

Laplace tranform and Fourier-Sato tranform

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Review on formal and moderate cohomology. Let M be a real manifold, and let $R\text{-Cons}(M)$ denote the category of R -constructible sheaves on M , $D_{R-c}^b(C_M)$ its derived category. Recall first the functors $\mathcal{T}hom(\cdot, \mathcal{D}b_M)$ of [4] and the dual functor $\cdot \overset{w}{\otimes} \mathcal{C}_M^\infty$ of [6], defined on the category $R\text{-Cons}(M)$, with values in the category $Mod(\mathcal{D}_M)$ of $\mathcal{D}_M$ -modules on M . (The first functor is contravariant).

They are characterized as follows. Denote by $\mathcal{D}b_M$ the sheaf of Schwartz's distributions on M and by $\mathcal{C}_M^\infty$ the sheaf of $\mathcal{C}^\infty$ functions on M . Let Z (resp. U) be a closed (resp. open) subanalytic subset of M . Then these two functors are exact and moreover:

$$\begin{aligned}\mathcal{T}hom(C_Z, \mathcal{D}b_M) &= \Gamma_Z(\mathcal{D}b_M), \\ C_U \overset{w}{\otimes} \mathcal{C}_M^\infty &= \mathcal{I}_{M \setminus U}^\infty,\end{aligned}$$

where $\Gamma_Z(\mathcal{D}b_M)$ denotes as usual the subsheaf of $\mathcal{D}b_M$ of sections supported by Z and $\mathcal{I}_{M \setminus U}^\infty$ denotes the ideal of $\mathcal{C}_M^\infty$ of sections vanishing up to order infinity on $M \setminus U$.

These functors being exact, they extend naturally to the derived category $D_{R-c}^b(C_X)$. We keep the same notations to denote the derived functors.

Now let X be a complex manifold and denote by $\bar{X}$ the complex conjugate manifold and by X_R the real underlying manifold. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X , let $\mathcal{D}_X$ be the sheaf of finite order holomorphic differential operators on X . The functors of moderate and formal cohomology (see [4], [6]) are defined for $F \in D_{R-c}^b(C_{X_R})$ by:

$$\begin{aligned}\mathcal{T}hom(F, \mathcal{O}_X) &= R\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, \mathcal{T}hom(F, \mathcal{D}b_{X_R})) \\ F \overset{w}{\otimes} \mathcal{O}_X &= R\mathcal{H}om_{\mathcal{D}_{\bar{X}}}(\mathcal{O}_{\bar{X}}, F \overset{w}{\otimes} \mathcal{C}_{X_R}^\infty).\end{aligned}$$

Laplace transform. Consider a complex vector space E of complex dimension n , and denote by $j : E \hookrightarrow P$ its projective compactification. Let $D_{R-c, R+}^b(C_E)$ denote the full triangulated subcategory of $D_{R-c}^b(C_E)$ consisting of R^+ -conic objects (i.e.

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objects whose cohomology is R -constructible and locally constant on the orbits of the action of R^+ on E).

Let $F \in D_{R-c, R^+}^b(C_E)$ and set for short

$$\mathrm{THom}(F, \mathcal{O}_E) = R\Gamma(P; \mathcal{T}hom(j_! F, \mathcal{O}_P))$$

$$\mathrm{WTens}(F, \mathcal{O}_E) = R\Gamma(P; j_! F \overset{w}{\otimes} \mathcal{O}_P)$$

These are objects of the bounded derived category $D^b(W(E))$ of the category of modules over the Weyl algebra $W(E)$. Let E^* denote the dual vector space to E . One denotes by $F^\wedge$ the Fourier-Sato transform of the sheaf F , an object of $D_{R-c, R^+}^b(C_{E^*})$. (see [5] for an exposition). One identifies $D^b(W(E^*))$ to $D^b(W(E))$ by the usual Fourier transform.

Theorem 0.1. *The classical Laplace transform extends naturally as isomorphisms in $D^b(W(E))$:*

$$L_E : \mathrm{THom}(F, \mathcal{O}_E) \simeq \mathrm{THom}(F^\wedge[n], \mathcal{O}_{E^*}) \quad (0.1)$$

$$L_E : \mathrm{WTens}(F, \mathcal{O}_E) \simeq \mathrm{WTens}(F^\wedge[n], \mathcal{O}_{E^*}). \quad (0.2)$$

Applications 1. Let M be a real vector space of dimension n such that E is a complexification of M . As a particular case of the theorem, we obtain a characterization of the Laplace transform of the space of distributions on M supported by (not necessarily convex) cones. Let γ denote a closed subanalytic cone in M and let $\Gamma_\gamma \mathcal{S}'_M$ denote the space of tempered distributions supported by γ . One has $\Gamma_\gamma \mathcal{S}'_M \simeq \mathrm{THom}(C_\gamma[-n], \mathcal{O}_E)$. Hence, we get that the Laplace transform of distributions induces an isomorphism:

$$L_E : \Gamma_\gamma \mathcal{S}'_M \simeq \mathrm{THom}((C_\gamma)^\wedge, \mathcal{O}_{E^*}) = .$$

When γ is proper and convex, this result is well known, since $(C_\gamma)^\wedge \simeq C_U$ where U is the open convex tube $\mathrm{int}\gamma^0 \times \sqrt{-1}M^*$, the interior of the polar cone to γ , and the right hand side denotes the space of holomorphic functions in this tube, tempered up to infinity. When $\gamma = M$, one recovers the isomorphism between $\mathcal{S}'_M$ and $\mathcal{S}'_{\sqrt{-1}M^*}$.

Let us consider now the case where γ is a non degenerate quadratic cone. Let (x', x'') denote the coordinates on $M = \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ with $p, q \geq 1$, and let $\gamma = \{x; x'^2 - x''^2 \leq 0\}$. Let (u', u'') denote the dual coordinates on M^* , and let $\lambda = \{(u', u''); u'^2 - u''^2 \geq 0\}$. One checks easily that $(C_\gamma)^\wedge \simeq C_\lambda[-q]$. We get the isomorphism:

$$L_E : \Gamma_\gamma \mathcal{S}'_M \simeq H^q \mathrm{THom}(C_{\lambda \times \sqrt{-1}M^*}, \mathcal{O}_{E^*}).$$

This last result is essentially due to Faraut-Gindikin [2] (in a different langage and with a different proof).

Applications 2. Denote by $\mathcal{O}_E^t$ and $\mathcal{O}_E^w$ the conic sheaves on E associated to the presheaves $U \mapsto \mathrm{THom}(C_U, \mathcal{O}_E)$ and $U \mapsto \mathrm{WTens}(C_U, \mathcal{O}_E)$, respectively. One easily deduces from the main theorem that the Laplace transform induces isomorphism= s:

$$\begin{aligned} (\mathcal{O}_E^t)^\wedge[n] &\simeq \mathcal{O}_{E^*}^t, \\ (\mathcal{O}_E^w)^\wedge[n] &\simeq \mathcal{O}_{E^*}^w. \end{aligned}$$

This gives a new proof of a result of Hotta-Kashiwara [3] and Brylinski-Malgrange-Verdier [1] on the Fourier-Sato transform of the sheaf of holomorphic solutions of a monodromic $\mathcal{D}$ -module.

References

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