

A proof of the Gauss-Bonnet-Chern Theorem by the symbol calculus of pseudo-differential operators

Chisato IWASAKI (岩崎 千里)

Department of mathematics, Himeji Institute of Technology

§1. Introduction. The aim of this paper is to give an analytic proof of the Gauss-Bonnet-Chern theorem for a smooth orientable Riemannian manifold with boundary by means of symbol calculus of pseudo-differential operators. The similar attempts for a smooth Riemannian manifold without boundary are found in E.Getzler [6], H.L.Cycon-R.G.Froese-W.Kirsch-B.Simon[5] and N.Berline-E.Getzler-M.Vergne[2] .

Let M be a Riemannian manifold and let $\chi(M)$ be the Euler characteristic of M . Let dv and $d\sigma$ be a volume element of M and one of its boundary ∂M respectively.

Analytical proofs are based of the following formula

$$\chi(M) = \int_M \sum_{p=0}^n (-1)^p \operatorname{tr} e_p(t, x, x) dv,$$

where $e_p(t, x, y)$ is the kernel of the fundamental solution $E_p(t)$ for the Cauchy problem for the heat equation for Δ_p on differential p -forms $A^p(M) = \Gamma(\wedge^p T^*(M))$, that is

$$E_p(t)f(x) = \int_M e_p(t, x, y)\varphi(y)dv_y, \quad \varphi \in A^p(M)$$

satisfies

$$(1.1) \quad \begin{cases} (\frac{\partial}{\partial t} + \Delta_p)E_p(t) = 0 & \text{in } (0, T) \times M, \\ E_p(0) = I & \text{in } M. \end{cases}$$

If M has boundary ∂M , then $E_p(t)$ satisfies $(1.2)_p$ instean of (1.1).

$$(1.2)_p \quad \begin{cases} (\frac{\partial}{\partial t} + \Delta_p)E_p(t) = 0 & \text{in } (0, T) \times M, \\ B_p E_p(t) = 0 & \text{on } (0, T) \times \partial M, \\ E_p(0) = I & \text{in } M, \end{cases}$$

with some boundary condition B_p (See §6).

(Δ_p, B_p) is an elliptic boundary value problem. So it is well-known that $e_p(t, x, y)$ has singularity only at $x = y$ as follows.

$$\operatorname{tr} e_p(t, x, x) \sim c_0(x)t^{-\frac{n}{2}} + c_1(x)t^{-\frac{n}{2}+\frac{1}{2}} + \dots + \dots \quad t \rightarrow 0.$$

The vanishing of the singularity of super trace at a point of M defined by

$$stre(t, x, x) = \sum_{p=0}^n (-1)^p tre_p(t, x, x)$$

is due to algebraic theorem in §3 stated in [5] which is owing to V.K.Patodi [15]. The point of this paper is that according to this theorem and the method of construction of the fundamental solution for the mixed problem in C.Iwasaki[11], even if M has boundary, one can prove the Gauss-Bonnet-Chern theorem only by symbol calculus of the top term of the asymptotic of the fundamental solution, considering operators acting on $A^*(M) = \sum_{p=0}^n A^p(M)$

Main theorem . We get the Gauss-Bonnet-Chern theorem. Moreover we have that
(1)

$$\lim_{t \rightarrow 0} \int_M \sum_{p=0}^n (-1)^p tre_p(t, x, x) \psi(x) dv = \int_M C_n(x, M) \psi(x) dv + \int_{\partial M} D_{n-1}(x) \psi(x) d\sigma$$

for any $\psi(x) \in C^\infty(M)$.

(2) For M without boundary or for x contained in $M \setminus \partial M$

$$\sum_{p=0}^n (-1)^p tre_p(t, x, x) = C_n(x, M) + o(\sqrt{t}) \quad \text{as } t \rightarrow 0.$$

(3) If M has boundary,

$$\sum_{p=0}^n (-1)^p tre_p(t, x, x) = 2D_{n-1}(x) \frac{1}{\sqrt{t}} + o(1) \quad \text{as } t \rightarrow 0$$

for $x \in \partial M$, where

$$C_n(x, M) dv = \begin{cases} \text{the Euler form,} & \text{if } n \text{ is even (See (5.2) for the precise definition);} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

and

$$D_{n-1}(x) = \begin{cases} \frac{1}{2} C_{n-1}(x, \partial M), & \text{if } n \text{ is odd;} \\ \text{See Definition 7,} & \text{if } n \text{ is even.} \end{cases}$$

There are many studies to prove the Gauss- Bonnet-Chern theorem analytically. McKean-Singer [14] proved

$$stre(t, x, x) = C_n(x) + 0(t)$$

when M is a closed manifold of dimension 2. V.K.Patodi[15] extended this equation for a manifold of any dimension. Moreover P.B.Gilkey[7],[8] proved the Gauss-Bonnet-Chern theorem by invariant theory in case M has boundary. There are probabilistic proofs in N.Ikeda-S.Watanabe[9] and I.Shigekawa-N.Ueki-S.Watanabe[16].

§2. The representation of Δ . Let M be a smooth Riemannian manifold with a Riemannian metric g . Set $X_1, X_2, \dots, X_n$ be a local orthonormal frame of $T(M)$ in a local patch of chart U . And let $\omega^1, \omega^2, \dots, \omega^n$ be its dual.

The differential d and its dual ϑ acting on $A^*(M)$ are given as follow, using the Levi-Civita connection ∇ .

$$d = \sum_{j=1}^n e(\omega^j) \nabla_{X_j}, \quad \vartheta = - \sum_{j=1}^n i(X_j) \nabla_{X_j},$$

where we use the following notations.

Notations.

$$e(\omega^j)\omega = \omega^j \wedge \omega, \quad i(X_j)\omega(Y_1, \dots, Y_{p-1}) = \omega(X_j, Y_1, \dots, Y_{p-1}).$$

Let $c_{i,j}^k$ be the following function and let $R(X, Y)$ be the curvature transformation, that is

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]},$$

$$\nabla_{X_i} X_j = \sum_{\ell=1}^n c_{i,j}^\ell X_\ell.$$

From the fact that our connection is the Riemannian connection we have

Proposition 1. The coefficients $c_{i,j}^k$ of connection form satisfy

$$c_{i,j}^k = -c_{i,k}^j, \quad [X_i, X_j] = \sum_{k=1}^n (c_{i,j}^k - c_{j,i}^k) X_k.$$

We have the following representation for $\Delta = d\vartheta + \vartheta d$ which is known as Weitzenböck's formula.

Lemma 1. The Laplacian Δ on $A^*(M)$ is given by

$$\Delta = -\left\{\sum_{j=1}^n \nabla_{X_j} \nabla_{X_j} - \sum_{i,j=1}^n c_{i,i}^j \nabla_{X_j} + \sum_{i,j=1}^n e(\omega^i) \iota(X_j) R(X_i, X_j)\right\}.$$

We use the following notations in the rest of this paper.

$$\begin{aligned} e(\omega^j) &= a_j^*, & \iota(X_m) &= a_m, \\ a_I &= a_{i_1} a_{i_2} \cdots a_{i_p}, & a_I^* &= a_{i_p}^* \cdots a_{i_1}^* \quad \text{for } I = \{i_1 < i_2 < \cdots < i_p\}, \\ \omega^I &= \omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} & \text{for } I = \{i_1 < i_2 < \cdots < i_p\}, \\ R(X_i, X_j) X_k &= \sum_{\ell=1}^n R_{kij}^\ell X_\ell \quad 1 \leq i, j, k, \leq n. \end{aligned}$$

Then we have

$$\Delta = -\left\{\sum_{j=1}^n (X_j I - G_j)^2 - \sum_{i,j=1}^n c_{i,i}^j (X_j I - G_j) - \sum_{i,j,\ell,m=1}^n R_{\ell ij}^m a_i^* a_j a_\ell^* a_m\right\}$$

on $A^*(M)$. Here

$$G_j = \sum_{\ell,m=1}^n c_{j,\ell}^m a_\ell^* a_m$$

and I is an identity operator on $\wedge^*(T^*(M))$.

Take a local coordinate $\{x_1, \dots, x_n\}$ of U . Let $\{\xi_1, \dots, \xi_n\}$ be its dual. By the above Lemma 1 we have

Lemma 2. The symbol of Δ is given by

$$\sigma(\Delta) = -\left\{\sum_{j=1}^n (\alpha_j I - G_j)^2 - \sum_{i,j,\ell,m=1}^n R_{\ell ij}^m a_i^* a_j a_\ell^* a_m\right\} + r_1,$$

where

$$r_1 = \sum_{k,j=1}^n i \left\{ \left(\frac{\partial}{\partial \xi_k} \right) \alpha_j \left(\frac{\partial}{\partial x_k} \right) \alpha_j - \left(\frac{\partial}{\partial \xi_k} \right) \alpha_j \left(\frac{\partial}{\partial x_k} \right) G_j \right\} I + \sum_{j,k=1}^n c_{k,k}^j (\alpha_j I - G_j)$$

and

$$\sigma(X_j) = \alpha_j.$$

The following proposition is fundamental for a_i, a_j^* .

Proposition 2.

$$\begin{aligned}
a_i a_j + a_j a_i &= 0, \\
a_i^* a_j^* + a_j^* a_i^* &= 0, \\
a_i a_j^* + a_j^* a_i &= \delta_{ij}.
\end{aligned}$$

§3. Berezin-Patodi formula. Let V be a vector space of dimension n with inner product and let $\wedge^p(V)$ be its anti-symmetric p tensors. Set $\wedge^*(V) = \sum_{p=0}^n \wedge^p(V)$. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis for V . Set a_i^* be a linear transformation on $\wedge^*(V)$ defined by $a_i^* v = v_i \wedge v$ and set a_i be an adjoint operator of a_i^* on $\wedge^*(V)$. Then $\{a_i^*, a_j\}$ satisfy Proposition 2. The following Theorem 1 was shown in [5] under the above assumptions.

Theorem 1(Berezin-Patodi[5]). For any linear operator A on $\wedge^*(V)$, we can write uniquely in the form $A = \sum_{I,J} \alpha_{I,J} a_I^* a_J$ and

$$\sum_{p=0}^n \text{tr}[(-1)^p A_p] = (-1)^n \alpha_{\{1,2,\dots,n\}\{1,2,\dots,n\}},$$

where $A_p = A|_{\wedge^p(V)}$.

§4. Construction of the asymptotics of the fundamental solution for the Cauchy problem.

Now let us consider the Cauchy problem on $\mathbf{R}^n$.

$$(4.1) \quad \begin{cases} (\frac{\partial}{\partial t} + R(x, D))U(t) = 0 & \text{in } (0, T) \times \mathbf{R}^n, \\ U(0) = I & \text{in } \mathbf{R}^n, \end{cases}$$

where $R(x, D)$ is a differential operator of which symbol $r(x, \xi) = p_2(x, \xi)I + p_1(x, \xi)$ satisfies $p_j \in S_{1,0}^j$ and $p_2 \geq \delta|\xi|^2$.

Definition 1. (1) Let $(\mathcal{A})_{ij} = a_i^* a_j$ $1 \leq i, j \leq n$.

(2) Let K^m be a subset of $S_{1,0}^m$ as follows.

$K^m = \{p(x, \xi : \mathcal{A}); \text{polynomial with respect to } \xi \text{ and } \mathcal{A} \text{ of order } m \text{ with coefficients } \mathcal{B}(\mathbf{R}^n)\}$.

(3) We define a pseudo-differential operator action on $A^*(M)$ by $P = p(x, D : \mathcal{A})$ of a symbol $\sigma(P) = p(x, \xi : \mathcal{A}) = \sum_{I,J} p_{I,J}(x, \xi) a_I^* a_J \in K^m$ as follows.

$$p(x, D : \mathcal{A})(\varphi_K \omega^K) = \sum_{I,J} p_{I,J}(x, D) \varphi_K a_I^* a_J(\omega^K)$$

Definition 2. (1) For a real number m , K_m is the set of all polynomials with respect to t of degree d with coefficient of K^{m+2d} .
 (2) For a real number ℓ , R_ℓ is the subset of $\cup_m B_t(S_{1,0}^m)$ which satisfies the following inequality for nonnegative constants $C_{\alpha,\beta}$, C and $\ell_{\alpha,\beta}$

$$\|(\frac{\partial}{\partial t})^k (\frac{\partial}{\partial \xi})^\alpha (\frac{\partial}{\partial x})^\beta q(t, x, \xi)\| \leq C_{\alpha,\beta} e^{-p_2 t + C \langle \xi \rangle^2} (t \langle \xi \rangle^2 + 1)^{\ell_{\alpha,\beta}} (\frac{1}{\sqrt{t}} + \langle \xi \rangle)^{\ell + 2k - |\alpha|}.$$

Now assume (4.2) for the symbol $r(x, \xi)$ of $R(x, D)$ in (4.1)

$$(4.2) \quad r(x, \xi : \mathcal{A}) = r_2(x, \xi : \mathcal{A}) + r_1(x, \xi : \mathcal{A}), \quad r_j \in K^j \quad (j = 1, 2),$$

$$r_2(x, \xi : \mathcal{A}) - p_2(x, \xi)I \in S_{1,0}^1.$$

Let

$$u_0 = e^{-tr_2(x, \xi : \mathcal{A})}.$$

Theorem 2. For any non negative integer N we have the asymptotics u^N of the fundamental solution for (4.1) of the form $u^N = \sum_{j=0}^N u_j$, $u_j = v_j u_0$ with $v_j \in K_{-j}$ in the sense

$$\begin{cases} (\frac{\partial}{\partial t} + r) \circ u^N = 0 & \text{mod } R_{-N+1}, \\ u^N(0) = I. \end{cases}$$

§5. The proof of the Gauss-Bonnet-Chern theorem without boundary.

We will construct the asymptotics of the fundamental solution for the Cauchy problem on M , that is,

$$\begin{cases} (\frac{\partial}{\partial t} + \Delta)U(t) = 0 & \text{in } (0, T) \times M, \\ U(0) = I & \text{in } M, \end{cases}$$

where the operator $U(t)$ is considered acting on $A^*(M) = \sum_{p=0}^n A^p(M)$. Owing to the fact that the fundamental solution has the pseudo-local property, it is sufficient to consider the fundamental solution in a local chart. We have

$$\text{str } e(t, x, x) = \text{str } \tilde{u}^N(t, x, x) + O(t^{-\frac{n}{2} + \frac{N}{2}}),$$

where

$$\tilde{u}^N(t, x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} u^N(t, x, \xi) d\xi.$$

In our case $r = r_2 + r_1$, where r_1 is given in Lemma 2 and

$$r_2 = - \sum_{j=1}^n (\alpha_j I - G_j)^2 + R.$$

Here

$$(5.1) \quad R = \sum_{i,j,\ell,m=1}^n R_{\ell ij}^m a_i^* a_j a_\ell^* a_m.$$

The principal symbol of Δ is equal to $p_2 = - \sum_{j=1}^n (\alpha_j)^2 I$.

By Theorem 1 , we have

$$\text{str } \tilde{u}_0(t, x, x) = \begin{cases} (\frac{1}{2\sqrt{\pi t}})^n \sqrt{\det g} \text{str} \{ \frac{(-1)^m}{m!} R^m t^m \} + 0(t), & \text{if } n = 2m ; \\ 0(\sqrt{t}), & \text{if } n \text{ is odd.} \end{cases}$$

$$\text{str } \tilde{u}_j(t, x, x) = 0(\sqrt{t}^j).$$

Noting (5.1), we have

Lemma 3. If $n = 2m$,

$$(\frac{1}{2\sqrt{\pi}})^n \text{str} \{ \frac{(-1)^m}{m!} R^m \} = C_n(x, M),$$

where

$$(5.2) \quad C_n(x, M) = (\frac{1}{2\sqrt{\pi}})^n \frac{1}{m!} \sum_{\pi, \sigma \in S_n} (\frac{1}{2})^m \text{sign}(\pi) \text{sign}(\sigma) \\ \times R_{\pi(1)\pi(2)\sigma(1)\sigma(2)} \cdots R_{\pi(n-1)\pi(n)\sigma(n-1)\sigma(n)}.$$

§6. Asymptotics of the fundamental solution for intial-boundary value problems. The study in [11] is applicable for the construction of the fundamental solution for our intial-boundary value problem . But as we have studied in §5 , the lower parts of the asymptotics of the fundamental solution play the important part for the proof of Gauss-Bonnet-Chern theorem . So in this case we introduce new class $\mathcal{J}_s$ instead of

$\mathcal{H}_s$ in [11], as we used K^m instead of $S_{1,0}^m$ in §4. The main part of the construction of the fundamental solution or its asymptotics is how to construct these ones in a local chart (cf. [11]).

We will write down the boundary operator B_p in a local coordinate. Take a local patch Ω near ∂M such that ∂M is defined by $\{\rho = 0\}$ in Ω and $M \cap \Omega \subset \{\rho \geq 0\}$. Assume that $\omega^n = cd\rho$ with some function c on M .

Choose a local coordinate $\{x_1, \dots, x_n\}$ in Ω such that $M \cap \Omega = \{(x', x_n); x' \in \mathcal{U}, x_n \geq 0\}$, $\Gamma = \partial M \cap \Omega = \{(x', 0); x' \in \mathcal{U}\}$ and

$$X_n|_{\Gamma} = \frac{\partial}{\partial x_n}.$$

The boundary operator B_p is as follows.

$$\varphi \in \text{Dom}(\vartheta), \quad d\varphi \in \text{Dom}(\vartheta),$$

where $\text{Dom}(\vartheta) = \{\varphi = \sum_J \varphi_J \omega^J, \varphi_J|_{\Gamma} = 0 \text{ for } n \in J\}$. So we obtain the equation for the boundary condition

$$\frac{\partial}{\partial x_n} \varphi|_{\Gamma} = 0$$

for $\varphi \in A^0(M)$ and for $\sum_J \varphi_J \omega^J \in A^p(M)$, $p \geq 1$

$$\left\{ \begin{array}{l} \varphi_J|_{\Gamma} = 0 \text{ if } n \in J, \\ \{(\frac{\partial}{\partial x_n} - \gamma + b) \sum_{n \notin J} \varphi_J \omega^J\}|_{\Gamma} = 0, \end{array} \right.$$

where γ and b are given in the following (6.1).

Definition 3. (1) We define $h^* = h^*(t, x', \xi) = h(t, x', 0, \xi)$ for a function $h(t, x, \xi)$ given in $\mathbf{R}^{2n+1}$.
(2) Set

$$(6.1) \quad \gamma = \gamma(x' : \mathcal{A}) = \sum_{1 \leq j, k \leq n} (c_{n,k}^j)^* a_k^* a_j,$$

$$b = b(x' : \mathcal{A}) = - \sum_{1 \leq j, k \leq n-1} (c_{j,k}^n)^* a_j^* a_k + \sum_{j=n \text{ or } k=n} (c_{n,k}^j)^* a_k^* a_j.$$

$$(3) \quad \mathcal{P} = a_n^* a_n, \quad \mathcal{Q} = a_n a_n^* = I - \mathcal{Q}, \quad B = \frac{\partial}{\partial x_n} - \gamma + b.$$

As the argument in [11], it is enough to construct the fundamental solution in $\mathbf{R}_+^n$. Suppose that the fundamental solution is in the form $U_B(t) = U(t) + V(t)$, where $U(t)$ is

the fundamental solution for the Cauchy problem. Then we consider the following problem in $\mathbf{R}_+^n$.

$$\left\{ \begin{array}{l} (\frac{d}{dt} + R(x, D))V(t) = 0 \quad \text{in } I \times \mathbf{R}_+^n, \\ \mathcal{P}V(t) = -\mathcal{P}U(t) \quad \text{on } I \times \mathbf{R}^{n-1} \times \{x_n = 0\}, \\ BQV(t) = -BQU(t) \quad \text{on } I \times \mathbf{R}^{n-1} \times \{x_n = 0\}, \\ \lim_{t \rightarrow 0} V(t) = 0 \quad \text{in } \mathbf{R}_+^n. \end{array} \right.$$

Definition 4. Let $\{q_j\}_{j \leq 2}$ be defined as

$$\begin{aligned} q_2 &= r_2(x', 0, \xi', \xi_n : \mathcal{A}) = r_2^*, \\ q_{2-j} &= \sum_{\ell+k=j, 0 \leq k \leq 2} ((\frac{\partial}{\partial x_n})^\ell r_{2-k})^* \frac{x_n^\ell}{\ell!}, \quad j \geq 1. \end{aligned}$$

For any fixed N we set

$$\hat{q} = \sum_{j=2}^{-N+1} q_j.$$

We have by Definition 3 and 4

$$q_2 = (\xi_n + i\gamma)^2 + \beta(x', \xi' : \mathcal{A}),$$

where

$$\beta = - \sum_{j=1}^{n-1} ((\alpha_j)^* I - (G_j)^*)^2 + R^* \in K^2.$$

Let $\{\tilde{w}_{j,k}\}$ be symbols defined in Definition 7 of [11] and let $\{W_{j,k}\}$ be operators defined by $\{\tilde{w}_{j,k}\}$.

Definition 5. For a pair (j, k) of integer j and nonpositive integer k we define a function

$$\{\tilde{v}_{j,k}(t, x_n, y_n; b, \gamma)\}_{j,k} = e^{\gamma(x_n - y_n)} \tilde{w}_{j,k}(t, x_n + y_n; b).$$

An operator $V_{j,k}(t; b, \gamma)$ corresponding to $\tilde{v}_{j,k}$ is defined as follows for a function $\varphi(y_n)$ defined on $\mathbf{R}_+^1$.

$$(V_{j,k}(t; b, \gamma)\varphi)(x_n) = \int_0^\infty \tilde{v}_{j,k}(t, x_n, y_n; b, \gamma)\varphi(y_n)dy_n.$$

Here

$$w_{0,0}(t, \xi_n) = \exp(-t\xi_n^2)$$

$$w_{j,0}(t, \xi_n) = (i\xi_n)^j w_{0,0}(t, \xi_n), j \geq 0,$$

$$\tilde{w}_{j,0}(t, \omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\omega \cdot \xi_n} \tilde{w}_{j,0}(t, \xi_n) d\xi_n, j \geq 0,$$

$$\tilde{w}_{j,0}(t, \omega; b) = -\frac{1}{\sqrt{\pi}} \left(\frac{1}{2\sqrt{t}}\right)^{j+1} \int_0^{\infty} e^{-(\sigma + \frac{\omega}{2\sqrt{t}})^2} \frac{(-\sigma)^{-j-1}}{(-j-1)!} d\sigma, j \leq -1,$$

$$\text{for } k \leq -1 \quad \tilde{w}_{j,k}(t, \omega; b) =$$

$$\begin{cases} -\frac{1}{\sqrt{\pi}} \left(\frac{1}{2\sqrt{t}}\right)^{j+k+1} \int_0^{\infty} e^{-(\sigma + \frac{\omega}{2\sqrt{t}})^2 + 2b\sqrt{t}\sigma} \frac{(-\sigma)^{-k-1}}{(-k-1)!} h_j(\sigma + \frac{\omega}{2\sqrt{t}}) d\sigma, & \text{if } j \geq 0; \\ \frac{1}{\sqrt{\pi}} \left(\frac{1}{2\sqrt{t}}\right)^{j+k+1} \int_0^{\infty} \frac{(-\tau)^{-j-1}}{(-j-1)!} d\tau \int_0^{\infty} e^{-(\sigma + \tau + \frac{\omega}{2\sqrt{t}})^2 + 2b\sqrt{t}\sigma} \frac{(-\sigma)^{-k-1}}{(-k-1)!} d\sigma, & \text{if } j \leq -1, \end{cases}$$

$$\text{where } h_j(\sigma) = \{(\frac{\partial}{\partial \sigma})^j e^{-\sigma^2}\} e^{\sigma^2}.$$

Definition 6.

(1) $\mathcal{J}_s$ is the set of all finite sum of the following functions

$$\{g(t, x_n, y_n, x', \xi' : \mathcal{A}) = t^d(x_n)^\ell q(x', \xi' : \mathcal{A}) \tilde{v}_{j,k}(t, x_n, y_n; b(x' : \mathcal{A}), \gamma(x' : \mathcal{A})) e^{-\beta(x', \xi' : \mathcal{A})t};$$

$$q \in K^m(\mathbf{R}^{n-1}), d \geq 0, \ell \geq 0, k \leq 0, m = s + 2d + \ell - j - k\}.$$

(2) $\tilde{R}_\ell$ is the set of all matrices which belong to $B([0, T] \times [0, \infty) \times [0, \infty); S_{1,0}^m(\mathbf{R}^{n-1}))$ and satisfy for any α, β, a, b, k

$$\begin{aligned} & \|(\frac{\partial}{\partial \xi'})^\alpha (\frac{\partial}{\partial x'})^\beta (\frac{\partial}{\partial x_n})^a (\frac{\partial}{\partial y_n})^b (\frac{\partial}{\partial t})^k g\| \\ & \leq C_{\alpha, \beta} \min(|\xi'|^{-|\alpha|}, \sqrt{t}^{|\alpha|}) \left(\frac{1}{\sqrt{t}}\right)^{\ell+1+2k+a+b} \exp(-\delta \frac{(x_n + y_n)^2}{4t} - c_0 |\xi'|^2 t) \end{aligned}$$

for any $\delta < 1$ and some $c_0 > 0$.

(3) For a symbol $g(t, x_n, y_n, x', \xi', \mathcal{A}) \in \mathcal{J}_s$ we define a integral-pseudodifferential operator as follows.

$$(G\varphi)(t, x', x_n : \mathcal{A}) = \int_0^\infty g(t, x_n, y_n, x', D' : \mathcal{A}) \varphi(\cdot, y_n) dy_n.$$

Theorem 3. (1) For any $g(t) \in \mathcal{J}_s$ and $h(t) \in \mathcal{J}_{s-1}$ there exists $v(t) \in \mathcal{J}_{s-2}$ such that

$$\begin{cases} (\frac{\partial}{\partial t} + \hat{q}) \circ v(t) = g(t) \mod \mathcal{J}_{s-1} + \tilde{R}_{-N} & \text{in } I \times \mathbf{R}_+^n, \\ Bv(t)|_{x_n=0} = h(t) \mod \mathcal{J}_{s-2} + \tilde{R}_{-N-1} & \text{in } I \times \mathbf{R}^{n-1}. \end{cases}$$

(2) For any $g(t) \in \mathcal{J}_s$ and $h(t) \in \mathcal{J}_{s-2}$ there exists $v(t) \in \mathcal{J}_{s-2}$ such that

$$\begin{cases} (\frac{\partial}{\partial t} + \hat{q}) \circ v(t) = g(t) \mod \mathcal{J}_{s-1} + \tilde{R}_{-N} & \text{in } I \times \mathbf{R}_+^n, \\ v(t)|_{x_n=0} = h(t) \mod \mathcal{J}_{s-3} + \tilde{R}_{-N-2} & \text{in } I \times \mathbf{R}^{n-1}. \end{cases}$$

Now we discuss our boundary value problem .

For a function $h(x)$ defined in $\bar{\mathbf{R}}_+^n$, we set a function $h^+(x)$ defined in $\mathbf{R}^n$ as follows.

$$h^+(x) = \begin{cases} h(x', x_n), & \text{if } x_n \geq 0; \\ 0, & \text{if } x_n < 0. \end{cases}$$

Also we set

$$\varphi^+ = \sum_J \varphi^+ \omega^J \text{ for } \varphi = \sum_J \varphi_J \omega^J.$$

Theorem 4. For any N the asymptotics of the fundamental solution $U_B(t)$ for the boundary problem (7.2) in the sense

$$\begin{cases} (\frac{\partial}{\partial t} + \hat{q}) \circ v(t) = -(\frac{\partial}{\partial t} + \hat{q}) \circ u(t) \mod \tilde{R}_{-N+2} & \text{in } I \times \mathbf{R}_+^n, \\ \mathcal{P}(u(t) + v(t))|_{x_n=0} = 0 \mod \tilde{R}_{-N} & \text{in } I \times \mathbf{R}^{n-1}, \\ B\mathcal{Q}(u(t) + v(t))|_{x_n=0} = 0 \mod \tilde{R}_{-N+1} & \text{in } I \times \mathbf{R}^{n-1}. \end{cases}$$

is obtained in the form $U_B(t)\varphi = U(t)\varphi^+ + V(t)\varphi$, where $V(t)$ is the operator defined by a symbol $v(t) \in \mathcal{J}_1$ such that $v(t) = \sum_{j=0}^N v_{1-j}(t)$, $v_j(t) \in \mathcal{J}_{-j}$, $v_1(t) = 2\mathcal{Q}\tilde{v}_{1,-1}e^{-t\beta}$.

For a function $h(x)$ defined in $\bar{\mathbf{R}}_+^n$, we set a function $h^+(x)$ defined in $\mathbf{R}^n$ as follows.

$$h^+(x) = \begin{cases} h(x', x_n), & \text{if } x_n \geq 0; \\ 0, & \text{if } x_n < 0. \end{cases}$$

Also we set

$$\varphi^+ = \sum_J \varphi^+ \omega^J \text{ for } \varphi = \sum_J \varphi_J \omega^J.$$

Theorem 4. For any N the asymptotics of the fundamental solution $U_B(t)$ for the boundary problem (7.2) in the sense

$$\left\{ \begin{array}{l} (\frac{\partial}{\partial t} + \hat{q}) \circ v(t) = -(\frac{\partial}{\partial t} + \hat{q}) \circ u(t) \mod \tilde{R}_{-N+2} \quad \text{in } I \times \mathbf{R}_+^n, \\ \mathcal{P}(u(t) + v(t))|_{x_n=0} = 0 \mod \tilde{R}_{-N} \quad \text{in } I \times \mathbf{R}^{n-1}, \\ BQ(u(t) + v(t))|_{x_n=0} = 0 \mod \tilde{R}_{-N+1} \quad \text{in } I \times \mathbf{R}^{n-1}. \end{array} \right.$$

is obtained in the form $U_B(t)\varphi = U(t)\varphi^+ + V(t)\varphi$, where $V(t)$ is the operator defined by a symbol $v(t) \in \mathcal{J}_1$ such that $v(t) = \sum_{j=0}^N v_{1-j}(t)$, $v_j(t) \in \mathcal{J}_{-j}$, $v_1(t) = 2Q\tilde{v}_{1,-1}e^{-t\beta}$.

§7. The proof of Gauss-Bonnet-Chern theorem with boundary. Let $\hat{R}(W, Z, X, Y)$ be the Riemannian curvature tensors induced on Γ . From Equation of Gauss we have

$$R(X_i, X_j, X_k, X_\ell) = \hat{R}(X_i, X_j, X_k, X_\ell) + c_{k,j}^n c_{\ell,i}^n - c_{\ell,j}^n c_{k,i}^n,$$

$$1 \leq i, j, k, \ell \leq n-1 \quad \text{on } \Gamma.$$

Definition 7.

$$D_{n-1}(x) = \left(\frac{1}{2}\right) \left(\frac{1}{2\sqrt{\pi}}\right)^{n-1} \frac{1}{m!} \sum_{\pi, \sigma \in S_{n-1}} \left(\frac{1}{2}\right)^m \text{sign}(\pi) \text{sign}(\sigma) \hat{R}_{\pi(1)\pi(2)\sigma(1)\sigma(2)} \cdots \\ \cdots \hat{R}_{\pi(n-2)\pi(n-1)\sigma(n-2)\sigma(n-1)}$$

if n is odd ($n-1 = 2m$).

$$D_{n-1}(x) = \sum_{k=0}^{m-1} \frac{1}{2^{m+k} \pi^m k! 1 \cdot 3 \cdot 5 \cdots (2m-2k-1)} \left(\frac{1}{2}\right)^k \sum_{\pi, \sigma \in S_{n-1}} \text{sign}(\pi) \text{sign}(\sigma) \\ \times R_{\pi(1)\pi(2)\sigma(1)\sigma(2)}^* \cdots R_{\pi(2k-1)\pi(2k)\sigma(2k-1)\sigma(2k)}^* \\ \times c_{\pi(2k+1), \sigma(2k+1)}^n c_{\pi(2k+2), \sigma(2k+2)}^n \cdots c_{\pi(n-1), \sigma(n-1)}^n$$

if n even ($n = 2m$).

By Theorem 4 asymptotic of the fundamental solution for the mixed problem is given by $U_0 + U_1 + \cdots + U_N + V_1 + V_0 + \cdots + V_{-N}$, $v_j \in \mathcal{J}_j$, $v_j = g_j e^{-t\beta}$, $g_1 = 2Q\tilde{v}_{1,-1}$.

For the supertrace of kernel $\tilde{v}_j(t, x, y)$ of operator V_j , we have the following lemma.

Lemma 4. For any integer N we have

$$\text{str } \tilde{v}_j(t, x, x) = \begin{cases} 0(\sqrt{t}^N), & \text{if } x_n \neq 0; \\ 0((\sqrt{t})^{-j}), & \text{if } x_n = 0. \end{cases}$$

$$\int_0^\epsilon \text{str } \tilde{v}_j(t, x, x) \psi(x) dx_n = 0((\sqrt{t})^{-j+1}).$$

Moreover we have

$$\text{str } \tilde{v}_1(t, x', 0, x', 0) = \frac{2}{\sqrt{t}} D_{n-1}(x') \sqrt{\det g} + 0(1).$$

$$\int_0^\epsilon \text{str } \tilde{v}_1(t, x, x) \psi(x) dx_n = \psi(x', 0) D_{n-1}(x') \sqrt{\det \hat{g}} + 0(\sqrt{t}),$$

where $\hat{g}$ is the Riemannian metric induced on ∂M .

Theorem 5. For any N we have

$$\text{str } \tilde{v}(t, x, x) = \begin{cases} 0(\sqrt{t}^N) & \text{if } x_n > 0 \\ \frac{2}{\sqrt{t}} D_{n-1}(x') \sqrt{\det g} + 0(1) & \text{if } x_n = 0. \end{cases}$$

$$\int_0^\epsilon \text{str } \tilde{v}(t, x, x) \psi(x) dx_n = D_{n-1}(x') \sqrt{\det \hat{g}} \psi(x', 0) + 0(\sqrt{t}).$$

Proof of Main Theorem. It is sufficient to consider the fundamental solution locally if we study the asymptotic behavior of the fundamental solution. In a local patch we have

$$e(t, x, x) dv = \tilde{u}(t, x, x) dx + \tilde{v}(t, x, x) dx.$$

Then for any N we get by Theorem 5

$$\text{str } e(t, x, x) = \text{str } \tilde{u}(t, x, x) + 0(t^N), \quad x \in M \setminus \partial M$$

$$\text{str } e(t, x, x) = C_n(x) + 0(\sqrt{t}), \quad x \in M \setminus \partial M$$

$$\text{str } e(t, x, x) = \frac{2}{\sqrt{t}} D_{n-1}(x) + 0(1), \quad x \in \partial M.$$

We remark that the induced volume element of ∂M is defined by $d\sigma = (-1)^n \sqrt{\det \tilde{g}} dx_1 dx_2 \cdots dx_n$ in a local chart. In our case $D_{n-1}(x')d\sigma$ is independent of orientation of M . So we have

$$\int_M \text{str } e(t, x, x) \psi(x) dv = \int_M C_n(x) \psi(x) dv + \int_{\partial M} D_{n-1}(x') \psi(x') d\sigma + o(\sqrt{t}).$$

The proof is complete.

q. e. d.

REFERENCES

- [1] M.F. Atiyah, R. Bott and V.K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279-330.
- [2] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, 1991, Springer-Verlag.
- [3] S. Chern, *A simple intrinsic proof of the Gauss Bonnet formula for closed Riemannian manifolds*, Ann. of Math. **45** (1944), 747-752.
- [4] S. Chern, *On the curvature integrals in a Riemannian manifold*, Ann. of Math. **45** (1944), 747-752.
- [5] H.L. Cycon, R.G. Froese, W. Kirsch and B. Simon, *Schrödinger operators*, Texts and Monographs in Physics, 1987, Springer.
- [6] E. Getzler, *The local Atiyah-Singer index theorem*, Critical phenomena, random systems, gauge theories, K. Osterwalder and R. Stora, eds. Les Houches, Session XLIII, (1984), 967-974, North-Holland.
- [7] P.B. Gilkey, *The boundary integrand in the formula for the signature and Euler characteristic of a Riemannian manifold with boundary*, Adv. in Math., **15** (1975), 334-360.
- [8] P.B. Gilkey, *Invariance Theory, The Heat Equation, and the Atiyah-Singer Index Theorem*, 1984, Publish or Perish, Inc..
- [9] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion process*, Kodansha/North-Holland, Tokyo/Amsterdam, 1981, Second Ed. 1989.
- [10] C. Iwasaki, *The fundamental solution for pseudo-differential operators of parabolic type*, Osaka J. Math. **14** (1977), 569-592.
- [11] C. Iwasaki, *The asymptotic expansion of the fundamental solution for initial-boundary value problems and its application*, preprint.
- [12] C. Iwasaki, *A proof of the Gauss-Bonnet-Chern Theorem by the symbol calculus of pseudo-differential operators*, preprint.
- [13] S. Kobayashi-K. Nomizu, *Foundations of Differential Geometry I, II*, 1963, John Wiley & Sons.

- [14] H.P.McKean and I.M.Singer, *Curvature and the eigenvalues of the Laplacian*, J.Differential Geometry **1** (1967),43-69.
- [15] V.K.Patodi, *Curvature and the eigenforms of the Laplace operator*, J.Differential Geometry **5** (1971), 233-249.
- [16] I.Shigekawa,N.Ueki and S.Watanabe, *A probabilistic proof of the Gauss-Bonnet-Chern Theorem for manifolds with boundary*, Osaka J. Math. **26** (1989),897-930
- [17] C.Tsutsumi, *The Fundamental Solution for a Degenerate Parabolic Pseudo- Differential Operator*.Proc.Japan Acad. **50** (1974),11-15.