text{else,} \end{cases}$ then*

$$H_*^{[a,b]}(\varepsilon, \delta) \cong H_{*-m_0(d-2)}^{\text{dR}}(\Sigma_{m_0}^b, \Sigma_{m_0}^a).$$

Proof. Let $E_{(\varepsilon, \delta)}^{[a,b]}$ be the spectral sequence determined by a filtration $\{\mathcal{F}_{\varepsilon, m}^{<b}/\mathcal{F}_{\varepsilon, m}^{<a}\}_{m \in \mathbb{Z}}$. We apply Proposition 3.7 to the first page. For the first case, $(E_{(\varepsilon, \delta)}^{[a,b]})_{p, q}^1 = 0$ for every $p, q \in \mathbb{Z}$, so the assertion is trivial. For the second case, $(E_{(\varepsilon, \delta)}^{[a,b]})_{p, q}^1 = 0$ for every $p \neq -m_0$, so all differentials are the zero map. Therefore, $H_q^{[a,b]}(\varepsilon, \delta) \cong (E_{(\varepsilon, \delta)}^{[a,b]})_{-m_0, q+m_0}^1$ and the assertion follows from

$$(E_{(\varepsilon, \delta)}^{[a,b]})_{-m_0, q+m_0}^1 = H_{q-m_0(d-2)}^{\text{dR}}(\Sigma_{m_0}^{b+m_0\varepsilon}, \Sigma_{m_0}^{a+m_0\varepsilon}) \cong H_{q-m_0(d-2)}^{\text{dR}}(\Sigma_{m_0}^b, \Sigma_{m_0}^a).$$

Here, the last isomorphism comes from Proposition 3.7. \square

4.2 Variants from $[-1, 1]$ and $[-1, 1]^2$ -modeled de Rham chains

In Section 3.6, we introduced $[-1, 1]$ -modeled and $[-1, 1]^2$ -modeled de Rham chains. In this section, we define chain complexes as Section 4.1 by using these types of chains. Their constructions and some of computations are parallel to the previous section, so we often omit proofs.

First, we deal with $[-1, 1]$ -modeled chains. For $a \in \mathbb{R}_{>0}$ and $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, we consider a graded \mathbb{R} -vector space

$$\bar{C}_*^{<a}(\varepsilon) := \bigoplus_{m=0}^{\infty} \bar{C}_{*-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

For $\bar{\delta} \in \bar{C}_{n-d}^{\text{dR}}(S_\varepsilon)$, we define a degree (-1) map $\bar{D}_{\bar{\delta}}: \bar{C}_*^{<a}(\varepsilon) \rightarrow \bar{C}_{*-1}^{<a}(\varepsilon)$ by

$$\bar{D}_{\bar{\delta}}(x) := \partial x + \sum_{k=1}^m (-1)^{p+1+kd} \bar{f}_{k,\bar{\delta}}(x)$$

for $x \in \bar{C}_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$.

Proposition 4.6. *If $\bar{\delta} \in \bar{C}_{n-d}^{\text{dR}}(S_\varepsilon)$ satisfies $\partial\bar{\delta} = 0$, then $\bar{D}_{\bar{\delta}} \circ \bar{D}_{\bar{\delta}} = 0$ holds.*

This is analogous to Proposition 4.1 and can be deduced from the two equations of Proposition 3.19. From this proposition, for $\bar{\delta} \in \bar{C}_{n-d}^{\text{dR}}(S_\varepsilon)$ with $\partial\bar{\delta} = 0$, we obtain a chain complex $(\bar{C}_*^{<a}(\varepsilon), \bar{D}_{\bar{\delta}})$. Let $\bar{H}_*^{<a}(\varepsilon, \bar{\delta})$ denote its homology.

Let us consider a relation to the chain complex defined in Section 4.1. The linear maps (18) for $X = \Sigma_m^{a+m\varepsilon}$

$$e_+, e_-: \bar{C}_*^{\text{dR}}(\Sigma_m^{a+m\varepsilon}) \rightarrow C_*^{\text{dR}}(\Sigma_m^{a+m\varepsilon}) \quad (m \in \mathbb{Z}_{\geq 0})$$

naturally induce linear maps

$$e_{\varepsilon,+}, e_{\varepsilon,-}: \bar{C}_*^{<a}(\varepsilon) \rightarrow C_*^{<a}(\varepsilon),$$

and these are chain maps from $(\bar{C}_*^{<a}(\varepsilon), \bar{D}_{\bar{\delta}})$ to $(C_*^{<a}(\varepsilon), D_{e_+\bar{\delta}})$ and $(C_*^{<a}(\varepsilon), D_{e_-\bar{\delta}})$ respectively.

We define a filtration $\{\bar{\mathcal{F}}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ by

$$\bar{\mathcal{F}}_{\varepsilon,p}^{<a} := \bigoplus_{m \geq -p} \bar{C}_{*-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

Let $\bar{E}_{(\varepsilon,\bar{\delta})}^{<a}$ be the spectral sequence determined by this filtration.

Lemma 4.7. *$e_{\varepsilon,+}$ and $e_{\varepsilon,-}$ are quasi-isomorphisms.*

Proof. We prove this assertion for only $e_{\varepsilon,+}$. The proof for $e_{\varepsilon,-}$ is parallel. Since $e_{\varepsilon,+}$ preserves filtrations $\{\bar{\mathcal{F}}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathcal{F}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$, this induces a map on the first page $(e_{\varepsilon,+})_*: (\bar{E}_{(\varepsilon,\bar{\delta})}^{<a})_{p,q}^1 \rightarrow (E_{(\varepsilon,e_+\bar{\delta})}^{<a})_{p,q}^1$. For $p = -m \leq 0$, this coincides with

$$(e_+)_*: \bar{H}_{q-m(d-1)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0) \rightarrow H_{q-m(d-1)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

From Lemma 3.16, this map is an isomorphism. Now the assertion follows from Lemma 4.2. \square

For $\bar{\varepsilon}, \hat{\varepsilon} \in (0, \varepsilon_0/(5C_0)]$ with $\bar{\varepsilon} \leq \hat{\varepsilon}$, the linear map $(j_{\bar{\varepsilon},\hat{\varepsilon}})_*: \bar{C}_*^{<a}(\bar{\varepsilon}) \rightarrow \bar{C}_*^{<a}(\hat{\varepsilon})$, induced by the inclusion maps $j_{\bar{\varepsilon},\hat{\varepsilon}}: \Sigma_m^{a+m\bar{\varepsilon}} \rightarrow \Sigma_m^{a+m\hat{\varepsilon}}$ for all $m \in \mathbb{Z}_{\geq 0}$, is a chain map from $(\bar{C}_*^{<a}(\bar{\varepsilon}), \bar{D}_{\bar{\delta}})$ to $(\bar{C}_*^{<a}(\hat{\varepsilon}), \bar{D}_{(i_{\bar{\varepsilon},\hat{\varepsilon}})_*\bar{\delta}})$.

Lemma 4.8. *Suppose that $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$. Then the induced map on homology*

$$(j_{\bar{\varepsilon}, \hat{\varepsilon}})_* : \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}) \rightarrow \bar{H}_*^{<a}(\hat{\varepsilon}, (i_{\bar{\varepsilon}, \hat{\varepsilon}})_* \bar{\delta})$$

is an isomorphism if $\hat{\varepsilon}$ satisfies $[a, a + \frac{2a}{\varepsilon_0} \hat{\varepsilon}] \cap \mathcal{L}(K) = \emptyset$.

Proof. The proof is parallel to that of Lemma 4.3. The chain map $(j_{\bar{\varepsilon}, \hat{\varepsilon}})_*$ preserves the filtrations $\{\bar{\mathcal{F}}_{\bar{\varepsilon}, p}^{<a}\}_{p \in \mathbb{Z}}$ and $\{\bar{\mathcal{F}}_{\hat{\varepsilon}, p}^{<a}\}_{p \in \mathbb{Z}}$. This induces an isomorphism on the first page since for every $m \in \mathbb{Z}_{\geq 0}$,

$$\bar{H}_*(\Sigma_m^{a+m\hat{\varepsilon}}, \Sigma_m^{a+m\bar{\varepsilon}}) = 0$$

holds by Proposition 3.18. Now the assertion follows from Lemma 4.2. \square

The above lemma leads us to the following definition.

Definition 4.9. Let $C \geq 1$ be the constant of Lemma 3.8. We define $\bar{\mathcal{T}}_a$ for $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ to be the set of pairs $(\bar{\varepsilon}, \bar{\delta})$ of $\bar{\varepsilon} \in (0, \varepsilon_0/(5C^3)]$ and $\bar{\delta} \in \bar{C}_{n-d}^{\text{dR}}(S_{\bar{\varepsilon}})$ such that:

1. $[a, a + \frac{2a}{\varepsilon_0} \hat{\varepsilon}] \cap \mathcal{L}(K) = \emptyset$ for $\hat{\varepsilon} := C^2 \bar{\varepsilon}$.
2. $\partial \bar{\delta} = 0$ and $(\text{ev}_0)_*[e_+ \bar{\delta}] = \text{Th}_{\bar{\varepsilon}} \in H_{n-d}^{\text{dR}}(N_{\bar{\varepsilon}}^{\text{reg}})$.

Next, we deal with $[-1, 1]^2$ -modeled chains. For $a \in \mathbb{R}_{>0}$ and $\varepsilon \in (0, \varepsilon_0/(5C_0)]$, we consider a graded \mathbb{R} -vector space

$$\hat{C}_*^{<a}(\varepsilon) := \bigoplus_{m=0}^{\infty} \hat{C}_{*-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

For $\hat{\delta} \in \hat{C}_{n-d}^{\text{dR}}(S_{\varepsilon})$, we define a degree (-1) map $\hat{D}_{\hat{\delta}}: \hat{C}_*^{<a}(\varepsilon) \rightarrow \hat{C}_{*-1}^{<a}(\varepsilon)$ by

$$\hat{D}_{\hat{\delta}}(x) := \partial x + \sum_{k=1}^m (-1)^{p+1+kd} \hat{f}_{k, \hat{\delta}}(x)$$

for $x \in \hat{C}_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$.

Proposition 4.10. *If $\hat{\delta} \in \hat{C}_{n-d}^{\text{dR}}(S_{\varepsilon})$ satisfies $\partial \hat{\delta} = 0$, then $\hat{D}_{\hat{\delta}} \circ \hat{D}_{\hat{\delta}} = 0$ holds.*

This is analogous to Proposition 4.1 and can be deduced from the two equations of Proposition 3.20. From this proposition, for $\hat{\delta} \in \hat{C}_{n-d}^{\text{dR}}(S_{\varepsilon})$ with $\partial \hat{\delta} = 0$, we obtain a chain complex $(\hat{C}_*^{<a}(\varepsilon), \hat{D}_{\hat{\delta}})$. Let $\hat{H}_*^{<a}(\varepsilon, \hat{\delta})$ denote its homology.

Let us consider a relation to the chain complex defined by $[-1, 1]$ -modeled de Rham chains. For $j = 1, 2$, the linear maps of (19) for $X = \Sigma_m^{a+m\varepsilon}$

$$e_{+, -}^j : \hat{C}_*^{\text{dR}}(\Sigma_m^{a+m\varepsilon}) \rightarrow \bar{C}_*^{\text{dR}}(\Sigma_m^{a+m\varepsilon}) \quad (m \in \mathbb{Z}_{\geq 0})$$

naturally induce linear maps

$$e_{\varepsilon, +}^j, e_{\varepsilon, -}^j : \hat{C}_*^{<a}(\varepsilon) \rightarrow \bar{C}_*^{<a}(\varepsilon),$$

and these are chain maps from $(\hat{C}_*^{<a}(\varepsilon), \hat{D}_{\hat{\delta}})$ to $(\bar{C}_*^{<a}(\varepsilon), \bar{D}_{e_{+, \hat{\delta}}^j})$ and $(\bar{C}_*^{<a}(\varepsilon), \bar{D}_{e_{-, \hat{\delta}}^j})$ respectively.

Lemma 4.11. *$e_{\varepsilon, +}^j$ and $e_{\varepsilon, -}^j$ ($j = 1, 2$) are quasi-isomorphisms.*

Proof. The proof is parallel to that of Lemma 4.7. This time, we use the spectral sequence determined by a filtration $\{\hat{\mathcal{F}}_{\varepsilon, p}^a\}_{p \in \mathbb{Z}}$, which is defined by

$$\hat{\mathcal{F}}_{\varepsilon, p}^a := \bigoplus_{m \geq -p} \hat{C}_{*-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

$e_{\varepsilon, +}^j, e_{\varepsilon, -}^j$ ($j = 1, 2$) preserve filtrations $\{\hat{\mathcal{F}}_{\varepsilon, p}^a\}_{p \in \mathbb{Z}}$ and $\{\bar{\mathcal{F}}_{\varepsilon, p}^a\}_{p \in \mathbb{Z}}$. By Lemma 3.17, they induce an isomorphism on the first page. Now the assertion follows from Lemma 4.2. \square

4.3 The limit of $\varepsilon \rightarrow 0$

In this section, we define a transition map

$$k_{(\varepsilon', \delta'), (\varepsilon, \delta)}: H_*^{<a}(\varepsilon', \delta') \rightarrow H_*^{<a}(\varepsilon, \delta)$$

for every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ and $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$, by using $(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_a$ satisfying

$$\varepsilon \leq \bar{\varepsilon}, e_+ \bar{\delta} = (i_{\varepsilon, \bar{\varepsilon}})_* \delta, e_- \bar{\delta} = (i_{\varepsilon', \bar{\varepsilon}})_* \delta'. \quad (24)$$

In fact, $k_{(\varepsilon', \delta'), (\varepsilon, \delta)}$ is an isomorphism. We also prove that $(\{H_*^{<a}(\varepsilon, \delta)\}_{(\varepsilon, \delta) \in \mathcal{T}_a}, \{k_{(\varepsilon', \delta'), (\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon})$ forms an inverse system.

4.3.1 Construction of transition maps

Let us first prove the existence of the above $(\bar{\varepsilon}, \bar{\delta})$.

Lemma 4.12. *For $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$, there exists $(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_a$ satisfying (24).*

Proof. Let us take $\bar{\varepsilon} := C\varepsilon$ for the constant C in Lemma 3.8, and rewrite $\delta_+ := (i_{\varepsilon, \bar{\varepsilon}})_* \delta$ and $\delta_- := (i_{\varepsilon', \bar{\varepsilon}})_* \delta'$ for short. Since

$$(\text{ev}_0)_*[\delta - (i_{\varepsilon', \varepsilon})_* \delta'] = \text{Th}_\varepsilon - (i_{\varepsilon', \varepsilon})_* \text{Th}_{\varepsilon'} = 0 \in H_{n-d}^{\text{dR}}(N_\varepsilon),$$

Proposition 3.9 shows that there exists $\theta \in C_{n-d+1}^{\text{dR}}(S_{\bar{\varepsilon}})$ such that

$$\partial\theta = (i_{\varepsilon, \bar{\varepsilon}})_*(\delta - (i_{\varepsilon', \varepsilon})_* \delta') = \delta_+ - \delta_-.$$

Let $\kappa: \mathbb{R} \rightarrow [0, 1]$ be a C^∞ function such that $\kappa(r) = \begin{cases} 1 & \text{if } 1 \leq r, \\ 0 & \text{if } r \leq -1. \end{cases}$ We take chains $\beta_+, \beta_- \in \bar{C}_0^{\text{dR}}(\{0\})$ defined by

$$\begin{cases} \beta_+ := [\mathbb{R}, \text{id}_{\mathbb{R}}, (\text{id}_{\mathbb{R}_{\geq 1}}, \text{id}_{\mathbb{R}_{\leq -1}}), \kappa], \\ \beta_- := [\mathbb{R}, \text{id}_{\mathbb{R}}, (\text{id}_{\mathbb{R}_{\geq 1}}, \text{id}_{\mathbb{R}_{\leq -1}}), 1 - \kappa]. \end{cases}$$

Now we define $\bar{\delta}$ by

$$\bar{\delta} := \beta_+ \times (\bar{i}\delta_+) + \beta_- \times (\bar{i}\delta_-) + (\partial\beta_+) \times (\bar{i}\theta) \in \bar{C}_{n-d}^{\text{dR}}(S_{\bar{\varepsilon}}).$$

This satisfies the condition (24). Moreover, $\partial\bar{\delta} = 0$ and $(\text{ev}_0)_*[e_+ \bar{\delta}] = (\text{ev}_0)_*[\delta_+] = \text{Th}_{\bar{\varepsilon}}$ hold. Now, it is clear that $(\bar{\varepsilon}, \bar{\delta}) = (C\varepsilon, \bar{\delta})$ satisfies the two conditions to be an element of $\bar{\mathcal{T}}_a$. \square

From Lemma 4.3 and Lemma 4.7, we can define isomorphisms

$$\begin{aligned} f_{(\bar{\varepsilon}, \bar{\delta}), +} &:= (j_{\varepsilon, \bar{\varepsilon}})_*^{-1} \circ (e_{\bar{\varepsilon}, +})_*: \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}) \rightarrow H_*^{<a}(\varepsilon, \delta), \\ f_{(\bar{\varepsilon}, \bar{\delta}), -} &:= (j_{\varepsilon', \bar{\varepsilon}})_*^{-1} \circ (e_{\bar{\varepsilon}, -})_*: \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}) \rightarrow H_*^{<a}(\varepsilon', \delta'), \end{aligned}$$

so that the following diagrams commute:

$$\begin{array}{ccc} \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}) & \xrightarrow{f_{(\bar{\varepsilon}, \bar{\delta}), +}} & H_*^{<a}(\varepsilon, \delta) & & \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}) & \xrightarrow{f_{(\bar{\varepsilon}, \bar{\delta}), -}} & H_*^{<a}(\varepsilon', \delta') \\ (e_{\bar{\varepsilon}, +})_* \downarrow & & \downarrow (j_{\varepsilon, \bar{\varepsilon}})_* & & (e_{\bar{\varepsilon}, -})_* \downarrow & & \downarrow (j_{\varepsilon', \bar{\varepsilon}})_* \\ H_*^{<a}(\bar{\varepsilon}, e_+ \bar{\delta}) & \xlongequal{\quad} & H_*^{<a}(\bar{\varepsilon}, (i_{\varepsilon, \bar{\varepsilon}})_* \delta), & & H_*^{<a}(\bar{\varepsilon}, e_- \bar{\delta}) & \xlongequal{\quad} & H_*^{<a}(\bar{\varepsilon}, (i_{\varepsilon', \bar{\varepsilon}})_* \delta'). \end{array}$$

We define an isomorphism

$$k_{(\bar{\varepsilon}, \bar{\delta})} := f_{(\bar{\varepsilon}, \bar{\delta}), +} \circ (f_{(\bar{\varepsilon}, \bar{\delta}), -})^{-1}: H_*^{<a}(\varepsilon', \delta') \rightarrow H_*^{<a}(\varepsilon, \delta).$$

Later, we will prove the independence on $(\bar{\varepsilon}, \bar{\delta})$ (Corollary 4.16), and this is the transition map $k_{(\varepsilon', \delta'), (\varepsilon, \delta)}$ we need.

Lemma 4.13. *When $(\varepsilon', \delta') = (\varepsilon, \delta) \in \mathcal{T}_a$, we may take $(\varepsilon, \bar{i}\delta) \in \bar{\mathcal{T}}_a$ as an element satisfying (24). In this case, we have*

$$k_{(\varepsilon, \bar{i}\delta)} = \text{id}_{H_*^{<a}(\varepsilon, \delta)}$$

Proof. To prove this assertion, let us introduce a chain map \bar{i}_ε from $(C_*^{<a}(\varepsilon), D_\delta)$ to $(\bar{C}_*^{<a}(\varepsilon), \bar{D}_{\bar{i}\delta})$ induced by $\bar{i}: C_{*-m}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}) \rightarrow \bar{C}_{*-m}^{\text{dR}}(\Sigma_m^{a+m\varepsilon})$ (see (17)) for all $m \in \mathbb{Z}_{\geq 0}$. This satisfies $e_{\varepsilon,+} \circ \bar{i}_\varepsilon = \text{id}_{C_*^{<a}(\varepsilon)} = e_{\varepsilon,-} \circ \bar{i}_\varepsilon$, and thus

$$(e_{\varepsilon,+})_* = (\bar{i}_\varepsilon)_*^{-1} = (e_{\varepsilon,-})_*: \bar{H}_*^{<a}(\varepsilon, \bar{i}\delta) \rightarrow H_*^{<a}(\varepsilon, \delta). \quad (25)$$

Therefore, $k_{(\varepsilon, \bar{i}\delta)} = (e_{\varepsilon,+})_* \circ (e_{\varepsilon,-})_*^{-1} = \text{id}_{H_*^{<a}(\varepsilon, \delta)}$. \square

4.3.2 Compositions

Next, we think about compositions of maps of $\{k_{(\bar{\varepsilon}, \bar{\delta})}\}_{(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_a}$. For $(\varepsilon, \delta), (\varepsilon', \delta'), (\varepsilon'', \delta'') \in \mathcal{T}_a$ with $\varepsilon'' \leq \varepsilon' \leq \varepsilon$, suppose that we have chosen $(\bar{\varepsilon}, \bar{\delta}), (\bar{\varepsilon}', \bar{\delta}'), (\tilde{\varepsilon}, \tilde{\delta}) \in \bar{\mathcal{T}}_a$ satisfying:

$$\begin{aligned} e_+ \bar{\delta} &= (i_{\varepsilon, \bar{\varepsilon}})_* \delta, & e_- \bar{\delta} &= (i_{\varepsilon', \bar{\varepsilon}})_* \delta', \\ e_+ \bar{\delta}' &= (i_{\varepsilon', \bar{\varepsilon}'})_* \delta', & e_- \bar{\delta}' &= (i_{\varepsilon'', \bar{\varepsilon}'})_* \delta'', \\ e_+ \tilde{\delta} &= (i_{\varepsilon, \tilde{\varepsilon}})_* \delta, & e_- \tilde{\delta} &= (i_{\varepsilon'', \tilde{\varepsilon}})_* \delta''. \end{aligned}$$

In this case, let us first prove the following lemma.

Lemma 4.14. *There exists $\hat{\varepsilon} \in (0, \varepsilon_0/(5C_0)]$ and $\hat{\delta} \in \hat{C}_{n-d}^{\text{dR}}(S_{\hat{\varepsilon}})$ such that $\partial \hat{\delta} = 0$ and*

$$e_+^1 \hat{\delta} = (i_{\tilde{\varepsilon}, \hat{\varepsilon}})_* \tilde{\delta}, \quad e_-^1 \hat{\delta} = (i_{\varepsilon', \hat{\varepsilon}})_* \bar{\delta}', \quad e_+^2 \hat{\delta} = (i_{\tilde{\varepsilon}, \hat{\varepsilon}})_* \bar{\delta}, \quad e_-^2 \hat{\delta} = (i_{\varepsilon'', \hat{\varepsilon}})_* (\bar{i}\delta'').$$

Proof. Let us take $\rho := C \cdot \max\{\bar{\varepsilon}, \bar{\varepsilon}', \tilde{\varepsilon}\}$, $\hat{\varepsilon} := C\rho$ and rewrite $\bar{\delta}_+ := (i_{\tilde{\varepsilon}, \rho})_* \tilde{\delta}$ and $\bar{\delta}_- := (i_{\varepsilon', \rho})_* \bar{\delta}'$ for short. Since

$$\begin{aligned} (\text{ev}_0)_* \circ (e_+)_* [(i_{\tilde{\varepsilon}, C-1\rho})_* \tilde{\delta} - (i_{\varepsilon', C-1\rho})_* \bar{\delta}'] &= (i_{\tilde{\varepsilon}, C-1\rho})_* (\text{Th}_{\tilde{\varepsilon}}) - (i_{\varepsilon', C-1\rho})_* (\text{Th}_{\varepsilon'}) \\ &= 0 \in H_*^{\text{dR}}(N_{C-1\rho}), \end{aligned}$$

Proposition 3.18 shows that there exists $\bar{\theta}_1 \in \bar{C}_{n-d+1}^{\text{dR}}(S_\rho)$ such that $\partial \bar{\theta}_1 = \bar{\delta}_+ - \bar{\delta}_-$. We define $\hat{\kappa}^1: \mathbb{R} \times \mathbb{R} \rightarrow [0, 1]: (r_1, r_2) \mapsto \kappa(r_1)$, where κ is the function appeared in the proof of Lemma 4.12. Then, we take chains $\hat{\beta}_+, \hat{\beta}_- \in \hat{C}_0^{\text{dR}}(\{0\})$ define by

$$\begin{cases} \hat{\beta}_+^1 := [\mathbb{R}^2, \text{id}_{\mathbb{R}^2}, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \hat{\kappa}], \\ \hat{\beta}_-^1 := [\mathbb{R}^2, \text{id}_{\mathbb{R}^2}, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), 1 - \hat{\kappa}], \end{cases}$$

where $\tau_+^j = \text{id}_{\{r_j \geq 1\}}$ and $\tau_-^j = \text{id}_{\{r_j \leq -1\}}$ for $j = 1, 2$. We define

$$\xi := \hat{\beta}_+^1 \times (\hat{i}^1 \bar{\delta}_+) + \hat{\beta}_-^1 \times (\hat{i}^1 \bar{\delta}_-) + (\partial \hat{\beta}_+^1) \times (\hat{i}^1 (\bar{\theta}_1 - \bar{i}e_- \bar{\theta}_1)) \in \hat{C}_{n-d}^{\text{dR}}(S_\rho).$$

This chain satisfies $\partial \xi = 0$ (note that $e_- (\partial \bar{\theta}_1) = e_- \bar{\delta}_+ - e_- \bar{\delta}_- = 0$). Moreover,

$$e_+^1 \xi = (i_{\tilde{\varepsilon}, \rho})_* \tilde{\delta}, \quad e_-^1 \xi = (i_{\varepsilon', \rho})_* \bar{\delta}', \quad e_-^2 \xi = (i_{\varepsilon'', \rho})_* (\bar{i}\delta'')$$

hold. The former two equations are easy to check. The third equation can be checked as follows: Since $e_-^2 \circ \hat{i}^1 = \bar{i} \circ e_-$ holds, we have $e_-^2 (\hat{i}^1 (\bar{\theta}_1 - \bar{i}e_- \bar{\theta}_1)) = 0$ and thus

$$e_-^2 \xi = (e_-^2 \hat{\beta}_+^1) \times (\bar{i}((i_{\varepsilon'', \rho})_* \delta'')) + (e_-^2 \hat{\beta}_-^1) \times (\bar{i}((i_{\varepsilon'', \rho})_* \delta'')) = (i_{\varepsilon'', \rho})_* (\bar{i}\delta'').$$

Let us write $\underline{\delta} := e_+^2 \xi \in \bar{C}_{n-d}^{\text{dR}}(S_\rho)$ and consider the difference of chains $\underline{\delta} - (i_{\bar{\varepsilon}, \rho})_* \bar{\delta}$. We claim that there exists $\bar{\theta}_2 \in \bar{C}_{n-d+1}^{\text{dR}}(S_{\bar{\varepsilon}})$ such that

$$\partial \bar{\theta}_2 = (i_{\rho, \hat{\varepsilon}})_* \underline{\delta} - (i_{\bar{\varepsilon}, \hat{\varepsilon}})_* \bar{\delta}, \quad e_+ \bar{\theta}_2 = e_- \bar{\theta}_2 = 0. \quad (26)$$

We prove this claim. Since $e_+ \circ e_+^2 = e_+ \circ e_+^1$ and $e_- \circ e_+^2 = e_+ \circ e_-^1$, we have

$$\begin{aligned} (e_+)_* [\underline{\delta} - (i_{\bar{\varepsilon}, \rho})_* \bar{\delta}] &= (e_+ \circ e_+^1)_* [\xi] - (i_{\varepsilon, \rho})_* [\delta] = 0 \in H_{n-d}^{\text{dR}}(S_\rho), \\ (e_-)_* [\underline{\delta} - (i_{\bar{\varepsilon}, \rho})_* \bar{\delta}] &= (e_+ \circ e_-^1)_* [\xi] - (i_{\varepsilon', \rho})_* [\delta'] = 0 \in H_{n-d}^{\text{dR}}(S_\rho). \end{aligned}$$

From Lemma 3.16, there exists $\bar{\theta}'_2 \in \bar{C}_{n-d+1}^{\text{dR}}(S_\rho)$ such that $\partial \bar{\theta}'_2 = \underline{\delta} - (i_{\bar{\varepsilon}, \rho})_* \bar{\delta}$ and $\partial(e_+ \bar{\theta}'_2) = \partial(e_- \bar{\theta}'_2) = 0$. Since $(\text{ev}_0)_*[e_+ \bar{\theta}'_2]$ and $(\text{ev}_0)_*[e_- \bar{\theta}'_2]$ are contained in $H_{n-d+1}^{\text{dR}}(N_\rho) = \{0\}$, Proposition 3.18 shows that there exist $\varphi_+, \varphi_- \in C_{n-d+2}^{\text{dR}}(S_{\hat{\varepsilon}})$ such that $\partial \varphi_+ = (i_{\rho, \hat{\varepsilon}})_*(e_+ \bar{\theta}'_2)$ and $\partial \varphi_- = (i_{\rho, \hat{\varepsilon}})_*(e_- \bar{\theta}'_2)$. Then, by using β_+, β_- in the proof of Lemma 4.12, we define

$$\bar{\theta}_2 := (i_{\rho, \hat{\varepsilon}})_* \bar{\theta}'_2 - \partial(\beta_+ \times i \varphi_+) - \partial(\beta_- \times i \varphi_-) \in \bar{C}_{n-d+1}^{\text{dR}}(S_{\hat{\varepsilon}}),$$

and this chain satisfies (26).

We take $\hat{\kappa}^2: \mathbb{R}^2 \rightarrow [0, 1]: (r_1, r_2) \mapsto \kappa(r_2)$ and

$$\hat{\beta}_+^2 := [\mathbb{R}^2, \text{id}_{\mathbb{R}^2}, (\tau_+^1, \tau_-^1), (\tau_+^2, \tau_-^2), \hat{\kappa}^2] \in \hat{C}_0^{\text{dR}}(\{0\}).$$

Finally, we define a chain

$$\hat{\delta} := (i_{\rho, \hat{\varepsilon}})_* \xi - \partial(\hat{\beta}_+^2 \times i^2 \bar{\theta}_2) \in \hat{C}_{n-d}^{\text{dR}}(S_{\hat{\varepsilon}}).$$

This satisfies $\partial \hat{\delta} = 0$ and the required four equations. \square

Lemma 4.14 is applied to prove the next proposition.

Proposition 4.15. $k_{(\bar{\varepsilon}, \bar{\delta})} \circ k_{(\bar{\varepsilon}', \bar{\delta}')} = k_{(\bar{\varepsilon}, \tilde{\delta})}: H_*^{<a}(\varepsilon'', \delta'') \rightarrow H_*^{<a}(\varepsilon, \delta)$.

Proof. Let $(\hat{\varepsilon}, \hat{\delta})$ be the pair of Lemma 4.14. From Lemma 4.8 and Lemma 4.11, we can define isomorphisms

$$\begin{aligned} f_{(\hat{\varepsilon}, \hat{\delta}), +}^1 &:= (j_{\bar{\varepsilon}, \hat{\varepsilon}})_*^{-1} \circ (e_{\hat{\varepsilon}, +}^1)_*: \hat{H}_*^{<a}(\hat{\varepsilon}, \hat{\delta}) \rightarrow \bar{H}_*^{<a}(\tilde{\varepsilon}, \tilde{\delta}), \\ f_{(\hat{\varepsilon}, \hat{\delta}), -}^1 &:= (j_{\bar{\varepsilon}', \hat{\varepsilon}})_*^{-1} \circ (e_{\hat{\varepsilon}, -}^1)_*: \hat{H}_*^{<a}(\hat{\varepsilon}, \hat{\delta}) \rightarrow \bar{H}_*^{<a}(\bar{\varepsilon}', \bar{\delta}'), \\ f_{(\hat{\varepsilon}, \hat{\delta}), +}^2 &:= (j_{\bar{\varepsilon}, \hat{\varepsilon}})_*^{-1} \circ (e_{\hat{\varepsilon}, +}^2)_*: \hat{H}_*^{<a}(\hat{\varepsilon}, \hat{\delta}) \rightarrow \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}). \end{aligned}$$

From the definitions of $k_{(\bar{\varepsilon}, \bar{\delta})}$, $k_{(\bar{\varepsilon}', \bar{\delta}')}$ and $k_{(\bar{\varepsilon}, \tilde{\delta})}$, it suffices to show that the following diagram commutes:

$$\begin{array}{ccccc} & & \bar{H}_*^{<a}(\tilde{\varepsilon}, \tilde{\delta}) & & \\ & & \uparrow f_{(\hat{\varepsilon}, \hat{\delta}), +}^1 & & \\ & & \hat{H}_*^{<a}(\hat{\varepsilon}, \hat{\delta}) & & \\ & f_{(\bar{\varepsilon}, \bar{\delta}), -} & & f_{(\bar{\varepsilon}, \bar{\delta}), +} & \\ & & & & \\ H_*^{<a}(\varepsilon'', \delta'') & & H_*^{<a}(\varepsilon', \delta') & & H_*^{<a}(\varepsilon, \delta) \\ & f_{(\hat{\varepsilon}, \hat{\delta}), -}^1 & & f_{(\hat{\varepsilon}, \hat{\delta}), +}^2 & \\ & & & & \\ & f_{(\bar{\varepsilon}', \bar{\delta}'), -} & & f_{(\bar{\varepsilon}, \bar{\delta}), -} & \\ & & \bar{H}_*^{<a}(\bar{\varepsilon}', \bar{\delta}') & & \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}). \end{array}$$

Note that all maps appearing in the diagram are isomorphisms. We need to prove the following three equations:

$$\begin{cases} f_{(\bar{\varepsilon}, \bar{\delta}), +} \circ f_{(\hat{\varepsilon}, \hat{\delta}), +}^1 = f_{(\bar{\varepsilon}, \bar{\delta}), +} \circ f_{(\hat{\varepsilon}, \hat{\delta}), +}^2, \\ f_{(\bar{\varepsilon}, \bar{\delta}), -} \circ f_{(\hat{\varepsilon}, \hat{\delta}), +}^2 = f_{(\bar{\varepsilon}', \bar{\delta}'), +} \circ f_{(\hat{\varepsilon}, \hat{\delta}), -}^1, \\ f_{(\bar{\varepsilon}, \bar{\delta}), -} \circ f_{(\hat{\varepsilon}, \hat{\delta}), +}^1 = f_{(\bar{\varepsilon}', \bar{\delta}'), -} \circ f_{(\hat{\varepsilon}, \hat{\delta}), -}^1. \end{cases} \quad (27)$$

Let us prove the first equation. Returning to the definition of $f_{(\bar{\varepsilon}, \bar{\delta}), +}$ and $f_{(\hat{\varepsilon}, \hat{\delta}), +}^1$,

$$\begin{aligned} f_{(\bar{\varepsilon}, \bar{\delta}), +} \circ f_{(\hat{\varepsilon}, \hat{\delta}), +}^1 &= (j_{\varepsilon, \bar{\varepsilon}})^{-1} \circ (e_{\bar{\varepsilon}, +})_* \circ (j_{\bar{\varepsilon}, \hat{\varepsilon}})^{-1} \circ (e_{\hat{\varepsilon}, +}^1)_* \\ &= (j_{\varepsilon, \hat{\varepsilon}})^{-1} \circ (e_{\hat{\varepsilon}, +} \circ e_{\bar{\varepsilon}, +}^1)_* \\ &= (j_{\varepsilon, \hat{\varepsilon}})^{-1} \circ (e_{\hat{\varepsilon}, +} \circ e_{\bar{\varepsilon}, +}^2)_* \\ &= (j_{\varepsilon, \bar{\varepsilon}})^{-1} \circ (e_{\bar{\varepsilon}, +})_* \circ (j_{\bar{\varepsilon}, \hat{\varepsilon}})^{-1} \circ (e_{\hat{\varepsilon}, +}^2)_* \\ &= f_{(\bar{\varepsilon}, \bar{\delta}), +} \circ f_{(\hat{\varepsilon}, \hat{\delta}), +}^2. \end{aligned}$$

Here, the second and fourth equality follow from obvious equations

$$(j_{\bar{\varepsilon}, \hat{\varepsilon}})_* \circ (e_{\bar{\varepsilon}, +})_* = (e_{\hat{\varepsilon}, +})_* \circ (j_{\bar{\varepsilon}, \hat{\varepsilon}})_*, \quad (j_{\bar{\varepsilon}, \hat{\varepsilon}})_* \circ (e_{\bar{\varepsilon}, +})_* = (e_{\hat{\varepsilon}, +})_* \circ (j_{\bar{\varepsilon}, \hat{\varepsilon}})_*.$$

The third equality follows from

$$e_{\hat{\varepsilon}, +} \circ e_{\bar{\varepsilon}, +}^1 = e_{\hat{\varepsilon}, +} \circ e_{\bar{\varepsilon}, +}^2,$$

which comes from the relation $e_+ \circ e_+^1 = e_+ \circ e_+^2$ of (20). The second equation of (27) can be proved similarly by applying

$$e_{\hat{\varepsilon}, -} \circ e_{\bar{\varepsilon}, +}^2 = e_{\hat{\varepsilon}, +} \circ e_{\bar{\varepsilon}, -}^1,$$

which comes from the relation $e_- \circ e_+^2 = e_+ \circ e_-^1$ of (20). To prove the third equation of (27), there is one issue: We need to apply

$$(e_{\hat{\varepsilon}, -} \circ e_{\bar{\varepsilon}, +}^1)_* = (e_{\hat{\varepsilon}, -} \circ e_{\bar{\varepsilon}, -}^1)_* : \hat{H}_*^{<a}(\hat{\varepsilon}, \hat{\delta}) \rightarrow H_*^{<a}(\hat{\varepsilon}, (i_{\varepsilon'', \hat{\varepsilon}})_* \bar{\delta}''),$$

which does not follow from (20) directly. To check this equation, we consider the following diagram including $\bar{H}_*^{<a}(\hat{\varepsilon}, (i_{\varepsilon'', \hat{\varepsilon}})_* \bar{\delta}'')$:

$$\begin{array}{ccc} \hat{H}_*^{<a}(\hat{\varepsilon}, \hat{\delta}) & \xrightarrow{(e_{\bar{\varepsilon}, +}^1)_*} & \bar{H}_*^{<a}(\hat{\varepsilon}, (i_{\bar{\varepsilon}, \hat{\varepsilon}})_* \bar{\delta}) \\ & \searrow (e_{\bar{\varepsilon}, -}^2)_* & \downarrow (e_{\hat{\varepsilon}, -})_* \\ & \bar{H}_*^{<a}(\hat{\varepsilon}, (i_{\varepsilon'', \hat{\varepsilon}})_* \bar{\delta}'') & \xrightarrow{(e_{\bar{\varepsilon}, +})_*} \\ (e_{\bar{\varepsilon}, -}^1)_* \downarrow & & \searrow (e_{\hat{\varepsilon}, -})_* \\ \bar{H}_*^{<a}(\hat{\varepsilon}, (i_{\bar{\varepsilon}', \hat{\varepsilon}})_* \bar{\delta}') & \xrightarrow{(e_{\bar{\varepsilon}, -})_*} & H_*^{<a}(\hat{\varepsilon}, (i_{\varepsilon'', \hat{\varepsilon}})_* \bar{\delta}''). \end{array}$$

Then

$$e_{\hat{\varepsilon}, -} \circ e_{\bar{\varepsilon}, +}^1 = e_{\hat{\varepsilon}, +} \circ e_{\bar{\varepsilon}, -}^2, \quad e_{\hat{\varepsilon}, -} \circ e_{\bar{\varepsilon}, -}^1 = e_{\hat{\varepsilon}, -} \circ e_{\bar{\varepsilon}, -}^2$$

follow directly from the relations $e_- \circ e_+^1 = e_+ \circ e_-^2$ and $e_- \circ e_-^1 = e_- \circ e_-^2$ of (20). If we rewrite $(i_{\varepsilon'', \hat{\varepsilon}})_* \bar{\delta}''$ by $\bar{\delta}^\#$, the equation (25) shows that

$$(e_{\hat{\varepsilon}, +})_* = (e_{\hat{\varepsilon}, -})_* : \bar{H}_*^{<a}(\hat{\varepsilon}, \bar{\delta}^\#) \rightarrow H_*^{<a}(\hat{\varepsilon}, \bar{\delta}^\#).$$

From the above diagram, we get $(e_{\hat{\varepsilon}, -})_* \circ (e_{\bar{\varepsilon}, +}^1)_* = (e_{\hat{\varepsilon}, -})_* \circ (e_{\bar{\varepsilon}, -}^1)_*$. This finishes the proof. \square

Corollary 4.16. For $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$, the isomorphism

$$k_{(\bar{\varepsilon}, \bar{\delta})}: H_*^{<a}(\varepsilon', \delta') \rightarrow H_*^{<a}(\varepsilon, \delta)$$

does not depend on the choice of $(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_a$ satisfying (24).

Proof. Let $(\bar{\varepsilon}, \bar{\delta})$ and $(\tilde{\varepsilon}, \tilde{\delta})$ be two arbitrary choices from $\bar{\mathcal{T}}_a$ satisfying (24). We apply Proposition 4.15 to the case where $(\varepsilon'', \delta'') = (\varepsilon', \delta')$ and $(\bar{\varepsilon}', \bar{\delta}') = (\varepsilon', \bar{i}\delta')$. By Lemma 4.13, $k_{(\varepsilon', \bar{i}\delta')}$ is equal to the identity map on $H_*^{<a}(\varepsilon', \delta')$, so we get an equation

$$k_{(\bar{\varepsilon}, \bar{\delta})} \circ \text{id}_{H_*^{<a}(\varepsilon', \delta')} = k_{(\tilde{\varepsilon}, \tilde{\delta})}.$$

□

From this result, we may rewrite $k_{(\bar{\varepsilon}, \bar{\delta})}: H_*^{<a}(\varepsilon', \delta') \rightarrow H_*^{<a}(\varepsilon, \delta)$ by $k_{(\varepsilon', \delta'), (\varepsilon, \delta)}$. The equations of Lemma 4.13 and Proposition 4.15 can be rewritten as

$$\begin{cases} k_{(\varepsilon, \delta), (\varepsilon, \delta)} = \text{id}_{H_*^{<a}(\varepsilon, \delta)}, \\ k_{(\varepsilon', \delta'), (\varepsilon, \delta)} \circ k_{(\varepsilon'', \delta''), (\varepsilon', \delta')} = k_{(\varepsilon'', \delta''), (\varepsilon, \delta)}. \end{cases}$$

Now, \mathcal{T}_a becomes a directed set by the relation $(\varepsilon', \delta') \leq (\varepsilon, \delta)$ if and only if $\varepsilon' \leq \varepsilon$. Then we obtain an inverse system

$$\left(\{H_*^{<a}(\varepsilon, \delta)\}_{(\varepsilon, \delta) \in \mathcal{T}_a}, \{k_{(\varepsilon', \delta'), (\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon} \right)$$

and its inverse limit

$$H_*^{<a}(Q, K) := \varprojlim_{\varepsilon \rightarrow 0} H_*^{<a}(\varepsilon, \delta)$$

is defined.

4.3.3 Spectral sequence

Lastly, we extend the above discussions to spectral sequences. For $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ with $\varepsilon' \leq \varepsilon$, we take $(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_a$ satisfying (24). Chain maps $(j_{\varepsilon, \bar{\varepsilon}})_*$, $e_{\bar{\varepsilon}, +}$, $e_{\bar{\varepsilon}, -}$ and $(j_{\varepsilon', \bar{\varepsilon}})_*$ induce a zig-zag of morphisms of spectral sequences

$$E_{(\varepsilon', \delta')}^{<a} \xrightarrow{(j_{\varepsilon', \bar{\varepsilon}})_*} E_{(\bar{\varepsilon}, (i_{\varepsilon', \bar{\varepsilon}})_* \delta')}^{<a} \xleftarrow{(e_{\bar{\varepsilon}, -})_*} \bar{E}_{(\bar{\varepsilon}, \bar{\delta})}^{<a} \xrightarrow{(e_{\bar{\varepsilon}, +})_*} E_{(\bar{\varepsilon}, (i_{\varepsilon, \bar{\varepsilon}})_* \delta)}^{<a} \xleftarrow{(j_{\varepsilon, \bar{\varepsilon}})_*} E_{(\varepsilon, \delta)}^{<a}. \quad (28)$$

All of them are isomorphism. Let $k_{(\varepsilon', \delta'), (\varepsilon, \delta)}: E_{(\varepsilon', \delta')}^{<a} \rightarrow E_{(\varepsilon, \delta)}^{<a}$ denote the composition of these maps. (The independence on (ε, δ) can be proved as Corollary 4.16.)

Proposition 4.17. *There exists a spectral sequence $E^{<a} = (\{(E^{<a})_{p,q}^r\}, \{(d^{<a})_{p,q}^r\})$ which converges to $H_*^{<a}(Q, K)$ in the sense of [31, Bounded Convergence 5.2.5] such that*

$$(E^{<a})_{-m,q}^1 = \begin{cases} H_{q-m(d-1)}^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) & \text{if } m \geq 0, \\ 0 & \text{if } m < 0 \end{cases}$$

Proof. On the first page, the middle two maps of (28) have the form

$$H_{q+p(d-1)}(\Sigma_{-p}^{a-p\bar{\varepsilon}}, \Sigma_{-p}^0) \xleftarrow{(e_{+})_*} \bar{H}_{q+p(d-1)}(\Sigma_{-p}^{a-p\bar{\varepsilon}}, \Sigma_{-p}^0) \xrightarrow{(e_{-})_*} H_{q+p(d-1)}(\Sigma_{-p}^{a-p\bar{\varepsilon}}, \Sigma_{-p}^0).$$

Since $(e_{-})_* = (\bar{i})_*^{-1} = (e_{+})_*$, this composition is equal to the identity map. Therefore, $k_{(\varepsilon', \delta'), (\varepsilon, \delta)}$ is equal to $(j_{\varepsilon', \bar{\varepsilon}})_*: H_{q+p(d-1)}(\Sigma_{-p}^{a-p\varepsilon'}, \Sigma_{-p}^0) \rightarrow H_{q+p(d-1)}(\Sigma_{-p}^{a-p\varepsilon}, \Sigma_{-p}^0)$ on the (p, q) -term ($p \leq 0$) of the first page.

By $\{k_{(\varepsilon', \delta'), (\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon}$, we define $(E^{<a})_{p,q}^r := \varprojlim_{\varepsilon \rightarrow 0} (E_{(\varepsilon, \delta)}^{<a})_{p,q}^r$. Moreover,

$$(d^{<a})_{p,q}^r : (E^{<a})_{p,q}^r \rightarrow (E^{<a})_{p-r, q+r-1}^r$$

is defined to be the map induced by $\{(d_{(\varepsilon, \delta)}^{<a})_{p,q}^r\}_{(\varepsilon, \delta) \in \mathcal{T}_a}$. Then the (p, q) -term of the first page is given by

$$(E^{<a})_{p,q}^1 = \varprojlim_{\varepsilon \rightarrow 0} H_{q+p(d-1)}(\Sigma_{-p}^{a-p\varepsilon}, \Sigma_{-p}^0) = H_{q+p(d-1)}(\Sigma_{-p}^a, \Sigma_{-p}^0),$$

for $p \leq 0$ and $(E^{<a})_{p,q}^1 = 0$ for $p > 0$. Since $\{k_{(\varepsilon', \delta'), (\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon}$ consists of isomorphisms, it is clear that $E^{<a}$ is a spectral sequence which converges to $H_*^{<a}(Q, K)$. \square

4.4 Definition of $H_*^{\text{string}}(Q, K)$

In this section, we define $I^{a,b} : H_*^{<a}(Q, K) \rightarrow H_*^{<b}(Q, K)$ for $a, b \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ with $a \leq b$ to get a direct system $(\{H_*^{<a}(Q, K)\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}, \{I^{a,b}\}_{a \leq b})$. After defining $H_*^{\text{string}}(Q, K)$ as its direct limit, we will give a structure of a unital graded \mathbb{R} -algebra.

4.4.1 The limit of $a \rightarrow \infty$

Let a, b be the above real numbers. For any $(\varepsilon, \delta) \in \mathcal{T}_a \cap \mathcal{T}_b$, we have considered in Section 4.1 a linear map

$$(I_{\varepsilon}^{a,b})_* : H_*^{<a}(\varepsilon, \delta) \rightarrow H_*^{<b}(\varepsilon, \delta),$$

which is induced by the inclusion maps $I_{\varepsilon}^{a,b} : \Sigma_m^{a+m\varepsilon} \rightarrow \Sigma_m^{b+m\varepsilon}$ for all $m \in \mathbb{Z}_{\geq 0}$.

Lemma 4.18. *Suppose that $(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_a \cap \bar{\mathcal{T}}_b$ satisfies (24) for $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a \cap \mathcal{T}_b$ with $\varepsilon' \leq \varepsilon$. Then, the following diagram commutes:*

$$\begin{array}{ccc} H_*^{<a}(\varepsilon, \delta) & \xrightarrow{(I_{\varepsilon}^{a,b})_*} & H_*^{<b}(\varepsilon, \delta) \\ \uparrow k_{(\varepsilon', \delta'), (\varepsilon, \delta)} & & \uparrow k_{(\varepsilon', \delta'), (\varepsilon, \delta)} \\ H_*^{<a}(\varepsilon', \delta') & \xrightarrow{(I_{\varepsilon'}^{a,b})_*} & H_*^{<b}(\varepsilon', \delta') \end{array}$$

Proof. $I_{\bar{\varepsilon}}^{a,b}$ induces a linear map $(I_{\bar{\varepsilon}}^{a,b})_* : \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}) \rightarrow \bar{H}_*^{<b}(\bar{\varepsilon}, \bar{\delta})$. It suffices to show that

$$\begin{aligned} (I_{\bar{\varepsilon}}^{a,b})_* \circ f_{(\bar{\varepsilon}, \bar{\delta}), +} &= f_{(\bar{\varepsilon}, \bar{\delta}), +} \circ (I_{\bar{\varepsilon}}^{a,b})_* : \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}) \rightarrow H_*^{<b}(\varepsilon, \delta), \\ (I_{\bar{\varepsilon}}^{a,b})_* \circ f_{(\bar{\varepsilon}, \bar{\delta}), -} &= f_{(\bar{\varepsilon}, \bar{\delta}), -} \circ (I_{\bar{\varepsilon}}^{a,b})_* : \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}) \rightarrow H_*^{<b}(\varepsilon', \delta'). \end{aligned}$$

Let us check the first equation. Since $j_{\varepsilon, \bar{\varepsilon}} \circ I_{\bar{\varepsilon}}^{a,b} = I_{\bar{\varepsilon}}^{a,b} \circ j_{\varepsilon, \bar{\varepsilon}} : \Sigma_m^{a+m\varepsilon} \rightarrow \Sigma_m^{b+m\varepsilon}$,

$$\begin{aligned} (I_{\bar{\varepsilon}}^{a,b})_* \circ f_{(\bar{\varepsilon}, \bar{\delta}), +} &= (I_{\bar{\varepsilon}}^{a,b})_* \circ (j_{\varepsilon, \bar{\varepsilon}})_*^{-1} \circ (e_{\bar{\varepsilon}, +})_* \\ &= (j_{\varepsilon, \bar{\varepsilon}})_*^{-1} \circ (I_{\bar{\varepsilon}}^{a,b})_* \circ (e_{\bar{\varepsilon}, +})_* \\ &= (j_{\varepsilon, \bar{\varepsilon}})_*^{-1} \circ (e_{\bar{\varepsilon}, +})_* \circ (I_{\bar{\varepsilon}}^{a,b})_* \\ &= f_{(\bar{\varepsilon}, \bar{\delta}), +} \circ (I_{\bar{\varepsilon}}^{a,b})_*. \end{aligned}$$

The second equation can be proved by replacing ε and “+” in the above computation by ε' and “-”. \square

This lemma implies that, after taking the limits of $(\varepsilon, \delta) \in \mathcal{T}_a \cap \mathcal{T}_b$ with $\varepsilon \rightarrow 0$, we get a linear map

$$I^{a,b} := \varprojlim_{\varepsilon \rightarrow 0} (I_{\varepsilon}^{a,b})_* : H_*^{<a}(Q, K) \rightarrow H_*^{<b}(Q, K).$$

Furthermore, for $b \geq a$ and $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a \cap \mathcal{T}_b$ with $\varepsilon' \leq \varepsilon$, $k_{(\varepsilon', \delta'), (\varepsilon, \delta)}$ induces an isomorphism from $H_*^{[a,b]}(\varepsilon', \delta')$ to $H_*^{[a,b]}(\varepsilon, \delta)$. Thus, we can also define

$$H_*^{[a,b]}(Q, K) := \varinjlim_{\varepsilon \rightarrow 0} H_*^{[a,b]}(\varepsilon, \delta).$$

Note that a long exact sequence

$$\dots \longrightarrow H_*^{<a}(Q, K) \xrightarrow{I^{a,b}} H_*^{<b}(Q, K) \longrightarrow H_*^{[a,b]}(Q, K) \longrightarrow H_{*-1}^{<a}(Q, K) \xrightarrow{I^{a,b}} \dots \quad (29)$$

is induced by (23).

From the direct system $(\{H_*^{<a}(Q, K)\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}, \{I^{a,b}\}_{a \leq b})$, we finally define a graded \mathbb{R} -vector space

$$H_*^{\text{string}}(Q, K) := \varinjlim_{a \rightarrow \infty} H_*^{<a}(Q, K).$$

4.4.2 Product structure

Let us see that $H_*^{\text{string}}(Q, K)$ has a structure of a unital associative graded \mathbb{R} -algebra. For any $a, a' \in \mathbb{R}_{>0} \cup \{\infty\}$, $m, m' \in \mathbb{Z}_{\geq 0}$ and $\varepsilon \in (0, \varepsilon_0/5]$, there is a map

$$\Pi: \Sigma_m^{a+m\varepsilon} \times \Sigma_{m'}^{a'+m'\varepsilon} \rightarrow \Sigma_{m+m'}^{(a+a')+(m+m')\varepsilon}: ((\gamma_k)_{k=1,\dots,m}, (\gamma'_l)_{l=1,\dots,m'}) \mapsto (\gamma_1, \dots, \gamma_m, \gamma'_1, \dots, \gamma'_{m'}).$$

When $m = 0$ or $m' = 0$, Π is identified with the identity map. We define a linear map

$$C_p^{<a}(\varepsilon) \otimes C_q^{<a'}(\varepsilon) \rightarrow C_{p+q}^{<a+a'}(\varepsilon): x \otimes y \mapsto x \star y \quad (30)$$

so that for $x \in C_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$ and $y \in C_{q-m'(d-2)}^{\text{dR}}(\Sigma_{m'}^{a'+m'\varepsilon}, \Sigma_{m'}^0)$,

$$x \star y = (-1)^{mqd} \Pi_*(x \times y)$$

We note that the associative relation $(x \star y) \star z = x \star (y \star z)$ holds. Suppose that $a, a', a+a' \notin \mathcal{L}(K)$. For any $(\varepsilon, \delta) \in \mathcal{T}_a \cap \mathcal{T}_{a'} \cap \mathcal{T}_{a+a'}$, the above map is compatible with the differential D_δ . Indeed, for $x \in C_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$ and $y \in C_{q-m'(d-2)}^{\text{dR}}(\Sigma_{m'}^{a'+m'\varepsilon}, \Sigma_{m'}^0)$,

$$\begin{aligned} (-1)^{mqd} D_\delta(x \star y) &= \partial(\Pi_*(x \times y)) + \sum_{k=1}^{m+m'} (-1)^{p+q+1+kd} f_{k,\delta}(\Pi_*(x \times y)) \\ &= \Pi_*(\partial x \times y) + \sum_{k=1}^m (-1)^{p+q+1+kd+s_0} \Pi_*(f_{k,\delta}(x) \times y) \\ &\quad + (-1)^{p-m(d-2)} \Pi_*(x \times \partial y) + \sum_{l=1}^{m'} (-1)^{p+q+1+(l+m)d} \Pi_*(x \times f_{l,\delta}(y)) \\ &= (-1)^{mqd} (D_\delta(x)) \star y + (-1)^{p+mqd} x \star (D_\delta(y)). \end{aligned}$$

Here, $s_0 := (q - m'(d-2))(d+1)$. This computation shows that

$$D_\delta(x \star y) = (D_\delta(x)) \star y + (-1)^p x \star (D_\delta(y))$$

holds for $x \in C_p^{<a}(\varepsilon)$ and $y \in C_q^{<a'}(\varepsilon)$. Therefore, (30) induces a linear map on homology

$$H_p^{<a}(\varepsilon, \delta) \otimes H_q^{<a'}(\varepsilon, \delta) \rightarrow H_{p+q}^{<a+a'}(\varepsilon, \delta)$$

for every $(\varepsilon, \delta) \in \mathcal{T}_a \cap \mathcal{T}_{a'} \cap \mathcal{T}_{a+a'}$.

Likewise, let us define $x \bar{\star} y := (-1)^{mqd} \Pi_*(x \times y) \in \bar{C}_{p+q}^{<a+a'}(\bar{\varepsilon})$ for $x \in \bar{C}_{p-m(d-2)}^{\text{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}, \Sigma_m^0)$ and $y \in \bar{C}_{q-m'(d-2)}^{\text{dR}}(\Sigma_{m'}^{a'+m'\bar{\varepsilon}}, \Sigma_{m'}^0)$. Then, $e_{\bar{\varepsilon},+}, e_{\bar{\varepsilon},-}$ intertwine the \star -operation and the $\bar{\star}$ -operation. This shows that the \star -operation is compatible with $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon' \leq \varepsilon}$. Therefore, on the limit of $\varepsilon \rightarrow 0$, a linear map

$$H_p^{<a}(Q, K) \otimes H_q^{<a'}(Q, K) \rightarrow H_{p+q}^{<a+a'}(Q, K)$$

is induced. The commutativity with $\{I^{a,b}\}_{a \leq b}$ naturally holds. As a result, we get an associative product structure on $H_*^{\text{string}}(Q, K)$. The element $1 \in H_0^{\text{string}}(Q, K)$, which comes from $1 \in \mathbb{R} = C_0^{\text{dR}}(\Sigma_0^a, \Sigma_0^0) \subset C_0^{<a}(\varepsilon, \delta)$, is the unit of this graded algebra.

4.4.3 Explicit choice of (ε, δ)

Lastly, let us introduce a condition on $(\varepsilon, \delta) \in \mathcal{T}_a$ so that δ can be written explicitly. The concrete computations in Section 5 and 6 become easier by choosing (ε, δ) satisfying this condition.

Suppose that there exists a fixed trivialization $\mathbb{R}^d \times K \rightarrow (TK)^\perp$ of the normal bundle of K which preserves orientations and fiber metrics. For every $\varepsilon \leq \varepsilon_0$, let us write $\mathcal{O}_\varepsilon := \{w \in \mathbb{R}^d \mid |w| < \varepsilon/2\}$. Composing with the exponential map (3), we obtain a diffeomorphism

$$h: \mathcal{O}_\varepsilon \times K \rightarrow N_\varepsilon,$$

which preserves orientations. In this case, we say $(\varepsilon, \delta) \in \mathcal{T}_a$ is *standard with respect to h* , if $\delta \in C_{n-d}^{\text{dR}}(S_\varepsilon)$ has the form

$$\delta = [N_\varepsilon, \psi_\varepsilon, h_*(\nu_\varepsilon \times 1)] \quad (31)$$

satisfying:

- $\psi_\varepsilon: N_\varepsilon \rightarrow S_\varepsilon: v \rightarrow (\sigma_j^v)_{j=1,2}$ is defined by

$$\sigma_1^v: [0, \varepsilon/2] \rightarrow N_\varepsilon: t \mapsto h\left(\frac{\varepsilon - 2t}{\varepsilon}w, x\right), \quad \sigma_2^v: [0, \varepsilon/2] \rightarrow N_\varepsilon: t \mapsto h\left(\frac{2t}{\varepsilon}w, x\right),$$

for $v = h(w, x) \in N_\varepsilon$.

- $h_*(\nu_\varepsilon \times 1) \in \Omega_c^{n-d}(N_\varepsilon)$ for some $\nu_\varepsilon \in \Omega_c^d(\mathcal{O}_\varepsilon)$ with $\int_{\mathcal{O}_\varepsilon} \nu_\varepsilon = 1$.

Suppose that $(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_a$ satisfies (24) for $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ ($\varepsilon' \leq \varepsilon$) which are given as above. Then, we say $(\bar{\varepsilon}, \bar{\delta})$ is *standard with respect to h* , if $\bar{\varepsilon} = \varepsilon$ and $\bar{\delta} \in \bar{C}_{n-d}^{\text{dR}}(S_\varepsilon)$ has the form

$$\bar{\delta} = (-1)^n [\mathbb{R} \times N_\varepsilon, \bar{\psi}_{\varepsilon', \varepsilon}, (\text{id}_{\mathbb{R}_{\geq 1} \times N_\varepsilon}, \text{id}_{\mathbb{R}_{\leq -1} \times N_\varepsilon}), \bar{\eta}_{\varepsilon', \varepsilon}] \quad (32)$$

such that for some C^∞ function $\kappa: \mathbb{R} \rightarrow [0, 1]$ with $\kappa(r) = \begin{cases} 0 & \text{if } r \leq -1, \\ 1 & \text{if } r \geq 1, \end{cases}$ the following hold:

- $\bar{\psi}_{\varepsilon', \varepsilon}(r, v) = (r, (\sigma_j^{(r,v)})_{j=1,2}) \in \mathbb{R} \times S_\varepsilon$ is defined by

$$\sigma_1^{(r,v)}: [0, \varepsilon_r/2] \rightarrow N_\varepsilon: t \mapsto h\left(\frac{\varepsilon_r - 2t}{\varepsilon_r}w, x\right), \quad \sigma_2^{(r,v)}: [0, \varepsilon_r/2] \rightarrow N_\varepsilon: t \mapsto h\left(\frac{2t}{\varepsilon_r}w, x\right)$$

for $r \in \mathbb{R}$, $v = h(w, x) \in N_\varepsilon$ and $\varepsilon_r := \kappa(r)\varepsilon + (1 - \kappa(r))\varepsilon'$.

- $(\text{id}_{\mathbb{R}} \times h)^* \bar{\eta}_{\varepsilon', \varepsilon} = \kappa \times (\nu_\varepsilon \times 1) + (1 - \kappa) \times (\nu_{\varepsilon'} \times 1) + (d\kappa) \times (\theta \times 1)$ for some $\theta \in \Omega_c^{d-1}(\mathcal{O}_\varepsilon)$ satisfying $d\theta = \nu_\varepsilon - \nu_{\varepsilon'}$.

In summary, in order to compute $H_*^{\text{string}}(Q, K)$ when a trivialization h is given, we only need to deal with $(\varepsilon, \delta) \in \mathcal{T}_a$ and $(\varepsilon, \bar{\delta}) \in \bar{\mathcal{T}}_a$ which are standard with respect to h .

4.5 Invariance

In this section, we prove the invariance of $H_*^{\text{string}}(Q, K)$ up to isomorphism by changing auxiliary data. More precisely, we consider the dependence of the construction on the following data (See Section 3.1):

1. a complete Riemannian metric g on Q .
2. a constant $C_0 \geq 1$ which bounds the speed of all $\gamma \in \Omega_K(Q)$.
3. a real number $\varepsilon_0 > 0$ which is the diameter (in the direction of fibers) of a tubular neighborhood N_{ε_0} of K .
4. a C^∞ function $\mu: [0, \frac{3}{2}] \rightarrow [0, 1]$ which is used to define con_k .

Notation. Let X be an arbitrary notation which we have defined in the former sections. As a rule in this section, if its definition depends on some auxiliary data S , we rewrite X by X_S when discussing the dependence on S .

Independence on μ . We choose a C^∞ family $\bar{\mu} := (\mu_r)_{r \in \mathbb{R}}$ such that each μ_r satisfy the same condition as μ , and $\mu_r = \begin{cases} \mu_{-1} & \text{if } r \leq -1, \\ \mu_1 & \text{if } r \geq 1. \end{cases}$ Then, a map con_{k, μ_r} is defined for each $r \in \mathbb{R}$.

For $(\bar{\varepsilon}, \bar{\delta}) \in \mathcal{T}_a$, let us define $\bar{f}_{k, \bar{\delta}, \bar{\mu}}: \bar{C}_*^{\text{dR}}(\Sigma_m^{a+m\bar{\varepsilon}}) \rightarrow \bar{C}_*^{\text{dR}}(\Sigma_{m+1}^{a+(m+1)\bar{\varepsilon}})$ by replacing con_k in the definition of Φ_k of (21) by $\text{con}_{k, \mu_{r(u)}}$. We also replace $\bar{f}_{k, \bar{\delta}}$ in the definition of $\bar{D}_{\bar{\delta}}$ by $\bar{f}_{k, \bar{\delta}, \bar{\mu}}$ to define a linear map

$$\bar{D}_{\bar{\delta}, \bar{\mu}}: \bar{C}_*^{<a}(\bar{\varepsilon}) \rightarrow \bar{C}_{*+1}^{<a}(\bar{\varepsilon}).$$

This satisfies $\bar{D}_{\bar{\delta}, \bar{\mu}} \circ \bar{D}_{\bar{\delta}, \bar{\mu}} = 0$, so we get a chain complex $(\bar{C}_*^{<a}(\bar{\varepsilon}), \bar{D}_{\bar{\delta}, \bar{\mu}})$. Let $\bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}, \bar{\mu})$ denote its homology group.

We rewrite $e_{\varepsilon, +}, e_{\varepsilon, -}: \bar{C}_*^{<a}(\bar{\varepsilon}) \rightarrow C_*^{<a}(\varepsilon)$ by $e_{\varepsilon, \bar{\mu}, +}, e_{\varepsilon, \bar{\mu}, -}$ respectively. They induce

$$\begin{aligned} (e_{\varepsilon, \bar{\mu}, +})_*: \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}, \bar{\mu}) &\rightarrow H_*^{<a}(\varepsilon, e_+\delta)_{\mu_1}, \\ (e_{\varepsilon, \bar{\mu}, -})_*: \bar{H}_*^{<a}(\bar{\varepsilon}, \bar{\delta}, \bar{\mu}) &\rightarrow H_*^{<a}(\varepsilon, e_-\delta)_{\mu_{-1}}. \end{aligned}$$

We can prove as Lemma 4.7 that they are isomorphisms. When $(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_{[a, b]}$ satisfies (24), we define an isomorphism $k_{(\bar{\varepsilon}, \bar{\delta}, \bar{\mu})}: H_*^{<a}(\varepsilon', \delta')_{\mu_{-1}} \rightarrow H_*^{<a}(\varepsilon, \delta)_{\mu_1}$ by

$$k_{(\bar{\varepsilon}, \bar{\delta}, \bar{\mu})} := ((j_{\varepsilon, \bar{\varepsilon}})_*^{-1} \circ (e_{\varepsilon, \bar{\mu}, +})_*) \circ ((j_{\varepsilon', \bar{\varepsilon}})_*^{-1} \circ (e_{\varepsilon, \bar{\mu}, -})_*)^{-1}.$$

It is proved as Proposition 4.15 that the two triangles in the following diagram commute:

$$\begin{array}{ccc} H_*^{<a}(\varepsilon, \delta)_{\mu_{-1}} & \xrightarrow{k_{(\varepsilon, \bar{\varepsilon}, \bar{\mu})}} & H_*^{<a}(\varepsilon, \delta)_{\mu_1} \\ \uparrow k_{(\varepsilon', \delta'), (\varepsilon, \delta)} & \nearrow k_{(\varepsilon, \bar{\delta}, \bar{\mu})} & \uparrow k_{(\varepsilon', \delta'), (\varepsilon, \delta)} \\ H_*^{<a}(\varepsilon', \delta')_{\mu_{-1}} & \xrightarrow{k_{(\varepsilon', \bar{\varepsilon}, \bar{\mu})}} & H_*^{<a}(\varepsilon', \delta')_{\mu_1} \end{array}$$

Therefore, $\{k_{(\varepsilon, \bar{\varepsilon}, \bar{\mu})}\}_{(\varepsilon, \delta) \in \mathcal{T}_a}$ induces an isomorphism on the limits of $\varepsilon \rightarrow 0$

$$k_{\bar{\mu}}^a: H_*^{<a}(Q, K)_{\mu_{-1}} \rightarrow H_*^{<a}(Q, K)_{\mu_1}.$$

It is easy to see as Lemma 4.18 that $\{k_{\bar{\mu}}^a\}_{a \in \mathbb{R}_{>0}} \setminus \mathcal{L}(K)$ commute with $\{I^{a, b}\}_{a \leq b}$, so we get an isomorphism from $H_*^{\text{string}}(Q, K)_{\mu_{-1}}$ to $H_*^{\text{string}}(Q, K)_{\mu_1}$. It is also possible to prove as Corollary 4.16 that this isomorphism does not depend on the choice of $\bar{\mu}$.

Independence on ε_0 . For $\varepsilon'_0 \leq \varepsilon_0$, we consider the inclusion maps $j_{\varepsilon'_0, \varepsilon_0}: \Sigma_{m, \varepsilon'_0}^{a+m\varepsilon} \rightarrow \Sigma_{m, \varepsilon_0}^{a+m\varepsilon}$ for all $m \in \mathbb{Z}_{\geq 0}$. They induce a chain map $(j_{\varepsilon'_0, \varepsilon_0})_*$ from $(C_*^{<a}(\varepsilon)_{\varepsilon'_0}, D_\delta)$ to $(C_*^{<a}(\varepsilon)_{\varepsilon_0}, D_\delta)$ for

every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ and $(\varepsilon, \delta) \in \mathcal{T}_{a, \varepsilon'_0} (\subset \mathcal{T}_{a, \varepsilon_0})$. This preserves filtrations $\{\mathcal{F}_{\varepsilon, p, \varepsilon_0}^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathcal{F}_{\varepsilon, p, \varepsilon'_0}^{<a}\}_{p \in \mathbb{Z}}$, so it induces an morphism of the spectral sequences. On the $(-m, q)$ -term ($m \geq 0$) of the first page, it is equal to the map

$$(j_{\varepsilon'_0, \varepsilon_0})_* : H_{q-m(d-1)}^{\text{dR}}(\Sigma_{m, \varepsilon'_0}^{a+m\varepsilon}, \Sigma_{m, \varepsilon'_0}^0) \rightarrow H_{q-m(d-1)}^{\text{dR}}(\Sigma_{m, \varepsilon_0}^{a+m\varepsilon}, \Sigma_{m, \varepsilon_0}^0),$$

which is an isomorphism by Lemma 3.11. Therefore, by Lemma 4.2, $(j_{\varepsilon'_0, \varepsilon_0})_* : H_*^{<a}(\varepsilon, \delta)_{\varepsilon'_0} \rightarrow H_*^{<a}(\varepsilon, \delta)_{\varepsilon_0}$ is also an isomorphism.

We can prove its commutativity with $\{k_{(\varepsilon', \delta'), (\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon}$ as Lemma 4.18, so we get an isomorphism on the limit of $\varepsilon \rightarrow 0$

$$\mathfrak{J}_{\varepsilon'_0, \varepsilon_0}^a : H_*^{<a}(Q, K)_{\varepsilon'_0} \rightarrow H_*^{<a}(Q, K)_{\varepsilon_0}.$$

It holds naturally that $\{\mathfrak{J}_{\varepsilon'_0, \varepsilon_0}^a\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}$ commutes with $\{I^{a,b}\}_{a \leq b}$. Therefore, on the limit of $a \rightarrow \infty$, we get an isomorphism from $H_*^{\text{string}}(Q, K)_{\varepsilon'_0}$ to $H_*^{\text{string}}(Q, K)_{\varepsilon_0}$.

Independence on C_0 . For $C'_0 \geq C_0 \geq 1$, we consider the inclusion maps $j_{C_0, C'_0} : \Sigma_{m, C_0}^a \rightarrow \Sigma_{m, C'_0}^a$ for all $m \in \mathbb{Z}_{\geq 0}$. Parallel to the proof of the independence on ε_0 , we apply Lemma 3.11 to show that an isomorphism

$$\mathfrak{J}_{C_0, C'_0}^a : H_*^{<a}(Q, K)_{C_0} \rightarrow H_*^{<a}(Q, K)_{C'_0}$$

is induced. It holds naturally that $\{\mathfrak{J}_{C_0, C'_0}^a\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}$ commutes with $\{I^{a,b}\}_{a \leq b}$. Therefore, on the limit of $a \rightarrow \infty$, we get an isomorphism from $H_*^{\text{string}}(Q, K)_{C_0}$ to $H_*^{\text{string}}(Q, K)_{C'_0}$.

Independence on g . First, let us introduce an graded algebra $\hat{H}_*^{\text{string}}(Q, K)_g$ which is isomorphic to $H_*^{\text{string}}(Q, K)_{(g, C_0, \varepsilon_0)}$, but whose definition does not depend on ε_0 and C_0 . For every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)_g$, we define the limit of $\varepsilon_0 \rightarrow 0$ and $C_0 \rightarrow \infty$

$$\hat{H}_*^{<a}(Q, K)_g := \varinjlim_{C_0 \rightarrow \infty} \varprojlim_{\varepsilon_0 \rightarrow 0} H_*^{<a}(Q, K)_{(g, C_0, \varepsilon_0)}$$

by $\{\mathfrak{J}_{\varepsilon'_0, \varepsilon_0}^a\}_{\varepsilon'_0 \leq \varepsilon_0}$ and $\{\mathfrak{J}_{C_0, C'_0}^a\}_{C_0 \leq C'_0}$ defined above. Then, $\{I^{a,b}\}_{a \leq b}$ induces a family of maps

$$\{\hat{I}_g^{a,b} : \hat{H}_*^{<a}(Q, K)_g \rightarrow \hat{H}_*^{<b}(Q, K)_g\}_{a \leq b},$$

and we can take the limit of $a \rightarrow \infty$ to define $\hat{H}_*^{\text{string}}(Q, K)_g := \varinjlim_{a \rightarrow \infty} \hat{H}_*^{<a}(Q, K)_g$.

Suppose that g and g' are complete Riemannian metrics on \hat{Q} . For $a > 0$, there exists a compact subset Z_a which contains the images of all $\gamma \in \bigcup_{C_0 \geq 1} \Omega_K(Q)_{(g, C_0)}$ with $\text{length}_g \gamma < a$ and the images of all $\gamma \in \bigcup_{C_0 \geq 1} \Omega_K(Q)_{(g', C_0)}$ with $\text{length}_{g'} \gamma < a$. For any $a \in \mathbb{R}_{>0} \setminus (\mathcal{L}(K)_g \cup \mathcal{L}(K)_{g'})$, there exists a constant $c_a \geq 1$ such that $|\cdot|_{g'} \leq c_a |\cdot|_g$ and $|\cdot|_g \leq c_a |\cdot|_{g'}$ on Z_a . We may additionally assume that $ac_a \notin \mathcal{L}(K)_g \cup \mathcal{L}(K)_{g'}$.

Let $a \in \mathbb{R}_{>0} \setminus (\mathcal{L}(K)_g \cup \mathcal{L}(K)_{g'})$. For any $C_0 \geq 1$ and $\varepsilon_0 > 0$, we have the inclusion map

$$j_{(C_0, \varepsilon_0)} : \Sigma_{m, (g, C_0, \varepsilon_0)}^a \rightarrow \Sigma_{m, (g', C_0 c_a, \varepsilon_0 c_a)}^{ac_a}.$$

In addition, let $S_{\varepsilon, g}^a$ be a subspace of $S_{\varepsilon, g}$ which consists of $(\sigma_i)_{i=1,2}$ satisfying $|(\sigma_i)'(t)|_g \leq c_a^{-1}$ for $i = 1, 2$. If ε is sufficiently small, we have the inclusion map

$$i_\varepsilon : S_{\varepsilon, g}^a \rightarrow S_{\varepsilon c_a, g'}.$$

These maps induces a linear map on the homology groups

$$(j_{(C_0, \varepsilon_0)})_* : H_*^{<a}(\varepsilon, \delta)_{(g, C_0, \varepsilon_0)} \rightarrow H_*^{<ac_a}(\varepsilon c_a, (i_\varepsilon)_* \delta)_{(g', C_0 c_a, \varepsilon_0 c_a)}$$

for $(\varepsilon, \delta) \in \mathcal{T}_{a,(g,C_0,\varepsilon_0)}$ such that $\delta \in C_{n-d}^{\text{dR}}(S_{\varepsilon,g}^a)$. Its commutativity with $\{k_{(\varepsilon',\delta'),(\varepsilon,\delta)}\}_{\varepsilon' \leq \varepsilon}$ can be proved as Lemma 4.18. Let us write the induced map on the limits of $\varepsilon \rightarrow 0$ by

$$\mathfrak{J}_{(C_0,\varepsilon_0)}^a: H_*^{<a}(Q, K)_{(g,C_0,\varepsilon_0)} \rightarrow H_*^{<ac_a}(Q, K)_{(g',C_0c_a,\varepsilon_0c_a)}.$$

Moreover, it holds naturally that the maps of $\{\mathfrak{J}_{(C,\varepsilon_0)}^a\}_{C_0 \geq 1, \varepsilon_0 > 0}$ commute with the maps of $\{\mathfrak{J}_{\varepsilon'_0, \varepsilon_0}^a\}_{\varepsilon'_0 \leq \varepsilon_0}$ and $\{\mathfrak{J}_{C_0, C'_0}^a\}_{C_0 \leq C'_0}$, so we get a map on the limit of $\varepsilon_0 \rightarrow 0$ and $C_0 \rightarrow \infty$

$$\mathfrak{J}^a := \varinjlim_{C_0 \rightarrow \infty} \varprojlim_{\varepsilon_0 \rightarrow 0} \mathfrak{J}_{(C_0,\varepsilon_0)}^a: \dot{H}_*^{<a}(Q, K)_g \rightarrow \dot{H}_*^{<ac_a}(Q, K)_{g'}.$$

Lastly, $\{\mathfrak{J}^a\}_{a \in \mathbb{R}_{>0} \setminus (\mathcal{L}(K)_g \cup \mathcal{L}(K)_{g'})}$ is compatible with $\{j_g^{a,b}\}_{a \leq b}$, so it induces a map on the limit of $a \rightarrow \infty$

$$\mathfrak{J}: \dot{H}_*^{\text{string}}(Q, K)_g \rightarrow \dot{H}_*^{\text{string}}(Q, K)_{g'}.$$

If we exchange g and g' , we can also define $(\mathfrak{J}')^a: \dot{H}_*^{<a}(Q, K)_{g'} \rightarrow \dot{H}_*^{<ac_a}(Q, K)_g$ for $a \in \mathbb{R}_{>0} \setminus (\mathcal{L}(K)_g \cup \mathcal{L}(K)_{g'})$. For $b := ac_a$, we have

$$(\mathfrak{J}')^b \circ \mathfrak{J}^a = j_g^{a,bc_b}, \quad \mathfrak{J}^b \circ (\mathfrak{J}')^a = j_{g'}^{a,bc_b}.$$

Therefore, $\varinjlim_{a \rightarrow \infty} (\mathfrak{J}')^a$ is the inverse map of \mathfrak{J} . This proves the independence on g .

We finish the proof of the independence on auxiliary data.

Finally, let us prove the invariance by changing the orientation of K . Suppose that $K = \sqcup_{\alpha \in A} K_\alpha$ for connected components $\{K_\alpha\}_{\alpha \in A}$. Then, $N_\varepsilon = \sqcup_{\alpha \in A} N_{\varepsilon,\alpha}$ and $S_\varepsilon = \sqcup_{\alpha \in A} S_{\varepsilon,\alpha}$, where $N_{\varepsilon,\alpha}$ is a tubular neighborhood of K_α and $S_{\varepsilon,\alpha}$ consists of pairs of paths in $N_{\varepsilon,\alpha}$. In addition, for every $\alpha_1, \dots, \alpha_{2m} \in A$, let $\Sigma_{m,(\alpha_1, \dots, \alpha_{2m})}^a$ be the subspace of Σ_m^a consisting of $(\gamma_k: [0, T_k] \rightarrow Q)_{k=1, \dots, m}$ such that $\gamma_k(0) \in K_{\alpha_{2k-1}}$ and $\gamma_k(T_k) \in K_{\alpha_{2k}}$ for $k = 1, \dots, m$. Then $C_{n-d}^{\text{dR}}(S_\varepsilon) = \bigoplus_{\alpha \in A} C_{n-d}^{\text{dR}}(S_{\varepsilon,\alpha})$ and $C_*^{\text{dR}}(\Sigma_m^a) = \bigoplus_{\alpha_1, \dots, \alpha_{2m} \in A} C_*^{\text{dR}}(\Sigma_{m,(\alpha_1, \dots, \alpha_{2m})}^a)$.

For any subset $B \subset A$, let K_B be an oriented submanifold obtained from K by reversing the orientations of $\{K_\alpha\}_{\alpha \in B}$. For every $\delta = \sum_{\alpha \in A} \delta_\alpha \in C_{n-d}^{\text{dR}}(S_\varepsilon)$ (δ_α is a chain in $S_{\varepsilon,\alpha}$), let us write $\delta_B := \sum_{\alpha \in A \setminus B} \delta_\alpha - \sum_{\alpha \in B} \delta_\alpha$. By using a notation

$$s(\alpha_1, \dots, \alpha_{2m}) := \#\{k \in \{1, \dots, m\} \mid \alpha_{2k} \in B\},$$

we define a linear map $F_\varepsilon^a: C_*^{<a}(\varepsilon) \rightarrow C_*^{<a}(\varepsilon)$ so that $F_\varepsilon^a(x) = (-1)^{s(\alpha_1, \dots, \alpha_{2m})}x$ for every $x \in C_{*-m(d-2)}^{\text{dR}}(\Sigma_{m,(\alpha_1, \dots, \alpha_{2m})}^a, \Sigma_{m,(\alpha_1, \dots, \alpha_{2m})}^0)$ ($m \geq 1$), and $F_\varepsilon^a(x) = x$ for $x \in C_*^{\text{dR}}(\Sigma_0^a, \Sigma_0^0)$. For every $(\varepsilon, \delta) \in \mathcal{T}_a$, (ε, δ_B) satisfies the conditions of Definition 4.4 for (Q, K_B) , and F_ε^a is a chain map from $(C_*^{<a}(\varepsilon), D_\delta)$ to $(C_*^{<a}(\varepsilon), D_{\delta_B})$. Moreover, F_ε^a is compatible with the \star -operation. By taking the limit of $\varepsilon \rightarrow 0$ and $a \rightarrow \infty$, we obtain an isomorphism of graded \mathbb{R} -algebras

$$F: H_*^{\text{string}}(Q, K) \rightarrow H_*^{\text{string}}(Q, K_B).$$

Remark 4.19. Similarly, one can prove the invariance of $H_*^{\text{string}}(Q, K)$ by changing the orientation of Q .

Proposition 4.20. $H_*^{\text{string}}(Q, K)$ is invariant by a C^∞ isotopy of K .

Proof. For two oriented compact submanifold K_0, K_1 of Q , suppose that there exists a C^∞ family of embedding maps $\{f_t: K_0 \rightarrow Q\}_{t \in [0,1]}$ such that f_0 is the inclusion map of K_0 and $f_1(K_0) = K_1$. Then, this isotopy can be extended to an ambient isotopy $\{F_t\}_{t \in [0,1]}$ such that $F_0 = \text{id}_Q$ and $F_1(K_0) = K_1$. Since F_1 is an isometry form $(Q, (F_1)^*g)$ to (Q, g) , it naturally induces an isomorphism

$$H_*^{\text{string}}(Q, K_0)_{(F_1)^*g} \rightarrow H_*^{\text{string}}(Q, K_1)_g.$$

The assertion follows from the independence on the Riemannian metric on Q . \square

5 Examples

In this section we determine the algebraic structure of $H_*^{\text{string}}(Q, K)$ for two examples when $Q = \mathbb{R}^{2d-1}$ ($d \geq 2$). These examples are higher dimensional generalizations of the Hopf link and the unlink in \mathbb{R}^3 .

The manifold $Q = \mathbb{R}^{2d-1}$ has the standard orientation. Let us use the coordinate $(z_0, z_1, z_2) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1} = \mathbb{R}^{2d-1}$. The unit sphere $S^{d-1} \subset \mathbb{R}^d$ is oriented as the boundary of the unit ball. We consider three ways of embedding of unit sphere $S^{d-1} \subset \mathbb{R}^d$ into \mathbb{R}^{2d-1} :

$$\begin{aligned} S^{d-1} \subset \mathbb{R}^d &= \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^{2d-1}: (z_0, z_1) \mapsto (z_0, z_1, 0), \\ S^{d-1} \subset \mathbb{R}^d &= \mathbb{R} \times \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{2d-1}: (z_1, z_2) \mapsto (0, z_1 + 1, z_2), \\ S^{d-1} \subset \mathbb{R}^d &= \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^{2d-1}: (z_0, z_1) \mapsto (z_0, z_1, z_2^*), \end{aligned}$$

for a fixed vector $z_2^* \in \mathbb{R}^{d-1} \setminus \{0\}$. Their images are written by K_0, K_1, K_2 in order. These submanifolds are oriented so that the diffeomorphisms $S^{d-1} \rightarrow K_i$ from the above maps change the sign of orientation by $(-1)^{d-1}$.

As a notation, given a set \mathcal{S} and a map $\mathcal{S} \rightarrow \mathbb{Z}: s \mapsto |s|$, let $\mathcal{A}_*(\mathcal{S})$ denote the unital non-commutative graded \mathbb{R} -algebra freely generated by \mathcal{S} such that $s \in \mathcal{A}_{|s|}(\mathcal{S})$ for every $s \in \mathcal{S}$.

5.1 Computation of $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_1)$

Let us define $\mathcal{A}_*^{\text{Hopf}} := \mathcal{A}_*(\mathcal{C} \cup \mathcal{D} \cup \mathcal{E})$ by the following three sets:

$$\begin{aligned} \mathcal{C} &:= \{c_{i,j}^0\}_{i \neq j} \cup \{c_{i,i}^1\}_i \cup \{c_{i,j}^1, \bar{c}_{i,j}^1\}_{i \neq j} \cup \{c_{i,j}^2\}_{i,j}, \\ \mathcal{D} &:= \{d_{i,i}^1\}_i \cup \{d_{i,j}^2\}_{i,j}, \\ \mathcal{E} &:= \{e_{i,i}^1\}_i \cup \{e_{i,j}^2\}_{i,j}, \end{aligned}$$

where i and j run over $\{0, 1\}$. The degree of each element is given by

$$\begin{aligned} |c_{i,j}^0| &= d - 2, \quad |c_{i,i}^1| = |c_{i,j}^1| = |\bar{c}_{i,j}^1| = 2d - 3 \text{ for } i \neq j, \\ |c_{i,i}^1| &= 2d - 3, \quad |c_{i,j}^2| = 3d - 4, \\ |d_{i,i}^1| &= 2d - 3, \quad |d_{i,j}^2| = 3d - 4, \\ |e_{i,i}^1| &= 2d - 4, \quad |e_{i,j}^2| = 3d - 5. \end{aligned}$$

We define a graded derivation $\partial: \mathcal{A}_*^{\text{Hopf}} \rightarrow \mathcal{A}_{*-1}^{\text{Hopf}}$ so that

$$\begin{aligned} \partial c_{i,j}^0 &= 0, \quad \partial c_{i,i}^1 = \partial c_{i,j}^1 = \partial \bar{c}_{i,j}^1 = 0, \quad \partial c_{i,j}^2 = 0, \\ \partial d_{i,i}^1 &= e_{i,i}^1, \quad \partial d_{i,j}^2 = e_{i,j}^2, \\ \partial e_{i,i}^1 &= 0, \quad \partial e_{i,j}^2 = 0, \end{aligned}$$

and they are extended by the Leibniz rule. We also define another graded derivation $F: \mathcal{A}_*^{\text{Hopf}} \rightarrow \mathcal{A}_{*-1}^{\text{Hopf}}$ so that

$$\begin{aligned} F c_{i,j}^0 &= F c_{i,j}^1 = F \bar{c}_{i,j}^1 = 0 \text{ for } i \neq j, \\ F c_{0,0}^1 &= (-1)^d e_{0,0}^1 + (-1)^d c_{0,1}^0 c_{1,0}^0, \quad F c_{1,1}^1 = (-1)^d e_{1,1}^1 + c_{1,0}^0 c_{0,1}^0, \\ F c_{0,0}^2 &= -e_{0,0}^2 - (\bar{c}_{0,1}^1 c_{1,0}^0 + (-1)^d c_{0,1}^1 c_{1,0}^0), \quad F c_{1,1}^2 = -e_{1,1}^2 - ((-1)^d \bar{c}_{1,0}^1 c_{0,1}^0 + c_{1,0}^1 c_{0,1}^0), \\ F c_{0,1}^2 &= -e_{0,1}^2, \quad F c_{1,0}^2 = -e_{1,0}^2, \\ F d_{i,i}^1 &= 0, \quad F d_{i,j}^2 = 0, \\ F e_{i,i}^1 &= 0, \quad F e_{i,j}^2 = 0, \end{aligned}$$

and they are extended by the Leibniz rule. It is easy to see that $\partial \circ \partial = 0$, $\partial \circ F + F \circ \partial = 0$ and $F \circ F = 0$ hold. Therefore, we obtain a differential graded \mathbb{R} -algebra $(\mathcal{A}_*^{\text{Hopf}}, \partial + F)$. Note that the differential graded \mathbb{R} -algebra $(\mathcal{A}_*^{\text{Hopf}}, \partial)$ is obtained from $(\mathcal{A}_*(\mathcal{C}), 0)$ by *stabilizations* (See [18, Definition 3.9]). Thus,

$$(\mathcal{A}_*(\mathcal{C}), 0) \xrightarrow{i} (\mathcal{A}_*^{\text{Hopf}}, \partial) \xleftarrow{\tau} (\mathcal{A}_*(\mathcal{C}), 0), \quad (33)$$

where i is the inclusion map and τ is the projection map, are quasi-isomorphisms. For the proof, see [18, Corollary 3.11].

The goal of this section is to prove the next theorem.

Theorem 5.1. *There exists an isomorphism of graded \mathbb{R} -algebras*

$$H_*(\mathcal{A}_*^{\text{Hopf}}, \partial + F) \cong H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_1).$$

To compute $H_*^{\text{string}}(Q, K_0 \cup K_1)$, we fix auxiliary data so that g is the standard Riemannian metric on \mathbb{R}^{2d-1} . The constant C_0 is required to be $C_0 > 3$. The other data, ε_0 and μ , are not specified. The proof is divided into three steps.

Step 1. We first observe de Rham chains in $C_*^{\text{dR}}(\Sigma_m^a, \Sigma_m^0)$. We define a map

$$\varphi: (K_0 \cup K_1)^2 \rightarrow \Omega_{K_0 \cup K_1}(\mathbb{R}^{2d-1})$$

so that each $(p, p') \in (K_0 \cup K_1)^2$ is mapped to a path of a segment

$$\varphi(p, p'): [0, 1] \rightarrow \mathbb{R}^{2d-1}: t \mapsto (1-t)p + tp'.$$

We fix two points $p_0 := (0, 1, 0) \in K_0$ and $p_1 := (0, 0, 0) \in K_1$. Then, we define de Rham chains

$$\begin{aligned} x_{i,j}^0 &:= [\{(p_i, p_j)\}, \varphi|_{\{(p_i, p_j)\}}, 1] \in C_0^{\text{dR}}(\Sigma_1^a, \Sigma_1^0) \quad (i \neq j), \\ x_{i,i}^1 &:= [\{p_i\} \times K_i, \varphi|_{\{p_i\} \times K_i}, 1] \in C_{d-1}^{\text{dR}}(\Sigma_1^a, \Sigma_1^0), \\ x_{i,j}^1 &:= [\{p_i\} \times K_j, \varphi|_{\{p_i\} \times K_j}, 1] \in C_{d-1}^{\text{dR}}(\Sigma_1^a, \Sigma_1^0) \quad (i \neq j), \\ \bar{x}_{i,j}^1 &:= [K_i \times \{p_j\}, \varphi|_{K_i \times \{p_j\}}, 1] \in C_{d-1}^{\text{dR}}(\Sigma_1^a, \Sigma_1^0) \quad (i \neq j), \\ x_{i,j}^2 &:= [K_i \times K_j, \varphi|_{K_i \times K_j}, 1] \in C_{2d-2}^{\text{dR}}(\Sigma_1^a, \Sigma_1^0). \end{aligned} \quad (34)$$

Here, $a > 3$ and i, j runs over $\{0, 1\}$. Obviously, they are cycle chains for ∂ . We write the set of these chains by

$$\mathcal{X} := \{x_{i,j}^0\}_{i \neq j} \cup \{x_{i,i}^1\}_i \cup \{x_{i,j}^1, \bar{x}_{i,j}^1\}_{i \neq j} \cup \{x_{i,j}^2\}_{i,j}.$$

If we define a function $l: \mathcal{X} \rightarrow \mathbb{R}_{>0}$ by

$$\begin{aligned} l(x_{i,j}^0) &= l(x_{i,i}^1) = l(\bar{x}_{i,j}^1) = 1 \text{ for } i \neq j, \\ l(x_{i,i}^1) &= l(x_{i,i}^2) = 2, \\ l(x_{i,j}^2) &= 3 \text{ for } i \neq j, \end{aligned}$$

then, each $x \in \mathcal{X}$ satisfies $x \in C_*^{\text{dR}}(\Sigma_1^{l(x)+\varepsilon}, \Sigma_1^0)$ for any $\varepsilon > 0$.

For every $a \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 1}$, let us consider a manifold

$$B_m^a := \{(q_1^0, q_1^1, \dots, q_m^0, q_m^1) \in (K_0 \cup K_1)^{2m} \mid \sum_{k=1}^m |q_k^0 - q_k^1| < a \text{ or } \min_{1 \leq k \leq m} |q_k^0 - q_k^1| < \varepsilon_0\}.$$

This is homotopy equivalent to Σ_m^a by two smooth maps

$$\begin{aligned} \pi_m: \Sigma_m^a &\rightarrow B_m^a: (\gamma_k: [0, T_k] \rightarrow \mathbb{R}^{2d-1})_{k=1, \dots, m} \rightarrow (\gamma_1(0), \gamma_1(T_1), \dots, \gamma_m(0), \gamma_m(T_m)), \\ i_m: B_m^a &\rightarrow \Sigma_m^a: (q_1^0, q_1^1, \dots, q_m^0, q_m^1) \mapsto (\varphi(q_k^0, q_k^1))_{k=1, \dots, m}, \end{aligned}$$

for which $\pi_m \circ i_m = \text{id}_{B_m^a}$ holds and $i_m \circ \pi_m$ is homotopic to $\text{id}_{\Sigma_m^a}$ (c.f. Lemma 3.2).

Notation. In this section, if N is a submanifold of a manifold M , then the inclusion map $N \rightarrow M$ is denoted by ι_N .

Lemma 5.2. *Let M be an oriented manifold and N be its open submanifold. Suppose that there exists an approximately smooth function $f: M \rightarrow \mathbb{R}$ such that $N = f^{-1}((-\infty, 0))$. In addition, we assume that $H_*^{\text{sing}}(M, N)$ has a finite dimension. Then there exists an isomorphism between $H_*^{\text{sing}}(M, N)$ and $H_*^{\text{dR}}(M, N)$ such that for every closed oriented k -dimensional submanifold K of M , the fundamental class $[K] \in H_k^{\text{sing}}(M, N)$ corresponds to $(-1)^{s(k)}[K, \iota_K, 1] \in H_k^{\text{dR}}(M, N)$, where $s(k) := (k - \dim M)(k - \dim M - 1)/2$.*

Proof. We consider the correspondence through the following isomorphisms:

$$H_*^{\text{sing}}(M, N) \cong H_{c, \text{dR}}^{\dim M - *}(M, N) \cong H_*^{\text{dR}}(M^{\text{reg}}, N^{\text{reg}}) \rightarrow H_*^{\text{dR}}(M, N).$$

The first isomorphism is defined by the Poincaré duality. The second isomorphism was given in Example 2.6. The last isomorphism is induced by $\text{id}_M: M^{\text{reg}} \rightarrow M$ [23, Proposition 5.2]. Let us identify the tubular neighborhood N_K of K with the normal bundle of K . Then, $[K] \in H_k^{\text{sing}}(M, N)$ corresponds to $[\eta] \in H_{c, \text{dR}}^{\dim M - k}(M, N)$, where $\eta \in \Omega_c^{\dim M - k}(M)$ has its support in N_K and represents the Thom class of the normal bundle. Recalling Example 2.6, we can see that this cohomology class corresponds to $(-1)^{s(k)}[M, \text{id}_M, \eta] = (-1)^{s(k)}[N_K, \text{id}_{N_K}, \eta] \in H_k^{\text{dR}}(M^{\text{reg}}, N^{\text{reg}})$. Let $\pi_{N_K}: N_K \rightarrow K$ be the bundle projection. Then, as a de Rham chain in $C_k^{\text{dR}}(M, N)$, $[N_K, \iota_{N_K}, \eta]$ is homologous to $[N_K, \iota_K \circ \pi_{N_K}, \eta]$ since $\iota_{N_K}: N_K \rightarrow M$ is homotopic to $\iota_K \circ \pi_{N_K}: N_K \rightarrow K \subset M$. Now the assertion follows since

$$[N_K, \iota_K \circ \pi_{N_K}, \eta] = [K, \iota_K, (\pi_{N_K})! \eta] = [K, \iota_K, 1] \in H_k^{\text{dR}}(M, N).$$

□

Let us see that a basis of $H_*^{\text{dR}}(\Sigma_m^a, \Sigma_m^0)$ is given by \mathcal{X} and the \star -operation. First, suppose that $m = 1$ and $a > 3$. Then $B_1^a = K \times K$. Through the isomorphism $H_*^{\text{dR}}(B_1^a, B_1^0) \cong H_*^{\text{sing}}(B_1^a, B_1^0)$ of Lemma 5.2, $\{(\pi_1)_*[x] \mid x \in \mathcal{X}\}$ corresponds to the set of singular homology classes

$$\{[(p_i, p_j)]\}_{i \neq j} \cup \{[p_i \times K_i]\}_i \cup \{[p_i \times K_j]\}_{i \neq j} \cup \{[K_i \times \{p_j\}]\}_{i \neq j} \cup \{[K_i \times K_j]\}_{i, j},$$

which is a basis of $H_*^{\text{sing}}(K \times K, B_1^0)$. Therefore, $\{[x] \mid x \in \mathcal{X}\}$ is a basis of $H_*^{\text{dR}}(\Sigma_1^a, \Sigma_1^0)$ for $a > 3$.

For $a \in (0, 3]$, we consider the deformation along the negative gradient vector field of a C^∞ function $E_{i, j}: (K_i \cup K_j)^2 \rightarrow \mathbb{R}: (p, p') \rightarrow |p - p'|^2$. If $i = j$, then $\max E_{i, j} = 4$, $\min E_{i, j} = 0$, and the subset $\{(p, p') \in K_i \times K_i \mid |p - p'| < a\}$ for $a \leq 2$ has $E_{i, i}^{-1}(0) = \{(p, p') \in K_i \times K_i \mid p = p'\}$ (the diagonal) as a deformation retract. If $i \neq j$, then $\max E_{i, j} = 9$, $\min E_{i, j} = 1$, and the subset $\{(p, p') \in K_i \times K_j \mid |p - p'| < a\}$ for $a \leq 3$ has $E_{i, j}^{-1}(0) = (K_i \times \{p_j\}) \cup (\{p_i\} \times K_j)$ (a bouquet) as a deformation retract. From these observations, we can see that $\{[x] \mid x \in \mathcal{X}, \mathfrak{l}(x) < a\}$ is a basis of $H_*^{\text{dR}}(\Sigma_1^a, \Sigma_1^0)$. In general, for any $m \in \mathbb{Z}_{\geq 1}$ and $a \in \mathbb{R}_{>0}$,

$$\{[x_1 \star \cdots \star x_m] \mid x_1, \dots, x_m \in \mathcal{X}, \mathfrak{l}(x_1) + \cdots + \mathfrak{l}(x_m) < a\} \quad (35)$$

is a basis of $H_*^{\text{dR}}(\Sigma_m^a, \Sigma_m^0)$.

We fix a trivialization $h: \mathcal{O}_{\varepsilon_0} \times (K_0 \cup K_1) \rightarrow N_{\varepsilon_0}$ so that $h(w, p_0) = p_0 + (0, w)$ and $h(w, p_1) = p_1 + (-w, 0)$ for every $w \in \mathcal{O}_{\varepsilon_0} \subset \mathbb{R}^d$. (The orientations of K_0, K_1 are chosen so that this map preserves orientations.)

Suppose that $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K_0 \cup K_1)$ ($= \mathbb{R}_{>0} \setminus \mathbb{Z}_{\geq 1}$). In the following series of three lemmata, we will observe the chains $f_{1, \delta}(x)$ for each $x \in \mathcal{X}$. Hereafter, we assume that $(\varepsilon, \delta) \in \mathcal{T}_a$ is standard with respect to h (See Subsection 4.4.3).

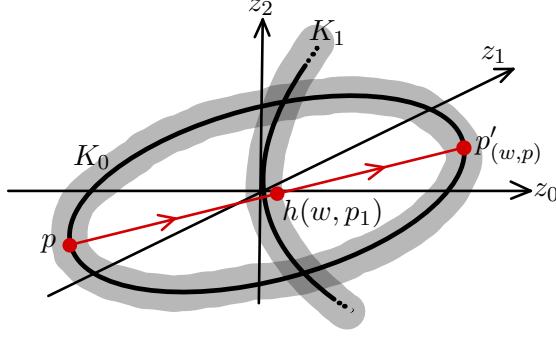


Figure. 3: The red path is $\varphi(p, p'_{(w,p)})$. The gray region is the tubular neighborhood N_ε of $K_0 \cup K_1$.

Lemma 5.3. For $i \neq j$, we have equations

$$f_{1,\delta}(x_{i,j}^0) = 0, \quad f_{1,\delta}(x_{i,j}^1) = f_{1,\delta}(\bar{x}_{i,j}^1) = 0.$$

Proof. For $(p, p') \in K_i \times K_j$ ($i \neq j$) such that either $p = p_i$ or $p' = p_j$, $\varphi(p, p')$ satisfies the condition (iii) of Lemma 3.12 for $m = k = 1$. The equations follow from Lemma 3.12. \square

Lemma 5.4. For $i \in \{0, 1\}$, there exist $y_{i,i}^1 \in C_1^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ and $y_{i,i}^2 \in C_d^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ such that

$$\begin{aligned} \partial y_{0,0}^1 &= f_{1,\delta}(x_{0,0}^1) - x_{0,1}^0 \star x_{1,0}^0, \\ \partial y_{1,1}^1 &= f_{1,\delta}(x_{1,1}^1) - (-1)^d x_{1,0}^0 \star x_{0,1}^0, \\ \partial y_{0,0}^2 &= f_{1,\delta}(x_{0,0}^2) - (\bar{x}_{0,1}^1 \star x_{1,0}^0 + (-1)^d x_{0,1}^1 \star x_{1,0}^0), \\ \partial y_{1,1}^2 &= f_{1,\delta}(x_{1,1}^2) - ((-1)^d \bar{x}_{1,0}^1 \star x_{0,1}^0 + x_{1,0}^1 \star x_{0,1}^0), \end{aligned} \quad (36)$$

and

$$(f_{1,\delta} + (-1)^d f_{2,\delta})(y_{i,i}^1) = 0, \quad (f_{1,\delta} + (-1)^d f_{2,\delta})(y_{i,i}^2) = 0. \quad (37)$$

Proof. We only show the existence of $y_{0,0}^2$ and $y_{0,0}^1$. Replacing (K_0, p_0) by (K_1, p_1) , $y_{1,1}^2$ and $y_{1,1}^1$ are constructed in a parallel way, except the difference of signs.

Since (ε, δ) is standard, δ has the form (31). Using the notations of (13), we can write $f_{1,\delta}(x_{0,0}^2) = [W_1, \Phi_1, \zeta_1]$, where

$$\begin{aligned} W_1 &= \{((p, p'), \tau, v) \in (K_0 \times K_0) \times \mathbb{R} \times N_\varepsilon \mid 2\varepsilon < \tau < 1 - 2\varepsilon, (1 - \tau)p + \tau p' = v\}, \\ \Phi_1: W_1 &\rightarrow \Sigma_2^{2+2\varepsilon}: ((p, p'), \tau, v) \mapsto \text{con}_1(\varphi(p, p'), (1, \tau), \psi_\varepsilon(v)), \\ \zeta_1 \in \Omega_c^d(W_1): &(\zeta_1)_{((p,p'), \tau, v)} = \rho_\varepsilon(1, \tau) \cdot (h_*(\nu_\varepsilon \times 1))_v. \end{aligned}$$

N_ε is the disjoint union of $h(\mathcal{O}_\varepsilon \times K_0)$ and $h(\mathcal{O}_\varepsilon \times K_1)$. Corresponding to this division, we define $W'_i := \{((p, p'), \tau, v) \in W_1 \mid v \in h(\mathcal{O}_\varepsilon \times K_i)\}$ for $i = 0, 1$. If $((p, p'), \tau, v) \in W'_0$, τ satisfies the condition (ii) of Lemma 3.12, so $[W'_0, \Phi_1, \zeta_1] = 0 \in C_{d-1}^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$. For $((p, p'), \tau, v) \in W'_1$, $\rho_\varepsilon(1, \tau) = 1$ holds. Moreover, there is an diffeomorphism

$$I: \mathcal{O}_\varepsilon \times K_0 \rightarrow W'_1: (w, p) \mapsto ((p, p'_{(w,p)}), |p - h(w, p_1)|, h(w, p_1)).$$

Here $p'_{(w,p)} \in K_0 \setminus \{p\}$ is determined by $h(w, p_1) \in \text{Im}(\varphi(p, p'_{(w,p)}))$, as described in Figure 3. The diffeomorphism I preserves orientations and $I^* \zeta_1 = \nu_\varepsilon \times 1$ holds.

Likewise, we consider an explicit description of $f_{1,\delta}(x_{0,0}^1)$. Then, we have

$$\begin{aligned} f_{1,\delta}(x_{0,0}^2) &= [\mathcal{O}_\varepsilon \times K_0, \Phi_1 \circ I, \nu_\varepsilon \times 1] \in C_{d-1}^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0), \\ f_{1,\delta}(x_{0,0}^1) &= [\mathcal{O}_\varepsilon \times \{p_0\}, \Phi_1 \circ I|_{\mathcal{O}_\varepsilon \times \{p_0\}}, \nu_\varepsilon \times 1] \in C_0^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0). \end{aligned}$$

Let us define $\tilde{\Phi}_1: \mathbb{R} \times (\mathcal{O}_\varepsilon \times K_0) \rightarrow \Sigma_2^{2+2\varepsilon}$ as follows: Choose a C^∞ function $\kappa: \mathbb{R} \rightarrow [0, 1]$ such that $\kappa(s) = \begin{cases} 0 & \text{if } s \leq \frac{1}{2}, \\ 1 & \text{if } s \geq 1. \end{cases}$ For $s \geq \frac{1}{2}$, we define $\tilde{\Phi}_1(s, (w, p)) := \Phi_1 \circ I(\kappa(s) \cdot w, p)$. Then the

first path (resp. the second path) of $\tilde{\Phi}_1(\frac{1}{2}, (w, p))$ is equal to $\varphi(p, p_1)$ (resp. $\varphi(p_1, -p)$) up to an reparameterization, so we define $\tilde{\Phi}(s, (w, p))$ for $s \leq \frac{1}{2}$ by interpolating the parameterizations so that $\tilde{\Phi}(s, (w, p)) = (\varphi(p, p_1), \varphi(p_1, -p))$ for $s \leq 0$. Now we define

$$\begin{aligned} \tilde{y}_{0,0}^2 &:= (-1)^d [\mathbb{R} \times (\mathcal{O}_\varepsilon \times K_0), \tilde{\Phi}_1, \chi \times (\nu_\varepsilon \times 1)] \in C_d^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0), \\ \tilde{y}_{0,0}^2 &:= (-1)^d [\mathbb{R} \times (\mathcal{O}_\varepsilon \times \{p_0\}), \tilde{\Phi}_1 \Big|_{\mathbb{R} \times (\mathcal{O}_\varepsilon \times \{p_0\})}, \chi \times (\nu_\varepsilon \times 1)] \in C_1^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0), \end{aligned}$$

where $\chi: \mathbb{R} \rightarrow [0, 1]$ is a C^∞ function with a compact support such that $\chi \equiv 1$ on $[0, 1]$.

From the constructions of $\tilde{y}_{0,0}^2$ and $\tilde{y}_{0,0}^1$, we can compute their boundary chains as follows: Let us introduce two maps $\varphi_0: K_0 \rightarrow K_0 \times K_0: p \mapsto (p, -p)$ and $\tilde{i}_2: K_0 \times K_0 \rightarrow \Sigma_2^{2+2\varepsilon}: (p, p') \mapsto (\varphi(p, p_1), \varphi(p_1, p'))$. Then, we have

$$\begin{aligned} \partial \tilde{y}_{0,0}^2 &= f_{1,\delta}(x_{0,0}^2) - (\tilde{i}_2)_* [\mathcal{O}_\varepsilon \times K_0, \varphi_0 \circ \text{pr}_{K_0}, \nu_\varepsilon] \\ &= f_{1,\delta}(x_{0,0}^2) - (\tilde{i}_2)_* [\varphi_0(K_0), \iota_{\varphi_0(K_0)}, 1], \\ \partial \tilde{y}_{0,0}^1 &= f_{1,\delta}(x_{0,0}^1) - (\tilde{i}_2)_* [\mathcal{O}_\varepsilon \times \{p_0\}, \varphi_0 \circ \text{pr}_{\{p_0\}}, \nu_\varepsilon \times 1] \\ &= f_{1,\delta}(x_{0,0}^1) - (\tilde{i}_2)_* [\{\varphi_0(p_0)\}, \iota_{\{\varphi_0(p_0)\}}, 1]. \end{aligned}$$

Here we used the condition that $\int_{\mathcal{O}_\varepsilon} \nu_\varepsilon = 1$. Moreover, we can check from the definition that $\tilde{\Phi}_1(s, (w, p))$ satisfies the condition (iii) of Lemma 3.12 for $m = 2$ and $k = 1, 2$. Therefore, $f_{k,\delta}(\tilde{y}_{0,0}^2) = 0$ and $f_{k,\delta}(\tilde{y}_{0,0}^1) = 0$ hold for $k = 1, 2$.

As homology classes in $H_*^{\text{sing}}(K_0 \times K_0)$,

$$\begin{aligned} [\varphi_0(K_0)] &= [K_0 \times \{p_0\}] + (-1)^d [\{p_0\} \times K_0] \in H_{d-1}^{\text{sing}}(K_0 \times K_0), \\ [\{\varphi_0(p_0)\}] &= [\{(p_0, p_0)\}] \in H_0^{\text{sing}}(K_0 \times K_0). \end{aligned}$$

Therefore, by Lemma 5.2, there exist $z_{0,0}^2 \in C_d^{\text{dR}}(K_0 \times K_0)$ and $z_{0,0}^1 \in C_1^{\text{dR}}(K_0 \times K_0)$ such that

$$\begin{aligned} \partial z_{0,0}^2 &= [\varphi_0(K_0), \iota_{\varphi_0(K_0)}, 1] - \left([K_0 \times \{p_0\}, \iota_{K_0 \times \{p_0\}}, 1] + (-1)^d [\{p_0\} \times K_0, \iota_{\{p_0\} \times K_0}, 1] \right), \\ \partial z_{0,0}^1 &= [\{\varphi_0(p_0)\}, \iota_{\{\varphi_0(p_0)\}}, 1] - [\{(p_0, p_0)\}, \iota_{\{(p_0, p_0)\}}, 1]. \end{aligned}$$

It is clear from the definition of each $x \in \mathcal{X}$ that

$$\begin{aligned} (\tilde{i}_2)_* [K_0 \times \{p_0\}, \iota_{K_0 \times \{p_0\}}, 1] &= \bar{x}_{0,1}^1 \star x_{1,0}^0, \quad (\tilde{i}_2)_* [\{p_0\} \times K_0, \iota_{\{p_0\} \times K_0}, 1] = x_{0,1}^1 \star x_{1,0}^0, \\ (\tilde{i}_2)_* [\{(p_0, p_0)\}, \iota_{\{(p_0, p_0)\}}, 1] &= x_{0,1}^0 \star x_{1,0}^0. \end{aligned}$$

Therefore, $y_{0,0}^2 := \tilde{y}_{0,0}^2 + (\tilde{i}_2)_* z_{0,0}^2 \in C_d^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ and $y_{0,0}^1 := \tilde{y}_{0,0}^1 + (\tilde{i}_2)_* z_{0,0}^1 \in C_0^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ satisfy the first and the second equations of (36). Moreover, any paths in the image of \tilde{i}_2 satisfies the condition (iii) of Lemma 3.12 for $m = 2$ and $k = 1, 2$. Therefore, $f_{k,\delta}((\tilde{i}_2)_* z_{0,0}^2) = 0$ and $f_{k,\delta}((\tilde{i}_2)_* z_{0,0}^1) = 0$ hold for $k = 1, 2$, and thus $y_{0,0}^2$ and $y_{0,0}^1$ satisfy (37). \square

Lemma 5.5. *There exist $y_{0,1}^2, y_{1,0}^2 \in C_{2d-1}^{\text{dR}}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0)$ such that*

$$\partial y_{0,1}^2 = f_{1,\delta}(x_{0,1}^2), \quad \partial y_{1,0}^2 = f_{1,\delta}(x_{1,0}^2), \quad (38)$$

and

$$(f_{1,\delta} + (-1)^d f_{2,\delta})(y_{0,1}^2) = 0, \quad (f_{1,\delta} + (-1)^d f_{2,\delta})(y_{1,0}^2) = 0. \quad (39)$$

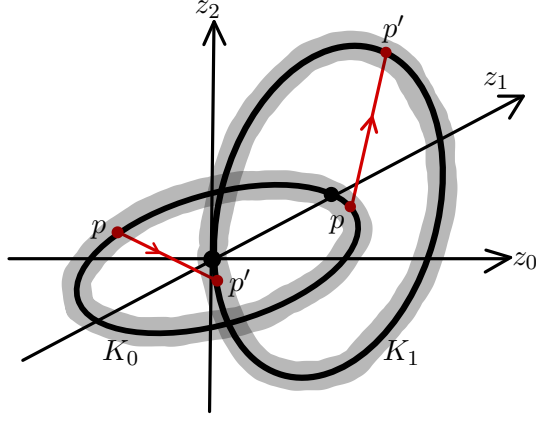


Figure. 4: The red path is $\varphi(p, p')$ when p is close to p_0 or p' is close to p_1 . The gray region is the tubular neighborhood N_ε of $K_0 \cup K_1$

Proof. We only show the existence of $y_{0,1}^2$. Exchanging K_0 and K_1 , $y_{1,0}^2$ is constructed in a parallel way.

Since (ε, δ) is standard, δ has the form (31). Using the notations of (13), we can write $f_{1,\delta}(x_{0,1}^2) = [W_1, \Phi_1, \zeta_1]$, where

$$\begin{aligned} W_1 &= \{((p, p'), \tau, v) \in (K_0 \times K_1) \times \mathbb{R} \times N_\varepsilon \mid 2\varepsilon < \tau < 1 - 2\varepsilon, (1 - \tau)p + \tau p' = v\}, \\ \Phi_1 &: W_1 \rightarrow \Sigma_2^{3+2\varepsilon}: ((p, p'), \tau, v) \mapsto \text{con}_1(\varphi(p, p'), (1, \tau), \psi_\varepsilon(v)), \\ \zeta_1 &\in \Omega_c^d(W_1): (\zeta_1)_{((p,p'),\tau,v)} = \rho_\varepsilon(1, \tau) \cdot (h_*(\nu_\varepsilon \times 1))_v. \end{aligned}$$

If $p \in K_0$ is sufficiently close to p_0 , then $\varphi(p, p')$ satisfy the condition (iii) of Lemma 3.12 for any $p' \in K_1$. Symmetrically, if $p' \in K_1$ is sufficiently close to p_1 , then $\varphi(p, p')$ satisfy the same condition for any $p \in K_0$. See Figure 4. Therefore, For any bump function $b: K_0 \times K_1 \rightarrow \mathbb{R}$ whose support is localized near $(K_0 \times \{p_1\}) \cup (\{p_0\} \times K_1)$, $[W_1, \Phi_1, \zeta'_1] = 0$ holds for

$$\zeta'_1 \in \Omega_c^d(W_1): (\zeta'_1)_{((p,p'),\tau,v)} = b(p, p') \cdot \rho_\varepsilon(1, \tau) \cdot (h_*(\nu_\varepsilon \times 1))_v.$$

We remark that $[W_1, \Phi_1, \zeta'_1]$ is an explicit description (13) of $f_{1,\delta}([K_0 \times K_1, \varphi|_{K_0 \times K_1}, b])$.

Now we choose b so that it is constant to 1 on a neighborhood of $(K_0 \times \{p_1\}) \cup (\{p_0\} \times K_1)$. Then, the above computation shows that

$$\begin{aligned} f_{1,\delta}(x_{0,1}^2) &= f_{1,\delta}([K_0 \times K_1, \varphi|_{K_0 \times K_1}, 1 - b]) + f_{1,\delta}([K_0 \times K_1, \varphi|_{K_0 \times K_1}, b]) \\ &= f_{1,\delta}([K'_0 \times K'_1, \varphi|_{K'_0 \times K'_1}, 1 - b|_{K'_0 \times K'_1}]), \end{aligned}$$

where K'_i ($i = 0, 1$) is the complement of a small closed ball containing p_i . Since $K'_0 \times K'_1$ is contractible, there exists a map $R: \mathbb{R} \times K'_0 \times K'_1 \rightarrow K'_0 \times K'_1$ such that $R(s, \cdot) = \text{id}_{K'_0 \times K'_1}$ for $s \geq 1$ and $R(s, \cdot)$ is constant to some point in $K'_0 \times K'_1$ for $s \leq 0$. Using the function χ in the proof of Lemma 5.4, let us define a chain

$$\tilde{x}_{0,1}^2 := [\mathbb{R} \times (K'_0 \times K'_1), \varphi \circ R, \chi \times (1 - b|_{K'_0 \times K'_1})] \in C_{2d-1}^{\text{dR}}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0).$$

This chain satisfies

$$\partial \tilde{x}_{0,1}^2 = [K'_0 \times K'_1, \varphi|_{K'_0 \times K'_1}, 1 - b|_{K'_0 \times K'_1}] + [\mathbb{R} \times (K'_0 \times K'_1), \varphi \circ R, \chi \times db|_{K'_0 \times K'_1}].$$

Note that $[K'_0 \times K'_1, \varphi \circ R(0, \cdot), 1 - b|_{K'_0 \times K'_1}] = 0$ since $\varphi \circ R(0, \cdot)$ is constant. The second chain of the RHS is mapped by $f_{1,\delta}$ to 0 since the support of db is localized near $(K_0 \times \{p_1\}) \cup (\{p_0\} \times K_1)$.

Now we define $y_{0,1}^2 := f_{1,\delta}(\tilde{x}_{0,1}^2) \in C_{2d-1}^{\text{dR}}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0)$. Then, the first equation of (38) holds since $\partial y_{0,1}^2 = f_{1,\delta}(\partial \tilde{x}_{0,1}^2) = f_{1,\delta}(x_{0,1}^2)$. The first equation (39) follows from $(f_{1,\delta} + (-1)^d f_{2,\delta}) \circ f_{1,\delta} = 0$ (See the proof of Proposition 4.1). \square

Step 2. We define a function $\mathfrak{l}: \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \rightarrow \mathbb{R}_{>0}$ by

$$\begin{aligned} \mathfrak{l}(c_{i,j}^0) &= \mathfrak{l}(c_{i,j}^1) = \mathfrak{l}(\bar{c}_{i,j}^1) = 1 \text{ for } i \neq j, \quad \mathfrak{l}(c_{i,i}^1) = 2, \\ \mathfrak{l}(c_{i,j}^2) &= \mathfrak{l}(d_{i,j}^2) = \mathfrak{l}(e_{i,j}^2) = \begin{cases} 2 & \text{if } i = j, \\ 3 & \text{if } i \neq j. \end{cases} \end{aligned}$$

For every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K_0 \cup K_1)$, let $\mathcal{A}_*^{<a}$ be an \mathbb{R} -subspace of $\mathcal{A}_*^{\text{Hopf}}$ spanned by words of elements in $\mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$ such that the sum of the values of \mathfrak{l} is less than a . Then, $(\partial + F)(\mathcal{A}_*^{<a}) \subset \mathcal{A}_*^{<a}$, so we get a subcomplex $(\mathcal{A}_*^{<a}, \partial + F)$.

We continue to use $(\varepsilon, \delta) \in \mathcal{T}_a$ which is standard with respect to h . The second step is to construct a chain map from $(\mathcal{A}_*^{<a}, \partial + F)$ to $(C_*^{<a}(\varepsilon), D_\delta)$, and prove that it is a quasi-isomorphism.

Let $y_{i,i}^1$ and $y_{i,j}^2$ be the chains of Lemma 5.4 and 5.5, which depends on (ε, δ) . We define a linear map $\Phi_{(\varepsilon, \delta)}^{<a}: \mathcal{A}_*^{<a} \rightarrow C_*^{<a}(\varepsilon)$ so that

$$\begin{aligned} \Phi_{(\varepsilon, \delta)}^{<a}(c_{i,j}^0) &:= x_{i,j}^0, \quad \Phi_{(\varepsilon, \delta)}^{<a}(c_{i,j}^1) := x_{i,j}^0, \quad \Phi_{(\varepsilon, \delta)}^{<a}(\bar{c}_{i,j}^1) := \bar{x}_{i,j}^1 \text{ for } i \neq j, \\ \Phi_{(\varepsilon, \delta)}^{<a}(c_{i,i}^1) &:= x_{i,i}^1, \quad \Phi_{(\varepsilon, \delta)}^{<a}(c_{i,j}^2) := x_{i,j}^2, \\ \Phi_{(\varepsilon, \delta)}^{<a}(d_{i,i}^1) &:= y_{i,i}^1, \quad \Phi_{(\varepsilon, \delta)}^{<a}(d_{i,j}^2) := y_{i,j}^2, \\ \Phi_{(\varepsilon, \delta)}^{<a}(e_{i,i}^1) &:= \partial y_{i,i}^1, \quad \Phi_{(\varepsilon, \delta)}^{<a}(e_{i,j}^2) := \partial y_{i,j}^1, \end{aligned} \tag{40}$$

and extend them naturally by the product map on $\mathcal{A}_*^{\text{Hopf}}$ and the \star -operation.

Proposition 5.6. $\Phi_{(\varepsilon, \delta)}^{<a}$ is a chain map from $(\mathcal{A}_*^{<a}, \partial + F)$ to $(C_*^{<a}(\varepsilon), D_\delta)$.

Proof. This follows immediately from the series of three lemmata in *Step 1*. For each $\xi \in \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$, $D_\delta \circ \Phi_{(\varepsilon, \delta)}^{<a}(\xi) = (\partial + F) \circ \Phi_{(\varepsilon, \delta)}^{<a}(\xi)$ is proved by:

- Lemma 5.3 if $\xi = c_{i,j}^0, c_{i,j}^1, \bar{c}_{i,j}^1$ ($i \neq j$).
- the equations (36) if $\xi = c_{i,i}^1, c_{i,i}^2$.
- the equations (38) if $\xi = c_{i,j}^2$ ($i \neq j$).
- the equations (37) if $\xi = d_{i,i}^1, d_{i,i}^2, e_{i,i}^1, e_{i,i}^2$.
- the equations (39) if $\xi = d_{i,j}^2, e_{i,j}^2$ ($i \neq j$).

□

Therefore, we have a linear map on the homology groups

$$(\Phi_{(\varepsilon, \delta)}^{<a})_*: H_*(\mathcal{A}_*^{<a}, \partial + F) \rightarrow H_*^{<a}(\varepsilon, \delta).$$

Proposition 5.7. $(\Phi_{(\varepsilon, \delta)}^{<a})_*$ is an isomorphism.

Proof. We introduce a function $\mathfrak{m}: \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \rightarrow \mathbb{Z}_{\geq 1}$ so that $\mathfrak{m}(\mathcal{C}) = \{1\}$ and $\mathfrak{m}(\mathcal{D}) = \mathfrak{m}(\mathcal{E}) = \{2\}$. For every $m \in \mathbb{Z}_{\geq 0}$, let $\mathcal{A}_*^{<a}(m)$ be an \mathbb{R} -subspace of $\mathcal{A}_*^{<a}$ generated by words of elements of $\mathcal{C} \cup \mathcal{D} \cup \mathcal{E}$ such that the sum of the values of \mathfrak{m} is equal to m . (When $m = 0$, $\mathcal{A}_*^{<a}(0) := \mathbb{R} \cdot 1$.) Then, the chain complex $(\mathcal{A}_*^{<a}, \partial + F)$ is filtered by subcomplexes $\{\mathcal{G}_p^{<a}\}_{p \in \mathbb{Z}}$ defined by $\mathcal{G}_p^{<a} := \bigoplus_{m \geq -p} \mathcal{A}_*^{<a}(m)$. Let us consider the spectral sequence determined by this filtration. The $(-m, q)$ -term ($m \geq 0$) of its first page is given by

$$H_{q-m}(\mathcal{A}_*^{<a}(m), \partial) \cong H_{q-m}(\mathcal{A}_*^{<a}(m, \mathcal{C}), 0) = \mathcal{A}_{q-m}^{<a}(m, \mathcal{C}).$$

Here, $\mathcal{A}_*^{<a}(m, \mathcal{C}) := \mathcal{A}_*^{<a}(m) \cap \mathcal{A}_*(\mathcal{C})$ and the first isomorphism is induced by restricting the quasi-isomorphisms (33). $\Phi_{(\varepsilon, \delta)}^{<a}$ preserves the filtrations $\{\mathcal{G}_p^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathcal{F}_{\varepsilon, p}^{<a}\}_{p \in \mathbb{Z}}$. The induced map on the $(-m)$ -th column ($m \geq 0$) of the first page has the form

$$(\Phi_{(\varepsilon, \delta)}^{<a})_*: \mathcal{A}_{*-m}^{<a}(m, \mathcal{C}) \rightarrow H_{*-m(d-1)}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0).$$

This map is an isomorphism since the basis $\{c_1 \cdots c_m \mid c_1, \dots, c_m \in \mathcal{C}, \mathfrak{l}(c_1) + \cdots + \mathfrak{l}(c_m) < a\}$ of $\mathcal{A}_{*-m}^{<a}(m, \mathcal{C})$ is mapped to the basis (35). In the $(-m)$ -th column for $m < 0$, $(\Phi_{(\varepsilon, \delta)}^{<a})_*$ is a map between the zero vector space. The proposition now follows from Lemma 4.2. \square

Step 3. The last step is to show that the family of maps

$$\{(\Phi_{(\varepsilon, \delta)}^{<a})_* \mid a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K), (\varepsilon, \delta) \in \mathcal{T}_a \text{ is standard with respect to } h\}$$

induces an isomorphism on the limit of $\varepsilon \rightarrow 0$ and $a \rightarrow \infty$.

Lemma 5.8. For $(\varepsilon, \delta), (\varepsilon', \delta') \in \mathcal{T}_a$ ($\varepsilon' \leq \varepsilon$) which are standard with respect to h ,

$$k_{(\varepsilon', \delta'), (\varepsilon, \delta)} \circ (\Phi_{(\varepsilon', \delta')}^{<a})_* = (\Phi_{(\varepsilon, \delta)}^{<a})_*.$$

Proof. We have defined $\Phi_{(\varepsilon, \delta)}^{<a}$ by the chains $\{y_{i,i}^1, y_{i,j}^2\}$ which depends on (ε, δ) . As a notation, let $\{(y_{i,i}^1)', (y_{i,j}^2)'\}$ be the corresponding chains constructed from (ε', δ') , by which $\Phi_{(\varepsilon', \delta')}^{<a}$ is defined.

We take $(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_a$ satisfying (24) for $(\varepsilon, \delta), (\varepsilon', \delta')$. We may assume that it is standard with respect to h , and thus $\bar{\varepsilon} = \varepsilon$. As in Lemma 5.3, $\bar{f}_{1, \bar{\delta}}(\bar{i}(x)) = 0$ holds for $x = x_{i,j}^0, x_{i,j}^1, \bar{x}_{i,j}^1$ ($i \neq j$), where \bar{i} is the map (17). We claim that there exist $[-1, 1]$ -modeled chains $\bar{y}_{i,i}^1 \in \bar{C}_1^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$, $\bar{y}_{i,i}^2 \in \bar{C}_d^{\text{dR}}(\Sigma_2^{2+2\varepsilon}, \Sigma_2^0)$ and $\bar{y}_{i,j}^2 \in \bar{C}_d^{\text{dR}}(\Sigma_2^{3+2\varepsilon}, \Sigma_2^0)$ ($i \neq j$) which satisfy the following equations:

- the variants of the equations (36), (37), (38) and (39) obtained by replacing $\{y_{i,i}^1, y_{i,j}^2, f_{k, \delta}, \star\}$ by $\{\bar{y}_{i,i}^1, \bar{y}_{i,j}^2, \bar{f}_{k, \delta}, \bar{\star}\}$, and $x \in \mathcal{X}$ by $\bar{i}(x)$.
- $e_+ \bar{y}_{i,i}^1 = y_{i,i}^1$, $e_+ \bar{y}_{i,j}^2 = y_{i,j}^2$, $e_- \bar{y}_{i,i}^1 = (j_{\varepsilon', \varepsilon})_*(y_{i,i}^1)'$, and $e_- \bar{y}_{i,j}^2 = (j_{\varepsilon', \varepsilon})_*(y_{i,j}^2)'$.

This claim is proved by rewriting the proof of Lemma 5.4 and 5.5 for $[-1, 1]$ -modeled chains. We omit the proof.

We define a linear map $\bar{\Phi}_\varepsilon^{<a}: \mathcal{A}_*^{<a} \rightarrow \bar{C}_*^{<a}(\varepsilon)$ as in (40) by replacing $x \in \mathcal{X}$ by $\bar{i}(x)$ and $\{y_{i,i}^1, y_{i,j}^2\}$ by $\{\bar{y}_{i,i}^1, \bar{y}_{i,j}^2\}$, and extend naturally by the product on $\mathcal{A}_*^{\text{Hopf}}$ and the $\bar{\star}$ -operation. The former equations about $\bar{y}_{i,i}^1, \bar{y}_{i,j}^2$ ensure that $\bar{\Phi}_\varepsilon^{<a}$ is a chain map from $(\mathcal{A}_*^{<a}, \partial + F)$ to $(\bar{C}_*^{<a}(\varepsilon), \bar{D}_\varepsilon)$, as in Proposition 5.6. The latter equations about $\bar{y}_{i,i}^1, \bar{y}_{i,j}^2$ ensure the commutativity of the following diagram:

$$\begin{array}{ccc}
& & H_*^{<a}(\varepsilon, \delta) \\
& \nearrow^{(\Phi_{(\varepsilon, \delta)}^{<a})_*} & \uparrow f_{(\varepsilon, \delta), +} \\
H_*(\mathcal{A}_*^{<a}, \partial + F) & \xrightarrow{(\bar{\Phi}_\varepsilon^{<a})_*} & \bar{H}_*^{<a}(\varepsilon, \bar{\delta}) \\
& \searrow_{(\Phi_{(\varepsilon', \delta')}^{<a})_*} & \downarrow f_{(\varepsilon, \delta), -} \\
& & H_*^{<a}(\varepsilon', \delta').
\end{array}$$

$k_{(\varepsilon', \delta'), (\varepsilon, \delta)}$

This proves the lemma. \square

Therefore, the family of maps $\{(\Phi_{(\varepsilon, \delta)}^{<a})_* \mid (\varepsilon, \delta) \in \mathcal{T}_a \text{ is standard with respect to } h\}$ induces an isomorphism on the limit of $\varepsilon \rightarrow 0$

$$\Phi_*^{<a}: H_*(\mathcal{A}_*^{<a}, \partial + F) \rightarrow H_*^{<a}(\mathbb{R}^{2d-1}, K_0 \cup K_1)$$

for every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K_0 \cup K_1)$. Moreover, $\{\Phi_*^{<a}\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K_0 \cup K_1)}$ is naturally compatible with $\{I^{a,b}\}_{a \leq b}$ and the family of linear maps

$$\{H_*(\mathcal{A}_*^{<a}, \partial + F) \rightarrow H_*(\mathcal{A}_*^{<b}, \partial + F)\}_{a \leq b}$$

which is induced by the inclusion map $\mathcal{A}_*^{<a} \rightarrow \mathcal{A}_*^{<b}$. Therefore, on the limit of $a \rightarrow \infty$, we obtain an isomorphism

$$H_*(\mathcal{A}_*^{\text{Hopf}}, \partial + F) = \varinjlim_{a \rightarrow \infty} H_*(\mathcal{A}_*^{<a}, \partial + F) \rightarrow H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_1).$$

This finishes the proof of Theorem 5.1.

5.2 Computation of $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_2)$

We define $\mathcal{A}_*^{\text{unlink}} := \mathcal{A}_*(\mathcal{C}')$ by the set

$$\mathcal{C}' := \{c_{i,j}^0\}_{i \neq j} \cup \{c_{i,i}^1\}_i \cup \{c_{i,j}^1, \bar{c}_{i,j}^1\}_{i \neq j} \cup \{c_{i,j}^2\}_{i,j},$$

where i and j run over $\{0, 2\}$. The degree of each element is given by

$$\begin{aligned} |c_{i,j}^0| &= d - 2, \quad |c_{i,j}^1| = |\bar{c}_{i,j}^1| = |c_{i,i}^1| = 2d - 3 \text{ for } i \neq j, \\ |c_{i,i}^1| &= 2d - 3, \quad |c_{i,j}^2| = 3d - 4. \end{aligned}$$

(Obviously, there exists an isomorphism $\mathcal{A}_*(\mathcal{C}) \cong \mathcal{A}_*^{\text{unlink}}$ as graded \mathbb{R} -algebras.) We define a graded derivation $\partial := 0: \mathcal{A}_*^{\text{unlink}} \rightarrow \mathcal{A}_{*-1}^{\text{unlink}}$. For a differential graded algebra $(\mathcal{A}_*^{\text{unlink}}, \partial)$, we have the following result.

Theorem 5.9. *There exists an isomorphism of graded \mathbb{R} -algebras*

$$H_*(\mathcal{A}_*^{\text{unlink}}, \partial) \cong H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_2).$$

To compute $H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_2)$, we fix auxiliary data so that g is the standard Riemannian metric on \mathbb{R}^{2d-1} . The constant C_0 is required to be $C_0 > \sqrt{|z_2^*|^2 + 4}$. The other data, ε_0 and μ , are not specified. The strategy of the proof is the same as Theorem 5.1, but it is much more simple. We only see the outline of each step.

Step 1. We may assume that $|z_2^*| > 2$. Let us fix points $p_0 := (0, 1, 0) \in K_0$ and $p_2 := (0, 1, z_2^*) \in K_2$, and define submanifolds of $(K_0 \cup K_2)^2$

$$\begin{aligned} K_{0,2} &:= \{(p, p') \in K_0 \times K_2 \mid p' = p + (0, 0, z_2^*)\}, \\ K_{2,0} &:= \{(p, p') \in K_2 \times K_0 \mid p' = p - (0, 0, z_2^*)\}. \end{aligned}$$

Let $\varphi: (K_0 \cup K_2)^2 \rightarrow \Omega_{K_0 \cup K_2}(\mathbb{R}^{2d-1})$ be the map defined as in Section 5.1 by replacing K_1 by K_2 . Then, we define the set of chains

$$\mathcal{X}' := \{x_{i,j}^0\}_{i \neq j} \cup \{x_{i,i}^1\}_i \cup \{x_{i,j}^1, \bar{x}_{i,j}^1\}_{i \neq j} \cup \{x_{i,j}^2\}_{i,j},$$

where i and j run over $\{0, 2\}$, as follows:

$$\begin{aligned} x_{i,j}^0 &:= [\{(p_i, p_j)\}, \varphi|_{\{(p_i, p_j)\}}, 1] \in C_0^{\text{dR}}(\Sigma_1^a, \Sigma_1^0) \quad (i \neq j), \\ x_{i,i}^1 &:= [\{p_i\} \times K_i, \varphi|_{\{p_i\} \times K_i}, 1] \in C_{d-1}^{\text{dR}}(\Sigma_1^a, \Sigma_1^0), \\ x_{i,j}^1 &:= [K_{i,j}, \varphi|_{K_{i,j}}, 1] \in C_{d-1}^{\text{dR}}(\Sigma_1^a, \Sigma_1^0) \quad (i \neq j), \\ \bar{x}_{i,j}^1 &:= [K_i \times \{p_j\}, \varphi|_{K_i \times \{p_j\}}, 1] \in C_{d-1}^{\text{dR}}(\Sigma_1^a, \Sigma_1^0) \quad (i \neq j), \\ x_{i,j}^2 &:= [K_i \times K_j, \varphi|_{K_i \times K_j}, 1] \in C_{2d-2}^{\text{dR}}(\Sigma_1^a, \Sigma_1^0). \end{aligned}$$

Here, $a > \sqrt{|z_2^*|^2 + 4}$. If we define $\mathfrak{l}: \mathcal{X}' \rightarrow \mathbb{R}_{>0}$ by

$$\begin{aligned} \mathfrak{l}(x_{i,j}^0) &= \mathfrak{l}(x_{i,j}^1) = |z_2^*| \text{ and } \mathfrak{l}(\bar{x}_{i,j}^1) = \mathfrak{l}(x_{i,j}^2) = \sqrt{|z_2^*|^2 + 4} \text{ for } i \neq j, \\ \mathfrak{l}(x_{i,i}^1) &= \mathfrak{l}(x_{i,i}^2) = 2, \end{aligned}$$

then, each $x \in \mathcal{X}'$ satisfies $x \in C_*^{\text{dR}}(\Sigma_1^{l(x)+\varepsilon}, \Sigma_1^0)$ for any $\varepsilon > 0$. Furthermore, a basis of $H_*^{\text{dR}}(\Sigma_m^a, \Sigma_m^0)$ for $a \in \mathbb{R}_{>0}$ and $m \in \mathbb{Z}_{\geq 1}$ is given by the set homology classes

$$\{[x_1 \star \cdots \star x_m] \mid x_1, \dots, x_m \in \mathcal{X}', \mathfrak{l}(x_1) + \cdots + \mathfrak{l}(x_m) < a\}.$$

The reason of simplicity in this case is the following: For any $x \in \mathcal{X}'$ and (ε, δ) which is standard with respect to h , if $\varepsilon > 0$ is sufficiently small, then the equation

$$f_{1,\delta}(x) = 0 \in C_*^{\text{dR}}(\Sigma_2^{l(x)+2\varepsilon}, \Sigma_2^0)$$

holds since the path $\varphi(p, p')$ satisfies the condition (iii) of Lemma 3.12 for any $(p, p') \in (K_0 \cup K_2)^2$ (c.f. Lemma 5.3).

Step 2. There exists a bijection $\mathcal{C}' \rightarrow \mathcal{X}'$ which maps $c_{i,j}^k$ to $x_{i,j}^k$ and $\bar{x}_{i,j}^1$ to $\bar{c}_{i,j}^1$ for $k \in \{0, 1, 2\}$ and $i, j \in \{0, 2\}$. Composing $\mathfrak{l}: \mathcal{X}' \rightarrow \mathbb{R}_{>0}$ with this bijection, a function $\mathfrak{l}: \mathcal{C}' \rightarrow \mathbb{R}_{>0}$ is defined. Similar to $\mathcal{A}_*^{\text{Hopf}}$, $\mathcal{A}_*^{\text{unlink}}$ is filtered by subcomplexes $(\mathcal{A}_*^{<a}, \partial)$ for all $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K_0 \cup K_2)$ which is defined by using $\mathfrak{l}: \mathcal{C}' \rightarrow \mathbb{R}_{>0}$.

Now $\Phi_\varepsilon^{<a}: \mathcal{A}_*^{<a} \rightarrow C_*^{<a}(\varepsilon)$ is defined so that $c \in \mathcal{C}'$ is mapped to $x \in \mathcal{X}'$ which corresponds to c via the above bijection, and extend it naturally via the product map on $\mathcal{A}_*^{\text{unlink}}$ and the \star -operation. It is clear in this case that $\Phi_\varepsilon^{<a}$ is a chain map from $(\mathcal{A}_*^{<a}, \partial = 0)$ to $(C_*^{<a}(\varepsilon), D_\delta)$. The fact that this map is a quasi-isomorphism is proved by a similar argument as Proposition 5.7 about spectral sequences.

Step 3. We check that the family of maps $(\Phi_\varepsilon^{<a})_*$ induces an isomorphism on the limit of $\varepsilon \rightarrow 0$ and $a \rightarrow \infty$. This finishes the proof of Theorem 5.9.

5.3 A corollary and its potential application

The next result is a corollary from the above computations

Corollary 5.10. *As graded \mathbb{R} -algebras,*

$$H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_1) \not\cong H_*^{\text{string}}(\mathbb{R}^{2d-1}, K_0 \cup K_2).$$

Proof. From Theorem 5.1 and 5.9, it suffices to show that $H_*(\mathcal{A}_*^{\text{Hopf}}, \partial + F)$ is not isomorphic to $\mathcal{A}_*^{\text{unlink}}$ as a graded \mathbb{R} -algebra. Let us rewrite $(c_{0,1}^0, c_{1,0}^0, e_{0,0}^1, e_{1,1}^1)$ by (a_0, a_1, b_0, b_1) and $(c_{0,2}^0, c_{2,0}^0)$ by (a'_0, a'_1) . In addition, we define $C_0 := b_0 + a_0 a_1$ and $C_1 := b_1 + (-1)^d a_1 a_0$.

If $d = 2$, $H_0(\mathcal{A}_*^{\text{Hopf}}, \partial + F)$ is the (a priori) non-commutative \mathbb{R} -algebra generated by $\{a_0, a_1, b_0, b_1\}$ modulo the ideal generated by $\{b_0, b_1, C_0, C_1\}$. This is isomorphic to the commutative algebra $\mathbb{R}[a_0, a_1]/(a_0 a_1)$. On the other hand, $\mathcal{A}_0^{\text{unlink}}$ is the non-commutative algebra freely generated by $\{a'_0, a'_1\}$. Therefore, $H_0(\mathcal{A}_*^{\text{Hopf}}, \partial + F) \not\cong \mathcal{A}_0^{\text{unlink}}$ as \mathbb{R} -algebras.

If $d \geq 3$, the lower degree parts are isomorphic as vector spaces. Indeed, for $p \leq 2d - 5$,

$$H_p(\mathcal{A}_*^{\text{Hopf}}, \partial + F) \cong \mathcal{A}_p^{\text{unlink}} \cong \begin{cases} \mathbb{R} & \text{if } p = 0, \\ \mathbb{R}a_0 \oplus \mathbb{R}a_1 & \text{if } p = d - 2, \\ 0 & \text{else .} \end{cases}$$

However, $H_{2d-4}(\mathcal{A}_*^{\text{Hopf}}, \partial + F)$ is the \mathbb{R} -vector space spanned by $\{a_i a_j \mid i, j \in \{0, 1\}\} \cup \{b_0, b_1\}$ modulo the subspace generated by $\{b_0, b_1, C_0, C_1\}$, so its dimension is equal to 2. On the other hand, $\mathcal{A}_{2d-4}^{\text{unlink}}$ is the \mathbb{R} -vector space spanned by $\{a'_i a'_j \mid i, j \in \{0, 1\}\}$, so its dimension is equal to 4. Therefore, $H_{2d-4}(\mathcal{A}_*^{\text{Hopf}}, \partial + F) \not\cong \mathcal{A}_{2d-4}^{\text{unlink}}$ as \mathbb{R} -vector spaces. \square

Let us see a potential application of this result. First we determine spin structures on unit conormal bundles

Proposition 5.11. *Let Q be an n -dimensional Riemannian manifold with a fixed spin structure. Then, for every submanifold K in Q , we can assign a spin structure on its unit conormal bundle Λ_K so that if K is isotopic to K' as a submanifold in Q , then Λ_K is isotopic to $\Lambda_{K'}$ as a Legendrian submanifold with a spin structure.*

Proof. Let us identify T^*Q with TQ via Riemannian metric. We also identify Q with the zero-section of TQ . Let L_K be the conormal bundle of K . Note that the tangent space of TQ at $(q, 0) \in Q$ is equal to $T_qQ \oplus T_qQ$, where the first component is the tangent space of the base space Q , and the second component is the tangent space of the fiber T_qQ . For every $q \in K$, $T_{(q,0)}(L_K) = T_qK \oplus (T_qK)^\perp$. Thus the vector bundle $T(L_K)|_K$ has a spin structure induced by $TQ|_K$. Since K is a deformation retract of L_K , this spin structure is extended to $T(L_K)$. By using a diffeomorphism $\mathbb{R}_{>0} \times \Lambda_K \rightarrow L_K \setminus K: (r, (q, p)) \mapsto (q, r \cdot p)$, we can determine a spin structure on $T(\Lambda_K)$ so that the spin structure on $\underline{\mathbb{R}} \oplus T\Lambda_K$, induced by the inclusion map $\text{Spin}(2n-1) \rightarrow \text{Spin}(2n)$, is equal to the spin structure on $T(L_K)|_{\Lambda_K} \cong \underline{\mathbb{R}} \oplus T\Lambda_K$. This spin structure on Λ_K for every submanifold K clearly satisfies the condition of this proposition. \square

Let us consider the unit conormal bundles of $K_0 \cup K_1$ and $K_0 \cup K_2$.

Proposition 5.12. *As a $(2d-2)$ -dimensional submanifold of $UT^*\mathbb{R}^{2d-1}$ with the spin structure determined by Proposition 5.11, $\Lambda_{K_0 \cup K_1}$ is isotopic to $\Lambda_{K_0 \cup K_2}$.*

Proof. For $s \in [0, 1]$, we define $K_1^s := \{q + (0, 2s, 0) \in \mathbb{R}^{2d-1} \mid q \in K_1\}$. We also choose a C^∞ function $[0, 1] \rightarrow [0, \pi]: s \mapsto \theta_s$ so that $\theta_0 = \theta_1 = 0$ and $\theta_{1/2} = \pi/2$, and define $R_s \in \text{SO}(2d-1)$ for $s \in [0, 1]$ by $R_s(v_0, v_1, v_2) := ((\cos \theta_s)v_0 - (\sin \theta_s)v_2, v_1, (\sin \theta_s)v_0 + (\cos \theta_s)v_2)$ for every $(v_0, v_1, v_2) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1}$. We then define an isotopy $(\Lambda_s)_{s \in [0,1]}$ from $\Lambda_{K_1^0} = \Lambda_{K_1}$ to $\Lambda_{K_1^1}$ by

$$\Lambda_s := \{(q, p) \in UT^*\mathbb{R}^{2d-1} \mid q \in K_s, p \circ R_s|_{T_qK_s^1} = 0\}.$$

Λ_s intersects Λ_{K_0} if and only if $s = 1/2$, and $\Lambda_{1/2} \cap \Lambda_{K_0} = \Lambda_{K_0} \cap UT_{p_0}^*\mathbb{R}^{2d-1}$, where $p_0 = (0, 1, 0) \in \mathbb{R}^{2d-1}$. We can slightly perturb $(\Lambda_s)_{s \in [0,1]}$ around $s = 1/2$ to an isotopy $(\Lambda'_s)_{s \in [0,1]}$ so that Λ'_s does not intersect Λ_{K_0} for every $s \in [0, 1]$. This isotopy is homotopic to an isotopy $(\Lambda_{K_1^s})_{s \in [0,1]}$, which preserves the spin structure of Proposition 5.11. In addition, $K_0 \cup K_1^1$ is isotopic to $K_0 \cup K_2$ in \mathbb{R}^{2d-1} . Therefore, as a C^∞ submanifold with a spin structure, $\Lambda_{K_0} \cup \Lambda_{K_1}$ is isotopic to $\Lambda_{K_0} \cup \Lambda_{K_2}$. \square

If Conjecture 1.4, which will be discussed in Section 7 in more detail, is true, then Corollary 5.10 can be applied to show that the unit conormal bundle $\Lambda_{K_0 \cup K_1}$ is not isotopic to $\Lambda_{K_0 \cup K_2}$ as a Legendrian submanifold with a spin structure in $UT^*\mathbb{R}^{2d-1}$, though they are isotopic as C^∞ submanifolds with spin structures by the above proposition.

6 Cord algebra and $H_0^{\text{string}}(Q, K)$

Throughout this section, we consider the case where the codimension of K is 2 (i.e. $d = 2$) and the normal bundle $(TK)^\perp$ is trivial. The purpose is to show that $H_0^{\text{string}}(Q, K)$ is isomorphic to an isotopy invariant of K , called *cord algebra*.

6.1 Cord algebra and string homology

In this section, we refer to [9, 27] and give a definition of cord algebra and string homology. Note that, in this paper, their coefficients are reduced from original $\mathbb{Z}[\pi_1(\partial N_{\varepsilon_0})]$ to \mathbb{R} .

We fix a frame of $(TK)^\perp$ to give an isomorphism $\mathbb{R}^2 \times K \cong (TK)^\perp$ of vector bundles over K which preserves their fiber metrics and orientations. Combining with the map (3), we obtain a diffeomorphism

$$h: \mathcal{O}_{\varepsilon_0} \times K \rightarrow N_{\varepsilon_0},$$

which preserves orientations. Here, $\mathcal{O}_\varepsilon = \{w \in \mathbb{R}^2 \mid |w| < \varepsilon/2\}$ for every $\varepsilon \leq \varepsilon_0$ as defined in Subsection 4.4.3.

First, we define an \mathbb{R} -algebra $\text{Cord}(Q, K; \mathbb{R})$. Its relation to the cord algebra defined in [9, 27] is discussed later in Remark 6.2. Let us prepare several notations. We fix $w_0 \in \mathcal{O}_{\varepsilon_0} \setminus \{0\}$ and define a submanifold disjoint from K

$$K' := \{h(w_0, x) \mid x \in K\} \subset N_{\varepsilon_0}.$$

For every $x \in K$, we define $c_x: [0, 1] \rightarrow Q \setminus K$ to be the constant path at $h(w_0, x) \in K'$. We also define $m_x: [0, 1] \rightarrow Q \setminus K$ to be a loop in a punctured disk $h((\mathcal{O}_{\varepsilon_0} \setminus \{0\}) \times \{x\}) \subset N_{\varepsilon_0} \setminus K$ based at $h(w_0, x) \in K'$, whose winding number around $h(0, x)$ is equal to 1. In addition, let $\pi_1(Q \setminus K, K')$ be the set of homotopy classes of continuous paths $\gamma: [0, 1] \rightarrow Q \setminus K$ such that $\gamma(\{0, 1\}) \subset K'$.

Definition 6.1. Let \mathcal{A} be the unital non-commutative \mathbb{R} -algebra freely generated by the set $\pi_1(Q \setminus K, K')$. We define the two-sided ideal \mathcal{I} generated by the following elements:

$$\begin{cases} [c_x], \\ [\gamma_1 \cdot \gamma_2] - [\gamma_1 \cdot m_x \cdot \gamma_2] - [\gamma_1][\gamma_2], \end{cases}$$

for all $x \in K$ and $\gamma_j: [0, 1] \rightarrow Q \setminus K$ ($j = 1, 2$) such that $\gamma_j(\{0, 1\}) \subset K'$ and $\gamma_1(1) = h(w_0, x) = \gamma_2(0)$. Then, we define an \mathbb{R} -algebra $\text{Cord}(Q, K; \mathbb{R}) := \mathcal{A}/\mathcal{I}$, and call it the *cord algebra* of (Q, K) .

Remark 6.2. When K is 1-dimensional and connected (i.e. K is an oriented knot in a 3-manifold Q), we fix a base point $* \in K'$. A *cord* is a path $\gamma: [0, 1] \rightarrow Q$ such that $\gamma([0, 1]) \cap K = \emptyset$ and $\gamma(0), \gamma(1) \in K' \setminus \{*\}$. The notion of cord algebra (or *cord ring*) of knots was defined in, for instance, [9, 26, 27]. The most refined one is [9, Definition 2.6], which is defined as a non-commutative algebra over \mathbb{Z} generated by the set of homotopy classes of cords and $\{\lambda^\pm, \mu^\pm\}$, modulo the relations about $\{\lambda^\pm, \mu^\pm\}$ and the ‘‘skein relations’’. If we substitute both λ and μ by $1 \in \mathbb{Z}$ and tensor this \mathbb{Z} -algebra with \mathbb{R} , we obtain an \mathbb{R} -algebra isomorphic to $\text{Cord}(Q, K; \mathbb{R})$. (The isomorphism is induced by a natural map from the set of homotopy classes of cords to $\pi_1(Q \setminus K, K')$.)

We should also note that in [27, Definition 2.1], the cord algebra over $\mathbb{Z}[H_1(\partial N_{\varepsilon_0})]$ was defined when K is a connected codimension 2 submanifold of an arbitrary manifold and its normal bundle is oriented. In our setting, we have an isomorphism $(h^{-1})_*: H_1(\partial N_{\varepsilon_0}) \rightarrow H_1(S^1 \times K) \cong H_1(S^1) \oplus H_1(K)$. There exists a ring homomorphism $\varphi: \mathbb{Z}[H_1(\partial N_{\varepsilon_0})] \rightarrow \mathbb{R}$ determined by $\varphi(h_*([S^1])) = -1$ and $\varphi(h_*(c)) = 1$ for every $c \in H_1(K)$. If the base change of the cord algebra of [27, Definition 2.1] is done by φ , we obtain an \mathbb{R} -algebra isomorphic to $\text{Cord}(Q, K; \mathbb{R})$.

Next, we refer to [9, Section 2.1] and define the string homology which is simplified for our purpose. For $m \in \mathbb{Z}_{\geq 1}$, $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $p = 0, 1$, we define an \mathbb{R} -subspace $C_p^{\text{th}}(m, a) \subset C_p^{\text{sing}}(\Sigma_m^a)$ consisting of generic singular p -chains satisfying jet transversality conditions. (Recall that we have fixed a topology of Σ_m^a in Subsection 3.3.4)

In the case of $p = 0$, $C_0^{\text{th}}(m, a)$ is generated by $(\gamma_k: [0, T_k] \rightarrow Q)_{k=1, \dots, m} \in \Sigma_m^a$ satisfying the following conditions:

- (0a) $(\gamma_k)'(0), (\gamma_k)'(T_k) \notin TK$ for every $k \in \{1, \dots, m\}$.
- (0b) $\gamma_k(t) \notin K$ for every $k \in \{1, \dots, m\}$ and $t \neq 0, T_k$.

In the case of $p = 1$, $C_1^\natural(m, a)$ is generated by 1-parameter families of paths

$$[0, 1] \rightarrow \Sigma_m^a : u \mapsto (\gamma_k^u : [0, T_k^u] \rightarrow Q)_{k=1, \dots, m}$$

such that $[0, 1] \rightarrow \mathbb{R}_{>0} : u \mapsto T_k^u$ is a C^∞ function and

$$\Gamma_k : \{(u, t) \mid 0 \leq u \leq 1, 0 \leq t \leq T_k^u\} \rightarrow Q$$

is a C^∞ map for every $k \in \{1, \dots, m\}$, and satisfies the following conditions:

(1a) $(\gamma_k^0)_{k=1, \dots, m}$ and $(\gamma_k^1)_{k=1, \dots, m}$ satisfy (0a), (0b).

(1b) $(\gamma_k^u)'(0), (\gamma_k^u)'(T_k^u) \notin TK$ for every $u \in [0, 1]$ and $\Gamma_k^{\text{int}} := \Gamma_k|_{\{(u, t) \mid t \neq 0, T_k^u\}}$ is transverse to K for every $k \in \{1, \dots, m\}$.

(1c) If $(u_*, t_*), (u'_*, t'_*) \in \prod_{k=1}^m (\Gamma_k^{\text{int}})^{-1}(K)$ are distinct points, then $u_* \neq u'_*$ holds.

Note that the condition (1b) implies that $(\Gamma_k^{\text{int}})^{-1}(K)$ is a finite set. In addition, we define $C_p^\natural(0, a) := C_p^{\text{sing}}(\Sigma_0^a)$ for $a \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $p = 0, 1$.

By (1a), the boundary operator of singular chain complex $\partial^{\text{sing}} : C_1^{\text{sing}}(\Sigma_m^a) \rightarrow C_0^{\text{sing}}(\Sigma_m^a)$ is restricted to the map

$$\partial^{\text{sing}} : C_1^\natural(m, a) \rightarrow C_0^\natural(m, a) : (\gamma_k^u)_{k=1, \dots, m}^{u \in [0, 1]} \mapsto (\gamma_k^1)_{k=1, \dots, m} - (\gamma_k^0)_{k=1, \dots, m}.$$

We also define a linear map $f_k^\natural : C_1^\natural(m, a) \rightarrow C_0^\natural(m+1, a)$ for $m \in \mathbb{Z}_{\geq 1}$ so that for $x = (\gamma_k^u : [0, T_k^u] \rightarrow Q)_{k=1, \dots, m}^{u \in [0, 1]} \in C_1^\natural(m, a)$,

$$f_k^\natural(x) := \sum_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} \text{sign}(u_*, t_*) \cdot (\gamma_1^{u_*}, \dots, \gamma_{k-1}^{u_*}, \widehat{\gamma_k^{u_*}}^1, \widehat{\gamma_k^{u_*}}^2, \gamma_{k+1}^{u_*}, \dots, \gamma_m^{u_*}).$$

Here

$$\widehat{\gamma_k^{u_*}}^1 := \gamma_k^{u_*}|_{[0, t_*]} : [0, t_*] \rightarrow Q, \quad \widehat{\gamma_k^{u_*}}^2 := \gamma_k^{u_*}|_{[t_*, T_k^{u_*}]}(\cdot - t_*): [0, T_k^{u_*} - t_*] \rightarrow Q,$$

and $\text{sign}(u_*, t_*) \in \{\pm 1\}$ is the orientation sign of an open embedding into $\mathcal{O}_{\varepsilon_0} \subset \mathbb{R}^2$

$$\Gamma_k^{\text{fiber}} := \text{pr}_{\mathbb{R}^2} \circ h^{-1} \circ \Gamma_k$$

defined on a small neighborhood of $(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K) \subset (0, 1) \times \mathbb{R}_{>0}$. For convenience, let us define for $p \notin \{0, 1\}$, $C_p^\natural(m, a) := 0$, $\partial^{\text{sing}} := 0 : C_p^\natural(m, a) \rightarrow C_{p-1}^\natural(m, a)$ and $f_k^\natural := 0 : C_p^\natural(m, a) \rightarrow C_{p-1}^\natural(m+1, a)$.

For $a \in \mathbb{R}_{>0} \cup \{\infty\}$ and $m \in \mathbb{Z}_{\geq 0}$, we define a quotient vector space

$$C_*^{\natural, <a}(m) := C_*^\natural(m, a) / C_*^\natural(m, 0).$$

Then ∂^{sing} and f_k^\natural induce linear maps

$$\partial^{\text{sing}} : C_*^{\natural, <a}(m) \rightarrow C_{* - 1}^{\natural, <a}(m), \quad f_k^\natural : C_*^{\natural, <a}(m) \rightarrow C_{* - 1}^{\natural, <a}(m+1).$$

Now we define a graded \mathbb{R} -vector space

$$C_*^{\natural, <a} := \bigoplus_{m=0}^{\infty} C_*^{\natural, <a}(m).$$

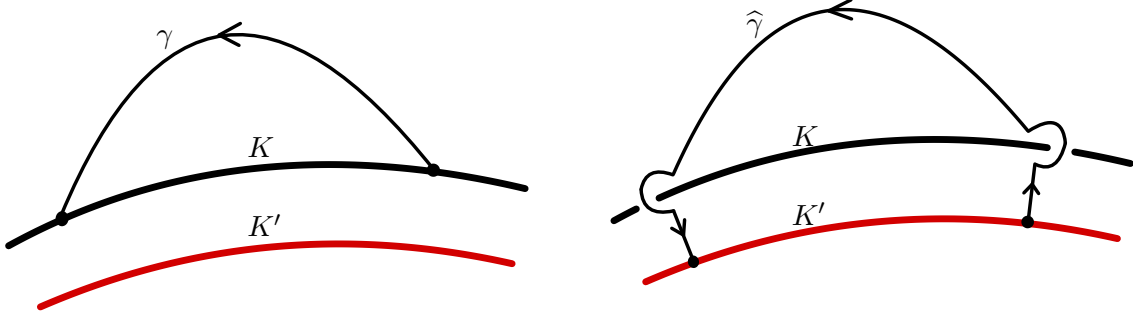


Figure. 5: The picture of $\hat{\gamma}$ defined for $(\gamma)_{k=1} \in \Sigma_1^\infty$.

For each $m \in \mathbb{Z}_{\geq 0}$, $C_*^{\mathfrak{h}, < a}(m)$ is considered to be its linear subspace in a natural way. Then, we define a degree (-1) map $D^{\mathfrak{h}}: C_*^{\mathfrak{h}, < a} \rightarrow C_{* - 1}^{\mathfrak{h}, < a}$ by

$$D^{\mathfrak{h}}(x) := \partial^{\text{sing}} x + \sum_{k=1}^m f_k^{\mathfrak{h}}(x). \quad (41)$$

for $x \in C_*^{\mathfrak{h}, < a}(m)$. For $x \in C_*^{\mathfrak{h}, < a}(0)$, the RHS is just equal to $\partial^{\text{sing}} x$. Then we obtain a chain complex $(C_*^{\mathfrak{h}, < a}, D^{\mathfrak{h}})$. Let $H_*^{\mathfrak{h}, < a}$ denote its homology group. In addition, for $a, b \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ with $a \leq b$, we define a linear map $J^{a, b}: H_*^{\mathfrak{h}, < a} \rightarrow H_*^{\mathfrak{h}, < b}$ induced by the inclusion maps $C_*^{\mathfrak{h}}(m, a) \rightarrow C_*^{\mathfrak{h}}(m, b)$ for all $m \in \mathbb{Z}_{\geq 0}$.

In this section, we call $H_*^{\mathfrak{h}, < \infty}$ the *string homology* of (Q, K) . Note that the direct limit $\varinjlim_{a \rightarrow \infty} H_*^{\mathfrak{h}, < a}$ defined by $\{J^{a, b}\}_{a \leq b}$ is isomorphic to $H_0^{\mathfrak{h}, < \infty}$. Furthermore, $H_0^{\mathfrak{h}, < \infty}$ has a associative product structure induced by $\Pi: \Sigma_m^\infty \times \Sigma_{m'}^\infty \rightarrow \Sigma_{m+m'}^\infty$. Thus $H_0^{\mathfrak{h}, < \infty}$ is a unital associative \mathbb{R} -algebra, whose unit comes from $* \in C_0^{\mathfrak{h}, < \infty}(0)$.

The next proposition is essentially proved in [9, Proposition 2.9].

Proposition 6.3. *As an \mathbb{R} -algebra, $\text{Cord}(Q, K; \mathbb{R})$ is isomorphic to $H_0^{\mathfrak{h}, < \infty}$.*

Proof. For every homotopy class $z \in \pi_1(Q \setminus K, K')$, we choose a C^∞ path γ which represents z . We then define a path $\bar{\gamma}$ as follows: For $x_0, x_1 \in K$ such that $\gamma(i) = h(w_0, x_i)$ for $i \in \{0, 1\}$,

$$\bar{\gamma}: [0, 3] \rightarrow Q: t \mapsto \begin{cases} h(t \cdot w_0, x_0) & \text{if } 0 \leq t \leq 1, \\ \gamma(t - 1) & \text{if } 1 \leq t \leq 2, \\ h((3 - t) \cdot w_0, x_1) & \text{if } 2 \leq t \leq 3. \end{cases}$$

We modify $\bar{\gamma}$ to $\tilde{\gamma}$ by a reparameterization so that $(\tilde{\gamma})_{k=1} \in \Sigma_1^\infty$ and satisfies (0a) and (0b). Then a homomorphism of unital \mathbb{R} -algebras

$$F: \mathcal{A} \rightarrow H_0^{\mathfrak{h}, < \infty}$$

is defined so that $z = [\gamma]$ is mapped to $[(\tilde{\gamma})_{k=1}]$. From the definition of $D^{\mathfrak{h}}$, it can be checked that F is a well-defined surjection and maps the ideal \mathcal{I} into 0. Therefore, we obtain a surjective homomorphism of unital \mathbb{R} -algebras $\bar{F}: \mathcal{A}/\mathcal{I} \rightarrow H_0^{\mathfrak{h}, < \infty}$.

We prove that \bar{F} is injective by describing its inverse map. For $(\gamma: [0, T] \rightarrow Q)_{k=1} \in \Sigma_1^\infty$ satisfying (0a) and (0b), let us define

$$\tilde{\gamma}: [0, 1] \rightarrow Q: t \mapsto \begin{cases} h((1 - 3t) \cdot w_0, \gamma(0)) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ \gamma(3Tt - T) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ h((3t - 2) \cdot w_0, \gamma(T)) & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

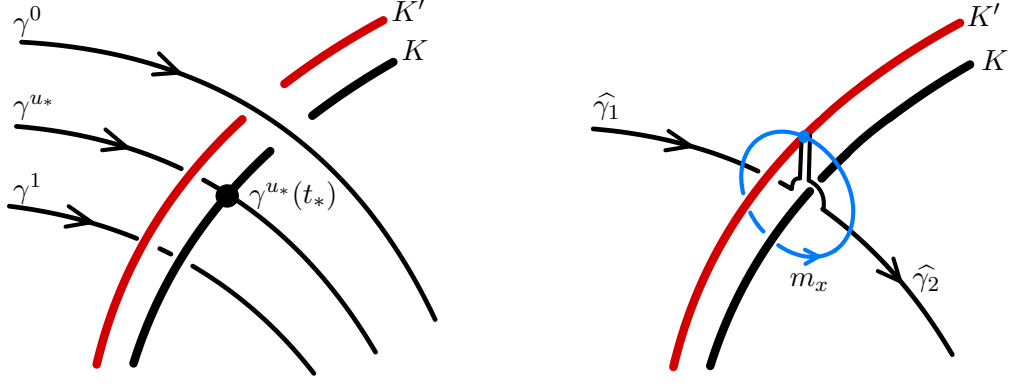


Figure. 6: The LHS describes the 1-chain $(\gamma^u)_{k=1}^{u \in [0,1]}$ such that $\text{sign}(u_*, t_*) = +1$ at $(u_*, t_*) \in (\Gamma_1^{\text{int}})^{-1}(K)$. From the RHS, we can see that $[\widehat{\gamma}^0] = [\widehat{\gamma}_1 \cdot \widehat{\gamma}_2]$ and $[\widehat{\gamma}^1] = [\widehat{\gamma}_1 \cdot m_x \cdot \widehat{\gamma}_2]$ as elements of $\pi_1(Q \setminus K, K')$.

As described in Figure 5, we change $\tilde{\gamma}$ into $\widehat{\gamma}$ by small perturbations inside N_{ε_0} around $t \in \{\frac{1}{3}, \frac{2}{3}\}$ so that $\widehat{\gamma}$ does not intersect K . We then obtain a homotopy class $[\widehat{\gamma}] \in \pi_1(Q \setminus K, K')$. Note that for any $\widehat{\gamma}^j$ ($j = 1, 2$) from two choices of perturbations, there exist $l_0, l_1 \in \{0, +1, -1\}$ such that

$$[\widehat{\gamma}^2] = [(m_{\gamma(0)})^{l_0} \cdot \widehat{\gamma}^1 \cdot (m_{\gamma(T)})^{l_1}] \in \pi_1(Q \setminus K, K').$$

Here, $(m_x)^1 := m_x$, $(m_x)^0 := c_x$, and $(m_x)^{-1}$ denotes the inverse path of m_x . (As a natural extension, $(m_x)^l$ for $l \in \mathbb{Z}$ is defined.) Thus, $[\widehat{\gamma}^1] = [\widehat{\gamma}^2]$ holds as an element of \mathcal{A}/\mathcal{I} . If $\gamma \in \Sigma_1^0$ (i.e. $\text{length} \gamma < \varepsilon_0$), $[\widehat{\gamma}] = [(m_x)^l] \in \pi_1(Q \setminus K, K')$ for some $x \in K$ and $l \in \mathbb{Z}$. In such case, $[\widehat{\gamma}] = 0$ holds as an element of \mathcal{A}/\mathcal{I} . Therefore, we have a well-defined linear map

$$G: C_0^{\natural, < \infty} \rightarrow \mathcal{A}/\mathcal{I}: \begin{cases} C^{\natural, < \infty}(m) \ni (\gamma_k)_{k=1, \dots, m} & \mapsto [\widehat{\gamma}_1] \cdots [\widehat{\gamma}_m] \quad (m \geq 1), \\ C^{\natural, < \infty}(0) \ni 1 & \mapsto \text{the unit.} \end{cases}$$

From the transversality condition (1b) together with (1c), it follows that $\text{Im } D^{\natural}$ is mapped into 0. Indeed, in a simple case, for $(\gamma^u)_{k=1}^{u \in [0,1]} \in C_1^{\natural}(1, \infty)$ such that $(\Gamma_1^{\text{int}})^{-1}(K) = \{(u_*, t_*)\}$, we can see that

$$G(D^{\natural}(x)) = \begin{cases} [\widehat{\gamma}_1 \cdot m_x \cdot \widehat{\gamma}_2] - [\widehat{\gamma}_1 \cdot \widehat{\gamma}_2] + \text{sign}(u_*, t_*)[\widehat{\gamma}_1][\widehat{\gamma}_2] & \text{if } \text{sign}(u_*, t_*) = +1, \\ [\widehat{\gamma}_1 \cdot \widehat{\gamma}_2] - [\widehat{\gamma}_1 \cdot m_x \cdot \widehat{\gamma}_2] + \text{sign}(u_*, t_*)[\widehat{\gamma}_1][\widehat{\gamma}_2] & \text{if } \text{sign}(u_*, t_*) = -1, \end{cases}$$

for $x := \gamma^{u_*}(t_*)$, $\gamma_1 := \gamma^{u_*}|_{[0, t_*]}$ and $\gamma_2 := \gamma^{u_*}|_{[t_*, T^{u_*}]}(\cdot - t_*)$. Figure 6 describes the case where $\text{sign}(u_*, t_*) = +1$. Thus $G(D^{\natural}((\gamma^u)_{k=1}^{u \in [0,1]})) = 0 \in \mathcal{A}/\mathcal{I}$. The condition (1c) implies that the general case can be reduced to this simple case. Therefore, we obtain a well-defined linear map $\bar{G}: H_0^{\natural, < \infty} \rightarrow \mathcal{A}/\mathcal{I}$. Finally, for any $[\gamma] \in \pi_1(Q \setminus K, K')$ such that $\gamma(i) = h(w_0, x_i)$ for $i \in \{0, 1\}$, there exist $l_0, l_1 \in \{0, +1, -1\}$ such that

$$\bar{G} \circ \bar{F}([\gamma]) = [(m_{x_0})^{l_0} \cdot \gamma \cdot (m_{x_1})^{l_1}],$$

which is equal to $[\gamma]$ in \mathcal{A}/\mathcal{I} . This implies that \bar{F} is injective. \square

6.2 Connecting string homology and $H_*^{\text{string}}(Q, K)$

The purpose of this section is to construct a linear map from $H_*^{\natural, < a}$ to $H_*^{< a}(Q, K)$ for all $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$. On the limit of $a \rightarrow \infty$, an \mathbb{R} -algebra homomorphism from $H_*^{\natural, < \infty}$ to $H_*^{\text{string}}(Q, K)$ is defined. Before constructing this map, we will prepare two things.

6.2.1 Preliminaries

First, we define a map Ψ which associates de Rham chains with singular chains. Let $\kappa, \chi: \mathbb{R} \rightarrow [0, 1]$ be C^∞ functions such that:

- $\kappa(u) = 0$ if $u \leq 0$ and $\kappa(u) = 1$ if $u \geq 1$. In addition, $\kappa'(u) > 0$ if $0 < u < 1$.
- $\chi: \mathbb{R} \rightarrow [0, 1]$ has a compact support and $\chi(s) = 1$ for every $s \in [0, 1]$.

For $p \in \mathbb{Z}$, a linear map

$$\Psi: C_p^{\text{fl}}(m, a) \rightarrow C_p^{\text{dR}}(\Sigma_m^a)$$

is defined by

$$\begin{cases} \Psi((\gamma_k)_{k=1, \dots, m}) := [\{0\}, c_0, 1] & \text{if } p = 0, \\ \Psi((\gamma_k^u)_{k=1, \dots, m}^{u \in [0, 1]}) := [\mathbb{R}, c_1, \chi] & \text{if } p = 1, \\ \Psi = 0 & \text{else,} \end{cases}$$

where c_0 is the constant map to $(\gamma_k)_{k=1, \dots, m}$ and $c_1: \mathbb{R} \rightarrow \Sigma_m^a: u \mapsto (\gamma_k^{\kappa(u)})_{k=1, \dots, m}$. These maps commute with boundary operators, namely,

$$\partial \circ \Psi = \Psi \circ \partial^{\text{sing}}: C_p^{\text{fl}}(m, a) \rightarrow C_{p-1}^{\text{dR}}(\Sigma_m^a).$$

Next, we define a filtration $\{C_p^{\text{fl}}(m, a, \varepsilon)\}_{\varepsilon > 0}$ of $C_p^{\text{fl}}(m, a)$. In the cases of $p = 0$, $C_0^{\text{fl}}(m, a, \varepsilon)$ is an \mathbb{R} -subspace generated by $(\gamma_k: [0, T_k] \rightarrow Q)_{k=1, \dots, m} \in \Sigma_m^a$ satisfying (0a), (0b) and the following condition:

- (0c) There exists $\tau_0 \in (0, \varepsilon_0/(5C_0)]$ such that $\gamma_k([\tau_0, T_k - \tau_0]) \cap N_\varepsilon = \emptyset$ for every $k \in \{1, \dots, m\}$.

In the case of $p = 1$, $C_1^{\text{fl}}(m, a, \varepsilon)$ is an \mathbb{R} -subspace generated by $(\gamma_k^u: [0, T_k^u] \rightarrow Q)_{k=1, \dots, m}^{u \in [0, 1]}$ satisfying (1a), (1b), (1c) and the following conditions:

- (1d) $(\gamma_k^0)_{k=1, \dots, m}$ and $(\gamma_k^1)_{k=1, \dots, m}$ satisfy (0c).

- (1e) There exist $\tau_0 \in (0, \varepsilon_0/(5C_0)]$ and an open neighborhood $U_{(u_*, t_*)}$ for each $(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)$ ($k = 1, \dots, m$) such that

$$(u_*, t_*) \in U_{(u_*, t_*)} \subset \{(u, t) \mid 0 < u < 1, \tau_0 < t < T_k^u - \tau_0\},$$

and the following hold:

- (1e-1) $U_{(u_*, t_*)} \subset \{(u, t) \mid |t - t_*| < \tau_0\}$.

- (1e-2) For any two distinct points $(u_*, t_*), (u'_*, t'_*) \in \coprod_{k=1}^m (\Gamma_k^{\text{int}})^{-1}(K)$, the projections of $U_{(u_*, t_*)}, U_{(u'_*, t'_*)} \subset [0, 1] \times \mathbb{R}_{>0}$ to $[0, 1]$

$$\text{pr}_{[0, 1]}(U_{(u_*, t_*)}), \text{pr}_{[0, 1]}(U_{(u'_*, t'_*)}) \subset [0, 1]$$

are disjoint.

- (1e-3) For every $k \in \{1, \dots, m\}$,

$$\left(\Gamma_k|_{\{(u, t) \mid \tau_0 \leq t \leq T_k^u - \tau_0\}} \right)^{-1}(N_\varepsilon) = \bigcup_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} U_{(u_*, t_*)}.$$

Moreover, for each $(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)$,

$$\Gamma_k^{\text{fiber}}: U_{(u_*, t_*)} \rightarrow \mathcal{O}_\varepsilon$$

is a diffeomorphism. Here, recall that $\Gamma_k^{\text{fiber}} = \text{pr}_{\mathbb{R}^2} \circ h^{-1} \circ \Gamma_k$.

In addition, we define $C_p^{\uparrow\uparrow}(m, a, \varepsilon) := C_p^{\uparrow\uparrow}(m, a)$ if $p \notin \{0, 1\}$ or $m = 0$.

Roughly speaking, (0c) means that $\gamma_k(t)$ is far from K by a distance at least $\varepsilon/2$, except when t is close to $\{0, T_k\}$. (1e) means that $\gamma_k^u(t)$ is far from K by a distance at least $\varepsilon/2$, except when t is close to $\{0, T_k^u\}$ or when (u, t) is close to some point in $(\Gamma_k^{\text{int}})^{-1}(K)$. Note that when $0 < \varepsilon' \leq \varepsilon$, $C_*^{\uparrow\uparrow}(m, a, \varepsilon) \subset C_*^{\uparrow\uparrow}(m, a, \varepsilon')$ holds, and

$$\bigcup_{\varepsilon > 0} C_*^{\uparrow\uparrow}(m, a, \varepsilon) = C_*^{\uparrow\uparrow}(m, a). \quad (42)$$

Moreover, ∂^{sing} and $f_k^{\uparrow\uparrow}$ ($k = 1, \dots, m$) are restricted to linear maps

$$\partial^{\text{sing}}: C_*^{\uparrow\uparrow}(m, a, \varepsilon) \rightarrow C_{* - 1}^{\uparrow\uparrow}(m, a, \varepsilon), \quad f_k^{\uparrow\uparrow}: C_*^{\uparrow\uparrow}(m, a, \varepsilon) \rightarrow C_{* - 1}^{\uparrow\uparrow}(m + 1, a, \varepsilon).$$

For every $\varepsilon > 0$, we define $C_*^{\uparrow\uparrow, < a}(m, \varepsilon) := C_*^{\uparrow\uparrow}(m, a, \varepsilon) / C_*^{\uparrow\uparrow}(m, 0, \varepsilon)$ and

$$C_*^{\uparrow\uparrow, < a}(\varepsilon) := \bigoplus_{m=0}^{\infty} C_*^{\uparrow\uparrow, < a}(m, \varepsilon).$$

A linear map $D_\varepsilon^{\uparrow\uparrow}: C_*^{\uparrow\uparrow, < a}(\varepsilon) \rightarrow C_{* - 1}^{\uparrow\uparrow, < a}(\varepsilon)$ is defined by the same form as (41). Then we obtain a chain complex $(C_*^{\uparrow\uparrow, < a}(\varepsilon), D_\varepsilon^{\uparrow\uparrow})$. Let $H_*^{\uparrow\uparrow, < a}(\varepsilon)$ denote its homology. When $0 < \varepsilon' \leq \varepsilon$, the inclusion maps $C_*^{\uparrow\uparrow}(m, a, \varepsilon) \rightarrow C_*^{\uparrow\uparrow}(m, a, \varepsilon')$ for all $m \in \mathbb{Z}_{\geq 0}$ induce a linear map

$$l_{\varepsilon, \varepsilon'}: H_*^{\uparrow\uparrow, < a}(\varepsilon) \rightarrow H_*^{\uparrow\uparrow, < a}(\varepsilon'),$$

and we have a direct system $(\{H_*^{\uparrow\uparrow, < a}(\varepsilon)\}_{\varepsilon > 0}, \{l_{\varepsilon, \varepsilon'}\}_{\varepsilon \geq \varepsilon'})$. From the relation (42), we have

$$\varinjlim_{\varepsilon \rightarrow 0} H_*^{\uparrow\uparrow, < a}(\varepsilon) = H_*^{\uparrow\uparrow, < a}.$$

6.2.2 Construction of chain map

With the above preparations, we consider the maps

$$C_p^{\uparrow\uparrow}(m, a, \varepsilon) \rightarrow C_p^{\text{dR}}(\Sigma_m^{a+m\varepsilon}): x \mapsto \Psi(x) \quad (m \in \mathbb{Z}_{\geq 0})$$

for $\varepsilon \in (0, \varepsilon_0/(5C_0)]$. They induce a linear map from $C_*^{\uparrow\uparrow, < a}(\varepsilon)$ to $C_*^{< a}(\varepsilon)$, but this is not a chain map. In order to fill in the gap, we need to prove the following lemma for $a \in \mathbb{R}_{> 0} \setminus \mathcal{L}(K)$.

Lemma 6.4. *Suppose that $(\varepsilon, \delta) \in \mathcal{T}_a$ is standard with respect to h . Then, for $m \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \dots, m\}$, there exists a linear map*

$$o_{k, (\varepsilon, \delta)}: C_1^{\uparrow\uparrow}(m, a, \varepsilon) \rightarrow C_1^{\text{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon})$$

such that the following hold for any $x \in C_1^{\uparrow\uparrow}(m, a, \varepsilon)$:

$$(i) \quad \partial(o_{k, (\varepsilon, \delta)}(x)) - (f_{k, \delta} \circ \Psi(x) - \Psi \circ f_k^{\uparrow\uparrow}(x)) \in C_0^{\text{dR}}(\Sigma_{m+1}^0).$$

$$(ii) \quad f_{l, \delta}(o_{k, (\varepsilon, \delta)}(x)) \in C_0^{\text{dR}}(\Sigma_{m+2}^0) \text{ for every } l \in \{1, \dots, m+1\}.$$

Proof. It suffices to define $o_{k, (\varepsilon, \delta)}(x)$ for $x = (\gamma_l^u)_{l=1, \dots, m}^{u \in [0, 1]}$ satisfying (1a), ..., (1e). The proof is divided into three steps: We define de Rham chains in the first two steps. In the last step, $o_{k, (\varepsilon, \delta)}(x)$ is defined as the sum of these chains and we check the conditions (i) and (ii).

Step 1. From the definitions of Ψ and $f_k^{\uparrow\uparrow}$,

$$\begin{aligned} & f_{k, \delta} \circ \Psi(x) - \Psi \circ f_k^{\uparrow\uparrow}(x) \\ &= f_{k, \delta}(\Psi(x)) - \sum_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} \text{sign}(u_*, t_*) \cdot [\{0\}, c_0(u_*, t_*), 1], \end{aligned}$$

where $c_0(u_*, t_*)$ is the constant map to $(\gamma_1^{u_*}, \dots, \widehat{\gamma_k^{u_*}}^{-1}, \widehat{\gamma_k^{u_*}}^{-2}, \dots, \gamma_m^{u_*})$. Since (ε, δ) is standard, it has the form (31). Using the notations of (13), we can explicitly write $f_{k,\delta}(\Psi(x)) = [W_k, \Phi_k, \zeta_k]$, where

$$\begin{aligned} W_k &= \{(u, \tau, v) \in \mathbb{R} \times \mathbb{R} \times N_\varepsilon \mid 2\varepsilon < \tau < T_k^{\mu(u)} - 2\varepsilon, \gamma_k^{\mu(u)}(\tau) = \sigma_1^v(0)\}, \\ \Phi_k &: W_k \rightarrow \Sigma_{m+1}^{a+(m+1)\varepsilon} : (u, \tau, v) \mapsto \text{con}_k((\gamma_l^{\mu(u)})_{l=1,\dots,m}, (T_k^{\mu(u)}, \tau), \psi_\varepsilon(v)), \\ \zeta_k &\in \Omega_c^2(W_k) : (\zeta_k)_{(u,\tau,v)} = \rho_\varepsilon(T_k^{\mu(u)}, \tau) \cdot \chi(u) \cdot (h_*(\nu_\varepsilon \times 1))_v. \end{aligned}$$

Recall the condition (1e-3) and note that $\sigma_1^v(0) = v$. We define $\bar{W}_k := W_k \cap \{\tau_0 < \tau < T_k^{\mu(u)} - \tau_0\}$. Then

$$\bar{W}_k \rightarrow \bigcup_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} U_{(u_*, t_*)} : (u, \tau, v) \mapsto (\mu(u), \tau)$$

is an orientation preserving diffeomorphism. Moreover, $\rho_\varepsilon(T_k^{\mu(u)}, \tau) \cdot \chi(u) = 1$ for $(u, \tau, v) \in \bar{W}_k$. On the other hand, it follows from (1e-1) and Lemma 3.12 that $\Phi_k(u, \tau, v) \in \Sigma_{m+1}^0$ for $(u, \tau, v) \in W_k \setminus \bar{W}_k$. Therefore, we have

$$\begin{aligned} & f_{k,\delta}(\Psi(x)) - \sum_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} [U_{(u_*, t_*)}, \Phi'_{(u_*, t_*)}, \zeta'_{(u_*, t_*)}] \\ &= f_{k,\delta}(\Psi(x)) - [\bar{W}_k, \Phi_k|_{\bar{W}_k}, \zeta_k|_{\bar{W}_k}] \in C_0^{\text{dR}}(\Sigma_{m+1}^0), \end{aligned}$$

where

$$\begin{aligned} \Phi'_{(u_*, t_*)} &: U_{(u_*, t_*)} \rightarrow \Sigma_{m+1}^{a+(m+1)\varepsilon} : (u, \tau) \mapsto \text{con}_k((\gamma_l^u)_{l=1,\dots,m}, (T_k^u, \tau), \psi_\varepsilon(\gamma_k^u(\tau))), \\ \zeta'_{(u_*, t_*)} &= (\Gamma_k^{\text{int}}|_{U_{(u_*, t_*)}})^*(h_*(\nu_\varepsilon \times 1)) \in \Omega_c^2(U_{(u_*, t_*)}). \end{aligned}$$

As a result, $f_{k,\delta} \circ \Psi(x) - \Psi \circ f_k^{\text{fl}}(x)$ is equal to the chain

$$\sum_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} \left([U_{(u_*, t_*)}, \Phi'_{(u_*, t_*)}, \zeta'_{(u_*, t_*)}] - \text{sign}(u_*, t_*) \cdot [\{0\}, c_0(u_*, t_*), 1] \right) \quad (43)$$

modulo $C_0^{\text{dR}}(\Sigma_{m+1}^0)$.

For each $(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)$, consider a diffeomorphism

$$\Gamma_k^{\text{fiber}} = \text{pr}_{\mathbb{R}^2} \circ h^{-1} \circ \Gamma_k^{\text{int}}|_{U_{(u_*, t_*)}} : U_{(u_*, t_*)} \rightarrow \mathcal{O}_\varepsilon$$

and a scalar multiplication $m_s : \mathcal{O}_\varepsilon \rightarrow \mathcal{O}_\varepsilon : w \mapsto \kappa(s) \cdot w$ for $s \in \mathbb{R}$. We define a deformation retraction to $\{(u_*, t_*)\}$

$$\mathbb{R} \times U_{(u_*, t_*)} \rightarrow U_{(u_*, t_*)} : (s, (u, \tau)) \mapsto (u_s, \tau_s) := (\Gamma_k^{\text{fiber}})^{-1} \circ m_s \circ (\Gamma_k^{\text{fiber}})(u, \tau).$$

Now we define a map

$$\tilde{\Phi}'_{(u_*, t_*)} : \mathbb{R} \times U_{(u_*, t_*)} \rightarrow \Sigma_{m+1}^{a+(m+1)\varepsilon} : (s, (u, \tau)) \mapsto \Phi'_{(u_*, t_*)}(u_s, \tau_s)$$

and a de Rham chain

$$o_k^1 := \sum_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} [\mathbb{R} \times U_{(u_*, t_*)}, \tilde{\Phi}'_{(u_*, t_*)}, \chi \times \zeta'_{(u_*, t_*)}] \in C_1^{\text{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon}).$$

If $s \geq 1$, $\tilde{\Phi}'_{(u_*, t_*)}(s, (u, \tau)) = \Phi'_{(u_*, t_*)}(u, \tau)$. If $s \leq 0$, $\tilde{\Phi}'_{(u_*, t_*)}(s, (u, \tau))$ is constant to

$$(\gamma_1^{u_*}, \dots, \widehat{\gamma_k^{u_*}}^{-1}, \widehat{\gamma_k^{u_*}}^{-2}, \dots, \gamma_m^{u_*}) =: c'_0(u_*, t_*),$$

which is defined by (12) for $\gamma_k = \gamma_k^{u_*}$, $(T_k, \tau) = (T_k^{u_*}, t_*)$ and $\sigma_i: [0, \varepsilon/2] \rightarrow \{\gamma_k^{u_*}(t_*)\} \subset Q$. Therefore, the boundary chain ∂o_k^1 is equal to

$$\sum_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} \left([U_{(u_*, t_*)}, \Phi'_{(u_*, t_*)}, \zeta'_{(u_*, t_*)}] - \left[\{0\}, c'_0(u_*, t_*), \int_{U_{(u_*, t_*)}} (\Gamma_k)^*(h_*(\nu_\varepsilon \times 1)) \right] \right).$$

Since $\int_{\mathcal{O}_\varepsilon} \nu_\varepsilon = 1$, we can compute that

$$\begin{aligned} \int_{U_{(u_*, t_*)}} (\Gamma_k)^*(h_*(\nu_\varepsilon \times 1)) &= \int_{U_{(u_*, t_*)}} (h^{-1} \circ \Gamma_k)^*(\nu_\varepsilon \times 1) \\ &= \int_{U_{(u_*, t_*)}} (\Gamma_k^{\text{fiber}})^* \nu_\varepsilon \\ &= \text{sign}(u_*, t_*) \in \{\pm 1\}. \end{aligned}$$

Thus, we have

$$\partial(o_k^1) = \sum_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} \left([U_{(u_*, t_*)}, \Phi'_{(u_*, t_*)}, \zeta'_{(u_*, t_*)}] - \text{sign}(u_*, t_*) \cdot [\{0\}, c'_0(u_*, t_*), 1] \right). \quad (44)$$

Step 2. For each $(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)$, $c'_0(u_*, t_*)$ coincides with $c_0(u_*, t_*)$ up to reparameterizations of the k -th and $(k+1)$ -th paths. We define by interpolating parameterizations

$$c_1(u_*, t_*): [0, 1] \rightarrow \Sigma_{m+1}^{a+(m+1)\varepsilon}: s \mapsto (\gamma_1^{u_*}, \dots, \gamma_{k-1}^{u_*}, \widehat{\gamma_k^{u_*}}^1 \circ \mu_s^1, \widehat{\gamma_k^{u_*}}^2 \circ \mu_s^2, \gamma_{k+1}^{u_*}, \dots, \gamma_m^{u_*}),$$

so that $c_1(u_*, t_*)(0) = c_0(u_*, t_*)$ and $c_1(u_*, t_*)(1) = c'_0(u_*, t_*)$. Then, we obtain a chain

$$o_{(u_*, t_*)}^2 := [\mathbb{R}, c_1(u_*, t_*) \circ \kappa, \chi] \in C_1^{\text{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon}),$$

which satisfies $\partial(o_{(u_*, t_*)}^2) = [\{0\}, c'_0(u_*, t_*), 1] - [\{0\}, c_0(u_*, t_*), 1]$.

Step 3. We define a chain

$$o_{k,(\varepsilon,\delta)}(x) := o_k^1 + \sum_{(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)} \text{sign}(u_*, t_*) \cdot o_{(u_*, t_*)}^2.$$

From (44), $\partial(o_{k,(\varepsilon,\delta)}(x))$ is equal to the chain of (43). Therefore, $o_{k,(\varepsilon,\delta)}(x)$ satisfies the condition (i). The condition (ii) can be checked as follows: From the conditions (1e-2) and (1e-4), those paths in $\Phi'_{(u_*, t_*)}(s, (u, \tau))$ and $c_1(u_*, t_*)(s)$ satisfy the condition (iii) of Lemma 3.12. Therefore, $f_{l,\delta}(o_k^1)$ and $f_{l,\delta}(o_{(u_*, t_*)}^2)$ belongs to $C_0^{\text{dR}}(\Sigma_{m+2}^0)$ for $l = 1, \dots, m+1$. This finishes the proof. \square

For $(\varepsilon, \delta) \in \mathcal{T}_a$ which is standard, we define a linear map $\Phi_{(\varepsilon,\delta)}^{<a} : C_*^{\text{fl}, <a}(\varepsilon) \rightarrow C_*^{<a}(\varepsilon)$ so that for $x \in C_p^{\text{fl}}(m, a, \varepsilon)$, the equivalence class $[x] \in C_p^{\text{fl}, <a}(m, \varepsilon)$ is mapped to

$$\Phi_{(\varepsilon,\delta)}^{<a}([x]) = \begin{cases} [\Psi(x)] & \text{if } p = 0, \\ [\Psi(x)] - \sum_{k=1}^m [o_{k,(\varepsilon,\delta)}(x)] & \text{if } p = 1, \\ 0 & \text{else.} \end{cases}$$

The two properties of $o_{k,(\varepsilon,\delta)}$ shows that $\Phi_{(\varepsilon,\delta)}^{<a}$ is a chain map from $(C_*^{\text{fl}, <a}(\varepsilon), D_\varepsilon^{\text{fl}})$ to $(C_*^{<a}(\varepsilon), D_\varepsilon)$. Therefore, we obtain a linear map on homology

$$(\Phi_{(\varepsilon,\delta)}^{<a})_* : H_*^{\text{fl}, <a}(\varepsilon) \rightarrow H_*^{<a}(\varepsilon, \delta).$$

6.2.3 Commutativity with transition maps

We need to check the relation of $\Phi_{(\varepsilon, \delta)}^{<a}$ with $\{k_{(\varepsilon', \delta), (\varepsilon, \delta)}\}_{\varepsilon' \leq \varepsilon}$ and $\{l_{\varepsilon, \varepsilon'}\}_{\varepsilon \geq \varepsilon'}$.

Proposition 6.5. *For $(\varepsilon, \delta), (\varepsilon', \delta) \in \mathcal{T}_a$ ($\varepsilon' \leq \varepsilon$) which are standard with respect to h , the following diagram commutes:*

$$\begin{array}{ccc} H_*^{\natural, <a}(\varepsilon) & \xrightarrow{(\Phi_{(\varepsilon, \delta)}^{<a})^*} & H_*^{<a}(\varepsilon, \delta) \\ l_{\varepsilon, \varepsilon'} \downarrow & & \uparrow k_{(\varepsilon', \delta), (\varepsilon, \delta)} \\ H_*^{\natural, <a}(\varepsilon') & \xrightarrow{(\Phi_{(\varepsilon', \delta)}^{<a})^*} & H_*^{<a}(\varepsilon', \delta). \end{array}$$

To prove this proposition, we return to the definition $k_{(\varepsilon', \delta), (\varepsilon, \delta)} = k_{(\bar{\varepsilon}, \bar{\delta})}$ by $(\bar{\varepsilon}, \bar{\delta}) \in \bar{\mathcal{T}}_a$ satisfying (24) for $(\varepsilon, \delta), (\varepsilon', \delta')$. We require that $(\bar{\varepsilon}, \bar{\delta})$ is standard with respect to h , and thus $\bar{\varepsilon} = \varepsilon$.

We set $\bar{\Psi} := \bar{i} \circ \Psi: C_*^{\natural}(m, a) \rightarrow \bar{C}_*^{\text{dR}}(\Sigma_m^a)$ for all $m \in \mathbb{Z}_{\geq 0}$. Again, the induced map from $C_*^{\natural, <a}(\varepsilon)$ to $\bar{C}_*^{<a}(\varepsilon)$ is not a chain map. To fill in the gap, we need the following lemma.

Lemma 6.6. *For $m \in \mathbb{Z}_{\geq 1}$ and $k \in \{1, \dots, m\}$, suppose that we have taken maps $o_{k, (\varepsilon, \delta)}, o_{k, (\varepsilon', \delta')}$ of Lemma 6.4 for $(\varepsilon, \delta), (\varepsilon', \delta')$. Then, there exists a linear map*

$$\bar{o}_{k, (\varepsilon, \delta)}: C_1^{\natural}(m, a, \varepsilon) \rightarrow \bar{C}_1^{\text{dR}}(\Sigma_{m+1}^{a+(m+1)\varepsilon})$$

such that the following hold for any $x \in C_1^{\natural}(m, a, \varepsilon)$:

- $\partial(\bar{o}_{k, (\varepsilon, \delta)}) - (\bar{f}_{k, \bar{\delta}} \circ \bar{\Psi}(x) - \bar{\Psi} \circ f_k^{\natural}(x)) \in \bar{C}_0^{\text{dR}}(\Sigma_{m+1}^0)$.
- $\bar{f}_{l, \bar{\delta}}(\bar{o}_{k, (\varepsilon, \delta)}(x)) \in \bar{C}_0^{\text{dR}}(\Sigma_{m+2}^0)$ for every $l \in \{1, \dots, m+1\}$.
- $e_+(\bar{o}_{k, (\varepsilon, \delta)}(x)) = o_{k, (\varepsilon, \delta)}(x)$ and $e_-(\bar{o}_{k, (\varepsilon, \delta)}(x)) = (j_{\varepsilon', \varepsilon})_*(o_{k, (\varepsilon', \delta')}(x))$.

Proof. We omit the detailed proof. Note that $\bar{\delta}$ has the form (32). For any $x = (\gamma_k)_{k=1, \dots, m}$, we can compute explicitly that the chain $\bar{f}_{k, \bar{\delta}} \circ \bar{\Psi}(x) - \bar{\Psi} \circ f_k^{\natural}(x)$ is equal to the sum of chains for all $(u_*, t_*) \in (\Gamma_k^{\text{int}})^{-1}(K)$

$$\begin{aligned} & [\mathbb{R} \times U_{(u_*, t_*)}, \bar{\Phi}'_{(u_*, t_*)}, (\text{id}_{\mathbb{R}_{\geq 1}} \times U_{(u_*, t_*)}, \text{id}_{\mathbb{R}_{\leq -1}} \times U_{(u_*, t_*)}), \bar{\zeta}'_{(u_*, t_*)}] \\ & - \text{sign}(u_*, t_*) \cdot [\mathbb{R}, c_0(u_*, t_*)], (\text{id}_{\mathbb{R}_{\geq 1}}, \text{id}_{\mathbb{R}_{\leq -1}}), 1] \end{aligned}$$

modulo $\bar{C}_0^{\text{dR}}(\Sigma_{m+1}^0)$. Here $\bar{\Phi}'_{(u_*, t_*)}: \mathbb{R} \times U_{(u_*, t_*)} \rightarrow \mathbb{R} \times \Sigma_{m+1}^{a+(m+1)\varepsilon}$ is determined by

$$\bar{\Phi}'_{(u_*, t_*)}(r, (u, \tau)) := (r, \text{con}_k((\gamma_l^u)_{l=1, \dots, m}, (T_k^u, \tau), \bar{\psi}_{\varepsilon', \varepsilon}(r, \gamma_k^u(\tau))))$$

and $(\bar{\zeta}'_{(u_*, t_*)}) := (\text{id}_{\mathbb{R}} \times \Gamma_k^{\text{int}}|_{U_{(u_*, t_*)}})^*(1 \times \bar{\eta}_{\varepsilon', \varepsilon}) \in \Omega^2(\mathbb{R} \times U_{(u_*, t_*)})$. The $[-1, 1]$ -modeled chain $\bar{o}_{k, (\varepsilon, \delta)}$ is defined in a similar way as $o_{k, (\varepsilon, \delta)}$ in Lemma 6.4, and we can check that this chain satisfies the required three conditions. \square

Proof of Proposition 6.5. We define a linear map $\bar{\Phi}_{(\varepsilon, \delta)}^{<a}: C_*^{\natural, <a}(\varepsilon) \rightarrow \bar{C}_*^{<a}(\varepsilon)$ so that for $x \in C_p^{\natural, <a}(m, a, \varepsilon)$,

$$\bar{\Phi}_{(\varepsilon, \delta)}^{<a}([x]) = \begin{cases} [\bar{\Psi}(x)] & \text{if } p = 0, \\ [\bar{\Psi}(x)] - \sum_{k=1}^m [\bar{o}_{k, (\varepsilon, \delta)}(x)] & \text{if } p = 1, \\ 0 & \text{else.} \end{cases}$$

The first two properties of $\bar{o}_{k,(\varepsilon,\bar{\delta})}$ shows that this is a chain map from $(C_*^{\mathfrak{h},<a}(\varepsilon), D_\varepsilon^{\mathfrak{h}})$ to $(\bar{C}_*^{<a}(\varepsilon), \bar{D}_{\bar{\delta}})$. Therefore, we get a linear map on homology

$$(\bar{\Phi}_{(\varepsilon,\bar{\delta})}^{<a})_*: H_*^{\mathfrak{h},<a}(\varepsilon) \rightarrow \bar{H}_*^{<a}(\varepsilon, \bar{\delta}).$$

The third property of $\bar{o}_{k,(\varepsilon,\bar{\delta})}$ implies that the following diagram commutes:

$$\begin{array}{ccccc} H_*^{\mathfrak{h},<a}(\varepsilon) & \xrightarrow{(\Phi_{(\varepsilon,\delta)}^{<a})_*} & H_*^{<a}(\varepsilon, \delta) & \xrightarrow{(j_{\varepsilon,\varepsilon})_* = \text{id}} & H_*^{<a}(\varepsilon, \delta) \\ & \searrow^{(\bar{\Phi}_{(\varepsilon,\bar{\delta})}^{<a})_*} & & \nearrow_{(e_{\varepsilon,+})_*} & \\ & & \bar{H}_*^{<a}(\varepsilon, \bar{\delta}) & & \\ & & & \searrow_{(e_{\varepsilon,-})_*} & \\ & & & & H_*^{<a}(\varepsilon, (i_{\varepsilon',\varepsilon})_* \delta'). \\ H_*^{\mathfrak{h},<a}(\varepsilon') & \xrightarrow{(\Phi_{(\varepsilon',\delta')}^{<a})_*} & H_*^{<a}(\varepsilon', \delta') & \xrightarrow{(j_{\varepsilon',\varepsilon})_*} & H_*^{<a}(\varepsilon, (i_{\varepsilon',\varepsilon})_* \delta'). \end{array}$$

The proposition is now proved since $k_{(\varepsilon,\bar{\delta})} = ((j_{\varepsilon,\varepsilon})_*^{-1} \circ (e_{\varepsilon,+})_*) \circ ((j_{\varepsilon',\varepsilon})_*^{-1} \circ (e_{\varepsilon,-})_*)^{-1}$. \square

Let $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$. Proposition 6.5 shows that the family of maps $\{(\Phi_{(\varepsilon,\delta)}^{<a})_* \mid (\varepsilon, \delta) \in \mathcal{T}_a\}$ is standard with respect to h induces a linear map on the limits of $\varepsilon \rightarrow 0$

$$\bar{\Phi}^{<a}: H_*^{\mathfrak{h},<a} = \varinjlim_{\varepsilon \rightarrow 0} H_*^{\mathfrak{h},<a}(\varepsilon) \rightarrow H_*^{<a}(Q, K) = \varinjlim_{\varepsilon \rightarrow 0} H_*^{<a}(\varepsilon, \delta).$$

Naturally, those maps of $\{\bar{\Phi}^{<a}\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}$ commutes with $\{I^{a,b}: H_*^{<a}(Q, K) \rightarrow H_*^{<b}(Q, K)\}_{a \leq b}$ and $\{J^{a,b}: H_*^{\mathfrak{h},<a} \rightarrow H_*^{\mathfrak{h},<b}\}_{a \leq b}$. Therefore, on the limit of $a \rightarrow \infty$, we have a linear map

$$\bar{\Phi}: H_*^{\mathfrak{h},\infty} \rightarrow H_*^{\text{string}}(Q, K).$$

Moreover, it is straightforward to check that $\bar{\Phi}$ is a homomorphism of unital \mathbb{R} -algebras.

6.3 Proof of isomorphism

In this section, we prove that for every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$, $\bar{\Phi}^{<a}$ is an isomorphism in the 0-th degree. As a consequence, it is shown that the cord algebra of (Q, K) is isomorphic to $H_0^{\text{string}}(Q, K)$.

For each $m \in \mathbb{Z}_{\geq 0}$, let $\partial_m^{\text{sing}}: C_1^{\mathfrak{h},<a}(m) \rightarrow C_0^{\mathfrak{h},<a}(m)$ denote the singular boundary operator. We also write

$$\begin{aligned} \Psi_{0,m}: \text{Coker } \partial_m^{\text{sing}} &\rightarrow H_0^{\text{dR}}(\Sigma_m^a, \Sigma_m^0): [x] \mapsto [\Psi(x)], \\ \Psi_{1,m}: \text{Ker } \partial_m^{\text{sing}} &\rightarrow H_1^{\text{dR}}(\Sigma_m^a, \Sigma_m^0): x \mapsto [\Psi(x)]. \end{aligned} \tag{45}$$

Lemma 6.7. $\Psi_{0,m}$ is an isomorphism and $\Psi_{1,m}$ is a surjection.

Proof. Naturally, there are two maps

$$\begin{aligned} \text{Coker } \partial_m^{\text{sing}} &\rightarrow H_0^{\text{sing}}(\Sigma_m^a, \Sigma_m^0), \\ \text{Ker } \partial_m^{\text{sing}} &\rightarrow H_1^{\text{sing}}(\Sigma_m^a, \Sigma_m^0). \end{aligned}$$

induced by the inclusion maps $C_p^{\mathfrak{h}}(m, a) \rightarrow C_p^{\text{sing}}(\Sigma_m^a)$ for $p = 0, 1$. The subset of Σ_m^a (resp. $C^0([0, 1], \Sigma_m^a)$) consisting of elements satisfying the conditions (0a), (0b) (resp. (1a), (1b), (1c)) is open dense. This fact implies that the first map is an isomorphism and the second map is a surjection. Then, we consider the following diagram for $p = 0, 1$:

$$\begin{array}{ccc} K_{p,m} & \xrightarrow{\Psi_{p,m}} & H_p^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) \xrightarrow{(6)} \varinjlim_{j \rightarrow \infty} H_p^{\text{dR}}(B_m^a(2^j), B_m^0(2^j)) \\ \downarrow & & \uparrow \\ H_p^{\text{sing}}(\Sigma_m^a, \Sigma_m^0) & \xrightarrow{(11)} & \varinjlim_{j \rightarrow \infty} H_p^{\text{sing}}(B_m^a(2^j), B_m^0(2^j)) \end{array}$$

Here, $K_{0,m} := \text{Coker } \partial_m^{\text{sing}}$ and $K_{1,m} := \text{Ker } \partial_m^{\text{sing}}$. The left vertical map is defined as above. The right vertical map is an isomorphism from Proposition 2.9. The horizontal maps are the isomorphisms of (6) and (11). The commutativity follows from the definition of the right vertical map. See [23, Section 4.7]. Now the assertion of the lemma follows from the diagram. \square

For the chain complexes $(C_*^{\mathfrak{h}, <a}, D^{\mathfrak{h}})$ and $(C_*^{\mathfrak{h}, <a}(\varepsilon), D_\varepsilon^{\mathfrak{h}})$, we define their filtrations $\{\mathcal{H}_p^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathcal{H}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ by

$$\mathcal{H}_p^{<a} := \bigoplus_{m \geq -p} C_*^{\mathfrak{h}, <a}(m), \quad \mathcal{H}_{\varepsilon,p}^{<a} := \bigoplus_{m \geq -p} C_*^{\mathfrak{h}, <a}(m, \varepsilon).$$

Let $E^{\mathfrak{h}, <a}$ and $E_\varepsilon^{\mathfrak{h}, <a}$ be the spectral sequences determined by $\{\mathcal{H}_p^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathcal{H}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ respectively. Their $(-m, q)$ -terms of the first pages are given by

$$(E^{\mathfrak{h}, <a})_{-m,q}^1 = \begin{cases} \text{Coker } \partial_m^{\text{sing}} & \text{if } q - m = 0 \text{ and } m \geq 0, \\ \text{Ker } \partial_m^{\text{sing}} & \text{if } q - m = 1 \text{ and } m \geq 0, \\ 0 & \text{else,} \end{cases}$$

$$(E_\varepsilon^{\mathfrak{h}, <a})_{-m,q}^1 = \begin{cases} \text{Coker } \partial_{\varepsilon,m}^{\text{sing}} & \text{if } q - m = 0 \text{ and } m \geq 0, \\ \text{Ker } \partial_{\varepsilon,m}^{\text{sing}} & \text{if } q - m = 1 \text{ and } m \geq 0, \\ 0 & \text{else.} \end{cases}$$

Here $\partial_{\varepsilon,m}^{\text{sing}}: C_1^{\mathfrak{h}, <a}(m, \varepsilon) \rightarrow C_0^{\mathfrak{h}, <a}(m, \varepsilon)$ also denotes the singular boundary operator. If $0 < \varepsilon' \leq \varepsilon$, there exists a morphism $l_{\varepsilon, \varepsilon'}: E_\varepsilon^{\mathfrak{h}, <a} \rightarrow E_{\varepsilon'}^{\mathfrak{h}, <a}$ induced by the inclusion maps $C_*^{\mathfrak{h}}(m, a, \varepsilon) \rightarrow C_*^{\mathfrak{h}}(m, a, \varepsilon')$ for all $m \in \mathbb{Z}_{\geq 0}$. Naturally, $\lim_{\varepsilon \rightarrow 0} E_\varepsilon^{\mathfrak{h}, <a} \cong E^{\mathfrak{h}, <a}$ holds.

For $(\varepsilon, \delta) \in \mathcal{T}_a$ which is standard with respect to h , the chain map $\Phi_{(\varepsilon, \delta)}^{<a}$ preserves the filtrations $\{\mathcal{H}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$ and $\{\mathcal{F}_{\varepsilon,p}^{<a}\}_{p \in \mathbb{Z}}$, so it induces a morphism of spectral sequences

$$(\Phi_{(\varepsilon, \delta)}^{<a})_*: E_\varepsilon^{\mathfrak{h}, <a} \rightarrow E_{(\varepsilon, \delta)}^{<a}.$$

Note that on the $(-m, q)$ -term ($m \geq 0$) of first pages, this can be written as follows:

$$\begin{aligned} (\Phi_{(\varepsilon, \delta)}^{<a})_*: \text{Coker } \partial_{\varepsilon,m}^{\text{sing}} &\rightarrow H_0^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0): [x] \rightarrow [\Psi(x)] \text{ if } q = m, \\ (\Phi_{(\varepsilon, \delta)}^{<a})_*: \text{Ker } \partial_{\varepsilon,m}^{\text{sing}} &\rightarrow H_1^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0): x \rightarrow [\Psi(x)] \text{ if } q = m + 1. \end{aligned} \quad (46)$$

Recall that we have defined $k_{(\varepsilon', \delta'), (\varepsilon, \delta)}: E_{(\varepsilon', \delta')}^{<a} \rightarrow E_{(\varepsilon, \delta)}^{<a}$ by the composition of the maps of (28). The next result is a variant of Proposition 6.5 for spectral sequences.

Proposition 6.8. *The following diagram commutes:*

$$\begin{array}{ccc} E_\varepsilon^{\mathfrak{h}, <a} & \xrightarrow{(\Phi_{(\varepsilon, \delta)}^{<a})_*} & E_{(\varepsilon, \delta)}^{<a} \\ l_{\varepsilon, \varepsilon'} \downarrow & & \uparrow k_{(\varepsilon', \delta'), (\varepsilon, \delta)} \\ E_{\varepsilon'}^{\mathfrak{h}, <a} & \xrightarrow{(\Phi_{(\varepsilon', \delta')}^{<a})_*} & E_{(\varepsilon', \delta')}^{<a}. \end{array}$$

This can be proved as Proposition 6.5 by taking $(\bar{\Phi}_{(\varepsilon, \delta)}^{<a})_*: E_\varepsilon^{\mathfrak{h}, <a} \rightarrow \bar{E}_{(\varepsilon, \delta)}^{<a}$ into consideration. We omit the proof.

We use the spectral sequence $E^{<a}$ of Proposition 4.17. The above proposition and (46) implies the existence of the following morphism of spectral sequences.

Proposition 6.9. *There exists a morphism of spectral sequences $\Phi^{<a}: E^{\mathfrak{h}, <a} \rightarrow E^{<a}$ such that on the $(-m, q)$ -term ($m \geq 0$) of the first page,*

$$\begin{aligned}\Phi^{<a} &= \Psi_{0,m}: (E^{\mathfrak{h}, <a})_{-m,q}^1 = \text{Coker } \partial_m^{\text{sing}} \rightarrow (E^{<a})_{-m,q}^1 = H_0^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) && \text{if } q = m, \\ \Phi^{<a} &= \Psi_{1,m}: (E^{\mathfrak{h}, <a})_{-m,q}^1 = \text{Ker } \partial_m^{\text{sing}} \rightarrow (E^{<a})_{-m,q}^1 = H_1^{\text{dR}}(\Sigma_m^a, \Sigma_m^0) && \text{if } q = m + 1.\end{aligned}$$

This property of $\Phi^{<a}: E^{\mathfrak{h}, <a} \rightarrow E^{<a}$ implies a result on the compatible map $\Phi^{<a}: H_p^{\mathfrak{h}, <a} \rightarrow H_p^{<a}(Q, K)$.

Proposition 6.10. *$\Phi^{<a}: H_p^{\mathfrak{h}, <a} \rightarrow H_p^{<a}(Q, K)$ is an isomorphism if $p = 0$ and a surjection if $p = 1$.*

Proof. By Lemma 6.7 and Proposition 6.9, $\Phi^{<a}: (E^{\mathfrak{h}, <a})_{p,q}^1 \rightarrow (E^{<a})_{p,q}^1$ is an isomorphism if $p + q \leq 0$ and a surjection if $p + q = 1$. Since $E^{\mathfrak{h}, <a}$ converges to $H_*^{\mathfrak{h}, <a}$ and $E^{<a}$ converges to $H_*^{<a}(Q, K)$, we can apply Lemma 4.2 to prove the above assertion for $\Phi^{<a}: H_*^{\mathfrak{h}, <a} \rightarrow H_*^{<a}(Q, K)$. \square

On their limits of $a \rightarrow \infty$, $\{\Phi^{<a}\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}$ induces an isomorphism

$$\Phi: H_0^{\mathfrak{h}, <\infty} \rightarrow H_0^{\text{string}}(Q, K).$$

Combining with Proposition 6.3, we finally obtain the following result.

Theorem 6.11. *As a unital \mathbb{R} -algebra, $\text{Cord}(Q, K; \mathbb{R})$ is isomorphic to $H_0^{\text{string}}(Q, K)$.*

7 Connection to Legendrian contact homology

In this section, we discuss intuitively how to construct an isomorphism from the Legendrian contact homology of the unit conormal bundle of K to $H_*^{\text{string}}(Q, K)$ when $Q = \mathbb{R}^n$.

7.1 Legendrian contact homology

We begin with basic definitions. Let M be a $(2n - 1)$ -dimensional manifold and $\alpha \in \Omega^1(M)$ be a contact form on M , that is, $\alpha_x \wedge (d\alpha)_x^{n-1} \neq 0$ for every $x \in M$. Then, $\xi := \text{Ker } \alpha$ is a contact distribution of M . The Reeb vector field R_α is a vector field on M characterized by $\alpha_x(R_\alpha) = 1$ and $(d\alpha)_x(R_\alpha, \cdot) = 0$ for every $x \in M$.

An $(n - 1)$ -dimensional submanifold Λ of M is called a *Legendrian submanifold* of (M, α) if $\alpha|_\Lambda = 0 \in \Omega^1(\Lambda)$. We define the set of Reeb chords of Λ by

$$\mathcal{R}(\Lambda) := \{c: [0, T] \rightarrow M \mid T > 0, \frac{dc}{dt} = R_\alpha \circ c, \text{ and } c(0), c(T) \in \Lambda\}.$$

We sketch the definition of the Legendrian contact homology a compact Legendrian submanifold Λ of (M, α) under several conditions on them. To define it, we shall use pseudo-holomorphic curves in the symplectization $(\mathbb{R} \times M, d(e^r \alpha))$ of (M, α) . (Here, r is the coordinate of \mathbb{R} .)

Remark 7.1. As far as the author knows, definitions of Legendrian contact homology can be found in literatures only for some restricted cases. One of such cases consists of contact manifolds of the form $(\mathbb{R} \times P, dz - \theta)$ for a Liouville manifold (P, θ) (Here, z is the coordinate of \mathbb{R}). In this case, a definition of Legendrian contact homology was given in [11, 14] by using pseudo-holomorphic curves in P . The correspondence between pseudo-holomorphic curves in P and those in $\mathbb{R} \times (\mathbb{R} \times P)$ was given in [10].

Referring to [14, Section 2.2], we impose the following condition on (M, Λ) :

- There exist a $(2n - 2)$ -dimensional Liouville manifold (P, θ) which is the completion of a compact Liouville domain and a diffeomorphism $\varphi: M \rightarrow \mathbb{R} \times P$ such that $\varphi^*(dz - \theta) = \alpha$. In this case, $\varphi_*(R_\alpha) = \partial_z$. We also assume that $c_1(TP) = 0$ and $H_1(P; \mathbb{Z})$ is free.
- Λ is a compact spin manifold. Let $\text{pr}_P: \mathbb{R} \times P \rightarrow P$ be the projection. As a generic condition on Λ , we require that $(\text{pr}_P)|_\Lambda: \Lambda \rightarrow P$ is an immersion and the self-intersection of $\text{pr}_P(\Lambda)$ consists only of transversal double points. In addition, the Maslov class of $\text{pr}_P(\Lambda)$ as an immersed Lagrangian submanifold is 0.

We fix a spin structure on Λ . Let us prepare several notations as follows:

- We define $\tau_s: \mathbb{R} \times M \rightarrow \mathbb{R} \times M: (r, x) \mapsto (r + s, x)$ for every $s \in \mathbb{R}$.
- Let $D := \{z \in \mathbb{C} \mid |z| \leq 1\}$ and denote the complex structure by $j \in \text{End}(TD)$. We orient ∂D clockwise, then the set $\partial D \setminus \{1\}$ is ordered along this orientation. For any finite subset \mathbf{p} of $\partial D \setminus \{1\}$, if we write $\mathbf{p} = \{p_1, \dots, p_m\}$, this means that $p_1 < p_2 < \dots < p_m$ in this order.
- For any subset $\mathbf{p} = \{p_1, \dots, p_m\}$ in $\partial D \setminus \{1\}$ (possibly \mathbf{p} is the empty set), we define a punctured disk $D_{\mathbf{p}} := D \setminus \{1, p_1, \dots, p_m\}$. Around $1 \in D$ and $p_k \in D$ ($k = 1, \dots, m$), we choose biholomorphic maps

$$\begin{cases} \psi_0 & : [0, \infty) \times [0, 1] \rightarrow D \setminus \{1\}, \\ \psi_k & : (-\infty, 0] \times [0, 1] \rightarrow D \setminus \{p_k\} \quad (k = 1, \dots, m), \end{cases}$$

such that $\begin{cases} \psi_0([0, \infty) \times \{0, 1\}) \subset \partial D \setminus \{1\}, \\ \psi_k((-\infty, 0] \times \{0, 1\}) \subset \partial D \setminus \{p_k\}, \end{cases}$ and $\begin{cases} \lim_{s \rightarrow \infty} \psi_0(s, t) = 1, \\ \lim_{s \rightarrow -\infty} \psi_k(s, t) = p_k, \end{cases}$ uniformly on $t \in [0, 1]$.

- For every Reeb chord $(c: [0, T] \rightarrow Q) \in \mathcal{R}(\Lambda)$, we write T by T_c .

With these preparations, let us summarize how to define the Legendrian contact homology of (M, Λ) .

1. Choose an almost complex structure J on $\mathbb{R} \times M$ such that:

- (a) $d(e^r \alpha)(\cdot, J \cdot)$ is a Riemannian metric on $\mathbb{R} \times M$.
- (b) $J(\xi) = \xi$ for the contact distribution ξ and $J(\partial_r) = R_\alpha$.
- (c) $(\tau_s)^* J = J$ for every $s \in \mathbb{R}$.

2. Fix $c, c_1, \dots, c_m \in \mathcal{R}(\Lambda)$. We consider the moduli space of pairs (u, \mathbf{p}) of $\mathbf{p} = \{p_1, \dots, p_m\} \in D \setminus \{1\}$, and a C^∞ map

$$u: D_{\mathbf{p}} \rightarrow \mathbb{R} \times M$$

satisfying the following conditions:

- $du + J \circ du \circ j = 0$. Namely, u is a pseudo-holomorphic curve with respect to J .
- $u(\partial D_{\mathbf{p}}) \subset \mathbb{R} \times \Lambda$.
- There exists $s_0, s_1, \dots, s_m \in \mathbb{R}$ such that for every $k = 1, \dots, m$

$$\begin{cases} \tau_{-s} \circ u \circ \psi_0(s, \cdot) \rightarrow (s_0, c(T_c \cdot)) & \text{if } s \rightarrow \infty, \\ \tau_{-s} \circ u \circ \psi_k(s, \cdot) \rightarrow (s_k, c(T_{c_k} \cdot)) & \text{if } s \rightarrow -\infty, \end{cases}$$

in C^∞ topology on $[0, 1]$.

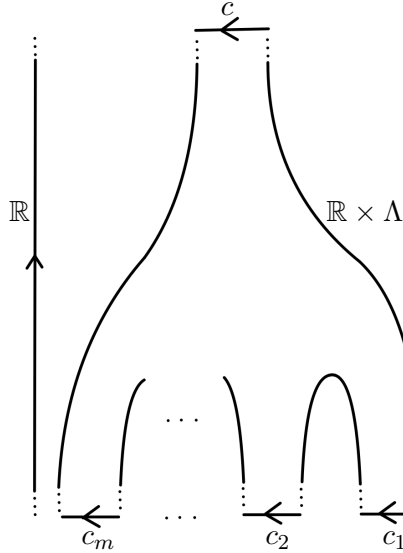


Figure. 7: A pseudo-holomorphic curve in $\mathbb{R} \times M$ with boundaries in $\mathbb{R} \times \Lambda$.

Figure 7 describes such pseudo-holomorphic curves. Let

$$\mathcal{M}_J(c; c_1, \dots, c_m)$$

denotes this moduli space of pseudo-holomorphic curves. (Precisely, it is the orbit space by the action of $\text{Aut}(D, 1)$, which is the group of automorphisms of D fixing $1 \in D$.) Since J is invariant by the translations τ_s for all $s \in \mathbb{R}$, this moduli space has a natural \mathbb{R} -action defined by $u \cdot s := \tau_s \circ u$. By [10, Theorem 2.1] together with [14, Lemma 4.5], we can choose J to be regular so that $\mathcal{M}_J(c; c_1, \dots, c_m)/\mathbb{R}$ has a structure of a finite dimensional manifold.

3. For every $c \in \mathcal{R}(\Lambda)$, we can assign an integer $|c|$ as in [10, Section 4.1] such that the dimension of $\mathcal{M}_J(c; c_1, \dots, c_m)/\mathbb{R}$ is equal to

$$|c| - \sum_{k=1}^m |c_k| - 1$$

when $(c_1, \dots, c_m) \neq (c)$. As a manifold, an orientation of $\mathcal{M}_J(c; c_1, \dots, c_m)/\mathbb{R}$ is determined, depending on the spin structure of Λ [14, Section 4.4]. (Precisely, $\mathcal{M}_J(c; c_1, \dots, c_m)/\mathbb{R}$ is oriented so that the diffeomorphism of [10, Theorem 2.1] preserves orientations.) When $|c| = \sum_{k=1}^m |c_k| + 1$, this is a compact oriented 0-dimensional manifold. It is a consequence of the Gromov compactness. See [12, Section 9] when $P = \mathbb{C}^n$ and [5, Section 11.3].

Let $\mathcal{A}_*(\Lambda)$ be the unital graded non-commutative \mathbb{R} -algebra freely generated by $\mathcal{R}(\Lambda)$ such that $c \in \mathcal{A}_{|c|}(\Lambda)$ for every $c \in \mathcal{R}(\Lambda)$. We define a derivation $\partial_J: \mathcal{A}_*(\Lambda) \rightarrow \mathcal{A}_{*-1}(\Lambda)$ by

$$\partial_J(c) := \sum_{|c_1| + \dots + |c_m| = |c| - 1} (-1)^{n(|c|+1)} \#_{\text{sign}}(\mathcal{M}_J(c; c_1, \dots, c_m)/\mathbb{R}) c_1 \cdots c_m.$$

for every $c \in \mathcal{R}(\Lambda)$, and extend it by the Leibniz rule. Here, $\#_{\text{sign}}$ is the number of the point counted with the signs induced by the orientation.

From [14, Theorem 1.1], $(\mathcal{A}_*(\Lambda), \partial_J)$ is a differential graded algebra over \mathbb{R} , namely, $\partial_J \circ \partial_J = 0$ holds. Moreover, its stable tame isomorphism class is independent on the choice of J and invariant under Legendrian isotopies of Λ preserving the spin structure. (The dependence on

the spin structure is discussed in [13, Theorem 4.31].) In particular, the isomorphism class of the homology of $(\mathcal{A}_*(\Lambda), \partial_J)$ as a unital graded \mathbb{R} -algebras is well-defined. This homology is called the *Legendrian contact homology* of (M, Λ) , which we denote by

$$\text{LCH}_*(M, \Lambda) := \text{Ker } \partial_J / \text{Im } \partial_J.$$

7.2 Conjecture

Let Q be a manifold and K be its submanifold. Let λ be a canonical Liouville form on T^*Q such that (T^*Q, λ) is a Liouville manifold. The *conormal bundle* of K defined by

$$L_K := \{(q, p) \in T^*Q \mid q \in K, p|_{T_q K} = 0\}$$

is an exact Lagrangian submanifold of T^*Q . Moreover, when Q is equipped with a Riemannian metric g , its unit cotangent bundle UT^*Q has a canonical contact form $\alpha := \lambda|_{UT^*Q}$. Then, the *unit conormal bundle* of K defined by

$$\Lambda_K := L_K \cap UT^*Q$$

is a Legendrian submanifold of (UT^*Q, α) . The next theorem is a well-known fact about the Reeb vector field R_α . Let UTQ be the unit tangent bundle of Q .

Theorem 7.2. ([22, Theorem 1.5.2]) *Via the identification*

$$\Psi_g: UTQ \rightarrow UT^*Q: v \mapsto g(v, \cdot),$$

*the flow of the Reeb vector field on UT^*Q coincides with the geodesic flow on UTQ .*

From the above fact, if we set $\pi_Q: UT^*Q \rightarrow Q$ to be the bundle projection, we obtain a bijection

$$\mathcal{R}(\Lambda_K) \rightarrow \mathcal{C}(K): c \mapsto \pi_Q \circ c.$$

Here, $\mathcal{C}(K)$ is the set defined in Section 3.1. It consists of geodesics in Q of unit speed with end points in K and perpendicular to K at the end points.

Suppose that \mathbb{R}^n is equipped with the standard Riemannian metric and K is a compact oriented submanifold of \mathbb{R}^n with $\text{codim } K = d$. As we have seen in Proposition 5.11, Λ_K has a spin structure induced by the standard spin structure on \mathbb{R}^n . We remark that $\varphi^*(dz - \lambda) = \alpha$ is satisfied by the diffeomorphism

$$\varphi: UT^*\mathbb{R}^n \rightarrow T^*S^{n-1} \times \mathbb{R}: (q, p) \mapsto ((p, q - \langle q, p \rangle \cdot p), \langle q, p \rangle),$$

and thus the Legendrian contact homology $\text{LCH}(UT^*\mathbb{R}^n, \Lambda_K)$ is defined as in Section 7.1.

Now our conjecture is stated as follows.

Conjecture 7.3. *For any compact oriented submanifold K of \mathbb{R}^n , there exists an isomorphism of unital graded \mathbb{R} -algebras*

$$\text{LCH}_*(UT^*\mathbb{R}^n, \Lambda_K) \cong H_*^{\text{string}}(\mathbb{R}^n, K).$$

Remark 7.4. The above construction of $\text{LCH}_*(M, \Lambda)$ will work in more general situations, such as where M has no contractible periodic Reeb orbit. The author expects that the conjecture can be extended, for instance, to the case where Q is spin and has a non-positive sectional curvature, imposing certain convexity conditions when it is non-compact.

7.3 Idea toward the proof of Conjecture 7.3

The rest of this section is devoted to sketching the idea toward proving the conjecture. The outline is similar to [9, Section 6, 7].

Caution 1. In the outline of the proof explained below, we does not concern with signs. In fact, from the current research for proving the conjecture, it seems better to define the Legendrian contact homology using the sign conventions different from [13] so that it is compatible with the signs which we fixed to define $H_*^{\text{string}}(Q, K)$.

Caution 2. At several points, we omit the issues about parametrizations of paths. In Remark 7.6, we will see what is the problem and also a possible solution is mentioned.

Let us prepare several notions. We require $K \subset \mathbb{R}^n$ to satisfy the following conditions which can be achieved by an arbitrarily small C^∞ perturbation of K :

- (i) All binormal chords are non-degenerate. In the present case, this is equivalent to that for every $(\gamma: [0, T] \rightarrow \mathbb{R}^n) \in \mathcal{C}(K)$, $(\gamma(0), \gamma(T)) \in K \times K$ is a non-degenerate critical point of the function

$$K \times K \rightarrow \mathbb{R}: (q, q') \mapsto |q - q'|^2.$$

- (ii) For every $m, m' \in \mathbb{Z}_{\geq 1}$ with $m \neq m'$, $\mathcal{L}_m(K) \cap \mathcal{L}_{m'}(K) = \emptyset$ holds.

The second condition can be achieved if the set $\{\text{length}(\gamma) \in \mathbb{R} \mid \gamma \in \mathcal{C}(K)\}$ is linearly independent over \mathbb{Q} . Under these conditions, it follows from [2, Corollary 4.2] that $\Lambda = \Lambda_K$ satisfies the conditions in Section 7.1 which we imposed to define $\text{LCH}_*(M, \Lambda)$.

For every $c \in \mathcal{R}(\Lambda_K)$, $T_c = \int c^*(\alpha) \in \mathbb{R}_{>0}$ is called the *action* of c . This satisfies $T_c = \text{length}(\pi_{\mathbb{R}^n} \circ c)$.

Definition 7.5. For any $a \in \mathbb{R}_{>0}$, let $(\mathcal{A}_*^{<a}, \partial_J)$ be the subcomplex of $(\mathcal{A}_*(\Lambda_K), \partial_J)$ spanned as an \mathbb{R} -vector space by words of Reeb chords whose sum of actions are less than a . Let $\text{LCH}_*^{<a}$ denote its homology. For $a, b \in \mathbb{R}_{>0}$ with $a \leq b$, the quotient complex $(\mathcal{A}_*^{[a,b]} := \mathcal{A}_*^{<b} / \mathcal{A}_*^{<a}, \partial_J)$ is defined. Let $\text{LCH}_*^{[a,b]}$ denote its homology.

In Subsection 7.3.1, 7.3.2 and 7.3.3, we will state Claim 1, 2 and 3 respectively. In Subsection 7.3.4, admitting these three claims, we give a proof of Conjecture 7.3.

7.3.1 Construction of de Rham chain via Kuranishi structure

Claim 1. Fix an arbitrary $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$. If $\varepsilon > 0$ is sufficiently small, there exists $\delta \in C_{n-d}^{\text{dR}}(S_\varepsilon)$ and a linear map

$$\Phi_\varepsilon^{<a}: \mathcal{A}_*^{<a} \rightarrow C_*^{<a}(\varepsilon)$$

such that $(\varepsilon, \delta) \in \mathcal{T}_a$ and $D_\delta \circ \Phi_\varepsilon^{<a} = \Phi_\varepsilon^{<a} \circ \partial_J$ holds.

Recall that in order to define $(C_*^{<a}(\varepsilon), D_\delta)$ and \mathcal{T}_a , we need to choose auxiliary data (see Section 3.1). We let g to be the standard metric on \mathbb{R}^n and do not specify C_0 , ε_0 and μ .

To prove this claim, we focus on pseudo-holomorphic curves in $T^*\mathbb{R}^n$ with switching Lagrangian boundary conditions explained below. We often identify the image of 0-section of $T^*\mathbb{R}^n$ with \mathbb{R}^n , and the symplectization $\mathbb{R} \times UT^*\mathbb{R}^n$ with $T^*\mathbb{R}^n \setminus \mathbb{R}^n$ via

$$\mathbb{R} \times UT^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n \setminus \mathbb{R}^n: (r, (q, p)) \mapsto (q, e^r p).$$

Let us choose an almost complex structure J' on $T^*\mathbb{R}^n$ satisfying all but the fourth conditions of [9, Definition 8.1]. (The fourth condition, which we have removed, is about the integrability of J' around K .) In particular, $d\lambda(\cdot, J\cdot)$ is a Riemannian metric and J' coincides with J on

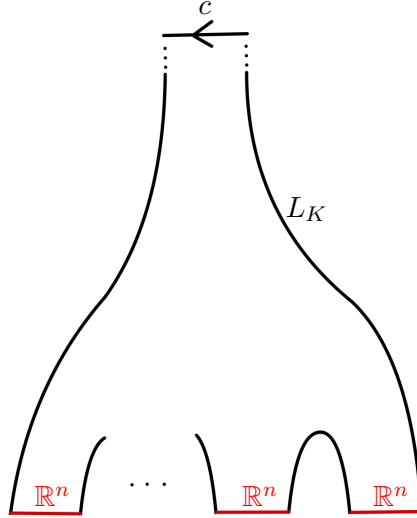


Figure. 8: A pseudo-holomorphic curve in $T^*\mathbb{R}^n$ with red boundaries in \mathbb{R}^n and black boundaries in L_K . The switching points of the boundary conditions are mapped to $K = \mathbb{R}^n \cap L_K$.

$\{(q, p) \in T^*\mathbb{R}^n \mid |p| \geq 1\}$. Explicitly, we can take it as follows: We choose a C^∞ function $\rho: [0, \infty) \rightarrow (0, \infty)$ such that $\rho(a) = \begin{cases} 1 & \text{if } a \leq \frac{1}{2}, \\ a & \text{if } a \geq 1. \end{cases}$ Fix $R > 0$ to be $|q| < R$ for every $q \in K$.

By using the coordinate $(q_1, \dots, q_n, p_1, \dots, p_n)$ of $T^*\mathbb{R}^n$, we define an almost complex structure J_ρ to be

$$(J_\rho)_{(q,p)}(\partial_{q_i}) = -\rho(|p|)\partial_{p_i}, (J_\rho)_{(q,p)}(\partial_{p_i}) = \rho(|p|)^{-1}\partial_{q_i}$$

for every $(q, p) \in T^*\mathbb{R}^n$. An almost complex structure J' satisfying the required conditions can be obtained by deforming $J_\rho|_\xi$ on $\{(q, p) \in T^*\mathbb{R}^n \mid |q| < R, |p| > \frac{1}{2}\}$, which is identified with an open subset of $\mathbb{R} \times UT^*\mathbb{R}^n$.

As a notation, for $\mathbf{p} = \{p_1, \dots, p_m\} \subset \partial D \setminus \{1\}$ with $m \neq 1$ and $k \in \{1, \dots, m\}$, let $\partial_k D_{\mathbf{p}}$ denote the connected component of $\partial D_{\mathbf{p}}$ whose closure in ∂D has $\{p_{k-1}, p_k\}$ as the boundary. (Here we set $p_0 = p_{m+1} = 1$.)

For any $c \in \mathcal{R}(\Lambda_K)$ and $m \in \mathbb{Z}_{\geq 0}$, we consider the moduli space of pairs (u, \mathbf{p}) of $\mathbf{p} = \{p_1, \dots, p_{2m}\} \in \partial D \setminus \{1\}$ and a C^∞ map

$$u: D_{\mathbf{p}} \rightarrow T^*\mathbb{R}^n$$

such that:

- $du + J' \circ du \circ j = 0$.
- For $k \in \{1, \dots, 2m+1\}$, $\begin{cases} u(\partial_k D_{\mathbf{p}}) \subset \mathbb{R}^n & \text{if } k \text{ is even,} \\ u(\partial_k D_{\mathbf{p}}) \subset L_K & \text{if } k \text{ is odd.} \end{cases}$
- If $s > 0$ is sufficiently large, $u \circ \psi_0(s, t) \in T^*\mathbb{R}^n \setminus \mathbb{R}^n \cong \mathbb{R} \times UT^*\mathbb{R}^n$. There exists $s'_0 \in \mathbb{R}$ and $x_k \in K = \mathbb{R}^n \cap L_K$ for $k = 1, \dots, 2m$ such that

$$\begin{cases} \tau_{-s} \circ u \circ \psi_0(s, \cdot) \rightarrow (s'_0, c(T_c \cdot)) & \text{if } s \rightarrow \infty, \\ u \circ \psi_k(s, \cdot) \rightarrow x_k & \text{if } s \rightarrow -\infty, \end{cases}$$

in C^∞ topology on $[0, 1]$.

Figure 8 describes such pseudo-holomorphic curves. Let $\mathcal{M}(c, m)$ denote this moduli space of pseudo-holomorphic curves.

In addition, we define \mathcal{N} to be the moduli space of pseudo-holomorphic curves with respect to J'

$$v: D \setminus \mathbf{p}' \rightarrow T^*\mathbb{R}^n \quad (\mathbf{p}' = \{p'_1, p'_2\} \in \partial D \setminus \{1\})$$

with one marked point at $1 \in \partial D$ such that:

- For any $z \in \partial D \setminus \mathbf{p}'$, $\begin{cases} v(z) \in \mathbb{R}^n & \text{if } z \text{ lies in the same component as } 1, \\ v(z) \in L_K & \text{else.} \end{cases}$
- For $k \in \{1, 2\}$, $\lim_{s \rightarrow -\infty} v \circ \psi_k(s, \cdot) = x_k$ in C^∞ topology on $[0, 1]$ for some point $x_k \in K = \mathbb{R}^n \cap L_K$.

Notation. Hereafter, we omit writing the set of boundary puncture \mathbf{p} or \mathbf{p}' from any element in $\mathcal{M}_J(c; c_1, \dots, c_m)$, $\mathcal{M}(c, m)$ and \mathcal{N} .

We remark that every $v \in \mathcal{N}$ is a constant map, since the Liouville form λ on $T^*\mathbb{R}^n$ vanishes when it is pulled back on the two Lagrangian submanifolds \mathbb{R}^n and L_K . In fact,

$$\text{ev}_0: \mathcal{N} \rightarrow K: v \mapsto v(1)$$

is a diffeomorphism. However, non-constant maps appear after a perturbation explained later.

For any $u \in \mathcal{M}(c, m)$ and $k \in \{1, \dots, m\}$, $u|_{\partial_{2k}D_{\mathbf{p}}}$ can be extended smoothly on the closure of $\partial_{2k}D_{\mathbf{p}}$. If we parameterize the domain by an interval $[0, T_k^u]$ so that p_{k-1} (resp. p_k) corresponds to 0 (resp. T_k^u), we get a sequence of m -paths

$$(\gamma_k^u: [0, T_k^u] \rightarrow \mathbb{R}^n)_{k=1, \dots, m}$$

with end points in K . (In Figure 8, they are the restrictions on the red boundary components.)

Remark 7.6. The moduli space $\mathcal{M}(c, m)$ is defined by taking the quotient by the action of $\text{Aut}(D, 1)$. Therefore, correctly speaking, u is a representative of an $\text{Aut}(D, 1)$ orbit and thus the path γ_k^u is well-defined only up to reparameterization. One might consider the parametrization by the arc length, but it does not work well with the later discussion. One way to solve this problem would be to consider pseudo-holomorphic curves with an arbitrary number of boundary marked points. This solution can be found in [24]. If we adopt this solution, we will need to change $H_*^{\text{string}}(\mathbb{R}^n, K)$ to the one consisting of de Rham chains of paths with an arbitrary number of interior marked points and also need to show that it is isomorphic to $H_*^{\text{string}}(\mathbb{R}^n, K)$ defined in this article.

Let us admit the following assertions without proofs:

- (A) For a generic J' , $\mathcal{M}(c, m)$ has a structure of an oriented manifold of dimension

$$|c| - m(d - 2).$$

- (B) For every $u \in \mathcal{M}(c, m)$,

$$\sum_{k=1}^m \text{length} \gamma_k^u \leq T_c.$$

The equality holds if and only if $m = 1$ and u is the trivial half strip over $\pi_{\mathbb{R}^n} \circ c$ characterized by $(du)_z \subset \text{Span}(\partial_r, R_\alpha)$ for every $z \in D_{\mathbf{p}} \setminus \partial_2 D_{\mathbf{p}}$ via the identification $T^*\mathbb{R}^n \setminus \mathbb{R}^n \cong \mathbb{R} \times UT^*\mathbb{R}^n$.

- (C) There exists a compactification $\overline{\mathcal{M}}(c, m)$ of $\mathcal{M}(c, m)$ whose boundary consists of pseudo-holomorphic buildings and pseudo-holomorphic curves with nodes. For the detailed explanation, see [8, Section 4].

It is important to note that if $u \in \mathcal{M}(c, 1)$ is the half strip over $\pi_{\mathbb{R}^n} \circ c$, then $\gamma_1^u = \pi_{\mathbb{R}^n} \circ c$.

Remark 7.7. (Comparison with [9]) The results corresponding to the above assertions can be found in [9]. About the assertion (A), we can refer to [9, Lemma 9.5] for the transversality, [9, Section 9.5] for orientations, and [8, Theorem A.1] for the computation of the dimension. The assertion (B) is proved in [9, Proposition 8.9]. For the assertion (C), we should note that in order to deal with $\overline{\mathcal{M}}(c, m)$ as a manifold with corners, [9] introduced the notion of a *winding number*, which is assigned to each switching point. This number is determined by the order of degeneracy. On the other hand, we will introduce later *Kuranishi structures* in order to deal with the compactified moduli space.

For every $m \in \mathbb{Z}_{\geq 1}$ and $c \in \mathcal{R}(\Lambda_K)$ with $T_c < a$, we consider a map

$$\mathcal{M}(c, m) \rightarrow \Sigma_m^a : u \mapsto (\gamma_1^u, \dots, \gamma_m^u),$$

and extend this map continuously on the compactification $\overline{\mathcal{M}}(c, m)$ to get a map

$$\varphi_{c,m} : \overline{\mathcal{M}}(c, m) \rightarrow \Sigma_m^a.$$

When $m = 0$, $\varphi_{c,0} : \overline{\mathcal{M}}(c, 0) \rightarrow \Sigma_0^a = \{*\}$ is a constant map. Likewise, let us define

$$\mathcal{N} \rightarrow S_{\varepsilon_0} : v \mapsto (\sigma_1^v, \sigma_2^v),$$

where σ_i^v for $i = 1, 2$ is a path obtained by extending $v|_{\partial_{2i-1}D_{\mathbf{p}'}}$ to the closure of $\partial_{2i-1}D_{\mathbf{p}'}$.

First, we intuitively regard $\varphi_{c,m}$ as a chain in Σ_m^a and observe its boundary chains. We do not need to deal with all boundary chains in the present argument for the following reasons:

- From the definition of de Rham chains, all degenerate chains are equal to 0. Here, we say a chain is *degenerate* if it has the form $[U \times V, \varphi \circ \text{pr}_V, \omega \times \eta]$ for some chain $[V, \varphi, \eta]$ and $\omega \in \Omega_c^p(U)$ with $\dim U - p > 0$.
- Any chain in Σ_m^0 is equal to 0 as an element of $C_*^{\text{dR}}(\Sigma_m^a, \Sigma_m^0)$. Recall that Σ_m^0 consists of sequences of m -paths one of whose length is smaller than ε_0 .

If we ignore chains which are degenerate or in Σ_m^0 , the remaining chains of $\varphi_{c,m}|_{\partial\overline{\mathcal{M}}(c,m)}$ consist of the following two types:

1. We have a chain defined on a pseudo-holomorphic building described in Figure 9

$$\begin{aligned} (\mathcal{M}_J(c; c_1, \dots, c_l)/\mathbb{R}) \times \prod_{\alpha=1}^l \mathcal{M}(c_\alpha, m_\alpha) &\rightarrow \Sigma_m^a \\ ([w], (u_\alpha)_{\alpha=1, \dots, l}) &\mapsto (\gamma_1^{u_1}, \dots, \gamma_{m_1}^{u_1}, \dots, \gamma_1^{u_l}, \dots, \gamma_{m_l}^{u_l}) \end{aligned} \quad (47)$$

for $c_1, \dots, c_l \in \mathcal{R}(\Lambda_K)$ and $m_1, \dots, m_l \in \mathbb{Z}_{\geq 0}$ with $\sum_{\alpha=1}^l m_\alpha = m$. If $\sum_{\alpha=1}^l |c_\alpha| > |c| - 1$ or $(|c_\alpha|, m_\alpha) \in \mathbb{Z}_{\geq 1} \times \{0\}$ for some $\alpha \in \{1, \dots, l\}$, then either $\mathcal{M}_J(a; c_1, \dots, c_l)$ or $\mathcal{M}(c_\alpha, m_\alpha)$ has positive dimension, so we think of (47) as a degenerate chain. Therefore, we only need to consider the case where $\sum_{\alpha=1}^l |c_\alpha| = |c| - 1$ and $|c_\alpha| = 0$ whenever $m_\alpha = 0$. If we extend Π , which was used to define the \star -operation, to a map $\Pi : \prod_{\alpha=1}^l \Sigma_{m_\alpha}^{a_\alpha} \rightarrow \Sigma_{m_1 + \dots + m_l}^{a_1 + \dots + a_l}$, (47) can be rewritten as a map

$$(([w], (u_\alpha)_{\alpha=1, \dots, l})) \mapsto \Pi(\varphi_{c_1, m_1}(u_1), \dots, \varphi_{c_l, m_l}(u_l)).$$

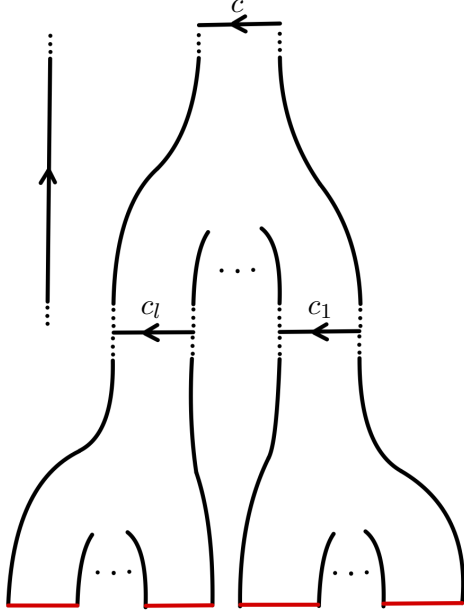


Figure. 9: A pseudo-holomorphic building consisting of pseudo-holomorphic curves in $\mathbb{R} \times UT^*\mathbb{R}^n$ and $T^*\mathbb{R}^n$.

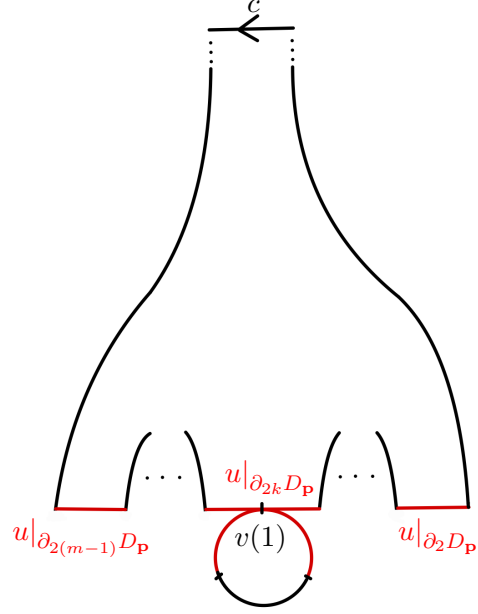


Figure. 10: A pseudo-holomorphic curve with a node on $\partial_{2k}D_{\mathbf{P}}$.

2. For $k = \{1, \dots, m-1\}$, let $\mathcal{M}_k(c, m-1)$ be the space of pairs (u, q) of $(u: D_{\mathbf{P}} \rightarrow T^*\mathbb{R}^n) \in \mathcal{M}(c, m-1)$ and a marked point $q \in \partial_{2k}D_{\mathbf{P}}$. We define an evaluation map

$$\text{ev}_k: \mathcal{M}_k(c, m-1) \rightarrow \mathbb{R}^n: (u, q) \mapsto u(q).$$

Then we have a chain defined on the fiber product of ev_k and $\text{ev}_0: \mathcal{N} \rightarrow \mathbb{R}^n: v \mapsto v(1)$ described in Figure 10

$$\begin{aligned} \mathcal{M}_k(c, m-1) \times_{\text{ev}_k \times \text{ev}_0} \mathcal{N} &\rightarrow \Sigma_m^a \\ ((u, q), v) &\mapsto (\gamma_1^u, \dots, (\gamma_k^u|_{[0, q]} \cdot \sigma_1^v), (\sigma_2^v \cdot \gamma_k^u|_{[q, T_k^u]}), \dots, \gamma_{m-1}^u). \end{aligned} \quad (48)$$

Here, we think of q as a point in $(0, T_k^u) \cong \partial_{2k}D_{\mathbf{P}}$. If we ignore the difference of parameterizations near end points, (48) can be rewritten as a map

$$((u, q), v) \mapsto \text{con}_k((\gamma_k^u)_{k=1, \dots, m-1}, (T_k^u, q), (\sigma_i^v)_{i=1, 2}).$$

More precisely, we need to extend these maps (47) and (48) to the compactification of the domains. The other chains consisting of $\varphi_{c, m}|_{\partial \overline{\mathcal{M}(c, m)}}$ are contained in Σ_m^0 . For examples, see Figure 11.

Remark 7.8. The following assertion, which holds when K is a knot, was crucial in [9] to define a chain map similar to $\Phi_\varepsilon^{<a}$:

- (D) We can choose J' such that $\mathcal{M}(c, m) = \emptyset$ for every $c \in \mathcal{R}(\Lambda_K)$ if $m \in \mathbb{Z}_{\geq 0}$ is sufficiently large.

In the present discussion, we do not need to rely on such a result since $\varphi_{c, m}$ vanishes as a chain in a pair of space (Σ_m^a, Σ_m^0) when m is sufficiently large. Indeed, if $m \geq \frac{2a}{\varepsilon_0}$, then $\Sigma_m^a = \Sigma_m^0$ holds. The reason of this simplification is that the paths in L_K obtained by restricting on the black boundary in Figure 8 are not necessary to define $\Phi_\varepsilon^{<a}$. This is not the case in [9], where the Legendrian contact homology with the coefficients in “chains in the based loop space of Λ_K ” are considered.

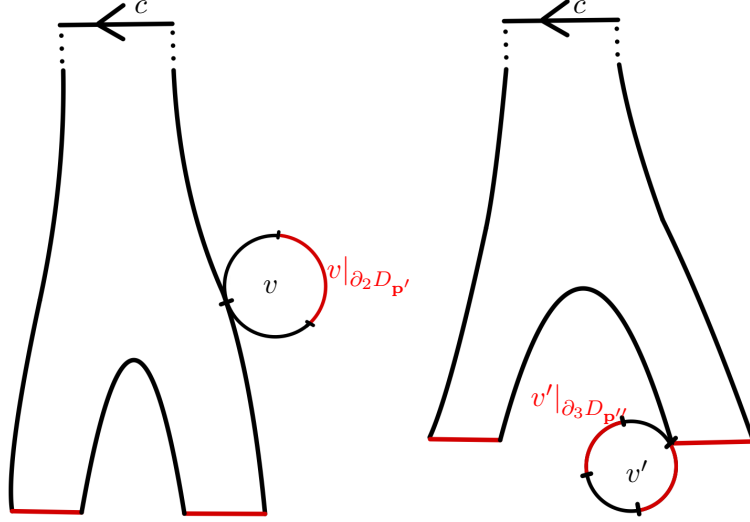


Figure. 11: These holomorphic curves with boundary nodes appear in the boundary of $\overline{\mathcal{M}}(c, 3)$. Here, $v: D_{\mathbf{p}'} \rightarrow T^*\mathbb{R}^n$ and $v': D_{\mathbf{p}''} \rightarrow T^*\mathbb{R}^n$ are pseudo-holomorphic disks, which must be constant by the boundary conditions. Therefore, $v|_{\partial_2 D_{\mathbf{p}'}}$ and $v'|_{\partial_3 D_{\mathbf{p}''}}$ are constant paths in \mathbb{R}^n . In particular, their length is less than ε_0 .

There is a problem about the transversality in order to define the fiber product in the second type of the boundary chains. To deal with this problem, we introduce a *Kuranishi structure* of \mathcal{N} . For simplicity, we assume that there exists a global Kuranishi chart $\mathcal{K} = (V, E, s, \psi)$ and a map $f: V \rightarrow S_{\varepsilon_0}$ which consist of the following data:

- an oriented manifold V .
- a finite rank oriented \mathbb{R} -vector bundle E over V such that $\dim V - \text{rk } E = \dim \mathcal{N} = n - d$.
- a C^∞ section $s: V \rightarrow E$.
- a homeomorphism $\psi: s^{-1}(0) \rightarrow \mathcal{N}$.
- (V, f) is a plot of S_{ε_0} such that $f \circ \psi^{-1}(v) = (\sigma_1^v, \sigma_2^v)$ for every $v \in \mathcal{N}$. In particular, $\text{ev}_0 \circ f: V \rightarrow N_{\varepsilon_0}$ is a submersion.

They are constructed from the Cauchy-Riemann operator $\bar{\partial}_{J'}$ defined as a section of a Banach bundle such that $(\bar{\partial}_{J'})^{-1}(0) = \mathcal{N}$. For details, we refer to [21, Chapter 3]. In the present case, $\bar{\partial}_{J'}$ is transverse to the zero section, so we may assume that s is transverse to the zero section and ψ is a diffeomorphism. By gluing arguments, for any compact subset of $\mathcal{M}_k(c, m-1)_{\text{ev}_k \times \text{ev}_0} \mathcal{N}$ we can construct a collection of Kuranishi charts [20, Definition 3.1] on $\overline{\mathcal{M}}(c, m)$ whose union covers this compact subset. Let us admit that, inductively on T_c and m , a Kuranishi structure of $\overline{\mathcal{M}}(c, m)$ is constructed, and there exists a strongly smooth map $\tilde{\varphi}_{c,m}$ (in a sense similar to [24, Definition 7.1.1]) on this Kuranishi space, whose underlying map is $\varphi_{c,m}$. In terms of this Kuranishi structure, we take a good coordinate system [20, Definition 3.15].

We also need to take a *CF-perturbation* $(\mathfrak{S}^r)_{r>0}$ of \mathcal{K} . For simplicity, we first assume that E is trivial. Identifying E with $\mathbb{R}^q \times V$ and s with a function $s: V \rightarrow \mathbb{R}^q$, we can take explicitly \mathfrak{S}^r which consists of the following data:

- an open neighborhood W of 0 in \mathbb{R}^q .
- a C^∞ section $s_r: W \times V \rightarrow \text{pr}_V^* E \cong \mathbb{R}^q \times (W \times V)$ defined by

$$s_r(w, v) := (s(v) - rw, (w, v))$$

for every $(w, v) \in W \times V$.

- $\eta \in \Omega_c^q(W)$ such that $\int_W \eta = 1$.

They satisfies the conditions of [20, Definition 7.4]. (Here we do not need to concern with the action of a finite group.) Indeed, s_r is transverse to the zero section and the sequence $(s_r)_{r>0}$ converges to $s \circ \text{pr}_V$ on $W \times V$ as $r \rightarrow 0$ in compact C^1 -topology. Moreover, since $\text{pr}_V|_{s_r^{-1}(0)} : s_r^{-1}(0) = \{(w, v) \mid s(v) = rw\} \rightarrow V$ is an open embedding,

$$\text{ev}_0 \circ f \circ \text{pr}_V|_{s_r^{-1}(0)} : s_r^{-1}(0) \rightarrow N_{\varepsilon_0}$$

is a submersion.

[24, Section 8] gave a construction of de Rham chain from a Kuranishi structure, a strongly smooth map and a CF-perturbation. (More precisely, those input data are required to be *admissible* in the sense of [20, Section 25].) We apply this to the present case. For the Kuranishi structure \mathcal{K} , the map $f : V \rightarrow S_{\varepsilon_0}$ and the CF-perturbation $(\mathfrak{S}^r)_{r>0}$, we choose a sufficiently small $r > 0$ which depends on $\varepsilon \in (0, \varepsilon_0]$ to define a de Rham chain

$$\delta := [s_r^{-1}(0), f \circ \text{pr}_V|_{s_r^{-1}(0)}, (\text{pr}_W^* \eta)|_{s_r^{-1}(0)}] \in C_{n-d}^{\text{dR}}(S_{\varepsilon}).$$

If E is not trivial, we need to take an open cover of trivializations. The de Rham chain is defined by using a partition of unity subordinated to this cover.

Let us admit that it is possible to define a de Rham chain

$$\Phi_{\varepsilon, m}^{<a}(c) \in C_{|c|-(m-2)d}^{\text{dR}}(\Sigma_m^{a+m\varepsilon})$$

constructed from the Kuranishi structure on $\overline{\mathcal{M}}(c, m)$, $\tilde{\varphi}_{c, m}$ and a CF-perturbation obtained by extending $(\mathfrak{S}^r)_{r>0}$ to $\overline{\mathcal{M}}(c, m)$ equipped with the good coordinate system. When $m = 0$, $\Phi_{\varepsilon, 0}^{<a}(c) \in C_{|c|}^{\text{dR}}(\Sigma_0^a)$ is defined as

$$\begin{cases} \#\text{sign} \mathcal{M}(c, 0) \in \mathbb{R} = C_0^{\text{dR}}(\Sigma_0^a) & \text{if } |c| = 0, \\ 0 & \text{else.} \end{cases}$$

Let us observe the properties of these de Rham chains. Since the inverse map of $\text{pr}_V|_{s_r^{-1}(0)}$ is given by $\text{pr}_V(s_r^{-1}(0)) \rightarrow s_r^{-1}(0) : v \mapsto (\frac{1}{r}s(v), v)$, we have $\delta = [V, f, (\frac{1}{r}s)^* \eta]$. Hence $(\text{ev}_0)_* \delta \in C_{n-d}^{\text{dR}}(N_{\varepsilon}^{\text{reg}}) \cong \Omega_c^d(N_{\varepsilon})$ is equal to

$$(\text{ev}_0 \circ f)_!((\frac{1}{r}s)^* \eta) \in \Omega_c^d(N_{\varepsilon}).$$

Note that if $r > 0$ is sufficiently small, then $(\frac{1}{r}s)^* \eta \in \Omega_c^q(V)$ has a support in a tubular neighborhood of $s^{-1}(0)$ and represents the Thom class of the normal bundle of $s^{-1}(0)$. We have the following diagram and the correspondence of (co)homology classes:

$$\begin{array}{ccc} H_{c, \text{dR}}^q(V) & \xrightarrow{(\text{ev}_0 \circ f)_!} & H_{c, \text{dR}}^d(N_{\varepsilon}) & & [(\frac{1}{r}s)^* \eta] & & [\eta_{TK^{\perp}}] \\ \cong \downarrow & & \cong \downarrow & & \downarrow & & \downarrow \\ H_{n-d}^{\text{sing}}(V) & \xrightarrow{(\text{ev}_0 \circ f)_*} & H_{n-d}^{\text{sing}}(N_{\varepsilon}) & & [s^{-1}(0)] \longmapsto & & (\text{ev}_0 \circ f)_*[s^{-1}(0)] = [K] \end{array}$$

Here the vertical isomorphisms come from the Poincaré duality and $\eta_{TK^{\perp}} \in \Omega_c^d(N_{\varepsilon})$ represents the Thom class of TK^{\perp} . Since $\text{Th}_{\varepsilon} \in H_{n-d}^{\text{dR}}(N_{\varepsilon}^{\text{reg}}) \cong H_{c, \text{dR}}^d(N_{\varepsilon})$ is the homology class corresponding to $[\eta_{TK^{\perp}}] \in H_{c, \text{dR}}^d(N_{\varepsilon})$, it follows that

$$(\text{ev}_0)_*[\delta] = \text{Th}_{\varepsilon} \in H_{n-d}^{\text{dR}}(N_{\varepsilon}^{\text{reg}}).$$

This shows that if $\varepsilon > 0$ is sufficiently small, then (ε, δ) is an element of \mathcal{T}_a .

In addition, the intuitive observation about the boundary chains of $\varphi_{c,m}$ implies that the following equation should hold in $C_{|c|-(m-2)d-1}^{\text{dR}}(\Sigma_m^{a+m\varepsilon}, \Sigma_m^0)$:

$$\begin{aligned} \partial\Phi_{\varepsilon,m}^{<a}(c) &= \sum_{|c_1|+\dots+|c_l|=|c|-1} \pm \langle \partial_J c, c_1 \dots c_l \rangle \left(\sum_{m_1+\dots+m_l=m} \pm \Phi_{\varepsilon,m_1}^{<a_1}(c_1) \star \dots \star \Phi_{\varepsilon,m_l}^{<a_l}(c_l) \right) \\ &\quad \pm \sum_{k=1}^{m-1} f_{k,\delta}(\Phi_{\varepsilon,m-1}^{<a}(c)), \end{aligned} \tag{49}$$

where $a_1, \dots, a_l \in \mathbb{R}_{>0}$ satisfy $a_\alpha > T_{c_\alpha}$ for every $\alpha \in \{1, \dots, l\}$ and $a_1 + \dots + a_l = a$. Here, $\langle \partial_J c, c_1 \dots c_l \rangle = \#\text{sign} \mathcal{M}_J(c; c_1, \dots, c_l)$ and each “ \pm ” means that the sign is uncertain in the present discussion. On the RHS, the first summand corresponds to the chains of (47) and the second summand corresponds to the chains of (48).

Now we define a linear map $\Phi_\varepsilon^{<a}: \mathcal{A}_*^{<a} \rightarrow C_*^{<a}(\varepsilon)$ so that for every $c \in \mathcal{R}(\Lambda_K)$,

$$\Phi_\varepsilon^{<a}(c) := (\Phi_{\varepsilon,m}^{<a}(c))_{m \in \mathbb{Z}_{>0}} \in C_*^{<a}(\varepsilon),$$

and it extends naturally via the product on $\mathcal{A}_*(\Lambda_K)$ and the \star -operation. The unit $1 \in \mathcal{A}_*^{<a}$ of $\mathcal{A}_*(\Lambda_K)$ is mapped to $1 \in \mathbb{R} = C_0^{\text{dR}}(\Sigma_0^a, \Sigma_0^0) \subset C_0^{<a}(\varepsilon)$. The equation (49) means that $\Phi_\varepsilon^{<a}$ is a chain map from $(\mathcal{A}_*^{<a}, \partial_J)$ to $(C_*^{<a}(\varepsilon), D_\delta)$, if the signs are determined as we expect. We finish the discussion about Claim 1.

If Claim 1 is proved, we obtain a linear map on homology

$$(\Phi_\varepsilon^{<a})_*: \text{LCH}_*^{<a} \rightarrow H_*^{<a}(\varepsilon, \delta).$$

If $b \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ satisfy $b > a$, $\Phi_\varepsilon^{<b}$ induces a chain map from $(\mathcal{A}_*^{[a,b]}, \partial_{\Lambda_K})$ to $(C_*^{[a,b]}(\varepsilon), D_\delta)$. Let us denote it by $\Phi_\varepsilon^{[a,b]}$. Then we obtain a linear map

$$(\Phi_\varepsilon^{[a,b]})_*: \text{LCH}_*^{[a,b]} \rightarrow H_*^{[a,b]}(\varepsilon, \delta).$$

7.3.2 Compatibility with the transition map

Claim 2. *If $(\varepsilon', \delta'), (\varepsilon, \delta) \in \mathcal{T}_a$ are the chains defined in Claim 1 for ε and ε' with $0 < \varepsilon' \leq \varepsilon$, then*

$$k_{(\varepsilon', \delta'), (\varepsilon, \delta)} \circ (\Phi_{\varepsilon'}^{<a})_* = (\Phi_\varepsilon^{<a})_*: \text{LCH}_*^{<a} \rightarrow H_*^{<a}(\varepsilon, \delta).$$

To prove this claim, we need to determine $\bar{\delta} \in \bar{C}_{n-d}^{\text{dR}}(S_\varepsilon)$ such that $e_+ \bar{\delta} = \delta$ and $e_- \bar{\delta} = (i_{\varepsilon', \varepsilon})_* \delta'$. This $[-1, 1]$ -modeled chain can be constructed by interpolating \mathfrak{S}^r and $\mathfrak{S}^{r'}$ which are used to define δ and δ' respectively. The point will be to construct a chain map $\bar{\Phi}_\varepsilon^{<a}$ from $(\mathcal{A}_*^{<a}, \partial_J)$ to $(\bar{C}_*^{<a}(\varepsilon), \bar{D}_{\bar{\delta}})$ by a similar process as Claim 1 such that the following diagram commutes:

$$\begin{array}{ccc} & & H_*^{<a}(\varepsilon, \delta) \\ & \nearrow^{(\Phi_\varepsilon^{<a})_*} & \uparrow \\ \text{LCH}_*^{<a} & \xrightarrow{(\bar{\Phi}_\varepsilon^{<a})_*} & \bar{H}_*^{<a}(\varepsilon, \bar{\delta}) \\ & \searrow_{(\Phi_{\varepsilon'}^{<a})_*} & \downarrow \\ & & H_*^{<a}(\varepsilon', \delta') \end{array} \quad \begin{array}{l} f_{(\varepsilon, \bar{\delta}), +} \\ f_{(\varepsilon, \bar{\delta}), -} \\ k_{(\varepsilon', \delta'), (\varepsilon, \delta)} \end{array}$$

We finish the discussion about Claim 2.

If Claim 2 is proved, we obtain a linear map on the limit of $\varepsilon \rightarrow 0$

$$\Phi_*^{<a} := \varprojlim_{\varepsilon \rightarrow 0} (\Phi_\varepsilon^{<a})_*: \text{LCH}_*^{<a} \rightarrow H_*^{<a}(\mathbb{R}^n, K) = \varprojlim_{\varepsilon \rightarrow 0} H_*^{<a}(\varepsilon, \delta).$$

Similarly, $\Phi_*^{[a,b]}: \text{LCH}_*^{[a,b]} \rightarrow H_*^{[a,b]}(\mathbb{R}^n, K)$ is defined.

From the construction of the chain maps in Claim 1, the maps of $\{\Phi_*^{<a}\}_{a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)}$ are naturally compatible with $I^{a,b}: H_*^{<a}(\mathbb{R}^n, K) \rightarrow H_*^{<b}(\mathbb{R}^n, K)$ and the linear map $i_*^{a,b}: \text{LCH}_*^{<a} \rightarrow \text{LCH}_*^{<b}$ induced by the inclusion maps $i^{a,b}: \mathcal{A}_*^{<a} \rightarrow \mathcal{A}_*^{<b}$ for all $a \leq b$. Then, on the limit of $a \rightarrow \infty$, we get a graded \mathbb{R} -linear map

$$\Phi_*: \text{LCH}_*(UT^*\mathbb{R}^n, \Lambda_K) \rightarrow H_*^{\text{string}}(\mathbb{R}^n, K).$$

Moreover, it would be straightforward to check that it is a homomorphism of graded \mathbb{R} -algebras.

7.3.3 Discussion using the length filtrations

Claim 3. *For every $a_0 \in \mathbb{R}_{>0}$, there exist $a, b \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ such that $a < a_0 < b$ and $\Phi_*^{[a,b]}: \text{LCH}_*^{[a,b]} \rightarrow H_*^{[a,b]}(\mathbb{R}^n, K)$ is an isomorphism.*

If $a_0 \notin \mathcal{L}(K)$, we take $a, b \in \mathbb{R}_{>0}$ so that $[a, b] \cap \mathcal{L}(K) = \emptyset$. Then $\text{LCH}_*^{[a,b]} = 0$ and $H_*^{[a,b]}(\mathbb{R}^n, K) = 0$ from Proposition 4.5, so the assertion is trivial. In the case where $a_0 \in \mathcal{L}(K)$, from the assumption on K that $\mathcal{L}_m(K) \setminus \mathcal{L}_{m'}(K) = \emptyset$ if $m \neq m'$, we can uniquely determine $l \in \mathbb{Z}_{\geq 1}$ such that $a_0 \in \mathcal{L}_l(K)$. We take $a, b \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$ so that $[a, b] \cap \mathcal{L}(K) = \{a_0\}$.

By Proposition 4.5, $H_*^{[a,b]}(\varepsilon, \delta) \cong H_{*-l(d-2)}^{\text{dR}}(\Sigma_l^b, \Sigma_l^a)$ for any $(\varepsilon, \delta) \in \mathcal{T}_a \cap \mathcal{T}_b$. On the other hand, $\partial_J: \mathcal{A}_*^{[a,b]} \rightarrow \mathcal{A}_{*-1}^{[a,b]}$ is the zero-map, so $\text{LCH}_*^{[a,b]} = \mathcal{A}_*^{[a,b]}$ and it is the vector space whose basis is the set of words

$$\{c_1 \cdots c_l \mid T_{c_1} + \cdots + T_{c_l} = a_0\}.$$

The linear map $(\Phi_\varepsilon^{<a})_*$ is described as

$$\mathcal{A}_*^{[a,b]} \rightarrow H_{*-l(d-2)}^{\text{dR}}(\Sigma_l^b, \Sigma_l^a): c_1 \cdots c_l \mapsto [\Phi_\varepsilon^{<a_1}(c_1) \star \cdots \star \Phi_\varepsilon^{<a_l}(c_l)],$$

where $a_k > T_{c_k}$ for $k = 1, \dots, l$ and $a_1 + \cdots + a_l = b$.

Let us introduce a C^∞ manifold

$$B_l^{a'} := \{(q_1^0, q_1^1, \dots, q_l^0, q_l^1) \in K^{2l} \mid \sum_{k=1}^l |q_k^1 - q_k^0| < a \text{ or } \min_{1 \leq k \leq l} |q_k^1 - q_k^0| < \varepsilon_0\}.$$

for every $a' \in \mathbb{R}_{>0}$. Then, we have an isomorphism $e_*: H_*^{\text{dR}}(\Sigma_l^b, \Sigma_l^a) \rightarrow H_*^{\text{dR}}(B_l^b, B_l^a)$ induced by

$$e: \Sigma_l^b \rightarrow B_l^b: (\gamma_k: [0, T_k] \rightarrow \mathbb{R}^n)_{k=1, \dots, l} \mapsto (\gamma_1(0), \gamma_1(T_1), \dots, \gamma_l(0), \gamma_l(T_l)).$$

In addition, by Proposition 2.9, there exists a canonical isomorphism

$$\Psi: H_*^{\text{sing}}(B_l^b, B_l^a) \rightarrow H_*^{\text{dR}}(B_l^b, B_l^a).$$

Let us denote $\bar{c} := (\pi_{\mathbb{R}^n} \circ c(0), \pi_{\mathbb{R}^n} \circ c(T_c)) \in K \times K$ for every $c \in \mathcal{R}(\Lambda_K)$.

We consider the C^∞ function

$$E: B_l^b \rightarrow \mathbb{R}: (q_1^0, q_1^1, \dots, q_l^0, q_l^1) \mapsto \sum_{k=1}^l |q_k^0 - q_k^1|^2.$$

The set of its critical points in $B_l^b \setminus B_l^a$ agrees with

$$\{(\bar{c}_1, \dots, \bar{c}_l) \mid c_1, \dots, c_l \in \mathcal{R}(\Lambda_K), T_{c_1} + \cdots + T_{c_l} = a_0\}.$$

and every critical point is non-degenerate. Let $D^u(c_1, \dots, c_l)$ be the embedded closed disk in B_l^b which is contained in the unstable submanifold of $(\bar{c}_1, \dots, \bar{c}_l)$ with respect to the negative

gradient flow of E and whose boundary is contained in B_l^a . Then, a basis of $H_*^{\text{sing}}(B_l^b, B_l^a)$ is given by the set of homology classes

$$\{[D^u(c_1, \dots, c_l)] \mid c_1, \dots, c_l \in \mathcal{R}(\Lambda_K), T_{c_1} + \dots + T_{c_l} = a_0\}.$$

It follows that

$$\{e_*^{-1} \circ \Psi([D^u(c_1, \dots, c_l)]) \mid c_1, \dots, c_l \in \mathcal{R}(\Lambda_K), T_{c_1} + \dots + T_{c_l} = a_0\}$$

is a basis of $H_{* - l(d-2)}^{\text{dR}}(\Sigma_l^b, \Sigma_l^a)$.

Therefore, Claim 3 is reduced to the following assertion: For every $c_1, \dots, c_l \in \mathcal{R}(\Lambda_K)$ with $T_{c_1} + \dots + T_{c_l} = a_0$,

$$e_* \circ (\Phi_\varepsilon^{<a})_*(c_1 \cdots c_l) = \pm \Psi([D(\bar{c}_1, \dots, \bar{c}_l)]) \quad (50)$$

holds in $H_{* - l(d-2)}^{\text{dR}}(B_l^b, B_l^a)$ if $\varepsilon > 0$ is sufficiently small.

For now, we explain intuitively why this equation should hold. Let us return to the intuitive observation about $\varphi_{c,m}$ in Subsection 7.3.1 and think of $\varphi_{c,m}$ as a chain in Σ_m^a . Then, the homology class in the left-hand side of (50) is represented by the sum of chains

$$e \circ \Pi \circ (\varphi_{c_1, m_1} \times \dots \times \varphi_{c_l, m_l}): \bar{\mathcal{M}}(c_1, m_1) \times \dots \times \bar{\mathcal{M}}(c_l, m_l) \rightarrow B_l^b \quad (51)$$

for all $m_1, \dots, m_l \in \mathbb{Z}_{\geq 0}$ with $m_1 + \dots + m_l = l$. For any $c'_1, \dots, c'_l \in \mathcal{R}(\Lambda_K)$ with $T_{c'_1} + \dots + T_{c'_l} = a_0$, let $W^s(c'_1, \dots, c'_l)$ be the stable manifold of the critical point $(\bar{c}'_1, \dots, \bar{c}'_l) \in B_l^b \setminus B_l^a$ with respect to the negative gradient flow of E . By Morse theory in terms of E , (50) is reduced to the following two assertions:

1. If $c_\alpha \neq c'_\alpha$ or $m_\alpha \geq 2$ for some $\alpha \in \{1, \dots, l\}$, then (51) does not intersect $W^s(c'_1, \dots, c'_l)$.
2. If $c_\alpha = c'_\alpha$ and $m_\alpha = 1$ for every $\alpha \in \{1, \dots, l\}$, then (51) intersects $W^s(c_1, \dots, c_l)$ transversely at one point.

To check them, we use the assertion (B) in Subsection 7.3.1 about the length estimate and the fact that the function

$$L_B: B_l^b \rightarrow \mathbb{R}: (q_1^0, q_1^1, \dots, q_l^0, q_l^1) \mapsto \sum_{k=1}^l |q_k^0 - q_k^1|$$

is decreasing along the negative gradient flow of E . The assertion (B) implies that the composite map

$$L_B \circ (51): \bar{\mathcal{M}}(c_1, m_1) \times \dots \times \bar{\mathcal{M}}(c_l, m_l) \rightarrow \mathbb{R}$$

is bounded from above by $T_{c_1} + \dots + T_{c_l} = a_0$. Moreover, its maximum is equal to a_0 if and only if $m_1 = \dots = m_l = 1$, in which case $L_B \circ (51)(u_1, \dots, u_l) = a_0$ if and only if u_α is the trivial half strip over $\pi_{\mathbb{R}^n} \circ c_\alpha$. On the other hand, The minimum of L_B on $W^s(c'_1, \dots, c'_l)$ is equal to $T_{c'_1} + \dots + T_{c'_l} = a_0$ and $(c'_1, \dots, c'_l) \in W^s(c'_1, \dots, c'_l)$ is the unique point which take the minimum. Therefore, (51) intersects $W^s(c'_1, \dots, c'_l)$ only if $m_1 = \dots = m_l$. Since the sequence of trivial half strips (u_1, \dots, u_l) is mapped by (51) to $(\bar{c}_1, \dots, \bar{c}_l) \in B_l^b$, (51) intersects $W^s(c'_1, \dots, c'_l)$ if and only if $c_\alpha = c'_\alpha$ and $m_\alpha = 1$ for every $\alpha \in \{1, \dots, l\}$, and moreover the intersection consists only of $\bar{c}_1, \dots, \bar{c}_l$. It remains to check the transversality, but here we omit the discussion. (A similar argument can be found in [4, Proposition 2].) We finish the discussion about Claim 3.

7.3.4 Conclusion assuming three claims

Assuming the above three claims, we shall prove Conjecture 7.3. We want to show that $\Phi_*^{<a}: \text{LCH}_*^{<a} \rightarrow H_*^{<a}(\mathbb{R}^n, K)$ is an isomorphism for every $a \in \mathbb{R}_{>0} \setminus \mathcal{L}(K)$. From Claim 3, we can choose a sequence $(r_l)_{l=0, \dots, k}$ in $\mathbb{R}_{>0} \setminus \mathcal{L}(K)$ such that $\varepsilon_0 = r_0 < r_1 < \dots < r_{k-1} < r_k = a$ and $\Phi_*^{[r_{l-1}, r_l]}: \text{LCH}_*^{[r_{l-1}, r_l]} \rightarrow H_*^{[r_{l-1}, r_l]}(\mathbb{R}^n, K)$ is an isomorphism for every $l = 1, \dots, k$. Since $[0, \varepsilon_0] \cap \mathcal{L}(K) = \emptyset$, it is obvious that $\Phi_*^{<r_0}: \text{LCH}_*^{<r_0} = \mathbb{R} \cdot 1 \rightarrow H_*^{<r_0}(\mathbb{R}^n, K) = \mathbb{R}$ is an isomorphism. We have the following commutative diagram whose horizontal sequences are exact:

$$\begin{array}{ccccccccc}
 \text{LCH}_{*+1}^{[r_{l-1}, r_l]} & \longrightarrow & \text{LCH}_*^{<r_{l-1}} & \longrightarrow & \text{LCH}_*^{<r_l} & \longrightarrow & \text{LCH}_*^{[r_{l-1}, r_l]} & \longrightarrow & \text{LCH}_{*-1}^{<r_{l-1}} \\
 \downarrow \Phi_*^{[r_{l-1}, r_l]} & & \downarrow \Phi_*^{<r_{l-1}} & & \downarrow \Phi_*^{<r_l} & & \downarrow \Phi_*^{[r_{l-1}, r_l]} & & \downarrow \Phi_*^{<r_{l-1}} \\
 H_{*+1}^{[r_{l-1}, r_l]} & \longrightarrow & H_*^{<r_{l-1}} & \longrightarrow & H_*^{<r_l} & \longrightarrow & H_*^{[r_{l-1}, r_l]} & \longrightarrow & H_{*-1}^{<r_{l-1}}.
 \end{array}$$

Here, we omit writing (\mathbb{R}^n, K) in the lower sequence. Inductively on $l = 1, \dots, k$, we can show that the middle vertical map is an isomorphism. Therefore, $\Phi_*^{<a} = \Phi_*^{<r_k}$ is an isomorphism. In the limit of $a \rightarrow \infty$, it follows that $\Phi_*: \text{LCH}_*(UT^*\mathbb{R}^n, \Lambda_K) \rightarrow H_*^{\text{string}}(\mathbb{R}^n, K)$ is an isomorphism. In summary, the conjecture can be proved by assuming the three claims.

References

- [1] A. Abbondandolo, A. Portaluri, and M. Schwarz. The homology of path spaces and Floer homology with conormal boundary conditions. *J. Fixed Point Theory Appl.*, 4(2):263–293, 2008.
- [2] A. Abbondandolo and M. Schwarz. On the Floer homology of cotangent bundles. *Comm. Pure Appl. Math.*, 59(2):254–316, 2006.
- [3] M. Abouzaid. Symplectic cohomology and Viterbo’s theorem. In *Free loop spaces in geometry and topology*, volume 24 of *IRMA Lect. Math. Theor. Phys.*, pages 271–485. Eur. Math. Soc., Zürich, 2015.
- [4] J. Asplund. Fiber Floer cohomology and conormal stops. *J. Symplectic Geom.*, 19(4):777–864, 2021.
- [5] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. Compactness results in symplectic field theory. *Geom. Topol.*, 7:799–888, 2003.
- [6] Y. Chekanov. Differential algebra of Legendrian links. *Invent. Math.*, 150(3):441–483, 2002.
- [7] K. T. Chen. On differentiable spaces. In *Categories in continuum physics (Buffalo, N.Y., 1982)*, volume 1174 of *Lecture Notes in Math.*, pages 38–42. Springer, Berlin, 1986.
- [8] K. Cieliebak, T. Ekholm, and J. Latschev. Compactness for holomorphic curves with switching Lagrangian boundary conditions. *J. Symplectic Geom.*, 8(3):267–298, 2010.
- [9] K. Cieliebak, T. Ekholm, J. Latschev, and L. Ng. Knot contact homology, string topology, and the cord algebra. *J. Éc. polytech. Math.*, 4:661–780, 2017.
- [10] G. Dimitroglou Rizell. Lifting pseudo-holomorphic polygons to the symplectisation of $P \times \mathbb{R}$ and applications. *Quantum Topol.*, 7(1):29–105, 2016.
- [11] T. Ekholm, J. Etnyre, and M. Sullivan. The contact homology of Legendrian submanifolds in \mathbb{R}^{2n+1} . *J. Differential Geom.*, 71(2):177–305, 2005.

- [12] T. Ekholm, J. Etnyre, and M. Sullivan. The contact homology of Legendrian submanifolds in \mathbb{R}^{2n+1} . *J. Differential Geom.*, 71(2):177–305, 2005.
- [13] T. Ekholm, J. Etnyre, and M. Sullivan. Orientations in Legendrian contact homology and exact Lagrangian immersions. *Internat. J. Math.*, 16(5):453–532, 2005.
- [14] T. Ekholm, J. Etnyre, and M. Sullivan. Legendrian contact homology in $P \times \mathbb{R}$. *Trans. Amer. Math. Soc.*, 359(7):3301–3335, 2007.
- [15] T. Ekholm, J. B. Etnyre, L. Ng, and M. G. Sullivan. Knot contact homology. *Geom. Topol.*, 17(2):975–1112, 2013.
- [16] T. Ekholm, L. Ng, and V. Shende. A complete knot invariant from contact homology. *Invent. Math.*, 211(3):1149–1200, 2018.
- [17] Y. Eliashberg. Invariants in contact topology. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 327–338, 1998.
- [18] J. B. Etnyre, L. L. Ng, and J. M. Sabloff. Invariants of Legendrian knots and coherent orientations. *J. Symplectic Geom.*, 1(2):321–367, 2002.
- [19] K. Fukaya. Application of Floer homology of Lagrangian submanifolds to symplectic topology. In *Morse theoretic methods in nonlinear analysis and in symplectic topology*, volume 217 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 231–276. Springer, Dordrecht, 2006.
- [20] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Kuranishi structures and virtual fundamental chains*. Springer Monographs in Mathematics. Springer, Singapore, [2020] ©2020.
- [21] K. Fukaya and K. Ono. Arnold conjecture and Gromov-Witten invariant. *Topology*, 38(5):933–1048, 1999.
- [22] H. Geiges. *An introduction to contact topology*, volume 109 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2008.
- [23] K. Irie. A chain level Batalin-Vilkovisky structure in string topology via de Rham chains. *Int. Math. Res. Not. IMRN*, (15):4602–4674, 2018.
- [24] K. Irie. Chain level loop bracket and pseudo-holomorphic disks. *J. Topol.*, 13(2):870–938, 2020.
- [25] J. Milnor. *Morse theory*. Annals of Mathematics Studies, No. 51. Princeton University Press, Princeton, N.J., 1963. Based on lecture notes by M. Spivak and R. Wells.
- [26] L. Ng. Knot and braid invariants from contact homology. II. *Geom. Topol.*, 9:1603–1637, 2005. With an appendix by the author and Siddhartha Gadgil.
- [27] L. Ng. Framed knot contact homology. *Duke Math. J.*, 141(2):365–406, 2008.
- [28] Y. Okamoto. On knot types of clean Lagrangian intersections in $T^*\mathbb{R}^3$. *arXiv preprint arXiv:2305.09912*, 2023.
- [29] V. Shende. The conormal torus is a complete knot invariant. *Forum Math. Pi*, 7:e6, 16, 2019.
- [30] C. Viterbo. Generating functions, symplectic geometry, and applications. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 537–547. Birkhäuser, Basel, 1995.

- [31] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.