

Structure of unbounded viscosity solutions to semilinear degenerate elliptic equations

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1. Introduction.

We consider the Dirichlet problem for a semilinear degenerate elliptic equation (DP):

$$(1) \quad -g(|x|)\Delta u + f(|x|, u) = 0 \quad \text{in } \mathbf{R}^N$$

$$(2) \quad \lim_{|x| \rightarrow \infty} \frac{u(x)}{h(|x|)} = 1,$$

where $N \geq 2$,

$$g(|x|) = |x|^{-a_1} |x|^{-\lambda_1} |x|^{-a_2} |x|^{-\lambda_2} \cdots |x|^{-a_k} |x|^{-\lambda_k} (|x| + 1)^{-\lambda^*},$$

$$0 < a_1 < a_2 < \cdots < a_k, \quad 0 < \lambda_i \quad (i = 1, 2, \dots, k) \quad \text{and} \quad \lambda^* \geq 0,$$

Δ is the Laplacian, and $h(|x|) \in C(|x| > a_k)$ will be determined later.

We discuss the problem (DP) under the following assumptions:

(A.1) $f(t, y) \in C([0, \infty) \times \mathbf{R})$ is locally Lipschitz continuous in (t, y) .

(A.2) For any $t > 0$ fixed, $f(t, y)$ is strictly increasing in y .

(A.3) For any $t \in [0, \infty)$, there exists a continuous function $\varphi(t)$ such that $f(t, \varphi(t)) = 0$.

EXAMPLE

$$g(|x|)\Delta u = u|u|^{p-1} - f(|x|).$$

In this paper, we study (DP) in case $N = 2$.

Our aim is to prove the following

A) For $\lambda_i > 0$ ($i = 1, 2, \dots, k$) and $\lambda^* \geq 0$, there exists a unique standard and radial

viscosity solution of (DP).

B) If $\lambda_i \geq 1$ for all i ($i = 1, 2, \dots, k$), there exists a unique viscosity solution of (DP).

From A), every viscosity solution is radial and standard.

C) If $0 < \lambda_i < 1$ for some i ($i = 1, 2, \dots, k$), there exist infinitely many viscosity solutions of (DP).

2. Structure of standard viscosity solutions.

Following Crandall and Huan [2], we call a viscosity solution u of (DP) a standard solution if $u(x) = \varphi(a_i)$ (i.e., $f(a_i, u(x)) = 0$) on $|x| = a_i$ ($i = 1, 2, \dots, k$).

In order to construct a standard viscosity solution we shall consider the following Dirichlet problems :

$$\begin{aligned}
 (\mathbf{P}_0) \quad & \begin{cases} -g(|x|)\Delta u + f(|x|, u) = 0 & \text{in } B_{a_1}, \\ u(x) = b_1 & \text{on } |x| = a_1; \end{cases} \\
 (\mathbf{P}_i) \quad & \begin{cases} -g(|x|)\Delta u + f(|x|, u) = 0 & \text{in } A(a_i, a_{i+1}), \\ u(x) = b_i & \text{on } |x| = a_i, \quad u(x) = b_{i+1} & \text{on } |x| = a_{i+1}; \end{cases} \\
 (\mathbf{P}_k) \quad & \begin{cases} -g(|x|)\Delta u + f(|x|, u) = 0 & \text{in } A(a_k, \infty), \\ u(x) = b_k & \text{on } |x| = a_k, \quad \lim_{|x| \rightarrow \infty} \frac{u(x)}{h(|x|)} = 1, \end{cases}
 \end{aligned}$$

where $A(a_i, a_{i+1}) = \{x \in \mathbf{R}^N : a_i < |x| < a_{i+1}\}$, $i = 1, 2, \dots, k-1$ and $b_i = \varphi(a_i)$ ($i = 1, 2, \dots, k$).

Let $u_0 \in C(\overline{B_{a_1}}) \cap C^2(B_{a_1})$ (resp. $u_i \in C(\overline{A(a_i, a_{i+1})}) \cap C^2(A(a_i, a_{i+1}))$) be a radial classical solution of $(\mathbf{P}_0)$ (resp. $(\mathbf{P}_i)$ ($i = 1, 2, \dots, k$)).

Put

$$(3) \quad \tilde{u}(x) = \begin{cases} u_0(x) & \text{for } x \in \overline{B_{a_1}}, \\ u_1(x) & \text{for } x \in \overline{A(a_1, a_2)} \\ \vdots \\ u_k(x) & \text{for } x \in \overline{A(a_k, \infty)}. \end{cases}$$

It is easy to verify, by the definition of viscosity solutions, that $\tilde{u}$ is a radial and standard

viscosity solution of (DP). An easy calculation shows that $u(x) = y(t)$ ($|x| = t$) is a radial classical solution of (P_1) if and only if $y(t)$ is a classical solution of the following boundary value problem (denoted by (BVP_1)):

$$(4) \quad g(t)\left(\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt}\right) = f(t, y) \quad \text{in } a_i < t < a_{i+1}$$

$$y(a_i) = b_i \quad \text{and} \quad y(a_{i+1}) = b_{i+1} \quad (i = 0, 1, \dots, k),$$

where $y(a_0) = b_0$ and $y(a_{k+1}) = b_{k+1}$ are replaced by $\frac{dy}{dt}(0) = 0$ and $\lim_{t \rightarrow \infty} \frac{y(t)}{h(t)} = 1$, respectively.

From now on we briefly explain that the existence and uniqueness of classical solutions of (BVP_1) ($i = 1, 2, \dots, k$) play an essential role to prove our assertion stated in Introduction. Assume $\lambda_i \geq 1$ for all $i = 1, 2, \dots, k$. Let $u(x)$ be an arbitrary viscosity solution of (DP). Define

$$\overline{U}(x) = \sup_{|y|=|x|} u(y) \quad \text{and} \quad \underline{U}(x) = \inf_{|y|=|x|} u(y).$$

We observe that $\overline{U}(x)$ (resp. $\underline{U}(x)$) is continuous and radial viscosity subsolution (resp. supersolution) and $\overline{U}(x) = \underline{U}(x) = b_i$ on $|x| = a_i$ (by $\lambda_i \geq 1$). By the well-known comparison theorem, we have

$$y_i(|x|) \leq \underline{U}(x) \leq \overline{U}(x) \leq y_i(|x|)$$

for $a_i \leq |x| \leq a_{i+1}$ ($i = 0, 1, 2, \dots, k$), where y_i is the unique solution of (BVP_1) .

3. Existence and uniqueness for (BVP_1) .

In order to study (BVP_1) , we introduce the following integral equations:

$$(5) \quad y(t) = \alpha + \int_0^t (\log t/s) s g(s)^{-1} f(s, y(s)) ds,$$

$$(6) \quad y(t) = \alpha + t_0 \beta \log(t/t_0) + \int_{t_0}^t \log(t/s) s g(s)^{-1} f(s, y(s)) ds,$$

where $0 < t_0 \notin \{a_1, a_2, \dots, a_k\}$, α and β are real parameters. Applying a fixed point theorem, we can prove the local existence of solutions of (5) and (6).

First, to solve (BVP_0) , we define

$$S_0^+ = \{\alpha \in \mathbf{R}; \lim_{t \rightarrow T_\alpha} y_\alpha(t) = +\infty\}$$

$$S_0 = \{\alpha \in \mathbf{R}; \lim_{t \rightarrow a_1} y_\alpha(t) = \text{exists}\}$$

$$S_0^- = \{\alpha \in \mathbf{R}; \lim_{t \rightarrow T_\alpha} y_\alpha(t) = -\infty\},$$

where y_α is a classical solution of (5) obtained by prolonging local solutions of (5) and (6).

We see that (i) in case $0 < \lambda_1 < 2$,

$$S_0^+ = [\bar{\alpha}, \infty), \quad S_0 = (\underline{\alpha}, \bar{\alpha}), \quad S_0^- = (-\infty, \underline{\alpha}]$$

$$\text{and } \{y_\alpha(a_1) = \lim_{t \rightarrow a_1} y_\alpha(t); \alpha \in S_0\} = \mathbf{R};$$

and (ii) in case $\lambda_1 \geq 2$,

$$S_0^+ = (\alpha_0, \infty), \quad S_0 = \{\alpha_0\}, \quad S_0^- = (-\infty, \alpha_0) \quad \text{and} \quad y_{\alpha_0}(a_1) = b_1.$$

Consequently we have

PROPOSITION 1. *There exists a unique classical solution y_0 of (BVP_0) .*

Next, to solve (BVP_i) ($i = 1, 2, \dots, k-1$), we fix $t_0 \in (a_i, a_{i+1})$ and define for each $\alpha \in \mathbf{R}$

$$B_i^+ = \{\beta \in \mathbf{R}; \lim_{t \downarrow T_{\alpha\beta}} y_{\alpha\beta}(t) = +\infty\}$$

$$B_i = \{\beta \in \mathbf{R}; \lim_{t \downarrow a_i} y_{\alpha\beta}(t) = \text{exists}\}$$

$$B_i^- = \{\beta \in \mathbf{R}; \lim_{t \downarrow T_{\alpha\beta}} y_{\alpha\beta}(t) = -\infty\},$$

where $y_{\alpha\beta}(t)$ is a solution of (6) on $(T_{\alpha\beta}, t_0]$ ($a_i \leq T_{\alpha\beta} < t_0$).

We can prove that (i) in case $0 < \lambda_i < 2$,

$$B_i^- = [\bar{\beta}, \infty), \quad B_i = (\underline{\beta}, \bar{\beta}), \quad B_i^+ = (-\infty, \underline{\beta}]$$

$$\text{and } \{y_{\alpha\beta}(a_i) = \lim_{t \downarrow a_i} y_{\alpha\beta}(t); \beta \in B_i\} = \mathbf{R};$$

and (ii) in case $\lambda_i \geq 2$,

$$B_i^- = (\beta_i, \infty), \quad B_i = \{\beta_i\}, \quad B_0^+ = (-\infty, \beta_i) \quad \text{for some } \beta_i = \beta(\alpha) \quad \text{and} \quad y_{\alpha\beta(\alpha)}(a_i) = b_i.$$

And then we solve

$$(7) \quad \begin{cases} g(t)(\frac{d^2 y}{dt^2} + \frac{1}{t} \frac{dy}{dt}) = f(t, y) & \text{in } [t_0, a_{i+1}) \\ y(t_0) = \alpha, \quad \frac{dy}{dt}(t_0) = \beta(\alpha). \end{cases}$$

Define

$$A_i^+ = \{\alpha \in \mathbf{R}; \lim_{t \uparrow T_\alpha} y_\alpha(t) = +\infty\}$$

$$A_i = \{\alpha \in \mathbf{R}; \lim_{t \uparrow a_{i+1}} y_\alpha(t) = \text{exists}\}$$

$$A_i^- = \{\alpha \in \mathbf{R}; \lim_{t \uparrow T_\alpha} y_\alpha(t) = -\infty\},$$

where $y_\alpha(t) := y_{\alpha\beta(\alpha)}(t)$ is a solution of (7) on $[t_0, T_\alpha)$ ($t_0 < T_\alpha \leq a_{i+1}$).

We observe that (i) in case $0 < \lambda_{i+1} < 2$,

$$A_i^+ = [\bar{\alpha}, \infty), \quad A_i = (\underline{\alpha}, \bar{\alpha}), \quad A_i^- = (-\infty, \underline{\alpha}]$$

$$\text{and } \{y_\alpha(a_{i+1}) = \lim_{t \uparrow a_{i+1}} y_\alpha(t); \alpha \in A_i\} = \mathbf{R};$$

and (ii) in case $\lambda_{i+1} \geq 2$,

$$A_i^+ = (\alpha_i, \infty), \quad A_i = \{\alpha_i\}, \quad A_i^- = (-\infty, \alpha_i) \quad \text{for some } \alpha_i, \quad \text{and} \quad y_{\alpha_i}(a_{i+1}) = b_{i+1}.$$

Therefore we have

PROPOSITION 2. *There exists a unique classical solution $y_i(t)$ of (BVP_i) ($i = 1, 2, \dots, k-1$).*

(Note that the uniqueness in Propositions 1 and 2 follows immediately from the maximum principle.)

Finally we shall prove the existence and uniqueness of solutions of (BVP_k). It should be noted that we have to introduce several boundary conditions at ∞ corresponding to the structure of (1). To state our result, we introduce some notation:

$$\ell := \lambda_1 + \lambda_2 + \dots + \lambda_k - \lambda^* \quad \text{and} \quad \gamma := (\ell - 2)/(p - 1),$$

where $p > 1$ is assumed. For (BVP_k) , we make the following assumptions:

$$(A.4) \quad \lim_{|x| \rightarrow \infty} \varphi(|x|) = \infty \text{ and } \lim_{t \rightarrow \infty} \frac{t^\ell(\ddot{\varphi}(t) + (1/t)\dot{\varphi}(t))}{\varphi(t)^p} = 0.$$

(A.5) There exist positive constants k_0 and K_0 such that

$$k_0(y_1 - y_2)(|y_1|^{p-1} + |y_2|^{p-1}) \leq f(t, y_1) - f(t, y_2)$$

$$\leq K_0(y_1 - y_2)(|y_1|^{p-1} + |y_2|^{p-1})$$

for every $y_1 > y_2$ and $t \gg 1$.

(A.6) $f(|x|, y)$ has the following form:

$$f(|x|, y) = y|y|^{p-1} - \varphi(|x|)|\varphi(|x|)|^{p-1}.$$

REMARK (i) It is easy to verify that $(A.6) \Rightarrow (A.5) \Rightarrow \{(A.1), (A.2)\}$.

(ii) If $\lim_{t \rightarrow \infty} \frac{t^\ell(\ddot{\varphi}(t) + (1/t)\dot{\varphi}(t))}{\varphi(t)^p} = \delta > 0$ and $\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t^\gamma} = \infty$, then $\varphi(t)$ blows up in a finite interval.

PROPOSITION 3. Let $\ell \leq 2$. Assume (A.4) and (A.5). Then there exists a unique solution of (BVP_k) with $h(t) \approx \varphi(t)$. Moreover, if $h(t) \not\approx \varphi(t)$ then (BVP_k) does not possess any solution, where $h(t) \approx \varphi(t)$ means that $\lim_{t \rightarrow \infty} \frac{h(t)}{\varphi(t)} = 1$.

Sketch of proof of Proposition 3. Let $y_\alpha(t)$ be a classical solution of (4) in $[a_k, T_\alpha)$ satisfying $y_\alpha(a_k) = b_k$. Then, it is important to note that $\lim_{t \rightarrow T_\alpha} y_\alpha(t) = +\infty$ or $\lim_{t \rightarrow T_\alpha} y_\alpha(t) = -\infty$. (In other words, equation (4) does not possess any bounded solution.) Therefore, as before, we define

$$A^+ = \{\alpha \in \mathbf{R}; \lim_{t \rightarrow T_\alpha} y_\alpha(t) = +\infty\}$$

$$A^- = \{\alpha \in \mathbf{R}; \lim_{t \rightarrow T_\alpha} y_\alpha(t) = -\infty\},$$

where $T_\alpha \leq \infty$. It is shown that (1) $A^+ \neq \emptyset$, (2) $A^- \neq \emptyset$, (3) $A^+ \cup A^- = \mathbf{R}$, and (4) $\alpha_1 < \alpha_2$ if $\alpha_1 \in A^-$ and $\alpha_2 \in A^+$. Hence, the cut $\bar{\alpha} = (A^-, A^+)$ is determined. Using (A.4), we have $A^- = (-\infty, \bar{\alpha})$, $A^+ = [\bar{\alpha}, \infty)$ and $T_{\bar{\alpha}} = \infty$. We can show that $\lim_{t \rightarrow \infty} \frac{y_{\bar{\alpha}}(t)}{\varphi(t)} = 1$ and the uniqueness of solutions of (BVP_k) with $h(t) \approx \varphi(t)$ holds.

In a similar spirit, we have

PROPOSITION 4. Let $\ell > 2$. Assume (A.4), (A.5) and $\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t^\gamma} = \infty$. Then the assertions as in Proposition 3 are valid.

Now, it remains to consider the case $\lim_{t \rightarrow \infty} \frac{\varphi(t)}{t^\gamma} = \kappa$ ($0 < \kappa < \infty$) under the assumptions (A.4) and (A.6). In this case we may assume

$$g(t)^{-1} = t^{-\ell} + g_1(t)t^{-\ell}, \quad |g_1(t)| \leq K_1/t$$

$$\varphi(t)^p = \kappa^p t^{\gamma p} + \varphi_1(t)t^{\gamma p}, \quad |\varphi_1(t)| \leq K_1/t$$

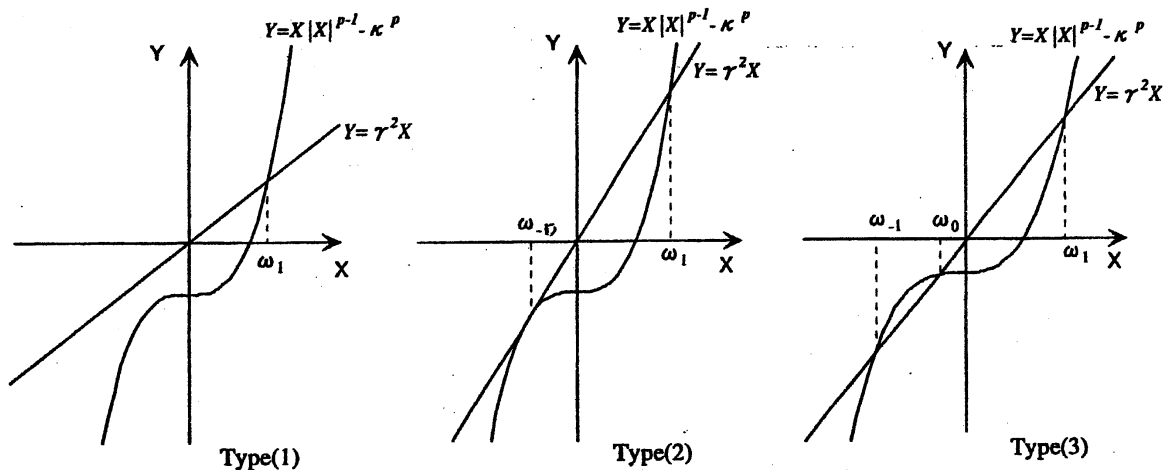
for every $t \gg 1$. Putting $y(t) = t^\gamma w(t)$, we get a new ODE for $w(t)$:

$$(8) \quad \frac{d^2 w}{dt^2}(t) + \frac{2\gamma + 1}{t} \frac{dw}{dt}(t) = \frac{1}{t^2} \{w|w|^{p-1} - \kappa^p - \gamma^2 w\} + (\text{lower term}),$$

where

$$\text{lower term} = \frac{1}{t^2} \{g_1(t)(w|w|^{p-1} - \kappa^p) - (1 + g_1(t))\varphi_1(t)\}.$$

Then we have 3 types such that



We have to introduce various boundary functions $h(|x|)$ corresponding to Type (1) - Type (3). In what follows, we will focus on Type (3), because Type (3) is the most interesting case. In this case, we first note that every solution $w(t)$ of (8) with infinite life span converges to the one of $\{\omega_{-1}, \omega_0, \omega_1\}$. From this it follows that if $y(t)$ is a

solution of (4) in (a_k, ∞) with infinite life span, then $y(t)/t^\gamma$ converges to the one of $\{w_{-1}, w_0, w_1\}$ as $t \rightarrow \infty$. Define

$$A^+ = \{\alpha \in \mathbf{R}; \lim_{t \uparrow T_\alpha} y_\alpha(t) = +\infty \text{ and } T_\alpha < \infty\}$$

$$A_1 = \{\alpha \in \mathbf{R}; \lim_{t \rightarrow \infty} y_\alpha(t)/t^\gamma = w_1\}$$

$$A_0 = \{\alpha \in \mathbf{R}; \lim_{t \rightarrow \infty} y_\alpha(t)/t^\gamma = w_0\}$$

$$A_{-1} = \{\alpha \in \mathbf{R}; \lim_{t \rightarrow \infty} y_\alpha(t)/t^\gamma = w_{-1}\}$$

$$A^- = \{\alpha \in \mathbf{R}; \lim_{t \uparrow T_\alpha} y_\alpha(t) = -\infty \text{ and } T_\alpha < \infty\}.$$

LEMMA 5. $A^- = (-\infty, \alpha_*)$, $A_{-1} = \{\alpha_*\}$, $A_0 = (\alpha_*, \alpha^*)$, $A_1 = \{\alpha^*\}$ and $A^+ = (\alpha^*, \infty)$.

Using this lemma, we have

PROPOSITION 6. Type (1) $\implies \exists$ a unique solution of (BVP_k) with $h(t) \approx w_1 t^\gamma$.

Type (2) $\implies \exists$ a unique solution of (BVP_k) with $h(t) \approx w_1 t^\gamma$ and $\exists$ infinitely many solutions of (BVP_k) with $h(t) \approx w_0 t^\gamma$.

Type (3) $\implies \exists$ a unique solution of (BVP_k) with $h(t) \approx w_1 t^\gamma$, $\exists$ infinitely many solutions of (BVP_k) with $h(t) \approx w_0 t^\gamma$, and $\exists$ a unique solution of (BVP_k) with $h(t) \approx w_{-1} t^\gamma$.

REMARK In the case where $\lim_{t \rightarrow \infty} y(t)/t^\gamma = w_0$, if the above boundary condition is replaced by stronger another condition, then we can prove the uniqueness of solutions of (BVP_k) . In fact, let $\alpha \in A_0 = (\alpha_*, \alpha^*)$ and y_α be a solution of (BVP_k) . Then, for every $\sigma \in \mathbf{R}$, there exists a unique $q_\sigma(t) = O(t^{\gamma-1})$ (as $t \rightarrow \infty$) such that

$$\lim_{t \rightarrow \infty} \frac{y_\alpha(t) - \{w_0 t^\gamma + q_\sigma(t)\}}{t^{\delta_1}} = \sigma,$$

where $\delta_1 = \sqrt{p|w_0|^{p-1}}$, $\delta_1 \neq 0$.

Let $\delta_1 = 0$. For every $\sigma \in \mathbf{R}$, there exists a unique $q_\sigma(t) = O(t^{\gamma-1})$ (as $t \rightarrow \infty$) such that

$$\lim_{t \rightarrow \infty} \frac{y_\alpha(t) - q_\sigma(t)}{\log t} = \sigma.$$

Main result

(I) Let $l \leq 2$ and $\lambda_i > 0$ ($i = 1, 2, \dots, k$). Assume (A.4) and (A.5). Then there exists a unique standard radial viscosity solution u of (DP) with boundary function

$h(|x|) \approx \varphi(|x|)$. Moreover, if $h(|x|) \not\approx \varphi(|x|)$ then (DP) does not possess any standard radial viscosity solution of (DP).

(II) Let $l > 2$ and $\lambda_i > 0$ ($i = 1, 2, \dots, k$). Assume (A.4), (A.5) and $\lim_{t \rightarrow \infty} \varphi(t)/t^\gamma = +\infty$. Then the assertions of (I) are also valid.

(III) Let $l > 2$ and $\lambda_i > 0$ ($i = 1, 2, \dots, k$). Assume (A.4), (A.6) and $\lim_{t \rightarrow \infty} \varphi(t)/t^\gamma = \kappa (> 0)$. Then the same results for standard radial viscosity solutions of (DP) as those in Proposition 6 hold. Of course, boundary functions are replaced by

$$h(|x|) \approx w_1 |x|^\gamma \quad \text{in case Type(1);}$$

$$h(|x|) \approx w_i |x|^\gamma \quad (i \in \{0, 1\}) \quad \text{in case Type(2);}$$

$$h(|x|) \approx w_i |x|^\gamma \quad (i \in \{-1, 0, 1\}) \quad \text{in case Type(3).}$$

In particular, in the case where $h(|x|) \approx w_0 |x|^\gamma$, the boundary condition at ∞ is replaced by

$$\lim_{|x| \rightarrow \infty} \frac{u(x) - \{w_0 |x|^\gamma + q_\sigma(|x|)\}}{|x|^{\delta_1}} = \sigma,$$

where $\delta_1 = \sqrt{p|w_0|^{p-1}}$, $\delta_1 \neq 0$.

If $\delta_1 = 0$, then the boundary condition at ∞ is represented with

$$\lim_{|x| \rightarrow \infty} \frac{u(x) - q_\sigma(|x|)}{\log |x|} = \sigma.$$

(IV) If $\lambda_i \geq 1$ for all $i \in \{1, 2, \dots, k\}$, then the uniqueness of viscosity solutions of (DP) holds. Hence, every viscosity solution of (DP) is radial.

References

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