

# On the structure of parabolic Humphreys-Verma modules

By

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Through an investigation into the bounded derived category  $D^b(\mathrm{coh}\mathcal{P})$  of the coherent sheaves on a projective homogeneous variety  $\mathcal{P}$  we have been led to study the parabolic Humphreys-Verma modules. Although these modules are defined only in positive characteristic, our geometric application appears effective characteristic-free.

Write  $\mathcal{P} = G/P$  with  $G$  a reductive algebraic group over an algebraically closed field of positive characteristic and  $P$  a parabolic subgroup of  $G$ . Let  $G_1$  be the Frobenius kernel of  $G$  and let  $\hat{\nabla}_P(\varepsilon)$  be the  $G_1P$ -module induced from 1-dimensional trivial  $P$ -module  $\varepsilon$ , a parabolic Humphreys-Verma module. We have recently found a way, though verified in only few limited cases yet, to parametrize certain components of the  $G_1T$ -socle series of  $\hat{\nabla}_P(\varepsilon)$  by the set  $W^P$  of distinguished coset representatives of the Weyl group of  $G$  by the Weyl group of  $P$  such that the associated coherent sheaves  $\mathcal{E}_w$ ,  $w \in W^P$ , on  $\mathcal{P}$  form a Karoubian complete strongly exceptional poset for  $D^b(\mathrm{coh}\mathcal{P})$ . Those sheaves are defined over  $\mathbb{Z}$  to verify Catanese's conjecture [Bö] transferring over to  $\mathbb{C}$ ; in some cases our constructions offer a new evidence to the conjecture in complex algebraic geometry. In this note, however, we will focus on the structure of Humphreys-Verma modules.

In order to be precise in which category the morphisms are taken, we will write  $\mathcal{C}(X, Y)$  for the set of morphisms in category  $\mathcal{C}$  from object  $X$  to  $Y$ . For an algebraic group  $H$  we let  $H\mathbf{Mod}$  denote the category of rational  $H$ -modules, and for a variety  $X$  the category of modules over the structure sheaf of  $X$  will be denoted by  $\mathbf{Mod}_X$ .

This is an expanded version of the author's talk at a RIMS meeting under the title of the present volume. Subsequent to the talk I have come up with a description of the structure of Humphreys-Verma modules for projective spaces, which is included in §3; in the talk I could merely exhibit the computations in the cases of  $\mathrm{GL}_2$  and  $\mathrm{GL}_3$ .

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## § 1. Humphreys-Verma modules

Let  $\mathbb{k}$  be an algebraically closed field of positive characteristic  $p$ . We will assume  $p$  is sufficiently large. Let  $G$  be a reductive algebraic group over  $\mathbb{k}$ ,  $P$  a parabolic subgroup of  $G$ , and  $G_1$  the Frobenius kernel of  $G$ . We call the functor

$$\mathrm{ind}_P^{G_1 P} = \mathbf{Sch}_{\mathbb{k}}(G_1 P, ?)^P : P\mathbf{Mod} \rightarrow G_1 P\mathbf{Mod}$$

parabolic Humphreys-Verma induction and write  $\hat{\nabla}_P$  for short.

In case  $P = B$  a Borel subgroup of  $G$  put  $\hat{\nabla} = \hat{\nabla}_B$ . Let  $\Lambda$  be the character group of  $B$ ,  $T$  a maximal torus of  $B$ ,  $R \subset \Lambda$  the root system of  $G$  relative to  $T$ . We choose a positive system  $R^+$  of  $R$  such that the roots of  $B$  are  $-R^+$ , and let  $R^s$  be the set of simple roots. Let  $W = N_G(T)/T$  the Weyl group of  $G$ ,  $W_p = W \ltimes p\mathbb{Z}R$ ,  $S_p = \{s_i, s_0 \mid \alpha_i \in R^s\}$  with  $s_i$  the reflexion associated to  $\alpha_i$  and  $s_0$  the reflexion in the wall  $\{v \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle v, \alpha_0^\vee \rangle = -p\}$ ,  $\alpha_0^\vee$  the highest coroot. Thus  $(W_p, S_p)$  forms a Coxeter system. We will consider the dot action of  $W_p$  on  $\Lambda$  such that  $x \bullet \lambda = x(\lambda + \rho) - \rho$ ,  $x \in W_p$ ,  $\lambda \in \Lambda$ ,  $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ . One has  $\hat{\nabla}(\lambda + p\mu) \simeq \hat{\nabla}(\lambda) \otimes p\mu \ \forall \mu \in \Lambda$ , and each  $\hat{\nabla}(\lambda)$  has

a  $G_1 T$ -simple socle  $\hat{L}(\lambda)$  of highest weight  $\lambda$  with all other composition factors having their highest weights  $< \lambda$ . If  $\Lambda_1 = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \in [0, p[ \ \forall \alpha \in R^s\}$  the set of restricted dominant weights, a simple  $G_1 T$ -module of highest weight  $\lambda \in \Lambda_1$  admits a structure of simple  $G$ -module  $L(\lambda)$ . Thus, the determination of the composition factors of all  $\hat{\nabla}(\lambda)$ ,  $\lambda \in \Lambda_1$ , will yield all irreducible characters for  $G$  by Steinberg's tensor product theorem. Moreover, let  $\mathcal{A}$  be the set of alcoves on  $\Lambda$  with respect to the dot action of  $W_p$ .  $\forall A \in \mathcal{A}$ ,  $\forall w \in W$ ,  $\forall \gamma \in \mathbb{Z}R$ , we will write  $Aw^{-1}t_{p\gamma}$  for the alcove  $(w \bullet A) + p\gamma$ . Let  $0_A$  be the image of  $0 \in \Lambda$  in  $A$  under the  $W_p$ -action. By the translation principle [J, II.7] the structure of  $\hat{\nabla}(0_A)$  describes that of all other  $\hat{\nabla}(\lambda)$ ,  $\lambda \in \Lambda \cap A$ , and the determination of irreducible characters of all  $L(0_A)$ ,  $0_A \in \Lambda_1$ , will obtain all irreducible characters for  $G$ , which has now been achieved for indefinitely large  $p$  by Andersen, Jantzen and Soergel [AJS] to verify Lusztig's conjecture. We will thus write  $\hat{\nabla}(A)$  for  $\hat{\nabla}(0_A)$ .

The Lusztig conjecture on the irreducible  $G$ -characters, in turn, determines the  $G_1 T$ -socle series of each  $\hat{\nabla}(A)$ ; let  $\mathrm{soc}_i \hat{\nabla}(A)$  be the  $i$ -th  $G_1 T$ -socle layer of  $\hat{\nabla}(A)$  and let  $\hat{L}(C)$  be the simple  $G_1 T$ -module of highest weight  $0_C$ ; by the linkage principle [J, II.9.15] the composition factors of  $\hat{\nabla}(A)$  are of the form  $\hat{L}(C)$ . Let  $w_0 \in W$  be the longest element of  $W$  with respect to  $\{s_i \mid \alpha_i \in R^s\}$ . Then

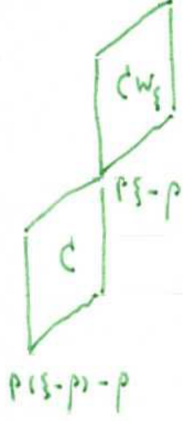
**Theorem 1.1** ([AK]/[RIMS]). *Let  $A \in \mathcal{A}$ .*

(i) *The Loewy length, that is the length of the socle series, of each  $\hat{\nabla}(A)$  is  $\ell(w_0) + 1$ .*

(ii) If  $0_C \in \Lambda_1 + p(\xi - \rho)$ ,  $\xi \in \Lambda$ , then with  $w_\xi = w_0 t_{p(\xi - w_0 \xi)}$

$$Q_{A, Cw_\xi} = \sum_{i \in \mathbb{N}} q^{\frac{1}{2}\{d(A, Cw_\xi) + i - \ell(w_0) - 1\}} [\text{soc}_i \hat{\nabla}(A) : \hat{L}(C)],$$

where  $Q_{A, C}$  is the periodic inverse Kazhdan-Lusztig polynomial  $[L]$  associated to  $A, C \in \mathcal{A}$  and  $d(A, Cw_\xi)$  is the distance from  $A$  to  $Cw_\xi$ .



## § 2. Parabolic Humphreys-Verma modules

We wish to obtain a formula to describe the socle series for general parabolic  $P$ . Let  $\Lambda_P = \mathbf{Grp}_{\mathbb{k}}(P, \text{GL}_1)$ ,  $L$  the standard Levi subgroup of  $P$ ,  $U_L$  the unipotent radical of  $B \cap L$ ,  $\text{Dist}(U_L)$  the algebra of distributions on  $U_L$ ,  $R_L \subseteq R$  the root system of  $L$ , and  $W_P = \langle s_\alpha \mid \alpha \in R_L \rangle \leq W$  the Weyl group of  $P$  and also of  $L$ .  $\forall \nu \in \Lambda_P$ , regarded as  $G_1 T$ -modules

$$\hat{\nabla}_P(\nu) = \text{ind}_P^{G_1 P}(\nu) \simeq \text{ind}_{P_1 T}^{G_1 T}(\nu),$$

where  $P_1$  is the Frobenius kernel of  $P$ . We call  $\hat{\nabla}_P(\nu)$  the parabolic Humphreys-Verma module of highest weight  $\nu$ ; if  $\rho_P = \frac{1}{2} \sum_{\alpha \in R^+ \setminus R_L} \alpha$ ,

$$\begin{aligned} \hat{\nabla}_P(\nu) &\simeq \text{ind}_{P_1}^{G_1}(\nu) \\ &\simeq \text{Dist}(G_1) \otimes_{\text{Dist}(P_1)} (\nu - 2(p-1)\rho_P) \quad \text{by [J, II.9.2]}. \end{aligned}$$

If  $U_P^+$  is the subgroup of  $G$  generated by the root subgroups  $U_\alpha$ ,  $\alpha \in R^+ \setminus R_L$ , and if  $U_{P,1}^+$  is the Frobenius kernel of  $U_P^+$ , then  $\text{Dist}(G_1) \simeq \text{Dist}(U_{P,1}^+) \otimes_{\mathbb{k}} \text{Dist}(P_1)$ , and hence

$$\hat{\nabla}(\nu) \simeq \text{Dist}(U_{P,1}^+) \otimes_{\mathbb{k}} (\nu - 2(p-1)\rho_P)$$

with  $\text{Dist}(U_{P,1}^+) \simeq \otimes_{\alpha \in R^+ \setminus R_L} \text{Dist}(U_{\alpha,1})$ . To relate a parabolic Humphreys-Verma module to ordinary Humphreys-Verma modules, one has at the character level

**Proposition 2.1** ([KY]).  $\forall \nu \in \Lambda$ ,

$$\text{ch} \hat{\nabla}_P(\nu) = e^\nu \prod_{\alpha \in R^+ \setminus R_L} \frac{1 - e^{-p\alpha}}{1 - e^{-\alpha}} = \sum_{\substack{w \in W_P \\ \gamma \in \mathbb{Z}R_L}} (-1)^{\ell(w)} \dim(\text{Dist}(U_L)_\gamma) \text{ch} \hat{\nabla}(w \bullet \nu + p\gamma).$$

Now let  $A \in \mathcal{A}$  with  $0_A \in \Lambda_P$ . As  $\hat{\nabla}_P(A) \leq \hat{\nabla}(A)$  as  $G_1B$ -modules, a naïve speculation on the  $G_1T$ -socle series of  $\hat{\nabla}_P(A)$  would be that  $\forall C \in \mathcal{A}$  with  $0_C \in \Lambda_1 + p(\xi - \rho)$ ,  $\xi \in \Lambda$ ,

$$\begin{aligned} (1) \quad \sum_{i \in \mathbb{N}} q^{\frac{1}{2}\{d(A, Cw_\xi) + i - \ell(w_0) - 1\}} [\text{soc}_i \hat{\nabla}_P(A) : \hat{L}(C)] &= \sum_{\substack{w \in W_P \\ \gamma \in \mathbb{Z}R_L}} (-1)^{\ell(w)} \dim(\text{Dist}(U_L)_\gamma) \\ &\quad \sum_{i \in \mathbb{N}} q^{\frac{1}{2}\{d(Aw^{-1}t_{p\gamma}, Cw_\xi) + i - \ell(w_0) - 1\}} [\text{soc}_i \hat{\nabla}(Aw^{-1}t_{p\gamma}) : \hat{L}(C)] \\ &= \sum_{\substack{w \in W_P \\ \gamma \in \mathbb{Z}R_L}} (-1)^{\ell(w)} \dim(\text{Dist}(U_L)_\gamma) Q_{Aw^{-1}t_{p\gamma}, Cw_\xi}. \end{aligned}$$

By design (1) holds under the specialization  $q \rightsquigarrow 1$ , and also in case  $P = B$ .

**Proposition 2.2** ([KY]). *(1) holds for  $G$  of rank  $\leq 2$ .*

If  $W^P = \{w \in W \mid wR_L^+ \subseteq R^+\}$ ,  $W = \bigsqcup_{w \in W^P} wW_P$ . Let  $w_{0,P} \in W_P$  with  $w_{0,P}R_L^+ = -R_L^+$ , and set  $w_0^P = w_0w_{0,P} \in W^P$ .  $\forall x \in W_p$ ,  $\exists! w \in W$ :  $x \bullet 0 \equiv w \bullet 0 \pmod{p\Lambda}$ .  $\forall w \in W$ , choose  $0_{P,w} \in \Lambda_1$  such that  $0_{P,w} \equiv w_0ww_{0,P} \bullet 0 \pmod{p\Lambda}$ . Then  $W_p \bullet 0 + p\Lambda = \bigsqcup_{w \in W} (0_{P,w} + p\Lambda)$ .

**Theorem 2.3** ([KY]). *Assume  $\text{rk } G \leq 2$ . Let  $A \in \mathcal{A}$  with  $0_A \in \Lambda_P$ .*

- (i) *The Loewy length of  $\hat{\nabla}_P(A)$  is  $\ell(w_0^P) + 1$ .*
- (ii)  *$\forall i \in [1, \ell(w_0^P) + 1]$ , there is a decomposition as  $G_1P$ -modules*

$$\text{soc}_i \hat{\nabla}_P(A) = \coprod_{w \in W^P} L(0_{P,w}) \otimes G_1 \mathbf{Mod}(L(0_{P,w}), \text{soc}_i \hat{\nabla}_P(A)).$$

- (iii) *In case  $A = A^+$  the bottom dominant alcove,*

$$G_1 \mathbf{Mod}(L(0_{P,w}), \text{soc}_{\ell(w_0^P)+1-\ell(w)} \hat{\nabla}_P(A^+)) \neq 0 \quad \forall w \in W^P.$$

**Remark 2.4.** (i) In case  $P = B$ , each  $L(0_{B,w})$ ,  $w \in W$ , appears as a  $G_1$ -composition factor of any  $\hat{\nabla}(A)$ .

To see that,  $\forall x, y \in W, \forall \mu \in \Lambda$ ,

$$\begin{aligned} [\hat{\nabla}(A^+x) : L(0_{B,y}) \otimes p\mu] &= [\hat{\nabla}(x^{-1} \bullet 0) : \hat{L}(0_{B,y} + p\mu)] = [\hat{\nabla}(x^{-1} \bullet 0) : \hat{L}(0_{B,y} + px^{-1}x \bullet \mu)] \\ &= [\hat{\nabla}(\varepsilon) : \hat{L}(0_{B,y} + px \bullet \mu)] \quad \text{by [J, II.9.16.4] with } \varepsilon \text{ denoting } 0 \in \Lambda. \end{aligned}$$

It is therefore enough to show that  $\forall w \in W, \exists \mu \in \Lambda: [\hat{\nabla}(\varepsilon) : \hat{L}(0_{B,w} + p\mu)] \neq 0$ . Write  $0_{B,w} = w_0w \bullet 0 + p\eta$  for some  $\eta \in \Lambda$ . Then

$$[\hat{\nabla}(\varepsilon) : \hat{L}(0_{B,w} + p\mu)] = [\hat{\nabla}(\varepsilon) : \hat{L}(w_0w \bullet 0 + p(\eta + \mu))] = [\hat{\nabla}(-p(\eta + \mu)) : \hat{L}(w_0w \bullet 0)].$$

Thus we have only to check  $\forall C \in \mathcal{A}, \exists \gamma \in \mathbb{Z}R: [\hat{\nabla}(A^+t_{p\gamma}) : \hat{L}(C)] \neq 0$ . If  $C \subseteq \Lambda_1 + p(\xi - \rho)$  for some  $\xi \in \Lambda$ , then by [Y]  $\forall w \in W, [\hat{\nabla}(A^+wt_{p\xi}) : \hat{L}(C)] \neq 0$ . Write  $A^+t_{p\xi} = A^+t_{p\gamma}y, \gamma \in \mathbb{Z}R, y \in W$ . Then

$$A^+y^{-1}t_{p\xi} = A^+t_{p\gamma}yt_{-p\xi}y^{-1}t_{p\xi} = A^+t_{p\gamma}t_{-py\xi}t_{p\xi} = A^+t_{p(\gamma-y\xi+\xi)}.$$

If  $y = y_1y_2$  with  $y_1, y_2 \in W$ , then  $y\xi - \xi = y_1y_2\xi - \xi = y_1(y_2\xi - \xi) + y_1\xi - \xi$ , and  $\forall \alpha \in R^+, s_\alpha\xi - \xi = \xi - \langle \xi, \alpha^\vee \rangle \alpha - \xi = -\langle \xi, \alpha^\vee \rangle \alpha \in \mathbb{Z}R$ . Thus  $\gamma - y\xi + \xi \in \mathbb{Z}R$ , as desired.

(ii) In application to the study of  $D^b(\text{coh } \mathcal{P})$ ,  $\mathcal{P} = G/P$ , the  $P$ -module structure on each  $G_1 \mathbf{Mod}(L(0_{P,w}), \text{soc}_{\ell(w_0^P)+1-\ell(w)} \hat{\nabla}_P(A^+))$ ,  $w \in W^P$ , appears to play an important role: as  $G_1$  acts trivially on those, untwisting the Frobenius, put  $\text{soc}_{P,w}^1 = G_1 \mathbf{Mod}(L(0_{P,w}), \text{soc}_{\ell(w_0^P)+1-\ell(w)} \hat{\nabla}_P(A^+))^{[-1]}$ . It appears from [KY] that each  $\text{soc}_{P,w}^1$  admits a direct summand  $E_w$  such that, writing  $\mathcal{L}_{\mathcal{P}}(E_w)$  for the locally free sheaf on  $\mathcal{P}$  associated to  $E_w$ ,  $\{\mathcal{L}_{\mathcal{P}}(E_w) \mid w \in W^P\}$  forms a Karoubian complete strongly exceptional poset such that  $\mathbf{Mod}_{\mathcal{P}}(\mathcal{L}_{\mathcal{P}}(E_x), \mathcal{L}_{\mathcal{P}}(E_y)) \neq 0$  iff  $x \leq y$ . Moreover, those  $E_w$  are defined over  $\mathbb{Z}$  to yield also a Karoubian complete strongly exceptional poset in characteristic 0.

### § 3. Projective spaces

Let  $E$  be a  $\mathbb{k}$ -linear space of basis  $e_1, \dots, e_{n+1}$ ,  $G = \text{GL}(E)$ , and  $P = N_G(\mathbb{k}e_{n+1})$ . Thus  $\mathcal{P} = G/P \simeq \mathbb{P}_{\mathbb{k}}^n$ . If  $F_{\mathcal{P}}$  (resp.  $F_{\mathbb{k}}$ ) is the absolute Frobenius morphism on  $\mathcal{P}$  (resp.  $\text{Spec}(\mathbb{k})$ ) and if  $q : G/P \rightarrow G/G_1P$  is a natural morphism, one has a commutative diagram of schemes

$$\begin{array}{ccccc} \mathcal{P} & \xrightarrow{F_{\mathcal{P}}} & \mathcal{P} & \xrightarrow{\text{structure}} & \text{Spec}(\mathbb{k}) \\ \downarrow q & \searrow F_{\mathcal{P}/\mathbb{k}} & \uparrow \phi & \square & \uparrow F_{\mathbb{k}} \\ G/G_1P & \xrightarrow{\sim} & \mathcal{P}^{(1)} & \longrightarrow & \text{Spec}(\mathbb{k}). \end{array}$$

If  $A^+$  is the bottom dominant alcove, one has from [Haa]

$$F_{\mathcal{P}*}\mathcal{O}_{\mathcal{P}} \simeq \phi_*\mathcal{L}_{G/G_1P}(\hat{\nabla}_P(A^+)).$$

On the other hand, we know from [HKR]

$$F_{\mathcal{P}*}\mathcal{O}_{\mathcal{P}} \simeq \prod_{i=0}^n \mathcal{O}_{\mathcal{P}}(-i) \otimes_{\mathbb{k}} V_i,$$

where  $V_i = \coprod_{\substack{j \in [0,p]^{n+1} \\ |j|=pi}} \mathbb{k}x^j$  in the polynomial algebra  $\mathbb{k}[x_1, \dots, x_{n+1}]$  with  $x^j = \prod_{i=1}^{n+1} x_i^{j_i}$  and  $|j| = \sum_{i=1}^{n+1} j_i$  if  $j = (j_1, \dots, j_{n+1})$ . Regarding  $x_1, \dots, x_{n+1}$  as the dual basis of  $e_1, \dots, e_{n+1}$ , let  $G$  act on  $\mathbb{k}[x_1, \dots, x_{n+1}]$  and also on  $\mathbb{k}[x_1, \dots, x_{n+1}]/(x_1^p, \dots, x_{n+1}^p)$  contragrediently. Then one can equip  $V_i$  with a structure of  $G$ -module by identifying it with its image in  $\mathbb{k}[x_1, \dots, x_{n+1}]/(x_1^p, \dots, x_{n+1}^p)$ .

Now let  $B$  be a Borel subgroup of  $P$  consisting of lower triangular matrices and  $T$  a maximal torus of  $B$  consisting of diagonal matrices. Identify  $\Lambda$  with  $\mathbb{Z}^{\oplus n+1}$  via the basis  $\varepsilon_i : \text{diag}(a_1, \dots, a_{n+1}) \mapsto a_i, i \in [1, n+1]$ , and  $W$  with the symmetric group  $\mathfrak{S}_{n+1}$  permuting the  $\varepsilon_i, i \in [1, n+1]$ . Then  $W^P = \{(i \ i+1 \ \dots \ n+1) \mid i \in [1, n+1]\}$ , and Serre's twisted sheaf on  $\mathbb{P}^n$  is given by  $\mathcal{O}_{\mathcal{P}}(1) = \mathcal{L}_{\mathcal{P}}(-\varepsilon_{n+1})$ .  $\forall i \in [1, n+1]$ , set  $\lambda_{(i \ i+1 \ \dots \ n+1)} = (i-1)\varepsilon_{n+1}$  and  $0_{P,(i \ i+1 \ \dots \ n+1)} = -(i-1)\varepsilon_{n+2-i} - (p-1)(\varepsilon_{n+3-i} + \dots + \varepsilon_{n+1}) \in \Lambda_1$ , which we agree to be 0 in case  $i = 1$ . Then  $V_{i-1} \simeq L(0_{P,(i \ i+1 \ \dots \ n+1)}) \ \forall i \in [1, n+1]$ , and hence

$$F_{\mathcal{P}*}\mathcal{O}_{\mathcal{P}} \simeq \prod_{i=1}^{n+1} \mathcal{L}_{\mathcal{P}}(\lambda_{(i \ i+1 \ \dots \ n+1)}) \otimes_{\mathbb{k}} L(0_{P,(i \ i+1 \ \dots \ n+1)}).$$

Confirming the pattern in Theorem 2.3, it holds that

**Theorem 3.1** ([K]). *Assume  $p \geq n+1$ .*

- (i) *The Loewy length of  $\hat{\nabla}_P(A^+)$  is  $n+1 = \ell(w_0^P) + 1$ .*
- (ii)  $\forall i \in [1, n+1], \text{soc}_i \hat{\nabla}_P(A^+) \simeq L(0_{P,(i \ i+1 \ \dots \ n+1)}) \otimes \lambda_{(i \ i+1 \ \dots \ n+1)}^{[1]}$ .

**Remark 3.2.** Regardless of characteristic  $\{\mathcal{L}_{\mathcal{P}}(\lambda_w) \mid w \in W^P\}$  forms a complete strongly exceptional poset on  $\mathbb{P}^n$  such that  $\mathbf{Mod}_{\mathbb{P}^n}(\mathcal{L}_{\mathcal{P}}(\lambda_x), \mathcal{L}_{\mathcal{P}}(\lambda_y)) \neq 0$  iff  $x \leq y$  [HKR]/[K08].

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