

WHAT IF λ IS A STRONG LIMIT SINGULAR CARDINAL ?

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1. BACKGROUND

Let κ denote a regular uncountable cardinal and λ a cardinal $\geq \kappa$. Let $\mathcal{P}_\kappa\lambda$ denote the set $\{x \subset \lambda \mid |x| < \kappa\}$. We refer the reader to Kanamori [6, Section 25] for basic facts about the combinatorics of $\mathcal{P}_\kappa\lambda$.

Suppose I is an ideal over $\mathcal{P}_\kappa\lambda$. Let $I^+ = \{X \subseteq \mathcal{P}_\kappa\lambda \mid X \notin I\}$. Let $\mathbb{P}_I$ denote the p.o. of members of I^+ ordered by $X \leq_{\mathbb{P}_I} Y \iff X \subseteq Y$.

Definition 1.1.

We say that an ideal I is precipitous if $\Vdash_{\mathbb{P}_I} \text{“Ult}(V; G) \text{ is wellfounded”}$.

Let $NS_{\kappa\lambda} = \{X \subseteq \mathcal{P}_\kappa\lambda \mid X \text{ is the non-stationary}\}$. $NS_{\kappa\lambda}$ is known as the non-stationary ideal over $\mathcal{P}_\kappa\lambda$. For a stationary $X \subseteq \mathcal{P}_\kappa\lambda$, let $NS_{\kappa\lambda} \restriction X$ denote the ideal over $\mathcal{P}_\kappa\lambda$ defined by $Y \in NS_{\kappa\lambda} \restriction X \iff Y \cap X \in NS_{\kappa\lambda}$.

Can $NS_{\kappa\lambda}$ or $NS_{\kappa\lambda} \restriction X$ be precipitous ?

Answer. : Yes (sometimes assuming ...).

Note The existence of a precipitous ideal has the strength of some large cardinal because it provides us with a “generic” elementary embedding of V .

Theorem 1.2 (Foreman, Magidor, Shelah, Goldring) [3][6].

If λ is regular and δ is a Woodin cardinal $> \lambda$, then $\Vdash_{\text{Coll}(\lambda, < \delta)} \text{“} NS_{\kappa\lambda} \text{ is precipitous”}$. ($\text{Coll}(\lambda, < \delta)$ is the Levy collapse of δ to λ^+ .)

Question. What if λ is singular ?

Burke and Matsubara [1] conjectured that $NS_{\kappa\lambda}$ cannot be precipitous if λ is singular.

Definition 1.3. Let δ be a cardinal. We say that an ideal I is δ -saturated if $\mathbb{P}_I$ satisfies the δ chain condition .

Fact. If I is a λ^+ -saturated κ -complete normal ideal over $\mathcal{P}_\kappa\lambda$, then I is precipitous.

Note. $NS_{\kappa\lambda}$ is the minimal κ -complete normal ideal over $\mathcal{P}_\kappa\lambda$.

Theorem 1.4 (Foreman-Magidor) [2].

Unless $\kappa = \lambda = \aleph_1$, $NS_{\kappa\lambda}$ cannot be λ^+ -saturated.

What about $NS_{\kappa\lambda} \restriction X$?

Menas' Conjecture. *Every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

It turned out that Menas' Conjecture is independent of ZFC.

Theorem 1.5. $L \models$ "Menas' Conjecture holds".

Theorem 1.6(Gitik) [5]. *Suppose that κ is supercompact and $\lambda > \kappa$. Then $\exists$ p.o. $\mathbb{P}$ that preserves cardinals $\geq \kappa$ such that $\Vdash_{\mathbb{P}}$ " κ is inaccessible and $\exists$ stationary $X \subseteq \mathcal{P}_\kappa\lambda$ such that X cannot be partitioned into κ^+ disjoint stationary sets".*

2.MAIN RESULTS

Theorem 2.1 (Matsubara-Shelah)[9]. *If λ is a strong limit singular cardinal then $NS_{\kappa\lambda}$ is nowhere precipitous (i.e. $NS_{\kappa\lambda} \restriction X$ is not precipitous for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$).*

Theorem 2.2 [9]. *If λ is a strong limit singular cardinal then every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

One of the ingredients of the proof is the following lemma.

Lemma 2.3. *If $2^{<\kappa} < \lambda^{<\kappa} = 2^\lambda$, then*

- (i) *every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets and*
- (ii) *$NS_{\kappa\lambda}$ is nowhere precipitous.(Matsubara-Shioya).*

Remark.

- (1) The hypothesis of Lemma 2.3 is satisfied if λ is a strong limit cardinal with $\text{cf}(\lambda) < \kappa$.
- (2) Under the hypothesis of Lemma 2.3, if $X \subseteq \mathcal{P}_\kappa\lambda$ has size $< 2^\lambda$ then X is bounded and therefore non-stationary.

For the proof of (i) see page 345 of Kanamori [8].

proof of (ii).

Consider the following game G_ω between two players, **Nonempty** and **Empty**.

Nonempty	X_1	X_2	$\dots$	X_n	$\dots$
Empty	Y_1	Y_2	$\dots$	Y_n	$\dots$

Nonempty and **Empty** alternately choose stationary sets $X_n, Y_n \subseteq \mathcal{P}_\kappa\lambda$ respectively so that $X_n \supseteq Y_n \supseteq X_{n+1}$ for $n=1,2,3,\dots$

After ω moves, **Empty** wins G_ω if $\bigcap_{n=1}^{\infty} X_n = \emptyset$

Fact. $NS_{\kappa\lambda}$ is nowhere precipitous iff **Empty** has a winning strategy in G_ω .

For the proof of this fact, see [4]. Let $\langle f_\alpha \mid \alpha < 2^\lambda \rangle$ enumerate functions from $\lambda^{<\omega}$ into $\mathcal{P}_\kappa\lambda$.

For a function $f : \lambda^{<\omega} \rightarrow \mathcal{P}_\kappa\lambda$, let

$$\underbrace{C(f)}_{\text{club set generated by } f} = \{s \in \mathcal{P}_\kappa\lambda \mid \bigcup f'' s^{<\omega} \subseteq s\}$$

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Fact. $X \subseteq \mathcal{P}_\kappa \lambda$ is stationary iff $\forall \alpha < 2^\lambda$ $C(f_\alpha) \cap X \neq \emptyset$.

We now describe **Empty**'s strategy. Suppose **Nonempty** plays X_1 . Choose a sequence $\langle s_\alpha^1 \mid \alpha < 2^\lambda \rangle$ from X_1 by induction on α as follow: Pick an element from $X_1 \cap C(f_0)$ and call it s_0^1 .

Given $\langle s_\alpha^1 \mid \alpha < \beta \rangle$ for some $\beta < 2^\lambda$, pick $s_\beta^1 \in X_1 \cap C(f_\beta) \setminus \underbrace{\{s_\alpha^1 \mid \alpha < \beta\}}_{\text{non-stationary}}$.

Let **Empty** play $Y_1 = \{s_\alpha^1 \mid \alpha < 2^\lambda\}$. Now suppose **Nonempty** plays X_n immediately following **Empty**'s move $Y_{n-1} = \{s_\alpha^{n-1} \mid \alpha < 2^\lambda\}$.

Choose $\langle s_\alpha^n \mid \alpha < 2^\lambda \rangle$ a sequence from X_n as follows:

Pick $s_0^n \in (X \cap C(f_\beta)) \setminus \underbrace{(\{s_\alpha^{n-1} \mid \alpha \leq \beta\} \cup \{s_\alpha^n \mid \alpha < \beta\})}_{\text{non-stationary}}$.

Let **Empty** play $Y_n = \{s_\alpha^n \mid \alpha < 2^\lambda\}$.

Claim. *This is a winning strategy for Empty*

proof: We want to show that $\bigcap_{n=1}^{\infty} Y_n = \emptyset$.

Suppose otherwise, say $t \in \bigcap_{n=1}^{\infty} Y_n$. For each $n < \omega$, $\exists ! \alpha_n < 2^\lambda$ such that $t = s_{\alpha_n}^n$.

It is easy to see that $\alpha_n > \alpha_{n+1}$ for each n . ($s_\beta^n \notin \{s_\alpha^n \mid \alpha \leq \beta\}$ etc ...)

We now prove Theorem 2.2 assuming Theorem 2.1 and Lemma 2.3 (i).

proof of Theorem 2.2. : Let λ be a strong limit singular cardinal . If $\text{cf}(\lambda) < \kappa$ then by Lemma 2.3 (i) we are done.

Assume $\text{cf}(\lambda) \geq \kappa$. In this case $\lambda^{<\kappa} = \lambda$. So it is enough to show that $NS_{\kappa\lambda} \restriction X$ is not λ -saturated for every stationary $X \subseteq \mathcal{P}_\kappa \lambda$.

But this is a consequence of $NS_{\kappa\lambda}$ being nowhere precipitous . In fact we know that $NS_{\kappa\lambda} \restriction X$ cannot be λ^+ -saturated for every stationary $X \subseteq \mathcal{P}_\kappa \lambda$.

proof of Theorem 2.1. : We now tamper with the definition of $\mathcal{P}_\kappa \lambda$.

From now on we let $\mathcal{P}_\kappa \lambda = \{s \subseteq \lambda \mid |s| < \kappa, s \cap \kappa \in \kappa\}$. This set is club in $\{s \subseteq \lambda \mid |s| < \kappa\}$. The following is the advantage of this change:

$X \subseteq \mathcal{P}_\kappa \lambda$ is stationary iff $\forall f : \lambda^{<\omega} \rightarrow \lambda$ $C[f] \cap X \neq \emptyset$

where $C[f] = \{s \in \mathcal{P}_\kappa \lambda \mid s \text{ is closed under } f\}$.

Let λ be a strong limit singular cardinal. By Lemma 2.3 (ii) we may assume that $\text{cf}(\lambda) \geq \kappa$. Let $\langle \lambda_i \mid i < \text{cf}(\lambda) \rangle$ be a continuous increasing sequence of strong limit singular cardinals converging to λ . Let $T = \{i < \text{cf}(\lambda) \mid \text{cf}(i) < \kappa\}$.

For each $i \in T$, let $E_i = \{s \in \mathcal{P}_\kappa \lambda \mid \sup(s) = \lambda_i, \lambda_i \notin s\}$

Note.

- (i) $|E_i| = 2^{\lambda_i}$
- (ii) $\bigcup_{i \in T} E_i$ is club in $\mathcal{P}_\kappa \lambda$.

For each $i \in T$, let $\langle f_\epsilon^i \mid \epsilon < 2^{\lambda_i} \rangle$ enumerate all of the functions whose domain $\subseteq \lambda_i^{<\omega}$ and range $\subseteq \lambda_i$.

Definition 2.4. $C^i[f_\epsilon^i] = \{s \in E_i \mid s^{<\omega} \subseteq \text{dom}(f_\epsilon^i) \text{ and } s \text{ is close}$

To show $NS_{\kappa\lambda}$ is nowhere precipitous we will present a win **Empty** in G_ω .

Suppose W_1 is **Nonempty**'s first move in G_ω . For each $i \in T$, we have a "local game" where each player alternately chooses subsets of E_i . **Nonempty**'s first move is $W_1 \cap E_i$.

Local game $G(i)$

For each $i \in T$, define a game $G(i)$ as follows:

Nonempty and **Empty** alternately choose $X_n, Y_n \subseteq E_i$ resp. $1, 2, \dots$, so that $X_n \supseteq Y_n \supseteq X_{n+1}$ and $\forall \epsilon < 2^{\lambda_i}$ ($|C^i[f_\epsilon^i] \cap C^i[f_\epsilon^i] \cap Y_n \neq \emptyset$).

Empty wins $G(i)$ iff $\bigcap_{n=1}^{\infty} X_n = \emptyset$.

Just as in the proof of Lemma 2.3 (ii) we can show that **Empty** has a winning strategy, say τ_i in G_i .

$$\begin{array}{ccccccc}
 G_\omega & \text{Nonempty} & W_1 & & W_2 & & \\
 & \text{Empty} & \downarrow & \bigcup_{i \in T} \tau_i(\langle W \cup E_i \rangle) & \downarrow & \bigcup_{i \in T} & \\
 G(i) & & W_1 \cap E_i & \uparrow & W_2 \cap E_i & & \\
 (i \in T) & & & \tau_i(\langle W \cap E_i \rangle) & & &
 \end{array}$$

The following lemma tells us that we can combine τ_i 's for $i \in T$ for G_ω .

Lemma 2.5. Suppose $W \subseteq \mathcal{P}_{\kappa\lambda}$ is stationary. If $U \subseteq \mathcal{P}_{\kappa\lambda}$ satisfies condition (#) then U is stationary.

(#) For each $i \in T$, $\forall \epsilon < 2^{\lambda_i}$ ($|C^i[f_\epsilon^i] \cap W| = 2^{\lambda_i} \longrightarrow C^i[f_\epsilon^i] \cap U \neq \emptyset$)

Now we describe **Empty**'s (combined) strategy σ in G_ω . Suppose W_1 is played.

Let **Empty** play $\bigcup_{i \in T} \tau_i(\langle W_1 \cap E_i \rangle) \stackrel{\text{def}}{=} \sigma(\langle W_1 \rangle)$.

Suppose

$$\begin{array}{ccccccc}
 W_1 & & W_2 & & \dots & & W_n \\
 & \sigma(\langle W_1 \rangle) & & \sigma(\langle W_1, W_2 \rangle) & & \dots &
 \end{array}$$

is the run of the game G_ω so far.

Let

$$\sigma(\langle W_1, W_2, \dots, W_n \rangle) \stackrel{\text{def}}{=} \bigcup_{i \in T} \tau_i(\langle W_1 \cap E_i, W_2 \cap E_i, \dots, W_n \cap E_i \rangle)$$

Lemma 2.5 guarantees that σ provides **Empty** a legal move i.e. strategy of **Nonempty**'s last move. This σ is a winning strategy for **Empty**. The proof of Lemma 2.5 depends upon the following lemma whose proof is by induction.

Lemma 2.6. Suppose $U \subseteq \mathcal{P}_{\kappa\lambda}$. If $\forall i \in T$ $|U \cap E_i| < 2^{\lambda_i}$, then U is not stationary.

To prove the last lemma, we need the following fact from pcf theory.

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pcf Fact. $\exists$ club $C \subseteq cf(\lambda)$ such that $pp(\lambda_i) = 2^{\lambda_i}$ for every $i \in C$.

See Shelah “Cardinal Arithmetic” [12] Conclusion 5.13 page 414 and Hotz, Steffens, Weitz “Introduction to Cardinal Arithmetic” [7] Theorem 9.1.3 page 271.

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