

LIFTING OF PAIRS OF ELLIPTIC MODULAR FORMS TO SIEGEL MODULAR FORMS OF HALF-INTEGRAL WEIGHT OF DEGREE TWO

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1. INTRODUCTION

The aim of this exposition is to explain our recent work on a certain lifting from pairs of two elliptic modular forms to Siegel modular forms of half-integral weight of degree two [H11a, H11b]. The existence of this lifting has been conjectured by Ibukiyama and the author [HI05] as follows.

Conjecture 1 ([HI05]). *Let k be an integer. Let $f \in S_{2k-2}(SL_2(\mathbb{Z}))$, $g \in S_{2k-4}(SL_2(\mathbb{Z}))$ be elliptic modular forms of weight $2k-2$ and $2k-4$, respectively. We assume that f and g are normalized Hecke eigenforms.*

Then, there exists $F \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$ such that F is an eigenform for any Hecke operators, and the L -function of F satisfies the identity

$$L(s, F) = L(s, f) L(s-1, g).$$

Here $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$ is a generalization of the Kohnen plus space for Siegel modular forms of degree two, and where $L(s, F)$ denotes the L -function of $F \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$, and where $L(s, f)$ and $L(s, g)$ denote the usual L -functions of f and g , respectively. We remark that the L -function of modular forms of half-integral weight was first introduced by Shimura [Sh73], and was generalized by Zhuravlev [Zh84] for Siegel modular forms of half-integral weight. The above L -function $L(s, F)$ contains the Euler 2-factor which is introduced in [HI05] for $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$. Some numerical examples of Euler factors have supported the above conjecture.

The following theorem is the main result of this exposition.

Theorem 2. *Let k be an even integer. Let f and g be as in the above conjecture. Then we obtain a Siegel modular form $\mathcal{F}_{f,g} \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$ from the pair of f and g . And if $\mathcal{F}_{f,g}$ does not vanish identically, then $\mathcal{F}_{f,g}$ satisfies the properties in the above conjecture.*

2. CONSTRUCTION OF $\mathcal{F}_{f,g}$

From now on we assume that k is an even integer, and we assume that $f \in S_{2k-2}(SL_2(\mathbb{Z}))$ and $g \in S_{2k-4}(SL_2(\mathbb{Z}))$ are normalized Hecke eigenforms, namely, the first Fourier coefficients of f and g are both 1.

The purpose of this section is to explain the construction of the Siegel modular form $\mathcal{F}_{f,g} \in S_{k-1/2}^+(\Gamma_0^{(2)}(4))$ which will satisfy the properties in Theorem 2. This construction was suggested by Prof. T. Ikeda to the author in 2001 at the Hakuba conference.

We denote by $S_k(\mathrm{Sp}_4(\mathbb{Z}))$ the space of Siegel cusp forms of weight k of degree 4, and denote by $S_{k-\frac{1}{2}}^+(\Gamma_0^{(3)}(4))$ the generalized plus space of weight $k - \frac{1}{2}$ of degree 3 (cf. [Ib92]), which is a certain subspace of Siegel modular forms of weight $k - \frac{1}{2}$ of degree three.

Let $I(g) \in S_k(\mathrm{Sp}_4(\mathbb{Z}))$ be the Duke-Imamoglu-Ikeda lift of g . We consider a Fourier-Jacobi expansion of $I(g)$:

$$I(g) \left(\begin{pmatrix} \tau_3 & z \\ z & \omega_1 \end{pmatrix} \right) = \sum_{n>0} \Psi_n(\tau_3, z) e^{2\pi\sqrt{-1}n\omega_1},$$

where $\tau_3 \in \mathfrak{H}_3$, $\omega_1 \in \mathfrak{H}_1$ and $z \in M_{3,1}(\mathbb{C})$. Here we denote by $\mathfrak{H}_n$ the Siegel upper half space of degree n and by $M_{n,m}(K)$ the matrices of size n by m with entries in a commutative ring K . We remark that the form Ψ_n is a Jacobi form of index n of weight k of degree 3. For the definition of Jacobi forms of higher degree the reader is referred to [Zi89].

By the isomorphism between the generalized plus space and the space of Jacobi forms of index one (cf. [Ib92]), there exists $G \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(3)}(4))$ which corresponds to the Jacobi form Ψ_1 of index 1.

We consider the following expansion of a pullback of G :

$$G \left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix} \right) = \sum_h \mathcal{F}_{h,g}(\tau_2) h(\omega_1),$$

where $\tau_2 \in \mathfrak{H}_2$ and $\omega_1 \in \mathfrak{H}_1$, and where h runs over all elements in a basis which consists of Hecke eigenforms in the Kohnen plus space $S_{k-\frac{1}{2}}^+(\Gamma_0^{(1)}(4))$. We see that $\mathcal{F}_{h,g}$ belongs to $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$ for any h . In particular there exists $\hat{f} \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(1)}(4))$ which corresponds to f by the Shimura correspondence. Therefore, we define

$$\mathcal{F}_{f,g} := \mathcal{F}_{\hat{f},g},$$

then the form $\mathcal{F}_{f,g}$ belongs to $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$.

To complete the proof of Theorem 2 we need to show that the form $\mathcal{F}_{f,g}$ is an eigenform for any Hecke operators under the assumption $\mathcal{F}_{f,g} \not\equiv 0$, and we need to show that the L-function $L(s, \mathcal{F}_{f,g})$ of $\mathcal{F}_{f,g}$ coincides with $L(s-1, g)L(s, f)$. We shall show these facts in the following sections.

3. FOURIER-JACOBI EXPANSION OF G

Let $G \in S_{k-\frac{1}{2}}^+(\Gamma_0^{(3)}(4))$ be the form defined in the previous section. We consider a Fourier-Jacobi expansion of G :

$$G\left(\begin{pmatrix} \tau_2 & z \\ z & \omega_1 \end{pmatrix}\right) = \sum_{m>0} \phi_m(\tau_2, z) e^{2\pi\sqrt{-1}m\omega_1},$$

where $\tau_2 \in \mathfrak{H}_2$, $\omega_1 \in \mathfrak{H}_1$ and $z \in M_{2,1}(\mathbb{C})$.

The purpose of this section is to give a certain relation among ϕ_m , which plays an important role in the proof of Theorem 2. We remark that the form ϕ_m is a Jacobi form of index m of weight $k - \frac{1}{2}$. Here, Jacobi forms of index m of weight $k - \frac{1}{2}$ of degree 2 are holomorphic functions on $\mathfrak{H}_2 \times M_{2,1}(\mathbb{C})$ which satisfy a certain transformation formula (cf. [H11a].) We denote by $J_{k-\frac{1}{2},m}^{(2)}$ the space of Jacobi forms of index m of weight $k - \frac{1}{2}$ of degree 2.

For any prime p we define two maps $V_{1,p}^{(2)}$ and $V_{2,p}^{(2)}$ which are maps from $J_{k-\frac{1}{2},m}^{(2)*}$ to $J_{k-\frac{1}{2},mp^2}^{(2)}$ (the reader is referred to [H11a] for the precise definition of $V_{i,p}^{(2)}$), where $J_{k-\frac{1}{2},m}^{(2)*}$ is a certain subspace of $J_{k-\frac{1}{2},m}^{(2)}$. These $V_{1,p}^{(2)}$ and $V_{2,p}^{(2)}$ are certain generalizations of the V_l -operator in [EZ85, p.43]. We also define the map $U_l : J_{k-\frac{1}{2},m}^{(2)} \rightarrow J_{k-\frac{1}{2},ml^2}^{(2)}$ defined by $(\phi|U_l)(\tau_2, z) := \phi(\tau_2, lz)$ for any $\phi \in J_{k-\frac{1}{2},m}^{(2)}$.

Now we obtain the following relations.

Theorem 3. *Let ϕ_m be as above. For any prime p we have*

$$\phi_m|V_{1,p}^{(2)} = pb(p)\phi_m|U_p + \phi_{mp^2} + \left(\frac{-m}{p}\right)p^{k-2}\phi_m|U_p + p^{2k-3}\phi_{\frac{m}{p^2}}|U_{p^2}$$

and

$$\begin{aligned} \phi_m|V_{2,p}^{(2)} &= b(p)\left(\phi_{mp^2} + \left(\frac{-m}{p}\right)p^{k-2}\phi_m|U_p + p^{2k-3}\phi_{\frac{m}{p^2}}|U_{p^2}\right) \\ &\quad + (p^{2k-4} - p^{2k-6})\phi_m|U_p, \end{aligned}$$

where $b(p)$ is the p -th Fourier coefficient of g .

We remark that the above identities can be translated to relations among Fourier coefficients of G . The outline of the proof of Theorem 3 will be given in Section 5. We also remark that the above identities can be regarded as certain generalizations of the Maass relation for Siegel modular forms of half-integral weight of degree three.

4. PROOF OF THEOREM 2

In this section we assume Theorem 3 and shall prove Theorem 2. We use the same symbols in Section 2, 3.

For a prime p , let $X_1(p)$ and $X_2(p)$ be Hecke operators which generate the local Hecke ring acting on $S_{k-\frac{1}{2}}^+(\Gamma_0^{(2)}(4))$. For the precise definition of these Hecke operators the reader is referred to [HI05, p.513]. By the virtue of the definitions of $V_{1,p}^{(2)}$ and $V_{2,p}^{(2)}$, we can obtain

$$\left(\phi|V_{1,p}^{(2)}\right)(\tau_2, 0) = p^{-k+\frac{7}{2}}\phi(\tau_2, 0)|X_1(p)$$

and

$$\left(\phi|V_{2,p}^{(2)}\right)(\tau_2, 0) = \phi(\tau_2, 0)|X_2(p)$$

for any $\phi \in J_{k-\frac{1}{2},m}^{(2)*}$, and where we regard $\phi(\tau_2, 0)$ as a function of $\tau_2 \in \mathfrak{H}_2$.

Therefore, due to Theorem 3, we obtain

$$\begin{aligned} & p^{-k+\frac{7}{2}}G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right)|X_1(p) \\ &= \sum_m p^{-k+\frac{7}{2}}((\phi_m(\tau_2, 0))|X_1(p))e^{2\pi\sqrt{-1}m\omega_1} \\ &= \sum_m \left((\phi_m|V_{1,p}^{(2)})(\tau_2, 0)\right)e^{2\pi\sqrt{-1}m\omega_2} \\ &= \sum_m \{pb(p)\phi_m(\tau_2, 0) + \phi_{mp^2}(\tau_2, 0) \\ &\quad + \left(\frac{-m}{p}\right)p^{k-2}\phi_m(\tau_2, 0) + p^{2k-3}\phi_{\frac{m}{p^2}}(\tau_2, 0)\}e^{2\pi\sqrt{-1}m\omega_1} \\ &= pb(p)G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) + G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right)|T_1(p^2). \end{aligned}$$

Similarly we have

$$\begin{aligned} & G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right)|X_2(p) \\ &= b(p)G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right)|T_1(p^2) + (p^{2k-4} - p^{2k-6})G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right). \end{aligned}$$

Because $G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) = \sum_h \mathcal{F}_{h,g}(\tau_2) h(\omega_1)$ and because of the above identities, we obtain

$$p^{-k+\frac{7}{2}} \mathcal{F}_{f,g}|X_1(p) = (pb(p) + a(p)) \mathcal{F}_{f,g}$$

and

$$\mathcal{F}_{f,g}|X_2(p) = (a(p)b(p) + p^{2k-4} - p^{2k-6}) \mathcal{F}_{f,g},$$

where $a(p)$ is the p -th Fourier coefficient of f . Hence $\mathcal{F}_{f,g}$ is an eigenform for any Hecke operators. Moreover the eigenvalues of $\mathcal{F}_{f,g}$ follow from the above identities. Let $\lambda(p) = p^{k-\frac{7}{2}}(pb(p) + a(p))$ and $\omega(p) = a(p)b(p) + p^{2k-4} - p^{2k-6}$ be the eigenvalues of $\mathcal{F}_{f,g}$ with respect to $X_1(p)$ and $X_2(p)$, respectively. Then we obtain

$$\begin{aligned} L(s, \mathcal{F}_{f,g}) &= \prod_p \left(1 - \lambda(p)p^{-k+\frac{7}{2}-s} + (p\omega(p) + p^{2k-5}(1+p^2))p^{-2s} \right. \\ &\quad \left. - \lambda(p)p^{k+\frac{1}{2}-3s} + p^{4k-6-4s} \right)^{-1} \\ &= L(s, f) L(s-1, g). \end{aligned}$$

□

5. PROOF OF THEOREM 3

In this section we shall give the outline of the proof of Theorem 3. The steps for the proof of Theorem 3 are as follows.

- (1) By the virtue of the Duke-Imamoglu-Ikeda lift, it is enough to show Theorem 3 for the case of generalized Cohen-Eisenstein series. Here, the generalized Cohen-Eisenstein series are certain Siegel modular forms of half-integral weight, which are not cusp forms (cf. [Co75] for degree one, and [Ar98] for general degree).
- (2) Show certain linear isomorphisms between the space of certain Jacobi forms of half-integral weight of integer index and the space of Jacobi forms of integral weight of matrix index.
- (3) Show a compatibility between the linear isomorphisms in (2) and certain operators which shift the indices of Jacobi forms.
- (4) We need a certain relation between Fourier-Jacobi coefficients of Siegel-Eisenstein series and Jacobi-Eisenstein series of matrix index shown by S. Boecherer [Bo83].
- (5) Calculate the action of shift operators on Jacobi-Eisenstein series of matrix index explicitly.
- (6) Show Theorem 3 for the case of generalized Cohen-Eisenstein series by using (2), (3), (4) and (5).

In this section we will explain the above steps more precisely.

5.1. Generalized Cohen-Eisenstein series. Let k' be an even integer. We denote by $\mathcal{H}_{k'-\frac{1}{2}}^{(3)}$ the generalized Cohen-Eisenstein series of weight $k' - \frac{1}{2}$ of degree 3, which is a certain Siegel modular form of weight $k' - \frac{1}{2}$ of degree 3 (cf. [Co75, Ar98]). More precisely, the form $\mathcal{H}_{k'-\frac{1}{2}}^{(3)}$ corresponds to the Jacobi-Eisenstein series $E_{k',1}^{(3)}$ of index 1 of weight k' of degree 3 by the isomorphism between Siegel modular forms of half-integral weight and Jacobi forms of index one (cf. [EZ85, Ib92].)

By the virtue of the Duke-Imamoglu-Ikeda lift, the Fourier coefficients of G satisfy the similar relation which the Fourier coefficients of $\mathcal{H}_{k'-\frac{1}{2}}^{(3)}$ satisfy. Hence, it is enough to show Theorem 3 for the case of $\mathcal{H}_{k'-\frac{1}{2}}^{(3)}$ for sufficiently many k' .

We take a Fourier-Jacobi expansion of $\mathcal{H}_{k-\frac{1}{2}}^{(3)}$:

$$\mathcal{H}_{k-\frac{1}{2}}^{(3)}\left(\begin{pmatrix} \tau_2 & z \\ z & \omega_1 \end{pmatrix}\right) = \sum_{m \geq 0} e_{k-\frac{1}{2},m}^{(2)}(\tau_2, z) e^{2\pi\sqrt{-1}m\omega_1},$$

where $\tau_2 \in \mathfrak{H}_2$, $\omega_1 \in \mathfrak{H}_1$ and $z \in M_{2,1}(\mathbb{C})$. The form $e_{k-\frac{1}{2},m}^{(2)}$ is a Jacobi form of index m of weight $k - \frac{1}{2}$ of degree 2, which belongs to $J_{k-\frac{1}{2},m}^{(2)*}$.

We need to show the following theorem.

Theorem 4. *Let $k > 5$ be an even integer and let $e_{k-\frac{1}{2},m}^{(2)}$ be as above. For any prime p we have*

$$\begin{aligned} e_{k-\frac{1}{2},m}^{(2)}|V_{1,p}^{(2)} &= p(1+p^{2k-5})e_{k-\frac{1}{2},m}^{(2)}|U_p + e_{k-\frac{1}{2},mp^2}^{(2)} \\ &\quad + \left(\frac{-m}{p}\right)p^{k-2}e_{k-\frac{1}{2},m}^{(2)}|U_p + p^{2k-3}e_{k-\frac{1}{2},\frac{m}{p^2}}^{(2)}|U_{p^2} \end{aligned}$$

and

$$\begin{aligned} e_{k-\frac{1}{2},m}^{(2)}|V_{2,p}^{(2)} &= (1+p^{2k-5})\left\{e_{k-\frac{1}{2},mp^2}^{(2)} + \left(\frac{-m}{p}\right)p^{k-2}e_{k-\frac{1}{2},m}^{(2)}|U_p \right. \\ &\quad \left. + p^{2k-3}e_{k-\frac{1}{2},\frac{m}{p^2}}^{(2)}|U_{p^2}\right\} + (p^{2k-4} - p^{2k-6})e_{k-\frac{1}{2},m}^{(2)}|U_p. \end{aligned}$$

Theorem 3 follows from this theorem.

5.2. An isomorphism between two spaces of Jacobi forms. Let $M_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ be the space of Siegel modular forms of weight k of even degree $n+2$, and let $M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ be the subspace of $M_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ which consists of all Duke-Imamoglu-Ikeda lifts in $M_k(\mathrm{Sp}_{n+2}(\mathbb{Z}))$. We remark that the subspace $M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z}))$ contains the Siegel-Eisenstein series.

Let $J_{k,1}^{(n+1)*}$ be the subspace of $J_{k,1}^{(n+1)}$ which consists of all Jacobi forms of index 1 obtained by the Fourier-Jacobi expansion of the Siegel modular forms in $M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z}))$. We denote by $M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4))$ the subspace of the generalized plus space $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$ of degree $n+1$ which corresponds to $J_{k,1}^{(n+1)*}$ by the isomorphism between two spaces $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n+1)}(4))$ and $J_{k,1}^{(n+1)}$. Now, we have the following diagram.

$$\begin{array}{ccc} M_k(\mathrm{Sp}_{n+2}(\mathbb{Z})) & \supset & M_k^*(\mathrm{Sp}_{n+2}(\mathbb{Z})) \\ \downarrow & & \downarrow \\ J_{k,1}^{(n+1)} & \supset & J_{k,1}^{(n+1)*} \cong M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4)) \end{array}$$

Let $\phi_1 \in J_{k,1}^{(n+1)}$. We regard the function $\phi_1(\tau_{n+1}, z) e^{2\pi\sqrt{-1}\omega_1}$ as a function on $\begin{pmatrix} \tau_{n+1} & z \\ t_z & \omega_1 \end{pmatrix} \in \mathfrak{H}_{n+2}$, where $\tau_{n+1} \in \mathfrak{H}_{n+1}$, $\omega_1 \in \mathfrak{H}_1$ and $z \in M_{n+1,1}(\mathbb{C})$. We consider the expansion:

$$\phi_1(\tau_{n+1}, z) e^{2\pi\sqrt{-1}\omega_1} = \sum_{\substack{S \in \mathrm{Sym}_2^* \\ S = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}}} \phi_S(\tau_n, z') e^{2\pi\sqrt{-1}\mathrm{tr}(S\omega_2)}$$

where Sym_2^* denotes the set of all half-integral symmetric matrices of size 2, and where $\begin{pmatrix} \tau_{n+1} & z \\ t_z & \omega_1 \end{pmatrix} = \begin{pmatrix} \tau_n & z' \\ t_{z'} & \omega_2 \end{pmatrix} \in \mathfrak{H}_{n+2}$, $\tau_n \in \mathfrak{H}_n$, $\omega_2 \in \mathfrak{H}_2$ and $z' \in M_{n,2}(\mathbb{C})$. We remark that $\phi_S \in J_{k,S}^{(n)}$, where $J_{k,S}^{(n)}$ is the space of Jacobi forms of weight k of index S of degree n (cf. [Zi89], [H11a]). We consider the map

$$\mathrm{FJ}_{1,S} : J_{k,1}^{(n+1)} \rightarrow J_{k,S}^{(n)} \quad \text{via} \quad \phi_1 \mapsto \phi_S.$$

We denote by $J_{k,S}^{(n)*} \subset J_{k,S}^{(n)}$ the image of $J_{k,1}^{(n+1)*}$ by the map $\mathrm{FJ}_{1,S}$. Namely, the Jacobi forms in $J_{k,S}^{(n)*}$ are obtained by the Duke-Imamoglu-Ikeda lifting and by the Fourier-Jacobi expansion.

Let m be an integer such that $-m \equiv 0, 1 \pmod{4}$. We take a matrix $\mathcal{M} \in \mathrm{Sym}_2^*$, such that $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}$ and $\det(2\mathcal{M}) = m$.

Lemma 5. *The space $J_{k,\mathcal{M}}^{(n)*}$ is linearly isomorphic to the space $J_{k-\frac{1}{2},m}^{(n)*}$ and this isomorphism is given by a correspondence between Fourier coefficients. We denote by $\iota_{\mathcal{M}}$ this map. Moreover, the following diagram*

is commutative.

$$\begin{array}{ccc}
 J_{k,1}^{(n+1)*} & \longrightarrow & M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4)) \\
 FJ_{1,\mathcal{M}} \downarrow & & \downarrow \\
 J_{k,\mathcal{M}}^{(n)*} & \xrightarrow{\iota_{\mathcal{M}}} & J_{k-\frac{1}{2},m}^{(n)*}
 \end{array}$$

where the map of the down arrow in the right hand side is given by the Fourier-Jacobi expansion.

Due to the above lemma the m -th Fourier-Jacobi coefficient $e_{k-\frac{1}{2},m}^{(n)}$ of $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)}$ corresponds to a certain Jacobi form $e_{k,\mathcal{M}}^{(n)}$ of index $\mathcal{M}$. Here $e_{k-\frac{1}{2},m}^{(n)} \in J_{k-\frac{1}{2},m}^{(n)*}$, $e_{k,\mathcal{M}}^{(n)} \in J_{k,\mathcal{M}}^{(n)*}$ and $\mathcal{H}_{k-\frac{1}{2}}^{(n+1)} \in M_{k-\frac{1}{2}}^*(\Gamma_0^{(n+1)}(4))$. In particular, we can say that the form $e_{k,\mathcal{M}}^{(n)}$ is the $\mathcal{M}$ -th Fourier-Jacobi coefficient of the Siegel-Eisenstein series of weight k of degree $n+2$ (see §5.4 below).

Now we want to translate the relation among $e_{k-\frac{1}{2},m}^{(2)}$ in Theorem 4 to the relation among $e_{k,\mathcal{M}}^{(2)}$. We need to show a compatibility between the above isomorphism $\iota_{\mathcal{M}} : J_{k,\mathcal{M}}^{(n)*} \xrightarrow{\sim} J_{k-\frac{1}{2},m}^{(n)*}$ and certain operators acting on each space. In the next subsection we will explain this compatibility.

5.3. Compatibility between isomorphisms of Jacobi forms and certain operators. We can define certain operators

$$V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)} : J_{k,\mathcal{M}}^{(n)} \rightarrow J_{k,\mathcal{M}[(\begin{smallmatrix} p & 1 \end{smallmatrix})]}^{(n)} \quad (i = 1, \dots, n).$$

Namely, the operator $V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)}$ changes the matrix of the index of Jacobi forms. The operator $V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)}$ is a generalization of V_l -operator in [EZ85, p.43]. For the precise definition of $V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)}$ the reader is referred to [H11a].

Lemma 6. *Let $\mathcal{M}$ be a matrix in Lemma 5. Then, the form*

$$\iota_{\mathcal{M}[(\begin{smallmatrix} p & 1 \end{smallmatrix})]}(e_{k,\mathcal{M}}^{(n)} | V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)})$$

is the same to $e_{k-\frac{1}{2},m}^{(n)} | V_{i,p}^{(n)}$ up to constant. Namely, the isomorphism $\iota_{\mathcal{M}} : J_{k,\mathcal{M}}^{(n)} \xrightarrow{\sim} J_{k-\frac{1}{2},m}^{(n)*}$ and the operators $V_{i,p}^{(n)}$, $V_{i,(\begin{smallmatrix} p & 1 \end{smallmatrix})}^{(n)}$ are compatible.*

We remark that the form $e_{k,\mathcal{M}}^{(n)}|V_{i,(p \ 1)}^{(n)} \in J_{k,\mathcal{M}[(p \ 1)]}^{(n)}$ may not be in $J_{k,\mathcal{M}[(p \ 1)]}^{(n)*}$, but we can extend the map $\iota_{\mathcal{M}[(p \ 1)]}$ to $J_{k,\mathcal{M}[(p \ 1)]}^{(n)}$.

Due to Lemma 5 and 6, we can translate the relation among $e_{k-\frac{1}{2},m}^{(2)}$ ($m \in \mathbb{Z}$) in Theorem 4 to the relation among $e_{k,\mathcal{M}}^{(2)}$ ($\mathcal{M} \in \text{Sym}_2^*$). Hence, it is enough to show the same relation in Theorem 4 for $e_{k,\mathcal{M}}^{(2)}$. For the calculation of $e_{k,\mathcal{M}}^{(2)}|V_{i,(p \ 1)}^{(2)}$, we need relations between $e_{k,\mathcal{M}}^{(2)}$ and Jacobi-Eisenstein series. We will explain this relation in the next subsection.

5.4. Fourier-Jacobi coefficients of Siegel-Eisenstein series. We denote by $E_k^{(n+2)}$ the Siegel-Eisenstein series of weight k of degree $n+2$. We take the Fourier-Jacobi expansion:

$$E_k^{(n+2)}\left(\begin{pmatrix} \tau_n & z \\ t & \omega_2 \end{pmatrix}\right) = \sum_{\mathcal{M} \in \text{Sym}_2^*} e_{k,\mathcal{M}}^{(n)}(\tau_n, z) e^{2\pi\sqrt{-1}\text{tr}(\mathcal{M}\omega_2)},$$

where $\tau_n \in \mathfrak{H}_n$, $\omega_2 \in \mathfrak{H}_2$ and $z \in M_{n,2}(\mathbb{C})$.

We denote by Sym_2^+ the set of all positive-definite half-integral symmetric matrices of size 2. We let $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$. We put $m = \det(2\mathcal{M})$, and let D_0 be the discriminant of $\mathbb{Q}(\sqrt{-m})$, and put $f = \sqrt{-\frac{m}{D_0}}$. We remark that f is a natural number.

We set $g_k(m) := \sum_{d|f} \mu(d) h_{k-\frac{1}{2}}\left(\frac{m}{d^2}\right)$, where $\mu(d)$ is the Möbius function and $h_{k-\frac{1}{2}}(d)$ denotes the d -th Fourier coefficient of the Cohen-Eisenstein series of weight $k - \frac{1}{2}$ (cf. [Co75].)

We denote by $E_{k,\mathcal{S}}^{(n)}$ the Jacobi-Eisenstein series of index $\mathcal{S} \in \text{Sym}_2^+$ of weight k of degree n (cf. [Zi89].) The following proposition follows from [Bo83, Satz 7].

Proposition 7. For $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$ we put $m = \det(2\mathcal{M})$. Let D_0, f be as above. If $k > n + 3$, then

$$e_{k,\mathcal{M}}^{(n)}(\tau, z) = \sum_{d|f} g_k\left(\frac{m}{d^2}\right) E_{k,\mathcal{M}[\iota W_d^{-1}]}^{(n)}(\tau, z W_d),$$

where we chose a matrix $W_d \in GL_2(\mathbb{Q}) \cap M_{2,2}(\mathbb{Z})$ for each d which satisfies the conditions $\det(W_d) = d$ and $W_d^{-1} \mathcal{M}^t W_d^{-1} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix} \in \text{Sym}_2^+$.

The above summation is well-defined, namely it does not depend on the choice of matrix W_d .

Hence, relations among $e_{k,\mathcal{M}}^{(2)}$ is translated to the relations among $E_{k,\mathcal{M}}^{(2)}$. Namely, for the calculation of $e_{k,\mathcal{M}}^{(2)}|V_{i,\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}^{(2)}$ it is enough to calculate $E_{k,\mathcal{M}[{}^tW_d^{-1}]}^{(2)}|V_{i,\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}^{(2)}$, because $V_{i,\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}^{(2)}$ is a linear map.

5.5. Jacobi-Eisenstein series $E_{k,\mathcal{M}}^{(2)}$ and operators $V^{(2)}$. Let $E_{k,\mathcal{M}}^{(2)}$ be the Jacobi-Eisenstein series of weight k of index $\mathcal{M} \in \text{Sym}_2^+$ of degree 2, which is a holomorphic function on $\mathfrak{H}_2 \times \mathbb{C}^2$.

Proposition 8. *We assume $\mathcal{M} = \begin{pmatrix} * & * \\ * & 1 \end{pmatrix}$. Then, the form $E_{k,\mathcal{M}}^{(2)}|V_{i,\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}^{(2)}$ ($i = 1, 2$) is written as a linear combination of three forms*

$$E_{k,\mathcal{M}}^{(2)}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}), \quad E_{k,\mathcal{M}[\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}]}^{(2)}(\tau, z),$$

and

$$E_{k,\mathcal{M}[X^{-1}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}]}^{(2)}(\tau, z \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} {}^tX \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}),$$

explicitly. Here, if $p|f$, then $X = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$ is a 2×2 matrix with an integer x , such that $\mathcal{M}[X^{-1}\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1}] \in \text{Sym}_2^+$, and if $p \nmid f$, then the third form of the above does not appear.

By Lemma 5, 6 and Proposition 7, 8, we obtain Theorem 4. Thus, we can conclude Theorem 3.

6. EXAMPLES OF NON-VANISHING OF $\mathcal{F}_{f,g}$

In Theorem 2 we need the assumption that $\mathcal{F}_{f,g}$ does not identically vanish. In this final section we shall give some examples of non-vanishing of $\mathcal{F}_{f,g}$.

For even integer $k \leq 24$, the dimensions of the spaces $S_{2k-4}(\text{SL}_2(\mathbb{Z}))$, $S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ and $S_{k-1/2}^+(\Gamma_0^{(2)}(4))$ are given as follows.

k	2, 4, 6	8	10	12	14	16	18	20	22	24
$\dim S_{2k-4}(\text{SL}_2(\mathbb{Z}))$	0	1	1	1	2	2	2	3	3	3
$\dim S_{2k-2}(\text{SL}_2(\mathbb{Z}))$	0	0	1	1	1	2	2	2	3	3
$\dim S_{k-1/2}^+(\Gamma_0^{(2)}(4))$	0	0	1	1	2	4	4	6	9	10

We remark that the dimension formula for $S_{k-1/2}^+(\Gamma_0^{(2)}(4))$ is given in [HI05].

Lemma 9. *Let $k = 10, 12$ or 14 . Then, the form $\mathcal{F}_{f,g} \in S_{k-1/2}^+(\Gamma_0^{(2)}(4))$ does not vanish identically for any normalized Hecke eigenforms $f \in S_{2k-2}(\text{SL}_2(\mathbb{Z}))$ and $g \in S_{2k-4}(\text{SL}_2(\mathbb{Z}))$.*

Proof. We assume $k = 10, 12$ or 14 . Then, $\dim S_{2k-2}(\mathrm{SL}_2(\mathbb{Z})) = 1$.

Let $g \in S_{2k-4}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, and let $G \in S_{k-1/2}^+(\Gamma_0^{(3)}(4))$ be the Siegel modular form of weight $k - 1/2$ of degree 3 which is constructed from g in §2. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, and let $h \in S_{k-1/2}^+(\Gamma_0^{(1)}(4))$ be the modular form of weight $k - \frac{1}{2}$ which corresponds to f by the Shimura correspondence. Here $S_{k-1/2}^+(\Gamma_0^{(1)}(4))$ denotes the Kohnen plus space.

Because $\dim S_{k-1/2}^+(\Gamma_0^{(1)}(4)) = 1$, we have

$$G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) = \mathcal{F}_{f,g}(\tau_2)h(\omega_1),$$

where $\tau_2 \in \mathfrak{H}_2$ and $\omega_1 \in \mathfrak{H}_1$. Hence, it is enough to show that $G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_2 \end{pmatrix}\right)$ does not vanish identically. We take the expansion

$$G\left(\begin{pmatrix} \tau_2 & 0 \\ 0 & \omega_1 \end{pmatrix}\right) = \sum_{N \in \mathrm{Sym}_2^+, m \in \mathbb{Z}} K(N, m) e^{2\pi\sqrt{-1}\mathrm{tr}(N\tau_2)} e^{2\pi\sqrt{-1}m\omega_2},$$

and we take the Fourier expansion of G :

$$G(Z) = \sum_{M \in \mathrm{Sym}_3^+} C(M) e^{2\pi\sqrt{-1}\mathrm{tr}(MZ)}.$$

Then

$$K(N, m) = \sum_{\substack{l \in M_{2,1}(\mathbb{Z}) \\ 4Nm - l^t l > 0}} C\left(\begin{pmatrix} N & \frac{1}{2}l \\ \frac{1}{2}l^t & m \end{pmatrix}\right).$$

The Fourier coefficients of G can be calculated by the formula of the Fourier coefficients of the Duke-Imamoglu-Ikeda lift. By numerical calculations we obtain $K\left(\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, 3\right) \neq 0$. Therefore, we conclude $\mathcal{F}_{f,g} \neq 0$. $\square$

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