

Ind-sheaves

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I Ind-lim and ind-objects (cf. SGA 4)

$\mathcal{U}$ a universe, $\mathcal{E}$ a $\mathcal{U}$ -category

$\mathcal{E}^\vee =$ the category of contravariant functors $\mathcal{E} \rightarrow \underline{\text{Set}}$

$h^\vee: \mathcal{E} \rightarrow \mathcal{E}^\vee, \quad x \mapsto \text{Hom}_{\mathcal{E}}(\cdot, x)$

$h^\vee$ is fully faithful

I : small filtered category, $\alpha: I \rightarrow \mathcal{E}$ a functor

Def

" $\varinjlim$ " $\alpha \in \mathcal{E}^\vee$ is given by:

$$(\varinjlim \alpha)(\gamma) = \varinjlim_i \text{Hom}_{\mathcal{E}}(\gamma, \alpha(i))$$

def $\text{Ind}(\mathcal{E})$ is the full subcategory of $\mathcal{E}^\vee$ defined by:

$$A \in \text{Ind}(\mathcal{E}) \iff \exists \alpha: I \rightarrow \mathcal{E}, \quad I \text{ small, filtered} \\ \text{s.t. } A \simeq \varinjlim \alpha$$

$$\text{Hence } \mathcal{E} \hookrightarrow \text{Ind}(\mathcal{E}) \hookrightarrow \mathcal{E}^\vee$$

Theorem Assume $\mathcal{E}$ is abelian.

- (i) $\text{Ind}(\mathcal{E})$ is abelian
- (ii) $\mathcal{E} \hookrightarrow \text{Ind}(\mathcal{E})$ is exact and thick
- (iii) $\text{Ind}(\mathcal{E})$ admits small $\varinjlim$ and $\varprojlim$ and $\varinjlim$ over a filtered category is exact
- (iv) Assume $\mathcal{E}$ has enough injectives. Then $D^b(\mathcal{E}) \simeq D^b_{\mathcal{E}}(\text{Ind}(\mathcal{E}))$.

II Ind-sheaves

X : topological space, Hausdorff, locally compact, countable at ∞

k : a field

$\text{Mod}(k_X)$: abelian category of sheaves on X of k -vector spaces

$\text{Mod}^c(k_X)$: subcategory of compactly supported sh.

$$\mathcal{I}^c(k_X) = \text{Ind}(\text{Mod}^c(k_X))$$

Z : $U \mapsto \mathcal{I}^c(k_U)$ is a stack, not $U \mapsto \mathcal{I}(k_U)$.

$$\text{Mod}(k_X) \xrightarrow{i_X} \mathcal{I}^c(k_X) \xrightarrow{\alpha_X} \text{Mod}(k_X)$$

$$i_X(F) = \varinjlim_{U \subset\subset X} F_U$$

$$\alpha_X(\varinjlim_i F_i) = \varinjlim_i F_i$$

$$\text{Hom}_{k_X}(\alpha_X(G), F) \cong \text{Hom}_{\mathcal{I}^c(k_X)}(G, i_X(F)).$$

Restriction: $U \subset X$ open

$$F = \varinjlim_i F_i \in \mathcal{I}^c(k_X)$$

$$F|_U = \varinjlim_{i, U \subset\subset U} (F_i)_U \in \mathcal{I}^c(k_U)$$

Lemma/def: the presheaf on X

$U \mapsto \text{Hom}_{\mathcal{I}^c(k_U)}(F|_U, G|_U)$ is a sheaf

denoted $\text{Hom}(F, G)$.

$\text{Hom}: \mathcal{I}^c(k_X)^{\text{op}} \times \mathcal{I}^c(k_X) \rightarrow \text{Mod}(k_X)$

The α_X admits a left adjoint β_X

$$\text{Hom}_{\mathcal{I}^c(k_X)}(\beta_X(F), G) \simeq \text{Hom}_{k_X}(F, \alpha_X(G))$$

$$\beta_X(F): \text{Mod}^c(k_X) \rightarrow \underline{\text{Ab}}$$

$$G \mapsto \Gamma(X; F \otimes_{k_X} \text{Hom}_{k_X}(G, k_X)).$$

Notation $Z \subset X$ locally closed

$$\beta_X(k_Z) = \widetilde{k_Z}$$

Ex U open in X

$$\widetilde{k_U} = \varinjlim_{V \subset U} k_V \quad k_V \text{ } V \text{ open}$$

S closed in X

$$\widetilde{k_S} = \varinjlim_U k_{\bar{U}}, \quad S \subset U, U \text{ open}$$

Tens $\mathcal{I}^c(k_X) \times \mathcal{I}^c(k_X) \rightarrow \mathcal{I}^c(k_X)$

$$\varinjlim_i F_i \otimes \varinjlim_j G_j := \varinjlim_{i,j} (F_i \otimes G_j)$$

$$\underline{\text{IHom}} : \mathcal{I}^c(R_X)^{\text{op}} \times \mathcal{I}^c(R_X) \rightarrow \mathcal{I}^c(R_X)$$

$$\text{IHom} \left(\varprojlim_i F_i, \varprojlim_j G_j \right) := \varprojlim_i \varprojlim_j \text{Hom}_{R_X}(F_i, G_j)$$

Some formulas

$$i_X(F \otimes G) \cong i_X F \otimes i_X G$$

$$i_X \text{Hom}_{R_X}(F, G) \cong \text{IHom}(i_X F, i_X G)$$

$$\alpha_X \circ \text{IHom}(F, G) = \text{Hom}(F, G)$$

$$\text{IHom}(F \otimes G, H) \cong \text{IHom}(F, \text{IHom}(G, H))$$

$$\beta_X(F \otimes G) \cong \beta_X F \otimes \beta_X G$$

$$\beta_X F \otimes \text{IHom}(G, K) \cong \text{IHom}(G, \beta_X(F) \otimes K)$$

External operations
and derived category

$f^!, f_*, f^{!!}, f^!$ } not here
 $D^b(\mathcal{I}^c(R_X))$

III Construction of v.d. sheaves

X : real analytic m.f., $k = \mathbb{C}$

$\mathcal{R}C^c(k_X)$: $\mathcal{R}$ -constructible sheaves
with compact support

$$\mathcal{IR}^c(k_X) := \text{Ind}(\mathcal{R}C^c(k_X))$$

$$\begin{array}{ccccc} \mathcal{R}C^c(k_X) & \hookrightarrow & \text{mod}^c(k_X) & \hookrightarrow & \text{mod}(k_X) \\ \downarrow & & \downarrow & & \nearrow \\ \mathcal{IR}^c(k_X) & \hookrightarrow & \mathcal{I}^c(k_X) & & \end{array}$$

Let $\mathcal{O} \subset \text{OP}(X)$ be the family of open subanalytic relatively compact sets

Th Let $\alpha: \mathcal{O}^{\text{op}} \rightarrow \mathbf{Ab}$ s.t.

$$\alpha(\emptyset) = \{0\}$$

$$0 \rightarrow \alpha(U \cup V) \rightarrow \alpha(U) \oplus \alpha(V) \rightarrow \alpha(U \cap V)$$

is exact $\forall U, V \in \mathcal{O}$.

Then $\exists!$ $A \in \mathcal{IR}^c(k_X)$ s.t.

$$\text{Hom}(k_U, A) = \alpha(U) \quad \forall U \in \mathcal{O}.$$

Example $\mathcal{E}_X^{\text{an}} \in \mathcal{IR}^c(k_X)$

associated to $\alpha(U) = \text{Thom}(\pi_U, \mathcal{E}_X^{\text{an}})$
(see K-S, Moderate and formal cohomology)

Let X be a complex manifold

Def $\mathcal{O}_X^t = R\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \mathcal{E}_{X/R}^{\otimes t})$

Z: $\mathcal{O}_X^t \in \mathcal{D}^b(\mathcal{I}^c(k_X))$ is not concentrated in degree 0

ref. π real analytic map.
X a complexification

$$\mathcal{D}b_M^t = R\mathcal{H}om(\mathcal{O}_X' \otimes_M, \mathcal{O}_X^t)$$

$$\mathcal{D}b_M = \alpha_M(\mathcal{D}b_M^t)$$

$\mathcal{D}b_M$ is also called the sheaf of distributions on M by some authors.

Let $F \in \mathcal{D}_{R-C}^b(\mathcal{O}_X)$

$$R\mathcal{H}om(F, \mathcal{O}^t) = \mathcal{H}om(F, \mathcal{O}_X)$$

($\mathcal{H}om$ is Kashiwara's functor of tempered cohomology).