

Accumulation of periodic points for local uniformly quasiregular mappings

By

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Abstract

We consider accumulation of periodic points in local uniformly quasiregular dynamics. Given a local uniformly quasiregular mapping f with a countable and closed set of isolated essential singularities and their accumulation points on a closed Riemannian manifold, we show that points in the Julia set are accumulated by periodic points. If, in addition, the Fatou set is non-empty and connected, the accumulation is by periodic points in the Julia set itself. We also give sufficient conditions for the density of repelling periodic points.

§ 1. Introduction

Let M and N be oriented Riemannian n -manifolds for $n \geq 2$. A continuous mapping $f: M \rightarrow N$ is called K -quasiregular, $K \geq 1$, if f belongs to the Sobolev space $W_{\text{loc}}^{1,n}(M, N)$ and satisfies the distortion inequality

$$\|df\|^n \leq K J_f \quad \text{a.e. on } M,$$

where $\|df\|$ is the operator norm of the differential df of f and J_f the *Jacobian determinant* of f satisfying $f^*(\text{vol}_N) = J_f \text{vol}_M$, where vol_M and vol_N are the Riemannian volume forms on M and N , respectively.

A quasiregular self-map $f: M \rightarrow M$ is called *uniformly K -quasiregular* (K -UQR) if all iterates f^k for $k \geq 1$ are K -quasiregular. Similarly as quasiregular mappings have

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the rôle of holomorphic mappings in the n -dimensional Euclidean conformal geometry for $n \geq 3$, the dynamics of uniformly quasiregular mappings can be viewed as the counterpart of holomorphic dynamics in the n -dimensional conformal geometry. We refer to the seminal paper of Iwaniec and Martin [12] and Hinkkanen, Martin, Mayer [9] for the fundamentals in this theory.

In this article we consider dynamics of local UQR-mappings. Let M be an oriented Riemannian n -manifold and $\Omega \subset M$ an open set. Following the terminology in [9], we say a mapping $f: \Omega \rightarrow M$ is a *local uniformly K -quasiregular*, $K \geq 1$, if for every $k \in \mathbb{N}$, $\bigcap_{j=0}^{k-1} f^{-j}(\Omega) \neq \emptyset$ and $f^k: \bigcap_{j=0}^{k-1} f^{-j}(\Omega) \rightarrow M$ is K -quasiregular.

With slight modifications, the standard terminology from dynamics is at our disposal also in this local setting. Let

$$D_f := \text{the interior of } \bigcap_{k \geq 0} f^{-k}(\Omega) = M \setminus \overline{\bigcup_{k \geq 0} f^{-k}(M \setminus \Omega)}.$$

As usual, the Fatou set $F(f)$ of f is the maximal open subset in D_f where the family $\{f^k; k \in \mathbb{N}\}$ is normal, the Julia set of f is the set

$$J(f) := M \setminus F(f),$$

and the exceptional set of f is

$$\mathcal{E}(f) := \{x \in M; \# \bigcup_{k \geq 0} f^{-k}(x) < \infty\}.$$

A point $x \in M$ is a *periodic point of f in M* if $x \in \bigcap_{j=0}^{p-1} f^{-j}(\Omega)$ and $f^p(x) = x$ for some $p \in \mathbb{N}$. We call p a *period of x (under f)*. Note that periodic points always belong to the set $\overline{D_f}$.

A periodic point $x \in M$ with period $p \in \mathbb{N}$ is *(topologically) repelling* if $f: U \rightarrow f^p(U)$ is univalent and $U \Subset f^p(U)$ for some open neighborhood U of x in $\bigcap_{j=0}^{p-1} f^{-j}(\Omega)$. Note that, then $x \in J(f)$; see [9, §4].

In [9], Hinkkanen, Martin and Mayer gave a classification of cyclic Fatou components of f (see Theorem 2.12) as well as periodic points. We study both $J(f)$ and $\mathcal{E}(f)$ for a non-constant local uniformly quasiregular mapping

$$f: \mathbb{M} \setminus S_f \rightarrow \mathbb{M},$$

where $\mathbb{M}$ is a closed, oriented, and connected Riemannian n -manifold, $n \geq 2$, and S_f is a countable and closed subset in $\mathbb{M}$ consisting of isolated essential singularities of f and their accumulation points in $\mathbb{M}$. In our first main theorem, we also consider a subclass of non-elementary UQR-mappings. A non-constant local uniformly quasiregular mapping $f: \mathbb{M} \setminus S_f \rightarrow \mathbb{M}$ is *non-elementary* if it is non-injective and satisfies

$$J(f) \not\subset \mathcal{E}(f).$$

For comments on the non-injectivity and non-elementarity, see Section 5.

Recall that a point x in a topological space X is *accumulated by* a subset S in X if for every neighborhood N of x , $S \cap (N \setminus \{x\}) \neq \emptyset$, and that a subset S in X is *perfect* if S is non-empty, compact, and has no isolated points in X .

Theorem 1. *Let $\mathbb{M}$ be a closed, oriented, and connected Riemannian n -manifold, $n \geq 2$, and $f: \mathbb{M} \setminus S_f \rightarrow \mathbb{M}$ a non-constant local uniformly K -quasiregular mapping, $K \geq 1$, where S_f is a countable and closed subset in $\mathbb{M}$ and consists of isolated essential singularities of f and their accumulation points in $\mathbb{M}$. Then $J(f)$ is nowhere dense in $\mathbb{M}$ unless $J(f) = \mathbb{M}$. Furthermore, the following hold:*

- (a) *If f is non-injective, then $J(f) \neq \emptyset$ and $\#\mathcal{E}(f) < \infty$. Moreover, for every $x \in \mathbb{M} \setminus \mathcal{E}(f)$, points in $J(f)$ are accumulated by $\bigcup_{k \geq 0} f^{-k}(x)$.*
- (b) *If f is non-injective and $S_f = \emptyset$, then $\mathcal{E}(f) \subset F(f)$ and f is non-elementary.*
- (c) *If f is a priori non-elementary, then $J(f)$ is perfect and points in $J(f)$ are accumulated by periodic points of f .*

For non-constant and non-injective uniformly quasiregular endomorphisms of the n -sphere $\mathbb{S}^n$, the accumulation of periodic points to $J(f)$ in Theorem 1 is due to Siebert [21, 3.3.6 Theorem]; note that by a theorem of Fletcher and Nicks [6], $J(f)$ is in fact uniformly perfect in this case.

The proof of the accumulation of periodic points to the Julia set for non-elementary f is based on two rescaling principles (see Section 2). It is a generalization of Schwick's argument [19] (see also Bargmann [2] and Berteloot–Duval [3]), which is reminiscent to Julia's construction of (expanding) homoclinic orbits for rational functions ([14, §14]). Our argument simultaneously treats all the cases $S_f = \emptyset$, $0 < \#\bigcup_{k \geq 0} f^{-k}(S_f) < \infty$, and $\#\bigcup_{k \geq 0} f^{-k}(S_f) = \infty$, which are typically studied separately.

In the final assertion in Theorem 1, it would be natural and desirable to obtain the density of (repelling) periodic points in $J(f)$.

Our second main theorem gives sufficient conditions for those density results. The topological dimension of a subset E in $\mathbb{M}$ is denoted by $\dim E$ and the branch set of f by B_f ; the *branch set* B_f is the set of points at which f is not a local homeomorphism.

Theorem 2. *Let $\mathbb{M}$ be a closed, oriented, and connected Riemannian n -manifold, $n \geq 2$, and $f: \mathbb{M} \setminus S_f \rightarrow \mathbb{M}$ be a non-elementary local uniformly K -quasiregular mapping, $K \geq 1$, where S_f is a countable and closed subset in $\mathbb{M}$ and consists of isolated essential singularities of f and their accumulation points in $\mathbb{M}$. Then*

- (a) *If $F(f)$ is non-empty and connected, then points in $J(f)$ are accumulated by periodic points of f contained in $J(f)$.*

(b) *If one of the following four conditions*

- (i) $\# \bigcup_{k \geq 0} f^{-k}(S_f) < \infty$ and $\dim J(f) > n - 2$,
- (ii) f has a repelling periodic point in $D_f \setminus (\mathcal{E}(f) \cup \bigcup_{k \in \mathbb{N}} f^k(B_{f^k}))$,
- (iii) $J(f) \not\subset \bigcap_{j \in \mathbb{N}} \overline{\bigcup_{k \geq j} f^k(B_{f^k})}$, or
- (iv) $n = 2$

holds, then points in $J(f)$ are accumulated by repelling periodic points of f .

Theorem 2 combines and extends previous results of Hinkkanen–Martin–Mayer ([9]) and Siebert ([20]) for UQR-mappings and classical results of Fatou and Julia ([14, §14]), Baker [1], Bhattacharyya [4], and Bolsch [5] and Herring [8] in the holomorphic case.

For non-constant and non-injective uniformly quasiregular endomorphisms of $\mathbb{S}^n$, the repelling density in $J(f)$ is due to Hinkkanen, Martin and Mayer [9] when $F(f)$ is either empty or not connected. Under these conditions $S_f = \emptyset$ and $\dim J(f) > n - 2$. Siebert [20, 4.3.6 Satz] proved the repelling density under the assumption $J(f) \not\subset \overline{\bigcup_{k \in \mathbb{N}} f^k(B_{f^k})}$. In this case $J(f) \not\subset \bigcap_{j \in \mathbb{N}} \overline{\bigcup_{k \geq j} f^k(B_{f^k})}$.

In the holomorphic dynamics, i.e. for $\mathbb{M} = \mathbb{S}^2$ (so $n = 2$) and $K = 1$, every non-constant and non-injective holomorphic mapping $f: \mathbb{S}^2 \setminus S_f \rightarrow \mathbb{S}^2$ is non-elementary (see Section 5). For $S_f = \emptyset$, the repelling density in $J(f)$ is a classical result of Fatou and Julia (cf. [14, §14]). For $\# \bigcup_{k \geq 0} f^{-k}(S_f) = 1, 2$ and $\# S_f = \infty$, it is due to Baker [1], Bhattacharyya [4], Bolsch [5] and Herring [8]. Note that our proof covers also the case $\# \bigcup_{k \geq 0} f^{-k}(S_f) > 2$.

This paper is organized as follows. In Section 2, we give a unified treatment for normal families and isolated essential singularities of quasiregular mappings. We also recall the invariance of the dynamical sets $D_f, F(f), J(f)$, and $\mathcal{E}(f)$ under f and the Hinkkanen–Martin–Mayer classification for cyclic Fatou components of non-elementary local uniformly quasiregular mappings. In Sections 3 and 4, we prove Theorems 1 and 2. We finish, in Section 5, with comments on the non-injectivity and non-elementarity of non-constant local uniformly quasiregular dynamics.

§ 2. Preliminaries

We begin with notations and fundamental facts from the local degree theory. For each oriented n -manifold X , we fix a generator ω_X of $H_c^n(X; \mathbb{Z})$ representing the orientation of X , and for each subdomain $D \subset X$, a generator ω_D of $H_c^n(D; \mathbb{Z})$ satisfying $\omega_X = \iota_{D,X}(\omega_D)$, where $\iota_{D,X}: H_c^n(D; \mathbb{Z}) \rightarrow H_c^n(X; \mathbb{Z})$ is the canonical isomorphism.

Let $f: M \rightarrow N$ be a continuous mapping between oriented n -manifolds M and N . For each domain $D \subset M$ and each $y \in N \setminus f(\partial D)$, the *local degree of f at $y \in N$ with*

respect to D is the non-negative integer $\mu(y, f, D)$ satisfying

$$(2.1) \quad \mu(y, f, D)\omega_D = \iota_{V,D}((f|V)^*\omega_\Omega),$$

where Ω is the component of $N \setminus f(\partial D)$ containing y and $V = f^{-1}(\Omega) \cap D$. Indeed, we can take any open and connected neighborhood of y in $N \setminus f(\partial D)$ as Ω . If $\mu(y, f, D) > 0$, then $f^{-1}(y) \cap D \neq \emptyset$. For more details, see e.g., [7, Section I.2].

From now on, let $n \geq 2$ and $K \geq 1$. Let M and N be connected and oriented Riemannian n -manifolds, and $f: M \rightarrow N$ a non-constant quasiregular mapping. By Reshetnyak's theorem (see e.g. [18, I.4.1]), f is a *branched cover*, that is, an open and discrete mapping. Every $x \in M$ has a *normal neighborhood* with respect to f , that is, an open neighborhood U of x satisfying $f(\partial U) = \partial(f(U))$ and $f^{-1}(f(x)) \cap U = \{x\}$. We denote by $i(x, f)$ the *topological index of f at x* , that is, $i(x, f) = \mu(f(x), f, U)$. The branch set B_f of f is the set of all $x \in M$ satisfying $i(x, f) \geq 2$, and is closed in M . By the Chernavskii-Väisälä theorem [22], the topological dimensions $\dim B_f$ and $\dim f(B_f)$ are at most $n - 2$.

The local degree theory readily yields the following manifold version of the Minio-witz–Rickman argument principle or the Hurwitz-type theorem; see [15, Lemma 2]; note that we do not assume that mappings f_j to be quasiregular.

Lemma 2.1. *Let M and N be oriented Riemannian n -manifolds, $n \geq 2$. Suppose a sequence (f_j) of continuous mappings from M to N tends to a quasiregular mapping $f: M \rightarrow N$ locally uniformly on M as $j \rightarrow \infty$. Then for every domain $D \Subset M$ with $f(\partial D) = \partial(f(D))$ and every compact subset $E \subset N \setminus f(\partial D)$, there exists $j_0 \in \mathbb{N}$ such that $\mu(y, f_j, D) = \mu(y, f, D)$ for every $j \geq j_0$ and every $y \in E$.*

Proof. Let $\Omega \Subset f(D)$ be a domain containing E and set $V := f^{-1}(\Omega) \cap D$. Then $(f|V)^*(\omega_\Omega) \in H_c^n(V; \mathbb{Z})$. Set $V_j := f_j^{-1}(\Omega) \cap D$ for each $j \in \mathbb{N}$. Since $f(\partial D) \cap \Omega = \emptyset$, by the uniform convergence of (f_j) to f on ∂D , there exists $j_0 \in \mathbb{N}$ for which $f_j(\partial D) \cap \Omega = \emptyset$ for every $j \geq j_0$. Thus $(f_j|V_j)^*(\omega_\Omega) \in H_c^n(V_j; \mathbb{Z})$ for $j \geq j_0$. Furthermore, mappings $f|D$ and $f_j|D$ are properly homotopic with respect to Ω for every $j \in \mathbb{N}$ large enough, that is, there exists $j_1 \in \mathbb{N}$ so that for every $j \geq j_1$ there exists a homotopy $F_j: \overline{D} \times [0, 1] \rightarrow N$ from $f|D$ to $f_j|D$ and $F_j(\partial D \times [0, 1]) \cap \Omega = \emptyset$. Thus $\iota_{V,D}((f|V)^*\omega_\Omega) = \iota_{V_j,D}((f_j|V_j)^*\omega_\Omega)$ for $j \geq \max\{j_0, j_1\}$, and (2.1) completes the proof. $\square$

A point $x' \in M$ is a *non-normality point* of a family $\mathcal{F}$ of K -quasiregular mappings from M to N if $\mathcal{F}$ is not normal on any open neighborhood of x' . A point $x' \in M$ is an *isolated essential singularity* of a quasiregular mapping $f: M \setminus \{x'\} \rightarrow N$ if f does not extend to a continuous mapping from M to N .

From now on, suppose that N is closed. The following theorems are manifold

versions Miniowitz's Zalcman-type lemma ([15, Lemma 1]) and a Miniowitz–Zalcman-type rescaling principle for isolated essential singularities, respectively.

Theorem 2.2 ([13, Theorem 19.9.3]). *Let M be an oriented Riemannian n -manifold and N a closed and oriented Riemannian n -manifold, $n \geq 2$, and let $x' \in M$. Then a family $\mathcal{F}$ of K -quasiregular mappings, $K \geq 1$, from M to N is not normal at x' if and only if there exist sequences (x_j) , (ρ_j) , and (f_j) in $\mathbb{R}^n$, $(0, \infty)$, and $\mathcal{F}$, respectively, and a non-constant K -quasiregular mapping $g: \mathbb{R}^n \rightarrow N$ such that $\lim_{j \rightarrow \infty} x_j = \phi(x')$, $\lim_{j \rightarrow \infty} \rho_j = 0$ and*

$$(2.2) \quad \lim_{j \rightarrow \infty} f_j \circ \phi^{-1}(x_j + \rho_j v) = g(v)$$

locally uniformly on $\mathbb{R}^n$, where $\phi: D \rightarrow \mathbb{R}^n$ is a coordinate chart of M at x' .

Theorem 2.3 ([17, Theorem 1]). *Let M be an oriented Riemannian n -manifold and N a closed and oriented Riemannian n -manifold, $n \geq 2$, and let $x' \in M$. Then a K -quasiregular mapping $f: M \setminus \{x'\} \rightarrow N$, $K \geq 1$, has an essential singularity at x' if and only if there exist sequences (x_j) and (ρ_j) in $\mathbb{R}^n$ and $(0, \infty)$, respectively, and a non-constant K -quasiregular mapping $g: X \rightarrow N$, where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$, such that $\lim_{j \rightarrow \infty} x_j = \phi(x')$, $\lim_{j \rightarrow \infty} \rho_j = 0$, and*

$$(2.3) \quad \lim_{j \rightarrow \infty} f \circ \phi^{-1}(x_j + \rho_j v) = g(v)$$

locally uniformly on X , where $\phi: D \rightarrow \mathbb{R}^n$ is a coordinate chart of M at x' .

By the Holopainen–Rickman Picard-type theorem [10], for every $n \geq 2$ and every $K \geq 1$, there exists a non-negative integer q such that $\#(N \setminus f(\mathbb{R}^n)) \leq q$ for every closed and oriented Riemannian n -manifold N and every non-constant K -quasiregular mapping $f: \mathbb{R}^n \rightarrow N$. We use this Picard-type theorem in this article also in the following form.

Theorem 2.4. *For every $n \geq 2$ and every $K \geq 1$, there exists a non-negative integer q' such that $\#(N \setminus g(X)) \leq q'$ for every closed and oriented Riemannian n -manifold N and every non-constant K -quasiregular mapping $f: X \rightarrow N$, where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$.*

Proof. Let $Z_n: \mathbb{R}^n \rightarrow \mathbb{R}^n \setminus \{0\}$ be the Zorich mapping and $K_n \geq 1$ the distortion constant of Z_n ; see e.g. [18, I.3.3] for the construction of the Zorich map. Set $K' := K \cdot K_n \geq 1$. Replacing f with $f \circ Z_n$ if necessary, we may assume that f is a K' -quasiregular mapping from $\mathbb{R}^n$ to N . Now the Holopainen–Rickman Picard-type theorem [10] completes the proof. $\square$

Let $q'(n, K)$ be the smallest such $q' \in \mathbb{N} \cup \{0\}$ as in Theorem 2.4, which we call the *quasiregular Picard constant for parameters $n \geq 2$ and $K \geq 1$* .

Having a Hurwitz-type theorem (Lemma 2.1) and rescaling theorems for a non-normality point of a family of K -quasiregular mappings and for an essential isolated singularity of a quasiregular mapping (Theorems 2.2 and 2.3) at our disposal, a “from little to big by rescaling” argument deduces the following Montel-type and big Picard-type theorems; see [15] and [17, Theorem 2].

Theorem 2.5. *Let M be an oriented Riemannian n -manifold and N a closed and oriented Riemannian n -manifold, $n \geq 2$. Then a non-normality point $x' \in M$ of a family $\mathcal{F}$ of K -quasiregular mappings, $K \geq 1$, from M to N is contained in $\bigcup_{f \in \mathcal{F}} \overline{f^{-1}(y)}$ for every $y \in N$ except for at most $q'(n, K)$ points.*

Theorem 2.6. *Let M be an oriented Riemannian n -manifold and N a closed and oriented Riemannian n -manifold, $n \geq 2$. Then an essential singularity $x' \in M$ of a K -quasiregular mapping $f : M \setminus \{x'\} \rightarrow N$, $K \geq 1$, is accumulated by $f^{-1}(y)$ for every $y \in N$ except for at most $q'(n, K)$ points.*

The similarity Theorems 2.5 and 2.6 goes beyond the statements and we prove these results simultaneously. The argument can also be viewed as a prototype of the proofs of Theorems 1 and 2.

Proof of Theorems 2.5 and 2.6. Let $x' \in M$ be either a non-normality point in Theorem 2.5 or an isolated essential singularity in Theorem 2.6.

Let X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$ and let $g : X \rightarrow N$ be the non-constant quasiregular mapping $v \mapsto f_j \circ \phi^{-1}(x_j + \rho_j v)$ as in Lemma 2.2 or in Lemma 2.3, respectively, associated to this x' . Here $f_j \equiv f$ if x' is as in Lemma 2.6.

Then $g(X)$ is an open subset in N , and satisfies $\#(N \setminus g(X)) \leq q'(n, K)$ by Theorem 2.4.

Let $y \in g(X)$. Fix a subdomain U in N containing y for which some component V of $g^{-1}(U)$ is relatively compact in X . Then $g : V \rightarrow U$ is proper. By the locally uniform convergence and Lemma 2.1, for every $j \in \mathbb{N}$ large enough, there exists $v_j \in V$ such that $\phi^{-1}(x_j + \rho_j v_j) \in f_j^{-1}(y)$. By the uniform convergence, $\lim_{j \rightarrow \infty} \phi^{-1}(x_j + \rho_j v) = x'$ uniformly on $v \in \overline{V}$. Thus $\lim_{j \rightarrow \infty} \phi^{-1}(x_j + \rho_j v_j) = x'$ and $x' \in \bigcup_{j \in \mathbb{N}} \overline{f_j^{-1}(y)}$.

Moreover, if x' is an essential singularity of f , then $\phi^{-1}(x_j + \rho_j v_j) \neq x'$ for every $j \in \mathbb{N}$. Thus x' is accumulated by $\bigcup_{j \in \mathbb{N}} f_j^{-1}(y) = f^{-1}(y)$. $\square$

The following Nevanlinna’s four totally ramified value theorem is specific to the case $n = 2$. Theorem 2.7 reduces to the original case that $X = \mathbb{R}^2$ and $N = \mathbb{S}^2$ by lifting it to the (conformal) universal coverings of X and N , which are isomorphic to $\mathbb{R}^2$ and a subdomain in $\mathbb{S}^2$, respectively.

Theorem 2.7 (cf. [16, p. 279, Theorem]). *Let $g : X \rightarrow N$ be a non-constant quasiregular mapping from X to a closed, oriented and connected Riemannian 2-manifold N , where X is either $\mathbb{R}^2$ or $\mathbb{R}^2 \setminus \{0\}$. Then for every $E \subset N$ containing more than 4 points, $E \cap g(X \setminus B_g) \neq \emptyset$.*

Again, having a Hurwitz-type theorem (Lemma 2.1) and rescaling theorems for both a non-normality point of a family of K -quasiregular mappings and an isolated singularity of a quasiregular mapping (Theorems 2.2 and 2.3) at our disposal, a “from little to big by rescaling” argument deduces the following two big versions of Theorem 2.7.

Lemma 2.8. *Let M be an oriented Riemannian 2-manifold and N a closed and oriented Riemannian 2-manifold, $n \geq 2$. Then a non-normality point $x' \in M$ of a family $\mathcal{F}$ of K -quasiregular mappings, $K \geq 1$, from M to N is contained in $\bigcup_{f \in \mathcal{F}} \overline{f^{-1}(E) \setminus B_f}$ for every $E \subset N$ containing more than 4 points.*

Lemma 2.9. *Let M be an oriented Riemannian 2-manifold and N a closed and oriented Riemannian 2-manifold, $n \geq 2$. Then an essential singularity $x' \in M$ of a quasiregular mapping $f : M \setminus \{x'\} \rightarrow N$ is accumulated by $f^{-1}(E) \setminus B_f$ for every $E \subset N$ containing more than 4 points.*

Again, due the similarity of the statements we give a simultaneous proof.

Proof of Lemmas 2.8 and 2.9. Let $x' \in M$ be as in either Lemma 2.8 or Lemma 2.9, and let $g(v) = f_j \circ \phi^{-1}(x_j + \rho_j v)$ be a non-constant quasiregular mapping from X to N as in Lemmas 2.2 and 2.3, respectively, associated to this x' , where X is either $\mathbb{R}^2$ or $\mathbb{R}^2 \setminus \{0\}$, and $f_j \equiv f$ in the case that x' is as in Lemma 2.9.

Let E be a subset in N containing more than 4 points. Then by Nevanlinna’s four totally ramified values theorem (Theorem 2.7), $g^{-1}(E) \setminus B_g \neq \emptyset$. Fix subdomains U in N intersecting E small enough that some component V of $g^{-1}(U)$ is relatively compact in $X \setminus B_g$. Then $g : V \rightarrow U$ is univalent, and by the locally uniform convergence (2.2) or (2.3) on X and the Hurwitz-type theorem (Lemma 2.1), for every $j \in \mathbb{N}$ large enough, there exists $v_j \in V$ such that $\phi^{-1}(x_j + \rho_j v_j) \in f_j^{-1}(E) \setminus B_{f_j}$. Furthermore, $\lim_{j \rightarrow \infty} \phi^{-1}(x_j + \rho_j v) = x'$ uniformly on $v \in \overline{V}$. Thus $\lim_{j \rightarrow \infty} \phi^{-1}(x_j + \rho_j v_j) = x'$ and $x' \in \bigcup_{j \in \mathbb{N}} \overline{f_j^{-1}(E) \setminus B_{f_j}}$.

Moreover, in the case that x' is as in Lemma 2.9, then $\phi^{-1}(x_j + \rho_j v_j) \neq x'$ for every $j \in \mathbb{N}$, so x' is accumulated by $\bigcup_{j \in \mathbb{N}} f_j^{-1}(E) \setminus B_{f_j} = f^{-1}(E) \setminus B_f$. $\square$

Let $f : \Omega \rightarrow M$ be a non-constant local uniformly K -quasiregular mapping from an open subset Ω in a closed and oriented Riemannian n -manifold M , $n \geq 2$, to M . The following lemmas are elementary.

Lemma 2.10. $f^{-1}(\mathcal{E}(f)) \subset \mathcal{E}(f)$, $f^{-1}(D_f) \subset D_f$, $f(D_f) \subset D_f$, $f^{-1}(F(f)) \subset F(f)$, $f(F(f)) \subset F(f)$, $f^{-1}(J(f)) \subset J(f)$, and $f(J(f) \cap D_f) \subset J(f)$.

Proof. The first inclusion $f^{-1}(\mathcal{E}(f)) \subset \mathcal{E}(f)$ is obvious. The inclusion $f^{-1}(D_f) \subset D_f$ immediately follows by the continuity and openness of f . The inclusion $f(D_f) \subset D_f$ also follows by the continuity and openness of f .

The inclusion $f^{-1}(F(f)) \subset F(f)$ follows by the continuity and openness of f and the Arzelà-Ascoli theorem. Indeed, let $x \in f^{-1}(F(f))$. Then $\{f^k; k \in \mathbb{N}\}$ is equicontinuous at $f(x)$, so $\{f^k \circ f; k \in \mathbb{N}\}$ is equicontinuous at x . Hence $x \in F(f)$.

Similarly, the inclusion $f(F(f)) \subset F(f)$ also follows by the continuity and openness of f and the Arzelà-Ascoli theorem. Indeed, let $x \in f(F(f))$, i.e., $x = f(y)$ for some $y \in F(f)$. Then $\{f^k \circ f; k \in \mathbb{N}\}$ is equicontinuous at y , so $\{f^k; k \in \mathbb{N}\}$ is equicontinuous at $x = f(y)$. Hence $x \in F(f)$.

Let us show $f^{-1}(J(f)) \subset J(f)$. The inclusion $f^{-1}(J(f) \setminus D_f) \subset J(f)$ follows from $f(D_f) \subset D_f$, which is equivalent to $f^{-1}(M \setminus D_f) \subset M \setminus D_f$, and $M \setminus D_f \subset J(f)$. The inclusion $f^{-1}(J(f) \cap D_f) \subset J(f)$ follows from $J(f) \cap D_f = D_f \setminus F(f)$ and $f(F(f)) \subset F(f)$.

The final $f(J(f) \cap D_f) \subset J(f)$ follows from $f^{-1}(F(f)) \subset F(f)$, which implies $f(D_f \setminus F(f)) \subset D_f \setminus F(f)$, and $J(f) \cap D_f = D_f \setminus F(f)$. $\square$

Lemma 2.11. *The interior of $J(f) \cap D_f$ is empty unless $J(f) = M$.*

Proof. Let $x \in J(f)$ be an interior point of $J(f)$, and fix an open neighborhood U of x in M contained in $J(f)$. Then by the Montel-type theorem (Theorem 2.5), we have $\#(M \setminus \bigcup_{k \in \mathbb{N}} f^k(U)) < \infty$, so $M = \overline{\bigcup_{k \in \mathbb{N}} f^k(U)}$, which is in $J(f)$ by Lemma 2.10 and the closedness of $J(f)$. $\square$

A cyclic Fatou component of f is a component U of $F(f)$ such that $f^p(U) \subset U$ for some $p \in \mathbb{N}$, which is called a period of U (under f). The proof of the following is almost verbatim to the Euclidean case and we refer to Hinkkanen–Martin–Mayer [9, Proposition 4.9] for the details.

Theorem 2.12. *Let Ω be an open subset in a closed and oriented Riemannian n -manifold M , $n \geq 2$, and $f: \Omega \rightarrow M$ be a non-elementary local uniformly quasiregular mapping. Then a cyclic Fatou component U of f having a period $p \in \mathbb{N}$ is one of the following:*

- (i) *a singular (or rotation) domain of f , that is, $f^p: U \rightarrow f^p(U)$ is univalent and the limit of any locally uniformly convergent sequence $(f^{pk_i})_i$ on U , where $\lim_{i \rightarrow \infty} k_i = \infty$, is non-constant,*

- (ii) an immediate attractive basin of f , that is, the sequence $(f^{p_k})_k$ converges locally uniformly on U , the limit is constant, and its value is in U , or
- (iii) an immediate parabolic basin of f , that is, the limit of any locally uniformly convergent sequence $(f^{p_{k_i}})_i$ on U , where $\lim_{i \rightarrow \infty} k_i = \infty$, is constant and its value is in ∂U .

In the following sections, given a subset S in $\mathbb{R}^n$ and $a, b \in \mathbb{R}$, we denote by $aS + b$ the set $\{av + b \in \mathbb{R}^n; v \in S\}$.

§ 3. Proof of Theorem 1

Let $\mathbb{M}$ be a closed, oriented, and connected Riemannian n -manifold, $n \geq 2$, and $f: \mathbb{M} \setminus S_f \rightarrow \mathbb{M}$ be a non-constant local uniformly K -quasiregular mapping, $K \geq 1$, where S_f is a countable and closed subset in $\mathbb{M}$ and consists of isolated essential singularities of f and their accumulation points in $\mathbb{M}$.

Lemma 3.1. *The interior of $J(f)$ is empty unless $J(f) = \mathbb{M}$.*

Proof. By Lemma 2.11, the interior of $J(f) \cap D_f$ is empty unless $J(f) = \mathbb{M}$. On the other hand, $J(f) \setminus D_f = \overline{\bigcup_{k \geq 0} f^{-k}(S_f)}$, which is the closure of a countable subset in $\mathbb{M}$, has no interior by the Baire category theorem. $\square$

Set

$$J_1(f) := J(f) \setminus \overline{\bigcup_{k \geq 0} f^{-k}(S_f)} = J(f) \cap D_f \quad \text{and}$$

$$J_2(f) := \bigcup_{k \geq 0} f^{-k}(\{x \in S_f : x \text{ is isolated in } S_f\}).$$

The forthcoming arguments in this and the next sections rest on the following observation on the density of $J_1(f) \cup J_2(f)$ in $J(f)$.

Lemma 3.2. *The set $J_1(f) \cup J_2(f)$ is dense in $J(f)$. Furthermore,*

- (i) if $\#\bigcup_{k \geq 0} f^{-k}(S_f) < \infty$, then $J_1(f) \cup J_2(f) = J(f)$ and $\#J_2(f) < \infty$;
- (ii) if $\#\bigcup_{k \geq 0} f^{-k}(S_f) = \infty$, then $J_1(f) = \emptyset$ and $J(f) = \overline{J_2(f)}$.

Proof. The density in S_f of isolated points of S_f implies $\overline{\bigcup_{k \geq 0} f^{-k}(S_f)} = \overline{J_2(f)}$, so $J_1(f) \cup \overline{J_2(f)} = J(f)$. If $\#\bigcup_{k \geq 0} f^{-k}(S_f) < \infty$, then $J_2(f) = \bigcup_{k \geq 0} f^{-k}(S_f) = \overline{J_2(f)}$, so $J(f) = J_1(f) \cup J_2(f)$ and $\#J_2(f) < \infty$. If $\#\bigcup_{k \geq 0} f^{-k}(S_f) = \infty$, then by the Montel-type theorem (Theorem 2.5), we have $J_1(f) = \emptyset$, so $J(f) = J_1(f) \cup \overline{J_2(f)} = \overline{J_2(f)}$. $\square$

The following is a simple application of the rescaling theorems (Theorems 2.2 and 2.3) to points in the dense subset $J_1(f) \cup J_2(f)$ in $J(f)$. We leave the details to the interested reader.

Lemma 3.3. *Let $a \in J_1(f) \cup J_2(f)$ and let $\phi : D \rightarrow \mathbb{R}^n$ be a coordinate chart of $\mathbb{M}$ at a . Then there exist*

- (i) *sequences (x_m) in $\mathbb{R}^n$ and (ρ_m) in $(0, \infty)$, which respectively tend to $\phi(a)$ and 0 as $m \rightarrow \infty$,*
- (ii) *a sequence (k_m) in $\mathbb{N}$, which is constant when $a \in J_2(f)$, and*
- (iii) *a non-constant K -quasiregular mapping $g : X \rightarrow \mathbb{M}$, where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$, and $X = \mathbb{R}^n$ when $a \in J_1(f)$,*

such that

$$(3.1) \quad \lim_{m \rightarrow \infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v) = g(v)$$

locally uniformly on X .

We show the remaining assertions in Theorem 1 in separate lemmas. We continue to use the notation $q'(n, K)$ introduced in Section 2.

We first show both the non-triviality of the Julia set $J(f)$ and the finiteness of the exceptional set $\mathcal{E}(f)$ for non-injective f .

Lemma 3.4. *If $S_f \neq \emptyset$, then f is non-injective, $J(f) \neq \emptyset$, and $\#\mathcal{E}(f) \leq q'(n, K)$. If $S_f = \emptyset$ and f is not injective, then $J(f) \neq \emptyset$, $\mathcal{E}(f) \subset F(f)$, and $\#\mathcal{E}(f) \leq q'(n, K)$.*

Proof. If $S_f \neq \emptyset$, then by the big Picard-type theorem (Theorem 2.6), f is not injective and $\#\mathcal{E}(f) \leq q'(n, K)$, and by the definition of $J(f)$, we have $\emptyset \neq S_f \subset \bigcup_{k \geq 0} f^{-k}(S_f) \subset J(f)$.

From now on, suppose that $S_f = \emptyset$ and $f : \mathbb{M} \setminus S_f \rightarrow \mathbb{M}$ is non-injective. Then $\deg f \geq 2$. We show first that $J(f) \neq \emptyset$. Indeed, suppose $J(f) = \emptyset$. Then, by compactness of $\mathbb{M}$, there exists a sequence (k_m) in $\mathbb{N}$ tending to ∞ such that (f^{k_m}) tends to a K -quasiregular endomorphism $h : \mathbb{M} \rightarrow \mathbb{M}$ uniformly on $\mathbb{M}$. Then for every $m \in \mathbb{N}$ large enough, f^{k_m} is homotopic to h and $\deg h = \deg(f^{k_m}) = (\deg f)^{k_m} \rightarrow \infty$ as $m \rightarrow \infty$ by the homotopy invariance of the degree. This is a contradiction and $J(f) \neq \emptyset$.

We show now that $\mathcal{E}(f) \subset F(f)$. Let $a \in \mathcal{E}(f)$. Since $\#\bigcup_{k \geq 0} f^{-k}(a) < \infty$, f restricts to a permutation of $\bigcup_{k \geq 0} f^{-k}(a)$. Thus there exists $p \in \mathbb{N}$ for which $f^p(a) = a$ and $i(a, f^p) = \deg(f^p) \geq 2$. Fix a local chart $\phi : D \rightarrow \mathbb{R}^n$ at a and identify f^p with

$\phi \circ f^p \circ \phi^{-1}$ in a neighborhood of $a' := \phi(a)$ where the composition is defined. Then there exist a neighborhood U of a' and $C > 0$ such that for every $k \in \mathbb{N}$, f^{pk} is a K -quasiregular mapping from U onto its image, and that for every $k \in \mathbb{N}$ and every $x \in U$,

$$|f^{pk}(x) - f^{pk}(a')| \leq C|x - a'|^{(i(a', f^p)^k/K)^{1/(n-1)}}$$

by [18, Theorem III.4.7] (see also [9, Lemma 4.1]). Then $\lim_{k \rightarrow \infty} f^{pk} = a'$ locally uniformly on U . Hence $a \in F(f)$.

Finally, we show $\#\mathcal{E}(f) \leq q'(n, K)$. If $\#\mathcal{E}(f) > q'(n, K)$, we may fix $A \subset \mathcal{E}(f)$ such that $q'(n, K) < \#A < \infty$ and $A' := \bigcup_{k \geq 0} f^{-k}(A) \subset \mathcal{E}(f)$. Then $q'(n, K) < \#A' < \infty$, and by the above description of each point in $\mathcal{E}(f)$, $f^{-1}(A') = A'$. By $\#A' > q'(n, K)$ and Theorem 2.5, $J(f) \subset \overline{\bigcup_{k \in \mathbb{N}} f^{-k}(A')}$, which contradicts that $\overline{\bigcup_{k \in \mathbb{N}} f^{-k}(A')} = \overline{A'} = A' \subset \mathcal{E}(f) \subset F(f)$. $\square$

We show next the accumulation of the backward orbits under f of non-exceptional points to $J(f)$ for non-injective f , which implies the perfectness of $J(f)$ for non-elementary f .

Lemma 3.5. *Suppose f is not injective. Then, for every $z \in \mathbb{M} \setminus \mathcal{E}(f)$, each point in $J(f)$ is accumulated by $\bigcup_{k \geq 0} f^{-k}(z)$. Moreover, if f is non-elementary, then $J(f)$ is perfect.*

Proof. Fix $a \in J_1(f) \cup J_2(f)$. Let $g(v) = \lim_{m \rightarrow \infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a . Then $\#(\mathbb{M} \setminus g(X)) < \infty$ by Theorem 2.4.

Fix $z \in \mathbb{M} \setminus \mathcal{E}(f)$. Then we can choose subdomains U_1 and U_2 in $g(X)$ intersecting $\bigcup_{k \in \mathbb{N}} f^{-k}(z)$ and having pair-wise disjoint closures so that, for each $i \in \{1, 2\}$, some component V_i of $g^{-1}(U_i)$ is relatively compact in X .

For each $i \in \{1, 2\}$, $g: V_i \rightarrow U_i$ is proper. By the locally uniform convergence (3.1) on X and Lemma 2.1, $f^{k_m}(\phi^{-1}(x_m + \rho_m V_i))$ intersects $\bigcup_{k \geq 0} f^{-k}(z)$ for every $m \in \mathbb{N}$ large enough. Thus, for m large enough, we may fix $v_m^{(i)} \in V_i$ satisfying $y_m^{(i)} := \phi^{-1}(x_m + \rho_m v_m^{(i)}) \in \bigcup_{k \geq 0} f^{-k}(z)$.

Let $i \in \{1, 2\}$. By the uniform convergence $\lim_{m \rightarrow \infty} \phi^{-1}(x_m + \rho_m v) = a$ on $v \in \overline{V_i}$, we have $\lim_{m \rightarrow \infty} y_m^{(i)} = a$, and, by the uniform convergence (3.1) on $\overline{V_i}$, we have $\bigcap_{N \in \mathbb{N}} \overline{\{f^{k_m}(y_m^{(i)}); k \geq N\}} \subset g(\overline{V_i}) = \overline{U_i}$. Since $\overline{U_1} \cap \overline{U_2} = \emptyset$, $\{y_m^{(1)}, y_m^{(2)}\} \neq \{a\}$ for $m \in \mathbb{N}$ large enough.

Hence any point $a \in J_1(f) \cup J_2(f)$ is accumulated by $\bigcup_{k \in \mathbb{N}} f^{-k}(z)$, and so is any point in $J(f)$ by Lemma 3.2.

If f is non-elementary, then choosing $z \in J(f) \setminus \mathcal{E}(f)$, we obtain the perfectness of $J(f)$ by the former assertion and $f^{-1}(J(f)) \subset J(f)$. $\square$

We record the following consequence of Lemmas 3.2, 3.4, and 3.5 as a lemma.

Lemma 3.6. *For non-elementary f , $J(f)$ is perfect, $\mathcal{E}(f)$ is finite, and any point in $J(f)$ is accumulated by $(J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$.*

Finally, the following lemma completes the proof of Theorem 1.

Lemma 3.7. *If f is non-elementary, then any point in $J(f)$ is accumulated by the set of all periodic points of f .*

Proof. Fix an open subset U in $\mathbb{M}$ intersecting $J(f)$. Let $a \in (J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$, and let $g(v) = \lim_{m \rightarrow \infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a , where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$ and $\phi: D \rightarrow \mathbb{R}^n$ is a coordinate chart of $\mathbb{M}$ at a . By Lemma 3.5 and Theorem 2.4,

$$(U \cap \bigcup_{k \geq 0} f^{-k}(a)) \cap g(X) \neq \emptyset.$$

Hence we can choose $j_1 \in \mathbb{N} \cup \{0\}$ and a subdomain $D_1 \Subset D$ containing a such that some component U_1 of $f^{-j_1}(D_1)$ is relatively compact in U and that some component V_1 of $g^{-1}(U_1)$ is relatively compact in X . Then $f^{j_1} \circ g: V_1 \rightarrow D_1$ is proper.

Choose an open neighborhood $W \Subset X$ of $\overline{V_1}$ small enough that $f^{j_1} \circ g(W) \Subset D$. By the uniform convergence $\lim_{m \rightarrow \infty} \phi^{-1}(x_m + \rho_m v) = a \in D_1$ on $v \in \overline{W}$ and the uniform convergence (3.1) on $\overline{W}$, we can define a mapping $\psi: \overline{W} \rightarrow \mathbb{R}^n$ and mappings $\psi_m: \overline{W} \rightarrow \mathbb{R}^n$ for every $m \in \mathbb{N}$ large enough by

$$\begin{cases} \psi(v) := \phi \circ f^{j_1} \circ g(v) - \phi(a) & \text{and} \\ \psi_m(v) := \phi \circ f^{j_1} \circ f^{k_m} \circ \phi^{-1}(x_m + \rho_m v) - (x_m + \rho_m v), \end{cases}$$

so that $\lim_{m \rightarrow \infty} \psi_m = \psi$ uniformly on $\overline{W}$.

The limit $\psi: V_1 \rightarrow \psi(V_1)$ is non-constant, quasiregular, and proper, and satisfies $0 \in \psi(V_1)$ by $a \in D_1 = f^{j_1}(g(V_1))$. Although for each $m \in \mathbb{N}$ large enough, $\psi_m: V_1 \rightarrow \mathbb{R}^n$ is not necessarily quasiregular, we have $\lim_{m \rightarrow \infty} \mu(0, \psi_m, V_1) = \mu(0, \psi, V_1) > 0$ after applying Lemma 2.1 to (ψ_m) and ψ on $\overline{V_1}$. Thus $0 \in \psi_m(V_1)$.

Hence for every $m \in \mathbb{N}$ large enough, there exists $v_m \in V_1$ such that $y_m := \phi^{-1}(x_m + \rho_m v_m)$ is a fixed point of $f^{j_1} \circ f^{k_m}$. Hence also $f^{k_m}(y_m)$ is a fixed point of $f^{j_1} \circ f^{k_m}$. By the uniform convergence (3.1) on $\overline{V_1}$, we have $\bigcap_{N \in \mathbb{N}} \overline{\{f^{k_m}(y_m); k \geq N\}} \subset g(\overline{V_1}) = \overline{U_1} \subset U$, so $f^{k_m}(y_m) \in U$ for every $m \in \mathbb{N}$ large enough.

We conclude that $J(f)$ is in the closure of the set of all periodic points of f , so the perfectness of $J(f)$ completes the proof. $\square$

§ 4. Proof of Theorem 2

Let $\mathbb{M}$ be a closed, oriented, and connected Riemannian n -manifold, $n \geq 2$. Suppose $f: \mathbb{M} \setminus S_f \rightarrow \mathbb{M}$ is a non-elementary local uniformly K -quasiregular mapping, $K \geq 1$, where S_f is a countable and closed subset in $\mathbb{M}$ and consists of isolated essential singularities of f and their accumulation points in $\mathbb{M}$. We continue to use the notations $J_1(f)$ and $J_2(f)$ introduced in Section 3.

We first show the first assertion of Theorem 2.

Lemma 4.1. *If $F(f)$ is non-empty and connected, then every point in $J(f)$ is accumulated by the set of periodic points of f contained in $J(f)$.*

Proof. By the assumption, $F(f)$ is a fixed cyclic Fatou component of f . We show first that f is not univalent on $F(f)$.

We consider three cases separately. In the case $S_f \neq \emptyset$, by the big Picard-type theorem (Theorem 2.6), for every $y \in F(f)$ except for at most finitely many points, we have $\#f^{-1}(y) = \infty$. In the case that $S_f = \emptyset$ and $B_f \cap F(f) = \emptyset$, we have $\deg f \geq 2$, and also $f(B_f) \cap F(f) = \emptyset$ by $f^{-1}(F(f)) \subset F(f)$. Thus $\#f^{-1}(y) = \deg f \geq 2$ for every $y \in F(f)$. Since $f^{-1}(F(f)) \subset F(f)$, f is not univalent on $F(f)$ in these two cases.

Suppose now that $S_f = \emptyset$ and $B_f \cap F(f) \neq \emptyset$. By the classification of cyclic Fatou components (Theorem 2.12), $F(f)$ is a fixed immediate either attractive or parabolic basin of f . So all the periodic points constructed in Lemma 3.7, but at most one, are in $J(f) = \mathbb{M} \setminus F(f)$. □

Next, we give a useful criterion for the repelling density in $J(f)$.

Lemma 4.2. *Let $a \in (J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$ and suppose that a non-constant quasiregular mapping g in Lemma 3.3 associated to this a satisfies the unramification condition*

$$(4.1) \quad a \notin \bigcup_{k \in \mathbb{N}} f^k(B_{f^k}) \quad \text{and} \quad J(f) \cap g(X \setminus B_g) \neq \emptyset.$$

Then every point in $J(f)$ is accumulated by the set of all repelling periodic points of f .

Proof. Let $a \in (J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$ and let $g(v) = \lim_{m \rightarrow \infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a , where $\phi: D \rightarrow \mathbb{R}^n$ is a coordinate chart of $\mathbb{M}$ at a , and suppose that these a and g satisfy (4.1).

Fix an open subset U in $\mathbb{M}$ intersecting $J(f)$. By Lemma 3.5 and $\#\mathcal{E}(f) < \infty$, there exists $j_1 \in \mathbb{N} \cup \{0\}$ such that $(f^{-j_1}(a) \cap U) \setminus \mathcal{E}(f) \neq \emptyset$. By the latter condition

in (4.1), $g(X \setminus B_g)$ is an open subset in $\mathbb{M}$ intersecting $J(f)$. Thus, by Lemma 3.5, there exists $j_2 \in \mathbb{N} \cup \{0\}$ such that $f^{-j_2}((f^{-j_1}(a) \cap U) \setminus \mathcal{E}(f)) \cap g(X \setminus B_g) \neq \emptyset$. Hence by the first condition in (4.1), we can choose a subdomain $D_1 \Subset D \setminus f^{j_1+j_2}(B_{f^{j_1+j_2}})$ containing a such that some component U_1 of $f^{-j_1}(D_1)$ is relatively compact in U and that some component V_1 of $g^{-1}(f^{-j_2}(U_1))$ is relatively compact in $X \setminus B_g$. Then $f^{j_1+j_2} \circ g : V_1 \rightarrow D_1$ is univalent.

By the same argument as in the proof of Lemma 3.7, we may choose, for every $m \in \mathbb{N}$ large enough, a point $v_m \in V_1$ such that $y_m := \phi^{-1}(x_m + \rho_m v_m)$ is a fixpoint of $f^{j_1+j_2} \circ f^{k_m}$. By the uniform convergence (3.1) on $\overline{V_1}$, we have $\bigcap_{N \in \mathbb{N}} \{f^{j_2} \circ f^{k_m}(y_m); k \geq N\} \subset f^{j_2}(g(\overline{V_1})) = \overline{U_1} \subset U$. Thus $f^{j_2} \circ f^{k_m}(y_m) \in U$ for every $m \in \mathbb{N}$ large enough.

Moreover, by the locally uniform convergence (3.1) on X and Lemma 2.1, the mapping $v \mapsto f^{j_1+j_2} \circ f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ is a univalent mapping from V_1 onto its image for every $m \in \mathbb{N}$ large enough. Hence

$$f^{j_1+j_2} \circ f^{k_m} : \phi^{-1}(x_m + \rho_m V_1) \rightarrow f^{j_1+j_2} \circ f^{k_m}(\phi^{-1}(x_m + \rho_m V_1))$$

is univalent for $m \in \mathbb{N}$ large enough. By the uniform convergence

$$\lim_{m \rightarrow \infty} \phi^{-1}(x_m + \rho_m v) = a \in D_1 = f^{j_1+j_2} \circ g(V_1)$$

on $v \in \overline{V_1}$ and the uniform convergence (3.1) on $\overline{V_1}$,

$$\phi^{-1}(x_m + \rho_m V_1) \Subset f^{j_1+j_2} \circ f^{k_m}(\phi^{-1}(x_m + \rho_m V_1))$$

for every $m \in \mathbb{N}$ large enough. Hence for every $m \in \mathbb{N}$ large enough, y_m is a repelling fixed point of $f^{j_1+j_2} \circ f^{k_m}$.

We conclude that $J(f)$ is in the closure of the set of all repelling periodic points of f , so the perfectness of $J(f)$ completes the proof. $\square$

We show the latter assertion of Theorem 2 under the conditions given there, separately.

Condition (i). Suppose $\# \bigcup_{k \geq 0} f^{-k}(S_f) < \infty$. Then by Lemmas 3.2 and 3.4, we have $\#(J_2(f) \cup \mathcal{E}(f)) < \infty$ and $J_1(f) = J(f) \setminus J_2(f)$. Suppose also that $\dim J(f) \geq n-1$. For every $k \in \mathbb{N}$, $\dim f^k(B_{f^k}) \leq n-2$, and then $\dim(\bigcup_{k \in \mathbb{N}} f^k(B_{f^k})) \leq n-2$ ([11, §2.2, Theorem III]). Hence we can fix $a \in J(f) \setminus (J_2(f) \cup \mathcal{E}(f) \cup \bigcup_{k \in \mathbb{N}} f^k(B_{f^k})) = J_1(f) \setminus (\mathcal{E}(f) \cup \bigcup_{k \in \mathbb{N}} f^k(B_{f^k}))$, and let $g : \mathbb{R}^n \rightarrow \mathbb{M}$ be a non-constant quasiregular mapping as in Lemma 3.3 associated to this a . Then $\dim g(B_g) \leq n-2$, so $J(f) \cap g(\mathbb{R}^n \setminus B_g) \neq \emptyset$.

The unramification condition (4.1) is satisfied by these a and g , and Lemma 4.2 completes the proof in this case.

Condition (ii). Let a be a repelling periodic point of f having a period $p \in \mathbb{N}$ in $D_f \setminus (\mathcal{E}(f) \cup \bigcup_{k \in \mathbb{N}} f^k(B_{f^k}))$. Then $a \in (J(f) \setminus \mathcal{E}(f)) \cap D(f) = J_1(f) \setminus \mathcal{E}(f)$. Let

$g(v) = \lim_{m \rightarrow \infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from $\mathbb{R}^n$ to $\mathbb{M}$ as in Lemma 3.3 associated to this a , where $\phi: D \rightarrow \mathbb{R}^n$ is a coordinate chart of $\mathbb{M}$ at this a . By [9, Theorem 6.3], we may, in fact, assume that $x_m \equiv \phi(a)$ and $p|k_m$ for all $m \in \mathbb{N}$, and g is in this case usually called a *Koenigs mapping of f^p at a* . Then $g(0) = a$, and by the proof of [9, Theorem 6.3], we also have $0 \notin B_g$. Hence $a \in J(f) \cap g(\mathbb{R}^n \setminus B_g)$, and (4.1) is satisfied by these a and g . Lemma 4.2 completes the proof in this case.

Condition (iii). Suppose that $J(f) \not\subset \bigcap_{j \in \mathbb{N}} \overline{\bigcup_{k \geq j} f^k(B_{f^k})}$. By the closedness of $\bigcap_{j \in \mathbb{N}} \overline{\bigcup_{k \geq j} f^k(B_{f^k})}$ and Lemma 3.6, we indeed have $J(f) \not\subset (\mathcal{E}(f) \cup \bigcap_{j \in \mathbb{N}} \overline{\bigcup_{k \geq j} f^k(B_{f^k})})$. Hence we can fix $N \in \mathbb{N}$ so large that the open subset $U_N := \mathbb{M} \setminus (\mathcal{E}(f) \cup \bigcup_{k \geq N} f^k(B_{f^k}))$ in $\mathbb{M}$ intersects $J(f)$.

Let $a \in (J_1(f) \cup J_2(f)) \cap U_N \subset (J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$, and let $g(v) = \lim_{m \rightarrow \infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a . Then $\#(\mathbb{M} \setminus g(X)) < \infty$ by Theorem 2.4. We claim that $\#\bigcup_{k \geq N} f^{-k}(a) = \infty$. Indeed, in the case $\#\bigcup_{k=0}^{N-1} f^{-k}(a) < \infty$, this follows by $a \notin \mathcal{E}(f)$. In the case $\#\bigcup_{k=0}^{N-1} f^{-k}(a) = \infty$, we have $S_f \neq \emptyset$. By applying the big Picard-type theorem (Theorem 2.6) in at most N times, we obtain $\#f^{-N}(a) = \infty$. Hence we can fix $j_1 \geq N$ such that $f^{-j_1}(a) \cap g(X) \neq \emptyset$, and a subdomain $U \Subset U_N$ containing a so small that some component V of $(f^{j_1} \circ g)^{-1}(U)$ is relatively compact in X . Then $g: V \rightarrow g(V)$ is proper.

By the uniform convergence (3.1) on $\overline{V}$, for every $m \in \mathbb{N}$ large enough, $f^{j_1} \circ f^{k_m} \circ \phi^{-1}(x_m + \rho_m V) \Subset U_N$. Then by $j_1 \geq N$ and the definition of U_N , $f^{k_m}: \phi^{-1}(x_m + \rho_m V) \rightarrow f^{k_m}(\phi^{-1}(x_m + \rho_m V))$ is univalent, so the mapping $v \mapsto f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ from V onto its image is univalent. Hence by the locally uniform convergence (3.1) on X and the Hurwitz-type theorem (Lemma 2.1), $V \cap B_g = \emptyset$. Then $\emptyset \neq f^{-j_1}(a) \cap g(V) \subset J(f) \cap g(X \setminus B_g)$, and (4.1) is satisfied by these a and g . Lemma 4.2 completes the proof in this case.

Condition (iv). Suppose that $n = 2$. If $\#\bigcup_{k \geq 0} f^{-k}(S_f) < \infty$, then by Lemmas 3.2 and 3.6, $J_1(f) = J(f) \setminus J_2(f)$ is uncountable. Since $\#\mathcal{E}(f) < \infty$ (in Lemma 3.6) and $\bigcup_{k \geq 0} B_{f^k}$ is countable (when $n = 2$), we may fix $a \in J_1(f) \setminus (J_2(f) \cup \mathcal{E}(f) \cup \bigcup_{k \in \mathbb{N}} f^k(B_{f^k})) \subset J_1(f) \setminus \mathcal{E}(f)$. Let $g: \mathbb{R}^n \rightarrow \mathbb{M}$ be a non-constant quasiregular mapping as in Lemma 3.3 associated to this a . By the countability of B_g (when $n = 2$) and the uncountability of $g^{-1}(J(f))$, we also have $g^{-1}(J(f)) \not\subset B_g$. The unramification condition (4.1) is satisfied by these a and g , and Lemma 4.2 completes the proof in this case.

In the remaining case $\#\bigcup_{k \geq 0} f^{-k}(S_f) = \infty$, the argument similar to the above does not work. For $n = 2$, instead of Lemma 4.2, we rely on the big versions (Lemmas 2.8 and 2.9) of the Nevanlinna four totally ramified value theorem (Theorem 2.7) to show Theorem 2 under $n = 2$, which is independent of the above proof specific to the

case $\#\bigcup_{k \geq 0} f^{-k}(S_f) < \infty$.

Proof of Theorem 2 under $n = 2$. Set

$$J'(f) := \begin{cases} J_1(f) \setminus \{\text{all periodic points of } f\} & \text{if } \#\bigcup_{k \geq 0} f^{-k}(S_f) < \infty, \\ J_2(f) & \text{if } \#\bigcup_{k \geq 0} f^{-k}(S_f) = \infty. \end{cases}$$

We claim that $J'(f)$ is dense in $J(f)$. If $\#\bigcup_{k \geq 0} f^{-k}(S_f) = \infty$, we have $J(f) = \overline{J_2(f)} = \overline{J'(f)}$ by Lemma 3.2. Thus we may assume that $\#\bigcup_{k \geq 0} f^{-k}(S_f) < \infty$ and it suffices to show that $J(f) = \overline{J'(f)}$.

By Lemmas 3.2 and 3.6, the set $J_1(f)$ is uncountable. Since f has at most countably many periodic points, $J'(f)$ is non-empty. Let $y \in J'(f)$. If $J(f) \not\subset \overline{J'(f)}$, then every point in $J(f) \setminus \overline{J'(f)}$ is accumulated by $\bigcup_{k \geq 0} f^{-k}(y)$ by Lemma 3.5. On the other hand, by Lemma 3.2, $\#J_2(f) < \infty$. Since $J_1(f) = J(f) \setminus J_2(f)$, there exists $x \in \bigcup_{k \geq 0} f^{-k}(y) \cap (J_1(f) \setminus \overline{J'(f)})$. Thus x is a periodic point of f , and so is y , which is a contradiction. Hence $J(f) = \overline{J'(f)}$ in the case $\#\bigcup_{k \geq 0} f^{-k}(S_f) < \infty$.

Since $J(f)$ is perfect, $\#J'(f) = \infty$. Fix an open subset U in $\mathbb{M}$ intersecting $J(f)$. We claim that there exists $a \in J'(f)$ such that $\#(U \cap \bigcup_{k \geq 0} (f^{-k}(a) \setminus B_{f^k})) = \infty$. Indeed, let $E \subset J'(f)$ such that $4 < \#E < \infty$ and let $b' \in U \cap (J_1(f) \cup J_2(f))$. For $b' \in J_1(f)$, $\{f^k; k \geq N\}$ is not normal at b' for any $N \in \mathbb{N}$. Hence $b' \in \bigcap_{N \in \mathbb{N}} \overline{\bigcup_{k \geq N} (f^{-k}(E) \setminus B_{f^k})}$ by Lemma 2.8. Moreover, if $b' \in f^{-k}(E)$ for infinitely many $k \in \mathbb{N}$, then, by $\#E < \infty$, $f^{k_1}(b') = f^{k_2}(b') \in E$ for some $k_1 < k_2$. Thus $f^{k_1}(b') \in E$ is a periodic point of f , which contradicts $E \subset J'(f)$. Hence b' is accumulated by $\bigcup_{k \geq 0} (f^{-k}(E) \setminus B_{f^k})$. In the case $b' \in J_2(f)$, b' is an isolated essential singularity of f^{j_1} for some $j_1 \in \mathbb{N}$, so by Lemma 2.9, b' is accumulated by $f^{-j_1}(E) \setminus B_{f^{j_1}}$. In both cases, by $\#E < \infty$, we can choose $a \in E$ such that $\#(U \cap \bigcup_{k \geq 0} (f^{-k}(a) \setminus B_{f^k})) = \infty$.

Let $g(v) = f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a , where X is either $\mathbb{R}^2$ or $\mathbb{R}^2 \setminus \{0\}$ and $\phi : D \rightarrow \mathbb{R}^2$ is a coordinate chart of $\mathbb{M}$ at a . Then by the Nevanlinna four totally ramified value theorem (Theorem 2.7),

$$\left(U \cap \bigcup_{k \geq 0} (f^{-k}(a) \setminus B_{f^k}) \right) \cap g(X \setminus B_g) \neq \emptyset.$$

Hence we can choose $j_1 \in \mathbb{N} \cup \{0\}$ and a subdomain $D_1 \Subset D$ containing a such that some component U_1 of $f^{-j_1}(D_1)$ is relatively compact in $U \setminus B_{f^{j_1}}$ and that some component V_1 of $g^{-1}(U_1)$ is relatively compact in $X \setminus B_g$. Then $f^{j_1} \circ g : V_1 \rightarrow D_1$ is univalent.

By the same argument in the proof of Lemma 3.7, for every $m \in \mathbb{N}$ large enough, we can choose $v_m \in V_1$ such that $y_m := \phi^{-1}(x_m + \rho_m v_m)$ is a fixed point of $f^{j_1} \circ f^{k_m}$, and so is $f^{k_m}(y_m)$, and we also have $f^{k_m}(y_m) \in U$ for every $m \in \mathbb{N}$ large enough.

Moreover, by the locally uniform convergence (3.1) on X and Lemma 2.1, the mapping $v \mapsto f^{j_1} \circ f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ is also a univalent mapping from V_1 onto its image for every $m \in \mathbb{N}$ large enough. Hence

$$f^{j_1} \circ f^{k_m} : \phi^{-1}(x_m + \rho_m V_1) \rightarrow f^{j_1} \circ f^{k_m}(\phi^{-1}(x_m + \rho_m V_1))$$

is univalent for $m \in \mathbb{N}$ large enough. By the uniform convergence $\lim_{m \rightarrow \infty} \phi^{-1}(x_m + \rho_m v) = a \in D_1 = f^{j_1} \circ g(V_1)$ on $v \in \overline{V_1}$ and the uniform convergence (3.1) on $\overline{V_1}$,

$$\phi^{-1}(x_m + \rho_m V_1) \subseteq f^{j_1} \circ f^{k_m}(\phi^{-1}(x_m + \rho_m V_1)).$$

for every $m \in \mathbb{N}$ large enough. Hence y_m is a repelling fixed point of $f^{j_1} \circ f^{k_m}$ for every $m \in \mathbb{N}$ large enough.

We conclude that $J(f)$ is in the closure of the set of all repelling periodic points of f , so the perfectness of $J(f)$ completes the proof. $\square$

§ 5. On the non-injectivity and non-elementarity of f

In the setting of Theorem 1, we have the following result on the non-elementarity of non-injective UQR-mappings.

Lemma 5.1. *Let $\mathbb{M}$ and $f: \mathbb{M} \setminus S_f \rightarrow \mathbb{M}$ be as in Theorem 1. Suppose in addition that f is non-injective. Then f is non-elementary if either $S_f = \emptyset$ or $\#\bigcup_{k \geq 0} f^{-k}(S_f) > q'(n, K)$.*

Proof. For $S_f = \emptyset$ the claim follows from Theorem 1. Suppose $\#\bigcup_{k \geq 0} f^{-k}(S_f) > q'(n, K)$. By the big Picard-type theorem (Theorem 2.6), we have $\#\bigcup_{k \geq 0} f^{-k}(S_f) = \infty$. Thus, by Lemma 3.2, $J(f) = \overline{\bigcup_{k \geq 0} f^{-k}(S_f)}$. Hence $J(f) \not\subset \mathcal{E}(f)$ since $\#\mathcal{E}(f) < \infty$. $\square$

It seems an interesting problem whether a non-injective f is always non-elementary. This is the case in holomorphic dynamics, i.e., the case that $\mathbb{M} = \mathbb{S}^2$ and $K = 1$. Indeed, if $0 < \#\bigcup_{k \geq 0} f^{-k}(S_f) \leq q'(2, 1) = 2$, f can be normalized to be either a transcendental entire function on $\mathbb{C}$ or a holomorphic endomorphism of $\mathbb{C} \setminus \{0\}$ having essential singularities at $0, \infty$, both of which are known to be non-elementary.

References

- [1] I. Baker, *Repulsive fixpoints of entire functions*, Math. Z. **104** (1968), 252–256.
- [2] D. Bargmann, *Simple proofs of some fundamental properties of the Julia set*, Ergodic Theory Dynam. Systems **19** (1999), 553–558.

- [3] F. Berteloot and J. Duval, *Une démonstration directe de la densité des cycles répulsifs dans l'ensemble de Julia*, Complex analysis and geometry (Paris, 1997), volume 188 of *Progr. Math.*, pages 221–222. Birkhäuser, Basel, 2000.
- [4] P. Bhattacharyya, *Iteration of analytic functions*, PhD thesis, Imperial College London (University of London), 1969.
- [5] A. Bolsch, *Repulsive periodic points of meromorphic functions*, Complex Variables Theory Appl. **31** (1996), 75–79.
- [6] A. N. Fletcher and D. A. Nicks, *Julia sets of uniformly quasiregular mappings are uniformly perfect*. Math. Proc. Cambridge Philos. Soc. **151** (2011), 541–550.
- [7] J. Heinonen and S. Rickman, *Geometric branched covers between generalized manifolds*, Duke Math. J. **113** (2002), 465–529.
- [8] M. E. Herring, *An extension of the Julia-Fatou theory of iteration*, Ph.D. thesis, London, 1995.
- [9] A. Hinkkanen, G. J. Martin, and V. Mayer, *Local dynamics of uniformly quasiregular mappings*, Math. Scand. **95** (2004), 80–100.
- [10] I. Holopainen and S. Rickman, *Ricci curvature, Harnack functions, and Picard type theorems for quasiregular mappings*, Analysis and topology, pages 315–326. World Sci. Publ., River Edge, NJ, 1998.
- [11] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Mathematical Series, v. 4, Princeton University Press, Princeton, N. J., 1941.
- [12] T. Iwaniec and G. Martin, *Quasiregular semigroups*, Ann. Acad. Sci. Fenn. **21** (1996), 241–254.
- [13] ———, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2001.
- [14] J. Milnor, *Dynamics in one complex variable*, volume 160 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, third edition, 2006.
- [15] R. Miniowitz, *Normal families of quasimeromorphic mappings*, Proc. Amer. Math. Soc. **84** (1982), 35–43.
- [16] R. Nevanlinna, *Analytic functions*, Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162. Springer-Verlag, New York, 1970.
- [17] Y. Okuyama and P. Pankka, *Rescaling principle for isolated essential singularities of quasiregular mappings*, ArXiv e-prints, Nov. 2012.
- [18] S. Rickman, *Quasiregular mappings*, volume 26 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993.
- [19] W. Schwick, *Repelling periodic points in the Julia set*, Bull. London Math. Soc. **29** (1997), 314–316.
- [20] H. Siebert, *Fixpunkte und normale Familien quasiregulärer Abbildungen*, Dissertation, University of Kiel, available at http://e-diss.uni-kiel.de/diss_1260. 2004.
- [21] ———, *Fixed points and normal families of quasiregular mappings*, J. Anal. Math. **98** (2006), 145–168.
- [22] J. Väisälä, *Discrete open mappings on manifolds*, Ann. Acad. Sci. Fenn. Ser. A I , No. 392:10, 1966.