

# THE FORMAL DEGREE OF DISCRETE SERIES REPRESENTATIONS OF $GL_N$

( $GL_N$  の離散系列表現の形式的次数について)

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## INTRODUCTION

Let  $\mathbf{G}$  be a connected reductive group defined over a non-Archimedean local field  $F$ , and  $G = \mathbf{G}(F)$ . It is important to determine the formal degrees of the discrete series representations of  $G$  for the explicit Plancherel measure for  $G$ . In Aubert and Plymen [1], the explicit Plancherel measure for  $G = GL_N(D)$  is derived via the work of Silberger and Zink [13], where  $D$  is a division algebra over  $F$ .

For  $GL_N(F)$ , there are many works, e.g., [16], [8], [4], and [13] of computing formal degrees of discrete series representations. Indeed, the general formal degree formula of the discrete series representations are given in [4], and the explicit values of the formal degrees are computed in [8] in the tame case, and in [13] in the general case.

In this article, we improve the method of [13] by using the results of [4], and compute the values of the formal degrees of the discrete series representations of  $GL_N(F)$  (Theorem 1.4), which are expressed in terms of critical exponents (see 1.1 below). These expressions are implicit in the formula of [13, Theorem 1.1]. Thus, our formulas are not essentially new. But our improved method remains valid for some other classical groups. In fact, we obtained analogous results for a symplectic group  $Sp_N(F)$  and for a unramified unitary group  $U(V, h)$ , where  $N$  is an even integer  $\geq 4$  and  $h$  is a non-degenerate Hermitian form of an  $N$ -dimensional  $F$ -vector space.

The contents of this article are summarized as follows: In Section 1, we give the improvement of the method of Silberger and Zink [13] for  $GL_N(F)$ , and in Section 2, we present results (Theorem 2.7) obtained using the recent works on Hecke algebras of self-dual simple types of Kariyama and Miyauchi [11] (cf. [10]) for the unramified unitary group  $U(V, h)$ .

## 1. AN IMPROVEMENT ON THE METHOD OF SILBERGER-ZINK

**1.1. Preliminaries.** Let  $F$  be a non-Archimedean local field. Let  $\mathfrak{o}_F$  be the ring of integers of  $F$ ,  $\mathfrak{p}_F$  its maximal ideal, and  $k_F = \mathfrak{o}_F/\mathfrak{p}_F$  the residue field. We denote by  $q = |k_F|$  the cardinality of  $k_F$ .

Let  $N$  be an integer  $\geq 2$ , and  $V$  an  $N$ -dimensional vector space over  $F$ . We set  $A = \text{End}_F(V)$  and denote by  $G = A^\times$  the multiplicative group of  $A$ . By an appropriate  $F$ -basis of  $V$ , we identify  $A = M_N(F)$  and  $G = GL_N(F)$ .

We use the notations of Bushnell-Kutzko [4]. Let  $\mathfrak{A}$  be a hereditary  $\mathfrak{o}_F$ -order in  $A$  with Jacobson radical  $\mathfrak{P} = \text{rad}(\mathfrak{A})$ . We define a subgroup  $\mathcal{K}_{\mathfrak{A}}$  of  $G$  by  $\mathcal{K}_{\mathfrak{A}} = \{g \in G \mid g\mathfrak{A}g^{-1} = \mathfrak{A}\}$ . For an element  $\beta$  in  $A$ , the integer  $k_0(\beta, \mathfrak{A})$  is defined in [4, (1.4.5), (1.4.6)].

Following [4, (1.5)], a stratum in  $A$  is a 4-tuple  $[\mathfrak{A}, n, r, \beta]$ , where  $\mathfrak{A}$  is as above,  $n, r$  are integers such that  $n > r$ , and  $\beta \in A$  with  $\beta \in \mathfrak{P}^{-n}$ .

**Definition 1.1.** ([4, (1.5.5)]) A stratum  $[\mathfrak{A}, n, r, \beta]$  is called *pure*, if the following conditions are satisfied:

- (1) the algebra  $E = F[\beta]$  is a field,
- (2)  $E^\times \subset \mathcal{K}_{\mathfrak{A}}$ ,
- (3)  $\beta \in \mathfrak{P}^{-n} \setminus \mathfrak{P}^{1-n}$ .

It is called *simple* if, in addition,

- (4)  $r < -k_0(\beta, \mathfrak{A})$ .

Thus, for a simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$ , the integer  $k_0(\beta, \mathfrak{A})$  satisfies  $k_0(\beta, \mathfrak{A}) = -\min\{r \in \mathbb{Z} : [\mathfrak{A}, n, r, \beta] \text{ is not simple}\}$  of [15, (3.6)], and it is called the *critical exponent*.

**1.2. Simple types.** Hereafter, we assume that the hereditary  $\mathfrak{o}_F$ -order  $\mathfrak{A}$  in a simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$  is always principal, that is, there exists an element  $z$  in  $A$  such that  $\mathfrak{P} = z\mathfrak{A} = \mathfrak{A}z$ .

Let  $e_0 = e(\mathfrak{A}|\mathfrak{o}_F)$  be the  $\mathfrak{o}_F$ -period of  $\mathfrak{A}$ , that is,  $\mathfrak{P}^{e_0} = \mathfrak{p}_F\mathfrak{A}$ , and set

$$f_0 = N/e_0.$$

Then each element  $x$  of  $\mathfrak{A}$  has the block form  $x = (x_{ij})_{1 \leq i, j \leq e_0}$  with  $x_{ij} \in M_{f_0}(\mathfrak{o}_F)$  if  $i \leq j$ , and  $x_{ij} \in M_{f_0}(\mathfrak{p}_F)$  otherwise.

Let  $B$  be the  $A$ -centralizer of  $\beta$ , and  $\mathfrak{B} = \mathfrak{A} \cap B$ . Then it follows from Definition 1.1(2) that  $\mathfrak{B}$  is a hereditary  $\mathfrak{o}_E$ -order in  $B$ . Let  $e_1 = e(\mathfrak{B}|\mathfrak{o}_E)$  be the  $\mathfrak{o}_E$ -period of  $\mathfrak{B}$ , defined as is  $e_0$  above, where  $\mathfrak{o}_E$  is the maximal ideal of  $E$ .

Associated with a simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$ , the following three compact open subgroups

$$H^1 = H^1(\beta, \mathfrak{A}) \subset J^1 = J^1(\beta, \mathfrak{A}) \subset J = J(\beta, \mathfrak{A})$$

of  $G$  are defined in [4, (3.1)]. Via these groups, a *simple type* in  $G$ , denoted by  $(J, \lambda)$ , is constructed as follows: Take a simple character  $\theta$  of  $H^1$  (see [4, (3.2)] for the definition). Then it is known that there exists a unique irreducible representation  $\eta = \eta(\theta)$  of  $J^1$  containing  $\theta$ . We obtain an extension,  $\kappa$ , of  $\eta$  to  $J$ , which is called a  $\beta$ -*extension*.

Write  $G_E = B^\times$ . Then  $G_E$  is isomorphic to  $GL_{N/[E:F]}(E)$ . Set

$$f_1 = N/([E:F]e_1).$$

For  $U(\mathfrak{B}) = \mathfrak{B}^\times \supset U^1(\mathfrak{B}) = 1 + \text{rad}(\mathfrak{B})$ , it follows from [4, (3.1.15)] that  $J = U(\mathfrak{B})J^1$  and that

$$J/J^1 \simeq U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq GL_{f_1}(k_E)^{e_1},$$

where  $k_E$  denotes the residue field of  $E$ . Then  $J/J^1$  is isomorphic to a Levi subgroup of  $G_{N/[E:F]}(k_E)$ . Let  $\sigma_0$  be an irreducible cuspidal representation of  $GL_{f_1}(k_E)$ , and  $\sigma$  the inflation of the representation  $\sigma_0^{\otimes e_1} = \sigma_0 \otimes \cdots \otimes \sigma_0$  ( $e_1$ -times) of  $J/J^1$  to  $J$ . Now the simple type  $(J, \lambda)$  in  $G$  is defined by

$$\lambda = \kappa \otimes \sigma,$$

in [4, (5.5.10)(a)].

In particular, a simple type  $(J, \lambda)$  in  $G$  of *level zero* is defined in [4, (5.5.10)(b)]. This is a special case of [4, (5.5.10)(a)], by setting  $E = F$ ,  $\mathfrak{B} = \mathfrak{A}$ ,  $J^t = U^t(\mathfrak{A})(t = 0, 1)$ , and  $\theta, \eta, \kappa$  all trivial. Thus,  $J/J^1$  is isomorphic to  $GL_{N/e_1}(k_F)^{e_1}$  for  $e_1 = e(\mathfrak{A}|\mathfrak{o}_F)$ , and  $\lambda$  is the inflation of a representation  $\sigma_0^{\otimes e_1}$ , where we set  $f_1 = N/e_1$  and  $\sigma_0$  is an irreducible cuspidal representation of  $GL_{f_1}(k_F)$ , as above.

A simple type  $(J, \lambda)$  in  $G$  is called *maximal*, if  $e_1 = e(\mathfrak{A}|\mathfrak{o}_E) = 1$ .

**1.3. Discrete series representations of  $G = GL_N(F)$ .** Let  $e_1$  be a positive integer dividing  $N$ , and  $\rho$  an irreducible supercuspidal representation of  $G' = GL_{N/e_1}(F)$ . Then there exists a maximal simple type  $(J_0, \lambda_0)$  in  $G'$  containing  $\lambda_0$  by [4, (8.4.1)].

Let  $M$  be a Levi subgroup of  $G = GL_N(F)$  that is isomorphic to  $G'^{e_1} = G' \times \cdots \times G'$  ( $e_1$ -times), and  $P = MN$  a parabolic subgroup of  $G$  with Levi factor  $M$  and with unipotent radical  $N$ . Then  $\rho^{\otimes e_1}$  is an irreducible supercuspidal representation of  $M \simeq G'^{e_1}$ . Set  $J_M = J_0^{e_1}$ , and  $\lambda_M = \lambda_0^{\otimes e_1}$ . Then  $(J_M, \lambda_M)$  is a  $[M, \rho^{\otimes e_1}]_M$ -type in the sense of [5, (8.1)], and by [6, Proposition 1.4], there exists an irreducible representation  $\lambda_P$  of a compact open subgroup  $J_P$  of  $G$  associated with the parabolic subgroup  $P$  such that  $(J_P, \lambda_P)$  is a  $G$ -cover of  $(J_M, \lambda_M)$ . Thus,  $(J_P, \lambda_P)$  is a  $[M, \rho^{\otimes e_1}]_G$ -type. The pair  $(J_P, \lambda_P)$  is derived, as in [4, (7.2.17)], from a simple type  $(J, \lambda)$  in  $G$  associated with a simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$  as in Section 1.2, and satisfies  $\text{Ind}_{J_P}^J \lambda_P \simeq \lambda$ .

By [17], the induced representation

$$\text{Ind}_P^G(|\det|^{(1-e_1)/2} \rho \otimes \cdots \otimes |\det|^{(e_1-1)/2} \rho)$$

contains a unique irreducible discrete series representation, say  $(\pi, \mathcal{V})$ , of  $G$ . Hence, by [4, (7.3.14)],  $(\pi, \mathcal{V})$  contains  $(J, \lambda)$  and so  $(J_P, \lambda_P)$ .

**1.4. A formal degree formula.** Let  $(\pi, \mathcal{V})$  be the irreducible discrete series representation of  $G$  in the previous section that contains a simple type  $(J, \lambda)$  in  $G$  associated with a simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$  with  $E = F[\beta]$  as in Section 1.2. Let  $B$  be the  $A$ -centralizer of  $\beta$ , and  $\mathfrak{B} = B \cap \mathfrak{A}$ .

Let  $e_1 = e(\mathfrak{B}|\mathfrak{o}_E)$  and  $f_1 = N/([E:F]e_1)$  be as in Section 1.2. Let  $K/E$  be an unramified extension of degree  $f_1$ , and set  $C^\times = GL_{e_1}(K)$ . Let  $\mathcal{I}$  be an Iwahori subgroup of  $C^\times$ , and  $\mathbf{1}_{\mathcal{I}}$  the trivial representation of  $\mathcal{I}$ . Then the Hecke algebras  $\mathcal{H}(G, \lambda)$  and  $\mathcal{H}(C^\times, \mathbf{1}_{\mathcal{I}})$  for  $(J, \lambda)$  and

$(\mathcal{I}, \mathbf{1}_{\mathcal{I}})$ , respectively, are obtained (cf. [4, Section 4]). It follows from [4, (5.5.16), (5.6.6)] that there exists a support-preserving isomorphism

$$\Psi : \mathcal{H}(G, \lambda) \simeq \mathcal{H}(C^\times, \mathbf{1}_{\mathcal{I}})$$

(cf. [4, (7.6.18)]). Via this isomorphism  $\Psi$ , the equivalence class of the irreducible discrete series representation  $(\pi, \mathcal{V})$  of  $G$  corresponds to that of an unramified twist  $\phi \cdot \text{St}_{C^\times}$  of the Steinberg representation  $\text{St}_{C^\times}$  of  $C^\times$  (cf. [13, p.13]). In the notation of [4, (7.7.1)], we have

$$\pi \simeq \mathfrak{Ad}_\Psi(\phi \cdot \text{St}_{C^\times})$$

where  $\mathfrak{Ad}_\Psi$  is defined as in [4, (7.6.21)]. Denote by  $\text{St}_G$  the Steinberg representation of  $G$ , and by  $\deg(\text{St}_G, dx)$  the formal degree of  $\text{St}_G$  relative to a Haar measure  $dx$  on  $G$ . Since the formal degrees of  $\text{St}_{C^\times}$  and  $\phi \cdot \text{St}_{C^\times}$  are the same, by [4, (7.7.11)], we obtain the following result.

**Theorem 1.2.** (cf. [13, Proposition 3.6]) *Let notations and assumptions be as above. Then for arbitrary measures  $dx$  on  $G$  and  $dy$  on  $C^\times$ ,*

$$(1.1) \quad \frac{\dim(\lambda)}{e(E|F)} \text{vol}(K^\times \mathcal{I} / K^\times, dy) \deg(\text{St}_{C^\times}, dy) = \text{vol}(F^\times J / F^\times, dx) \deg(\pi, dx),$$

where  $e(E|F)$  denotes the index of ramification of  $E/F$ .

**1.5. A deformation of the formal degree formula.** There exists a hereditary  $\mathfrak{o}_F$ -order  $\mathfrak{A}_m$  in  $A$  such that  $U(\mathfrak{A}_m) = \mathfrak{A}_m^\times$  is an Iwahori subgroup of  $G$  that is contained in the parahoric subgroup  $U(\mathfrak{A})$ . Now, we first normalize  $dx$  on  $G$  so that  $\text{vol}(F^\times U(\mathfrak{A}_m) / F^\times, dx) = 1$ . Then

$$\begin{aligned} \text{vol}(F^\times U(\mathfrak{A}) / F^\times, dx) &= (U(\mathfrak{A}) : U(\mathfrak{A}_m)) \text{vol}(F^\times U(\mathfrak{A}_m) / F^\times, dx) \\ &= (U(\mathfrak{A}) : U(\mathfrak{A}_m)), \end{aligned}$$

and by [3, Proposition 5.3], the formal degree of the Steinberg representation  $\text{St}_G$  of  $G$  is given by

$$\deg(\text{St}_G, dx) = \frac{1}{N} \widetilde{W}_{A_{N-1}}(q^{-1})^{-1},$$

where  $\widetilde{W}_{A_{N-1}}(t)$  is the Poincaré series of type  $A_{N-1}$  (see [12]). Thus, we obtain Macdonald's formula:

$$\text{vol}(F^\times U(\mathfrak{A}) / F^\times, dx) \deg(\text{St}_G, dx) = \frac{1}{N} (U(\mathfrak{A}) : U(\mathfrak{A}_m)) \widetilde{W}_{A_{N-1}}(q^{-1})^{-1}$$

which is the same as that of [13, 3.7]. This formula holds for any Haar measure  $dx$  on  $G$ . We again normalize  $dx$  on  $G$  so that

$$\deg(\text{St}_G, dx) = 1.$$

Then

$$\text{vol}(F^\times U(\mathfrak{A}) / F^\times, dx) = \frac{1}{N} (U(\mathfrak{A}) : U(\mathfrak{A}_m)) \widetilde{W}_{A_{N-1}}(q^{-1})^{-1}$$

On the other hand, for  $C^\times = GL_{e_1}(K)$ , set

$$f = f(K|F) = [k_K : k_F],$$

where  $k_K$  is the residue field of  $K$ . Then, from Macdonald's formula for  $\text{St}_{C^\times}$ , we also obtain

$$\text{vol}(K^\times \mathcal{I}/K^\times, dy) \deg(\text{St}_{C^\times}, dy) = \frac{1}{e_1} \widetilde{W}_{A_{e_1-1}}(q^{-f})^{-1}.$$

Hence, since  $N/e_1 = [K : F] = fe(E|F)$  and

$$\text{vol}(F^\times J/F^\times, dx) = \text{vol}(F^\times U(\mathfrak{A})/F^\times, dx)(U(\mathfrak{A}) : J)^{-1},$$

we obtain

$$(1.2) \quad \frac{\deg(\text{St}_{C^\times}, dy) \text{vol}(K^\times \mathcal{I}/K^\times, dy)}{e(E|F) \text{vol}(F^\times J/F^\times, dx)} = f \frac{\widetilde{W}_{A_{N-1}}(q^{-1})}{\widetilde{W}_{A_{e_1-1}}(q^{-f})} \frac{(U(\mathfrak{A}) : J)}{(U(\mathfrak{A}) : U(\mathfrak{A}_m))}.$$

Since  $J = U(\mathfrak{B})J^1$  and  $J^1 \subset U^1(\mathfrak{A})$ , we moreover obtain

$$(1.3) \quad (U(\mathfrak{A}) : J) = \frac{(U(\mathfrak{A}) : U^1(\mathfrak{A}))}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))} (U^1(\mathfrak{A}) : J^1).$$

**Lemma 1.3.** *Let notations and assumptions be as above. Then the formal degree  $\deg(\pi, dx)$  is equal to*

$$\left( f \frac{\widetilde{W}_{A_{N-1}}(q^{-1})}{\widetilde{W}_{A_{e_1-1}}(q^{-f})} \right) (U(\mathfrak{A}_m) : U^1(\mathfrak{A})) \left( \frac{\dim(\sigma)}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))} \right) \times \left( (U^1(\mathfrak{A}) : J^1) \dim(\eta) \right).$$

*Proof.* By definition,  $\dim(\lambda) = \dim(\eta) \dim(\sigma)$ . Thus, the lemma follows from Eqs. (1.1) to (1.3).

**1.6. Calculation of the factors of  $\deg(\pi, dx)$ .** By the definition in [12],

$$(1.4) \quad f \frac{\widetilde{W}_{A_{N-1}}(q^{-1})}{\widetilde{W}_{A_{e_1-1}}(q^{-f})} = f \frac{q^N - 1}{q^{N/e} - 1} \frac{(q^f - 1)^{e_1}}{(q - 1)^N}$$

where we set  $e = e(E|F)$ .

Since  $U(\mathfrak{A})/U^1(\mathfrak{A}) \simeq GL_{f_0}(k_F)^{e_0}$  in Section 1.2, the quotient group  $U(\mathfrak{A}_m)/U^1(\mathfrak{A})$  is isomorphic to the product of  $e_0$ -copies of a Borel subgroup  $\overline{B}_0$  of  $GL_{f_0}(k_F)$ . Hence,

$$(1.5) \quad (U(\mathfrak{A}_m) : U^1(\mathfrak{A})) = |\overline{B}_0|^{e_0} = \{(q - 1)^{f_0} q^{\frac{1}{2}f_0(f_0-1)}\}^{e_0}.$$

By the definition of  $\sigma_0$ ,

$$\dim(\sigma) = (\dim(\sigma_0))^{e_1} = \prod_{i=1}^{f_1-1} (q^{f(E|F)i} - 1)^{e_1},$$

and  $(U(\mathfrak{B}) : U^1(\mathfrak{B})) = |GL_{f_1}(k_E)|^{e_1}$ . Hence,

$$(1.6) \quad \frac{\dim(\sigma)}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))} = \frac{q^{-\frac{1}{2}f(E|F)e_1f_1(f_1-1)}}{(q^f - 1)^{e_1}}.$$

**1.7. Calculation of the factor  $(U^1(\mathfrak{A}) : J^1) \dim(\eta)$ .** Since by definition  $U^1(\mathfrak{A}) = 1 + \mathfrak{P} \supset J^1 = 1 + \mathfrak{J}^1 \supset H^1 = 1 + \mathfrak{H}^1$ , it follows from [4, (5.1.1)] that the last factor  $(U^1(\mathfrak{A}) : J^1) \dim(\eta)$  is equal to

$$\begin{aligned} (\mathfrak{P} : \mathfrak{J}^1) \sqrt{(J^1 : H^1)} &= (\mathfrak{P} : \mathfrak{J}^1)(\mathfrak{J}^1 : \mathfrak{H}^1)^{1/2} \\ &= (\mathfrak{P} : \mathfrak{J}^1)^{1/2} (\mathfrak{P} : \mathfrak{H}^1)^{1/2}. \end{aligned}$$

This amounts to  $\dim \pi_\beta^1$  in [13, 7.1]. We compute this similarly to as done in [13].

For the simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$ ,  $[\mathfrak{A}, n, r, \beta]$  with  $r = -k_0(\beta, \mathfrak{A})$  is a pure stratum, and it determines a family  $[\mathfrak{A}, n, r_i, \gamma_i]$ ,  $0 \leq i \leq s$  of simple strata that satisfy the conditions [4, (2.4.2)]. This family is called a *defining sequence* for the pure stratum  $[\mathfrak{A}, n, r, \beta]$ . Hence we get a family of pairs  $(r_i, \gamma_i)$ ,  $0 \leq i \leq s$ , such that

- (1)  $[\mathfrak{A}, n, r_i, \gamma_i]$  is simple,  $0 \leq i \leq s$ ;
- (2)  $[\mathfrak{A}, n, r_0, \gamma_0] \sim [\mathfrak{A}, n, r, \beta]$ ;
- (3)  $0 < r = r_0 < r_1 < \cdots < r_s < r_{s+1} = n$ ;
- (4)  $r_{i+1} = -k_0(\gamma_i, \mathfrak{A})$ , and for  $0 \leq i \leq s-1$ ,

$$[\mathfrak{A}, n, r_{i+1}, \gamma_{i+1}] \sim [\mathfrak{A}, n, r_{i+1}, \gamma_i],$$

- (5)  $-k_0(\gamma_s, \mathfrak{A}) = n$  or  $\infty$ .

These conditions include all the conditions in [4, (2.4.2)] except condition (vi).

Denote by  $A_{\gamma_i}$  the  $A$ -centralizer of  $\gamma_i$ . Then by using the family  $(r_i, \gamma_i)$ ,  $0 \leq i \leq s$ , we define  $\mathfrak{J} = \mathfrak{J}(\beta) = \mathfrak{J}(\beta, \mathfrak{A})$  and  $\mathfrak{H} = \mathfrak{H}(\beta) = \mathfrak{H}(\beta, \mathfrak{A})$  inductively as follows:

$$\begin{aligned} \mathfrak{J}(\gamma_i) &= \mathfrak{A} \cap A_{\gamma_i} + \mathfrak{J}(\gamma_{i+1}) \cap \mathfrak{P}^{[(r_{i+1}+1)/2]}, \\ \mathfrak{H}(\gamma_i) &= \mathfrak{A} \cap A_{\gamma_i} + \mathfrak{H}(\gamma_{i+1}) \cap \mathfrak{P}^{[r_{i+1}/2]+1}, \end{aligned}$$

for  $-1 \leq i \leq s$ , where we set  $\gamma_{-1} = \beta$  and  $\mathfrak{J}(\gamma_{s+1}) = \mathfrak{H}(\gamma_{s+1}) = \mathfrak{A}$ , (see [4, (3.1.7), (3.1.8)]).

For a real number  $r$ , set

$$\bar{r} = [r/2] + 1, \quad \underline{r} = [(r+1)/2],$$

where  $[x]$  denotes the greatest integer  $\leq x$ , for a real number  $x$ . From a filtration

$$\mathfrak{P} = \mathfrak{P} + \mathfrak{J}^1 \supset \mathfrak{P}^{\underline{r}_1} + \mathfrak{J}^1 \supset \mathfrak{P}^{\underline{r}_2} + \mathfrak{J}^1 \supset \cdots \supset \mathfrak{P}^{\underline{r}_s} + \mathfrak{J}^1 \supset \mathfrak{P}^n + \mathfrak{J}^1 = \mathfrak{J}^1$$

in  $\mathfrak{P} \supset \mathfrak{J}^1$ , we get

$$\begin{aligned} (\mathfrak{P} : \mathfrak{J}^1) &= \prod_{i=-1}^s (\mathfrak{P}^{r_i} + \mathfrak{J}^1 : \mathfrak{P}^{r_{i+1}} + \mathfrak{J}^1) \\ &= \prod_{i=-1}^s \frac{(\mathfrak{P}^{r_i} : \mathfrak{P}^{r_{i+1}})}{(\mathfrak{P}^{r_i} \cap \mathfrak{J} : \mathfrak{P}^{r_{i+1}} \cap \mathfrak{J})} \\ &= \prod_{i=-1}^s \frac{(\mathfrak{P}^{r_i} : \mathfrak{P}^{r_{i+1}})}{(\mathfrak{P}^{r_i} \cap A_{\gamma_i} : \mathfrak{P}^{r_{i+1}} \cap A_{\gamma_i})}, \end{aligned}$$

where we set  $r_{-1} = 1$ . The last line follows from [4, (3.1.8), (3.1.10)].

For  $-1 \leq i \leq s$ , set

$$d_i = \underline{r_{i+1}} - \underline{r_i}.$$

Since  $\mathfrak{P}$  is principal,

$$(\mathfrak{P}^{r_i} : \mathfrak{P}^{r_{i+1}}) = (\mathfrak{A} : \mathfrak{P})^{d_i} = |M_{f_0}(k_F)|^{e_0 d_i} = q^{N f_0 d_i}.$$

Set  $f'_i = f(F[\gamma_i]|F)$ ,  $e'_i = e(F[\gamma_i]|F)$ , and

$$m_i = (N/[F[\gamma_i] : F])/(e_0/e'_i) = f_0/f'_i.$$

Then

$$(\mathfrak{P}^{r_i} \cap A_{\gamma_i} : \mathfrak{P}^{r_{i+1}} \cap A_{\gamma_i}) = |M_{m_i}(k_{F[\gamma_i]})|^{(e_0/e'_i)d_i} = q^{N f_0 d_i / [F[\gamma_i] : F]}.$$

Hence,

$$(\mathfrak{P} : \mathfrak{J}^1) = q^\mu; \quad \mu = \sum_{i=-1}^s N f_0 (1 - [F[\gamma_i] : F]^{-1}) d_i.$$

Similarly, setting  $d'_i = \overline{r_{i+1}} - \overline{r_i}$ , we get

$$(\mathfrak{P} : \mathfrak{H}^1) = q^{\mu'}; \quad \mu' = \sum_{i=-1}^s N f_0 (1 - [F[\gamma_i] : F]^{-1}) d'_i.$$

Finally, we obtain  $(\mathfrak{P} : \mathfrak{J}^1)^{1/2} (\mathfrak{P} : \mathfrak{H}^1)^{1/2} = q^{\nu/2}$  with

$$(1.7) \quad \nu = \sum_{i=-1}^s N f_0 (1 - [F[\gamma_i] : F]^{-1}) (r_{i+1} - r_i).$$

**1.8. Main Theorem for  $GL_N(F)$ .** We are now ready to determine the explicit formal degree formula for an irreducible discrete series representation  $(\pi, \mathcal{V})$  of  $G$  as in Section 1.3.

**Theorem 1.4.** ([13, Theorem 1.1]) *Let  $(\pi, \mathcal{V})$  be an irreducible discrete series representation of  $G$  that contains a simple type  $(J, \lambda)$  in  $G$  associated with a simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$  as in Section 1.3. Let*

$dx$  be a Haar measure on  $G$  such that  $\deg(\text{St}_G, dx) = 1$ . For a family  $(r_i, \gamma_i)$ ,  $0 \leq i \leq s$ , as in Section 1.7, set

$$\Delta = \frac{1}{e(\mathfrak{A}|\mathfrak{o}_F)} \sum_{i=-1}^s (1 - [F[\gamma_i] : F]^{-1})(r_{i+1} - r_i)$$

where we reset  $(r_{-1}, \gamma_{-1}) = (0, \beta)$  and  $r_{s+1} = n$ . Then this positive rational number  $\Delta$  does not depend on the choice of defining sequence, and

$$\deg(\pi, dx) = f \frac{q^N - 1}{q^{N/e} - 1} q^{\frac{1}{2}[N^2\Delta - N(1-1/e)]},$$

where  $f = f(K|F)$  and  $e = e(K|F) = e(E|F)$ .

*Proof.* Denote by  $\deg(\pi, dx)_{p'}$  the  $p$ -prime part of  $\deg(\pi, dx)/f$ . From Lemma 1.3 and from Eqs. (1.4) to (1.6), it follows immediately that

$$f \cdot \deg(\pi, dx)_{p'} = f \frac{q^N - 1}{q^{N/e} - 1}.$$

Since the  $q$ -power of (1.5)  $\times$  (1.6) is equal to

$$\begin{aligned} \frac{1}{2}e_0f_0(f_0 - 1) &= \frac{1}{2}f(E|F)e_1f_1(f_1 - 1) \\ &= \frac{1}{2}N\left(\frac{1}{e} - 1\right) + \frac{1}{2}Nf_0\left(1 - \frac{1}{[E : F]}\right), \end{aligned}$$

the sum of this value and the value  $\nu$  of (1.7) is reduced to the  $q$ -power of the right-hand side of the formula in the assertion.

It follows directly from [4, (2.1.4)] that  $\Delta$  does not depend on the choice of defining sequence. The proof is complete.

In Theorem 1.4, if  $e_1 = e(\mathfrak{B}|\mathfrak{o}_E) = 1$ , the irreducible discrete series representation  $(\pi, \mathcal{V})$  of  $G$  is supercuspidal. Thus, Theorem 1.4 contains the formal degree formula for an irreducible supercuspidal representation of  $G$  containing a maximal simple type  $(J, \lambda)$  as follows:

**Corollary 1.5.** *Let  $(\pi, \mathcal{V})$  be an irreducible supercuspidal representation of  $G$  containing a maximal simple type  $(J, \lambda)$  in  $G$ , and  $\{(r_i, \gamma_i) : 0 \leq i \leq s\}$  a family as in Theorem 1.3. Then*

$$\deg(\pi, dx) = f \frac{q^N - 1}{q^{N/e} - 1} q^{\frac{1}{2}[N^2\Delta - N(1-1/e)]},$$

for a Haar measure  $dx$  on  $G$  with  $\deg(\text{St}_G, dx) = 1$ , where

$$\Delta = \frac{1}{e} \sum_{i=-1}^s (1 - [F[\gamma_i] : F]^{-1})(r_{i+1} - r_i).$$

**Remarks 1.6.** (i) It is shown by [4, (6.2.2)] that an irreducible supercuspidal representation  $(\pi, \mathcal{V})$  of  $G$  containing a maximal simple type

$(J, \lambda)$  in  $G$  of positive level is equivalent to  $\text{c-Ind}_{E^\times J}^G \Lambda$  for an extension  $\Lambda$  of  $\lambda$ . From this fact, we obtain

$$\deg(\pi, dx) = \frac{\dim(\lambda)}{\text{vol}(E^\times J/F^\times, dx)}$$

for any Haar measure  $dx$  on  $G$ , (cf. [7, 5.9]), independently from Theorem 1.2, and can similarly show Corollary 1.5.

(ii) In Theorem 1.4, the level zero case is implicit. In this case,  $E = F$ ,  $\mathfrak{B} = \mathfrak{A}$ ,  $J^t = U^t(\mathfrak{A})(t = 0, 1)$  and  $\eta$  is trivial, as in Section 1.2. Thus, in the formula for  $\deg(\pi, dx)$  of Lemma 1.3, we have  $(U^1(\mathfrak{A}) : J^1) \dim(\eta) = 1$ , and can get

$$\deg(\pi, dx) = N/e(\mathfrak{A}|\mathfrak{o}_F)$$

similarly to as done in Section 1.6, for such a Haar measure  $dx$  on  $G$  as above.

(iii) This formula follows also from Theorem 1.4, by setting  $e = 1$ ,  $\Delta = 0$  and  $\lambda = \lambda_1$ , since  $E = F$ ,  $J = U(\mathfrak{A})$  and so the defining sequence is empty. Hence, by Theorem 1.4, the formal degrees of all discrete series representations of  $GL_N(F)$  are computed.

## 2. AN APPLICATION TO UNRAMIFIED $p$ -ADIC UNITARY GROUPS

**2.1. Unramified unitary groups.** Let  $F$  be a non-Archimedean local field of odd residual characteristic, with a non-trivial galois involution  $x \mapsto \bar{x}$  with fixed field  $F_0$ . Let  $N$  be an even integer  $\geq 4$ , and  $V$  an  $N$ -dimensional  $F$ -vector space equipped with a non-degenerate  $F/F_0$ -Hermitian form  $h$  with anisotropic part (0). Let  $G = U(V, h)$  be the unitary group of  $(V, h)$ . Hereafter, we assume that  $F/F_0$  is unramified.

Recently, in [9], we defined a *self-dual* simple type  $(J, \lambda)$  in  $G$  associated with a certain skew simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$ , and, in [11], we proved that the Hecke algebra  $\mathcal{H}(G, \lambda)$  is the affine Hecke algebra of type  $\tilde{C}_m$  for some positive integer  $m \geq 2$ , and determined the parameters of the Hecke algebra completely. Thanks to these results on  $G$ , we can apply the improved method of the previous section for  $GL_N(F)$  to the unramified unitary group  $G$ , and we obtain analogous results for the group  $G$ . Here we present a part of these results without proofs.

**2.2. Self-dual simple types.** We also denote by  $x \mapsto \bar{x}$  the adjoint (anti-)involution on  $A = \text{End}_F(V)$  induced by the Hermitian form  $h$ .

A simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$ , defined in Section 1.1, is called *skew* if  $\mathfrak{A}$  is defined by a self-dual strict  $\mathfrak{o}_F$ -lattice sequence  $\Lambda$  in  $V$  (cf. [14, 1.2]) and  $\beta$  is skew in  $A$ , i.e.,  $\bar{\beta} = -\beta$ . Assume that  $[\mathfrak{A}, n, 0, \beta]$  is a skew simple stratum in  $A$  with  $E = F[\beta]$ . Write  $E_0 = \{x \in E : \bar{x} = x\}$ . Then there exists a non-degenerate  $E/E_0$ -Hermitian form  $h_E$  on the  $E$ -vector space  $V$  such that, setting  $L^\# = \{v \in V : h_E(v, L) \subset \mathfrak{p}_E\}$ , for an  $\mathfrak{o}_E$ -lattice  $L$  in  $V$ , we have  $L^\# = \{v \in V : h(v, L) \subset \mathfrak{p}_F\}$  (cf. [15, Section 2]).

**Definition 2.1.** Following [9], we say that a skew simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$  is *good* if the following conditions are satisfied:

- (1)  $E/E_0$  is an unramified quadratic extension;
- (2)  $R = \dim_E(V)$  is even;
- (3) there exists an  $\mathfrak{o}_E$ -lattice  $L$  in  $\{\Lambda(n) : n \in \mathbb{Z}\}$  such that  $L^\# = \varpi_E L$ , where  $\Lambda$  is as above and  $\varpi_E$  is a uniformizer of  $E$ .

Hereafter, we assume that  $[\mathfrak{A}, n, 0, \beta]$  is a good skew simple stratum in  $A$  with  $E = F[\beta]$ . Let  $B$  be the  $A$ -centralizer of  $\beta$ ,  $\mathfrak{B} = B \cap \mathfrak{A}$ ,  $e_1 = e(\mathfrak{B}|\mathfrak{o}_E)$ , and  $f_1 = N/([E : F]e_1)$ , as before.

Similarly, we have compact open subgroups  $H^1(\beta, \mathfrak{A}) \subset J^1(\beta, \mathfrak{A}) \subset J(\beta, \mathfrak{A})$  of  $G$ . Denote simply by  $H^1 \subset J^1 \subset J$  these subgroups. As in Section 1.2, we begin with a skew simple character  $\theta$  of  $H^1$ , and there exists a unique irreducible representation  $\eta$  of  $J^1$  containing  $\theta$ . Moreover, we also have a  $\beta$ -extension  $\kappa$  of  $\eta$  to  $J$  by [15, 4.2]. Set

$$m = [e_1/2].$$

It follows from the conditions of Definition 2.1 that the quotient group  $J/J^1$  is isomorphic to

$$U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \begin{cases} GL_{f_1}(k_E)^m & \text{if } e_1 \text{ is even,} \\ GL_{f_1}(k_E)^m \times U_{f_1}(k_E/k_{E_0}) & \text{if } e_1 \text{ is odd,} \end{cases}$$

where by Definition 2.1(2),  $f_1$  is even and  $U_{f_1}(k_E/k_{E_0})$  denotes the unitary group of a non-degenerate  $k_E/k_{E_0}$ -Hermitian form on an  $f_1$ -dimensional  $k_E$ -vector space. Let  $\sigma_0$  and  $\sigma_1$  be irreducible cuspidal representations of  $GL_{f_1}(k_E)$  and  $U_{f_1}(k_E/k_{E_0})$ , respectively. Let  $\sigma$  be the inflation of  $\sigma_0^{\otimes m}$  to  $J$  if  $e_1$  is even, and that of  $\sigma_0^{\otimes m} \otimes \sigma_1$  if  $e_1$  is odd. A simple type  $(J, \lambda)$  in  $G$  (of positive level) is defined similarly by  $\lambda = \kappa \otimes \sigma$ . By [15, p.334], on each factor  $GL_{f_1}(k_E)$  of  $U(\mathfrak{B})/U^1(\mathfrak{B})$ , a certain Weyl group element of  $G_E = B \cap G$  induces an involution  $g \mapsto \bar{g}$ .

**Definition 2.2.** ([11, Definition 5.2]) A simple type  $(J, \lambda)$  in  $G$  is called *self-dual* if  $\sigma_0$  is equivalent to the representation  $g \mapsto \sigma_0(\bar{g})$ .

**2.3. A formal degree formula for unramified  $U(V, h)$ .** Recently, by [10] and [11], we determined the structure of the Hecke algebra  $\mathcal{H}(G, \lambda)$  for such a self-dual simple type  $(J, \lambda)$  above as follows:

**Proposition 2.3.** *Let  $[\mathfrak{A}, n, 0, \beta]$  be a good skew simple stratum in  $A$  with  $E = F[\beta]$ , and  $(J, \lambda)$  a self-dual simple type in  $G$  associated with  $[\mathfrak{A}, n, 0, \beta]$  in  $A$ . Let  $B$  be the  $A$ -centralizer of  $\beta$ ,  $e_1 = e(\mathfrak{B}|\mathfrak{o}_E)$ , and  $m = [e_1/2]$ . Assume that  $m \geq 2$ . Then the Hecke algebra  $\mathcal{H}(G, \lambda)$  for  $(J, \lambda)$  is an affine Hecke algebra of type  $\tilde{C}_m$  with parameter  $(q_1, q_2, q_3) = (q^{-N/e_0}, q^{-N/2e_0}, q^{-N/2e_0})$ , where  $e_0 = e(\mathfrak{A}|\mathfrak{o}_F)$  as in Section 1.1.*

Let  $K/E_0$  be a field extension such that  $K \supset E$ ,  $K \supset K_0 \supset E_0$ ,  $e(K|E_0) = 1$ , and  $[K : E] = [K_0 : E_0] = f_1$ . Let  $C^\times$  be the unitary

group of type  $C_m$  defined by a non-degenerate  $K/K_0$ -Hermitian form, and  $\mathcal{I}$  an Iwahori subgroup of  $C^\times$ . Immediately,  $|k_K| = |k_E|^{f_1} = q^{N/e_0}$ , and so the Iwahori-Hecke algebra  $\mathcal{H}(C^\times, \mathbf{1}_{\mathcal{I}})$  turns out to be also the affine Hecke algebra of type  $\tilde{C}_m$  with parameter  $(q_1, q_2, q_3) = (q^{-N/e_0}, q^{-N/2e_0}, q^{-N/2e_0})$  provided that  $m \geq 2$ .

When  $m = [e_1/2] = 1$ , again by [11],  $\mathcal{H}(G, \lambda)$  is isomorphic to the affine Hecke algebra with parameter  $|k_{K_0}| = q^{N/2e_0}$  over the infinite dihedral group, and so is the Iwahori-Hecke algebra  $\mathcal{H}(C^\times, \mathbf{1}_{\mathcal{I}})$  of unramified  $C^\times = U(1, 1)(K_0)$  relative to  $\mathbf{1}_{\mathcal{I}}$  as well (cf. [2, 3.d]). Hence, we obtain the following:

**Proposition 2.4.** *Let notations and assumptions be as above. In particular, let  $m \geq 1$ . Then there exists a canonical isomorphism*

$$\Psi : \mathcal{H}(G, \lambda) \simeq \mathcal{H}(C^\times, \mathbf{1}_{\mathcal{I}}),$$

*that is support-preserving.*

**Theorem 2.5.** *Via the Hecke isomorphism  $\Psi$  in Proposition 2.4, the Steinberg representation  $\text{St}_{C^\times}$  of  $C^\times$  corresponds to the equivalence class of an irreducible square-integrable representation, say  $(\pi, \mathcal{V})$ , of  $G$  that contains the self-dual simple type  $(J, \lambda)$  in  $G$  as above.*

*Proof.* This is the analogue of [5, (7.7.1)] for  $GL_N(F)$ . The method of proof remains valid for unramified  $G$ .

**Corollary 2.6.** *Let notations and assumptions be as in Theorem 2.5. Then*

$$\text{vol}(J, dx) \frac{\deg(\pi, dx)}{\dim(\lambda)} = \text{vol}(\mathcal{I}, dy) \deg(\text{St}_{C^\times}, dy)$$

*for any Haar measures  $dx$  on  $G$  and  $dy$  on  $C^\times$ .*

**2.4. A formal degree formula for unramified  $G$ .** We normalize  $dx$  on  $G$  so that the formal degree of the Steinberg representation  $\text{St}_G$  of  $G$  is equal to 1 relative to  $dx$ . Then, as in Section 1.5, the formal degree  $\deg(\pi, dx)$  is rewritten as

$$\begin{aligned} & \left( \frac{\widetilde{W}_{C_{N/2}}(q^{-1}, q^{-1/2}, q^{-1/2})}{\widetilde{W}_{C_m}(q^{-N/e_0}, q^{-N/2e_0}, q^{-N/2e_0})} \right) (U(\mathfrak{A}_m) : U^1(\mathfrak{A})) \\ & \times \left( \frac{\dim(\sigma)}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))} \right) ((U^1(\mathfrak{A}) : J^1) \dim(\eta)), \end{aligned}$$

where  $U(\mathfrak{A}_m)$  is an Iwahori subgroup of  $G$  that is contained in  $U(\mathfrak{A})$ , and, for example,  $\widetilde{W}_{C_m}(t_1, t_2, t_3)$  denotes the Poincaré series of type  $\tilde{C}_m$  (see [12, Section 3]). We note that this also holds in the case of  $m = 1$ . For, if we formally set  $m = 1$  in the Poincaré series  $\widetilde{W}_{C_m}(t_1, t_2, t_3)$ , we have

$$\frac{(1 - t_1)(1 + t_2)(1 + t_3)}{(1 - t_1)(1 - t_2 t_3)} = \frac{(1 + t_2)(1 + t_3)}{1 - t_2 t_3}.$$

This is nothing but the Poincaré series for the infinite dihedral group.

Similarly, we can easily compute the factors except for the last  $(U^1(\mathfrak{A}) : J^1) \dim(\eta)$  as in Section 1.6.

The calculation of this last factor is rather more laborious than that for  $GL_N$  in Section 1.7. We also have a defining sequence for the pure stratum  $[\mathfrak{A}, n, r, \beta]$  in  $A$  with  $r = -k_0(\beta, \mathfrak{A})$ , and get a family  $\{(r_i, \gamma_i) | 0 \leq i \leq s\}$ , together with  $(r_{-1}, \gamma_{-1}) = (1, \beta)$ , where  $\gamma_i$  is skew and simple in  $A$ , by [14, Section 3], as in Section 1.7. Set  $d_i = [(r_{i+1} + 1)/2] - [(r_i + 1)/2]$  and  $d'_i = [r_{i+1}/2] - [r_i/2]$ , for  $-1 \leq i \leq s$ , as before.

To present the main theorem, we need to define positive integers  $\delta_i$ , for  $-1 \leq i \leq s$ , as follows: set  $\delta_i = 0$  if  $e_0/e(F[\gamma_i]|F)$  is even, and otherwise,

$$\begin{aligned} \delta_i &= [d_i/2] + [(d'_i + 1)/2] && \text{if } r_i \equiv 0 \pmod{4}, \\ \delta_i &= [(d_i + 1)/2] + [d'_i/2] && \text{if } r_i \equiv 1 \pmod{4}, \\ \delta_i &= [(d_i + 1)/2] + [(d'_i + 1)/2] && \text{if } r_i \equiv 2 \pmod{4}, \\ \delta_i &= [d_i/2] + [d'_i/2] && \text{if } r_i \equiv 3 \pmod{4}. \end{aligned}$$

**Theorem 2.7.** *Let  $(\pi, \mathcal{V})$  be an irreducible square-integrable representation of  $G$  containing a self-dual simple type  $(J, \lambda)$  in  $G$  as in Theorem 2.5. Let  $\{(r_i, \gamma_i) | 0 \leq i \leq s\}$  be a family defined as above. Let  $e_0 = e(\mathfrak{A}|\mathfrak{o}_F)$ ,  $e = e(E|F)$ , and  $e_i^0 = e(F[\gamma_i]|F[\gamma_i]_0)$ , for  $0 \leq i \leq s$ , where each  $F[\gamma_i]_0$  is the fixed field of  $F[\gamma_i]$  under the involution induced by the adjoint one  $x \mapsto \bar{x}$  on  $A$  in Section 2.2. Set*

$$\begin{aligned} \Delta &= \frac{1}{e_0} \sum_{i=-1}^s \left(1 - \frac{1}{[F[\gamma_i] : F]}\right) (r_{i+1} - r_i), \\ \Delta' &= \frac{1}{e_0} \sum_{i=-1}^s (1 - e_i^0) \delta_i \end{aligned}$$

where each  $\delta_i$  is the integer defined above and we reset  $(r_{-1}, \gamma_{-1}) = (0, \beta)$  and  $r_{s+1} = n$ . Then these  $\Delta$  and  $\Delta'$  do not depend on the choice of defining sequence, and the formal degree  $\deg(\pi, dx)$  is given by

(1) when  $e_0/e = e(\mathfrak{B}|\mathfrak{o}_E)$  is even,

$$\begin{aligned} & q^{\frac{1}{4}[N^2\Delta - N(1-1/e) + N\Delta']} \\ & \times \prod_{i=0}^{e_0/2e-1} \frac{q^{N(i+e_0/2e)/e_0} - 1}{(q^{N(i+1)/e_0} - 1)(q^{N(i+1/2)/e_0} + 1)^2} \\ & \times \prod_{i=0}^{N/2-1} \frac{(q^{i+1} - 1)(q^{i+1} + 1)^2}{q^{N/2+i-1} - 1}, \end{aligned}$$

(2) when  $e_0/e = e(\mathfrak{B}|\mathfrak{o}_E)$  is odd,

$$\begin{aligned}
& \frac{q^{\frac{1}{4}[N^2\Delta - N(1-1/e) + N\Delta']}}{q^{N/2e_0} - 1} \\
& \times \prod_{i=0}^{(e_0/e-3)/2} \frac{q^{N(i+(e_0/e-1)/2)/e_0} - 1}{(q^{N(i+1)/e_0} - 1)(q^{N(i+1/2)/e_0} + 1)^2} \\
& \times \prod_{i=0}^{N/2-1} \frac{(q^{i+1} - 1)(q^{i+1} + 1)^2}{q^{N/2+i-1} - 1}.
\end{aligned}$$

- Remarks 2.8.** (1) We also obtained analogous results for  $Sp_N(F)$ . This formal degree formula is more complicated than that in Theorem 2.7 for the unramified unitary group  $G$ .
- (2) In [11], it is proved that Theorem 2.5 holds for a self-dual simple type  $(J, \lambda)$  in  $G$  associated with not only a skew simple stratum  $[\mathfrak{A}, n, 0, \beta]$  in  $A$  which is not good (see Definition 2.1), but also a skew (non-simple) semisimple stratum  $[\Lambda, n, 0, \beta]$  in  $A$ , that is,  $\beta$  is a skew semisimple element of  $A$  (cf. [14]).
- (3) We constructed a special skew semisimple stratum  $[\Lambda, n, 0, \beta]$  in  $A$ , and computed the corresponding formal degree as well.

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