

Some Applications of Quaternions.

By

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1. Riccati's Equation and Cayley's Formula.

The general solution of Riccati's equation

$$\frac{dz}{dt} = Lz^2 + Mz + N,$$

where L, M and N are functions of t alone, has the form

$$z = \frac{az_0 + \beta}{\gamma z_0 + \delta}, \quad \dots\dots\dots (1)$$

in which a, β, γ and δ are functions of t , and z_0 is an arbitrary constant. We can always assume $a\delta - \beta\gamma = 1$. Conversely, given a, β, γ and δ , we can build an equation which z satisfies. For that purpose, solving the equation (1) with respect to z_0 ,

$$z_0 = \frac{\delta z - \beta}{-\gamma z + a}.$$

Differentiating both sides with respect to t and noticing that

$$\frac{dz_0}{dt} = 0, \text{ and } a\delta - \beta\gamma = 1,$$

we have the equation

$$\frac{dz}{dt} = (\gamma\delta' - \gamma'\delta)z^2 + \{(\delta a' - \delta'a) + (\beta\gamma' - \beta'\gamma)\}z + (a\beta' - a'\beta), \quad (2)$$

where the dashes mean the differentiation with respect to t . We now consider the form of the equation (2) for Cayley's formula,

$$z = \frac{(a+ic)z_0 - (b-ia)^1}{(b+ia)z_0 + (d-ic)}$$

¹ Klein: Ikosaeder p. 34. (e.g.)

where a, b, c, d are real constants satisfying the condition

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Consider these constants as real functions of the real variable t ; in this case the equation (2) becomes

$$\frac{dz}{dt} = -irz + \frac{q-ip}{2} + \frac{q+ip}{2} z^2, \quad (3)$$

where $\gamma\delta' - \gamma'\delta = (b+ia)(d'-ic') - (b'+ia')(d-ic)$

$$\begin{aligned} &= bd' - b'd + ac' - a'c \\ &+ i(ad' - a'd + cb' - c'b) \\ &\equiv \frac{q+ip}{2}, \end{aligned}$$

$$\begin{aligned} a\beta' - a'\beta &= \bar{\delta}(-\bar{\gamma}') - \bar{\delta}'(-\bar{\gamma}) \\ &= \overline{\gamma\delta' - \gamma'\delta} \\ &= \frac{q-ip}{2}, \end{aligned}$$

lastly $\delta a' - \delta' a + \beta \gamma' - \beta' \gamma = 2i\{(dc' - d'c) + (ab' - a'b)\}$
 $\equiv -ir.$

Or

$$\begin{aligned} \frac{p}{2} &= ad' - a'd + b'c - bc' \\ \frac{q}{2} &= ac' - a'c + bd' - b'd \\ \frac{r}{2} &= ab - ab' + cd' - c'd \end{aligned} \quad (4)$$

Hence the function

$$z = \frac{(d+ic)z_0 - (b-ia)}{(b+ia)z_0 + (d-ic)}$$

is the general solution of such an equation as

$$\frac{dz}{dt} = -irz + \frac{q-ip}{2} + \frac{q+ip}{2} z^2,$$

where

$$a^2 + b^2 + c^2 + d^2 = 1.$$

2. Darboux-Riccati's Equation.

Consider two systems of rectangular axes, having the same origin; the one space-fixed and the other rotating. Let a, β, γ be the direction-cosines of any one of the space-fixed axes referred to the moving system; p, q, r , three components of rotation of the moving axes referred to themselves; then the kinematical equations are

$$\left. \begin{aligned} \frac{da}{dt} &= \beta r - \gamma q \\ \frac{d\beta}{dt} &= \gamma p - a r \\ \frac{d\gamma}{dt} &= a q - \beta p, \end{aligned} \right\} \quad (1)$$

where t is the time.

To discuss the solutions of the equations Prof. Darboux¹ proceeded as follows:—

We notice the condition

$$a^2 + \beta^2 + \gamma^2 = \text{constant}.$$

When the constant is not equal to zero, we may always assume

$$a^2 + \beta^2 + \gamma^2 = 1, \quad (2)$$

which is consistent with our assumption that a, β, γ are the direction-cosines of an axis. Put

$$\left. \begin{aligned} \frac{a+i\beta}{1-\gamma} &= \frac{1+\gamma}{a-i\beta} \equiv x \\ \frac{a-i\beta}{1-\gamma} &= \frac{1+\gamma}{a+i\beta} \equiv -\frac{1}{y} \end{aligned} \right\} \quad (3)$$

which give $a = \frac{1-xy}{x-y}$, $\beta = i \frac{1+xy}{x-y}$, $\gamma = \frac{x+y}{x-y}$;

then the equation (1) become

$$\begin{aligned} \frac{dx}{dt} &= -irx + \frac{q-ip}{2} + \frac{q+ip}{2} x^2, \\ \frac{dy}{dt} &= -iry + \frac{q-ip}{2} + \frac{q+ip}{2} y^2. \end{aligned}$$

¹ Lecons sur la Théorie Générale des Surfaces, I (first edition) pp. 21-22.

Or writing σ for x or y ,

$$\frac{d\sigma}{dt} = -i r \sigma + \frac{q - ip}{2} + \frac{q + ip}{2} \sigma^2, \quad (4)$$

which is the same equation as §I (3). Hence the general solution is of the form

$$\sigma = \frac{(d + ic) \sigma_0 - (b - ia)}{(b + ia) \sigma_0 + (d - ic)},$$

and the relations between p, q, r and a, b, c, d are given by the equations §I (4). Those equations may be transformed by some easy calculations into the following

$$\left. \begin{aligned} \frac{da}{dt} + \frac{p}{2} d + \frac{q}{2} c - \frac{r}{2} b &= 0, \\ \frac{db}{dt} - \frac{p}{2} c + \frac{q}{2} d + \frac{r}{2} a &= 0, \\ \frac{dc}{dt} + \frac{p}{2} b - \frac{q}{2} a + \frac{r}{2} d &= 0, \\ \frac{dd}{dt} - \frac{p}{2} a - \frac{q}{2} b - \frac{r}{2} c &= 0; \end{aligned} \right\} \quad (5)$$

noticing that $a^2 + b^2 + c^2 + d^2 = 1$, therefore $a \frac{da}{dt} + b \frac{db}{dt} + c \frac{dc}{dt} + d \frac{dd}{dt} = 0$.

3. Given p, q, r to find a, b, c, d , we have to solve the system of four differential equations (5) of the last section. If we put $d=0$, then the first three equations become as follows:

$$\begin{aligned} \frac{da}{dt} &= b \frac{r}{2} - c \frac{q}{2}, \\ \frac{db}{dt} &= c \frac{p}{2} - a \frac{r}{2}, \\ \frac{dc}{dt} &= a \frac{q}{2} - b \frac{p}{2}, \end{aligned}$$

which are of the same type as the system of equations (1) of the last section. Therefore the system (5) may be considered as a more general case and consequently we attempt, by some method, to obtain Riccati's equation which is equivalent to the system of differential equa-

tions (5). For that purpose we use Hamilton's quaternions I, i_1, i_2, i_3 which satisfy the relations

$$\begin{aligned} i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -I, \\ i_1 i_2 + i_2 i_1 = i_2 i_3 + i_3 i_2 = i_3 i_1 + i_1 i_3 = 0. \end{aligned}$$

Now multiplying I, i_1, i_2, i_3 into the equations (5) and summing them up, we obtain after some easy calculations

$$\frac{dx}{dt} = x \left(\frac{p}{2} i_3 + \frac{q}{2} i_2 - \frac{r}{2} i_1 \right), \quad (1)$$

where

$$x = a + bi_1 + ci_2 + di_3.$$

This equation may be considered a special Riccati's equation. In a space of two dimensions, the analogous system of equations obtained from (1) § 2 by the assumption $\gamma=0$, viz.,

$$\left. \begin{aligned} \frac{da}{dt} &= \beta r \\ \frac{d\beta}{dt} &= -ar. \end{aligned} \right\} \quad (2)$$

Multiplying I, i and summing them up, we have

$$\frac{dx}{dt} = -irx, \quad x = a + i\beta. \quad (3)$$

which is quite similar to (1).

4. As a more general case than § 2 (5), we consider the system of five equations

$$\left. \begin{aligned} \frac{da}{dt} &= \beta r - \gamma q - \delta p + \varepsilon q, \\ \frac{d\beta}{dt} &= \gamma p - \delta q + \varepsilon r - ar, \\ \frac{d\gamma}{dt} &= -\delta r + \varepsilon r + aq - \beta p, \\ \frac{d\delta}{dt} &= \varepsilon q + ap + \beta q + \gamma r, \\ \frac{d\varepsilon}{dt} &= -aq - \beta r - \gamma r - \delta q. \end{aligned} \right\} \quad (1)$$

From these equations we have

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 = \text{const.},$$

and this constant may be regarded as unity except in the case of zero; i.e.

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 = 1.$$

We put, after Prof. Darboux, § 2 (3),

$$\left. \begin{aligned} \frac{\alpha + \beta i_1 + \gamma i_2 + \delta i_3}{1 - \varepsilon} &= \frac{1 + \varepsilon}{\alpha - \beta i_1 - \gamma i_2 - \delta i_3} \equiv x, \\ \frac{\alpha - \beta i_1 - \gamma i_2 - \delta i_3}{1 - \varepsilon} &= \frac{1 + \varepsilon}{\alpha + \beta i_1 + \gamma i_2 + \delta i_3} \equiv -y^{-1}, \end{aligned} \right\} \quad (2)$$

whence we have

$$\alpha + \beta i_1 + \gamma i_2 + \delta i_3 = (1 - \varepsilon) x,$$

$$\alpha - \beta i_1 - \gamma i_2 - \delta i_3 = (1 + \varepsilon) x^{-1}.$$

Therefore

$$\frac{d\alpha}{dt} + \frac{d\beta}{dt} i_1 + \frac{d\gamma}{dt} i_2 + \frac{d\delta}{dt} i_3 = (1 - \varepsilon) \frac{dx}{dt} - \frac{d\varepsilon}{dt} x,$$

which, by the equations (1), is equal to the following sum

$$\begin{aligned} & \beta r - \gamma q - \delta p + \varepsilon q + (\gamma p - \delta q + \varepsilon r - \alpha r) i_1 \\ & + (-\delta r + \varepsilon r + \alpha q - \beta p) i_2 + (\varepsilon q + \alpha p + \beta q + \gamma r) i_3 \\ & = \varepsilon (q + r i_1 + r i_2 + q i_3) + (1 - \varepsilon) x (-r i_1 + q i_2 + p i_3). \end{aligned}$$

In the same way,

$$\begin{aligned} \frac{d\alpha}{dt} - \frac{d\beta}{dt} i_1 - \frac{d\gamma}{dt} i_2 - \frac{d\delta}{dt} i_3 &= -(1 + \varepsilon) x^{-1} \frac{dx}{dt} x^{-1} + \frac{d\varepsilon}{dt} x^{-1} \\ &= \varepsilon (q - r i_1 - r i_2 - q i_3) + (1 + \varepsilon) (r i_1 - q i_2 - p i_3) x^{-1}. \end{aligned}$$

Hence we have

$$\begin{aligned} & \frac{d\alpha}{dt} + \frac{d\beta}{dt} i_1 + \frac{d\gamma}{dt} i_2 + \frac{d\delta}{dt} i_3 + x \left(\frac{d\alpha}{dt} - \frac{d\beta}{dt} i_1 - \frac{d\gamma}{dt} i_2 - \frac{d\delta}{dt} i_3 \right) x \\ &= -2\varepsilon \frac{dx}{dt}^* \\ &= 2\varepsilon x (r i_1 - q i_2 - p i_3) + \varepsilon (q + r i_1 + r i_2 + q i_3) + \varepsilon x (q - r i_1 - r i_2 - q i_3) x, \end{aligned}$$

* The product of a scalar and a quaternion is permutable.

whence we have

$$\frac{dx}{dt} = -x(r_1 - qi_2 - pi_3) - \frac{q + ri_1 + ri_2 + qi_3}{2} - x \frac{q - ri_1 - ri_2 - qi_3}{2} x. \quad (3)$$

Starting from the equations

$$a + \beta i_1 + \gamma i_2 + \delta i_3 = -(1 + \epsilon)y,$$

$$a - \beta i_1 - \gamma i_2 - \delta i_3 = -(1 - \epsilon)y^{-1},$$

we arrive at the same equation for y ; hence x and y are the solutions of the following equation

$$\frac{d\sigma}{dt} = -\sigma(r_1 - qi_2 - pi_3) - \frac{q + ri_1 + ri_2 + qi_3}{2} - \sigma \frac{q - ri_1 - ri_2 - qi_3}{2} \sigma. \quad (4)$$

This equation is quite analogical with § 2 (4)

$$\frac{d\sigma}{dt} = -i r \sigma + \frac{q - ip}{2} + \frac{q + ip}{2} \sigma^2.$$

5. Serret-Frenet's Formula.

Serret-Frenet's formulas of a space curve are discussed as a special case of the kinematical equations § 2 (1). Formulas for hyperspace¹ e.g., of five dimensions are

$$\left. \begin{aligned} \frac{da}{ds} &= \frac{\beta}{R_1}, \\ \frac{d\beta}{ds} &= \frac{\gamma}{R_2} - \frac{a}{R_1}, \\ \frac{d\gamma}{ds} &= \frac{\delta}{R_3} - \frac{\beta}{R_2}, \\ \frac{d\delta}{ds} &= \frac{\epsilon}{R_4} - \frac{\gamma}{R_3}, \\ \frac{d\epsilon}{ds} &= -\frac{\delta}{R_4}, \end{aligned} \right\} \quad (1)$$

where s is the arc length, and $\frac{1}{R}$'s are curvatures. This system of equations does not admit of the application of quaternions. For that

¹ G. E. A. Brunnell, Math. Ann., 19 (1882). F. Meyer, Jahresbericht d. Deut. Math. Ver., 19 (1910); &c.

purpose, we must lead these equations to those § 4 (1). But the equations (1) are a special case of a kinematical equations for hyper-space; and we rather consider the reduction of the latter equations.

The kinematical equations for n dimensions are given by the following system of n equations

$$\frac{da_i}{dt} = \sum_{h=1}^n p_{ih} a_h, \quad i = 1, 2, \dots, n. \quad (2)$$

where

$$p_{ii} = 0, \quad p_{ih} = -p_{hi},$$

so that the coefficients p make a skew symmetric determinant. Now we transform these equations by the following orthogonal transformations.

$$\left. \begin{aligned} \beta_j &= \sum_{k=1}^n a_{jk} a_k, & |a_{jk}| &= +1, \\ & & j &= 1, 2, \dots, n, \\ \text{or} \quad a_k &= \sum_{j=1}^n a_{jk} \beta_j, & k &= 1, 2, \dots, n. \end{aligned} \right\} \quad (3)$$

Differentiating with respect to t ,

$$\frac{d\beta_j}{dt} = \sum_{k=1}^n \left(a_{jk} \frac{da_k}{dt} + a_k \frac{da_{jk}}{dt} \right),$$

$$\text{by (2)} \quad = \sum_{k=1}^n \left(a_{jk} \sum_{h=1}^n p_{kh} a_h + a_k \frac{da_{jk}}{dt} \right)$$

$$\begin{aligned} \text{by (3)} \quad &= \sum_{k=1}^n \left(a_{jk} \sum_{h=1}^n p_{kh} \sum_{r=1}^n a_{rh} \beta_r + \sum_{r=1}^n a_{rk} \beta_r \frac{da_{jk}}{dt} \right) \\ &= \sum_{r=1}^n \left(\sum_{k,h=1}^n p_{kh} a_{jk} a_{rh} + \sum_{k=1}^n a_{rk} \frac{da_{jk}}{dt} \right) \beta_r. \end{aligned}$$

$$\text{Put} \quad P_{jr}' \equiv \sum_{k,h=1}^n p_{kh} a_{jk} a_{rh},$$

* T. Craig, *Displacements depending on One, Two and Three Parameters in a space of Four Dimensions*, Amer. J. Math., **20** (1898); for n dimensions, N. J. Hatzidakis, *Ibid.*, **22** (1900).

$$P_{jr}'' \equiv \sum_{k=1}^n a_{rk} \frac{da_{jk}}{dt},$$

and $P_{jr} \equiv P_{jr}' + P_{jr}''.$

Then we have

$$P_{jr}' = - \sum_{k, h=1}^n p_{kh} a_{rk} a_{jh} = -P_{rj}',$$

$$P_{jr}'' = - \sum_{k=1}^n a_{jk} \frac{da_{rk}}{dt} = -P_{rj}'';$$

hence $P_{jr} = -P_{rj}$, and $P_{jj} = 0.$

Therefore the transformed equations are

$$\left. \begin{aligned} \frac{d\beta_j}{dt} &= \sum_{r=1}^n P_{jr} \beta_r, \\ P_{jr} &= -P_{rj}, \quad j = 1, 2, \dots, n, \\ P_{jj} &= 0, \quad r = 1, 2, \dots, n. \end{aligned} \right\} \quad (4)$$

where

When the transformed equations are required to be

$$\left. \begin{aligned} \frac{d\beta_j}{dt} &= \sum_{r=1}^n q_{jr} \beta_r, \\ q_{rj} &= -q_{jr}, \quad j = 1, 2, \dots, n, \\ q_{jj} &= 0, \quad r = 1, 2, \dots, n. \end{aligned} \right\} \quad (5)$$

Then considering a 's as unknown, we have to solve the system of n^2 differential equations

$$\begin{aligned} P_{jr} &= q_{jr}, \quad j = 1, 2, \dots, n, \\ & \quad r = 1, 2, \dots, n; \end{aligned}$$

or more completely

$$\sum_{k=1}^n a_{rk} \frac{da_{jk}}{dt} + \sum_{h=1}^n p_{kh} a_{jk} a_{rh} = q_{jr}, \quad j, r = 1, 2, \dots, n.$$

Or multiplying a_{rs} into both sides and summing with respect to r

$$\sum_{j=1}^n \sum_{k=1}^n a_{js} a_{rk} \frac{da_{jk}}{dt} + \sum_{r=1}^n \sum_{k=1}^n p_{rk} a_{rs} a_{jk} a_{rk} = \sum_{r=1}^n a_{rs} q_{jr},$$

$$\text{or} \quad \frac{da_{js}}{dt} + \sum_{k=1}^n p_{ks} a_{jk} = \sum_{j=1}^n a_{js} q_{jr},$$

$$\text{or} \quad \frac{da_{js}}{dt} = \sum_{k=1}^n (p_{sk} a_{jk} + q_{jk} a_{ks}), \quad j, s = 1, 2, \dots, n. \quad (6)$$

On conditions that

$$\left. \begin{aligned} \sum_{j=1}^n a_{ji}^2 &= 1, & i &= 1, 2, \dots, n, \\ \sum_{j=1}^n a_{ji} a_{jr} &= 0, & j &= 1, 2, \dots, n; i \neq j. \end{aligned} \right\} \quad (7)$$

By these conditions the number of the differential equations is reduced to $\frac{n(n-1)}{2}$. After Cayley these n^2 quantities a 's can be represented by $\frac{n(n-1)}{2}$ essential parameters; considering these parameters as new unknowns, the equations (6) will be transformed into a system of $\frac{n(n-1)}{2}$ differential equations with $\frac{n(n-1)}{2}$ unknowns. Any particular solution will suffice to obtain a 's and consequently the orthogonal transformations (3).

Specially for $n = 5$, we have

$$q_{12} = q_{25} = q_{35} = q_{45} = r,$$

$$q_{15} = q_{31} = q_{42} = q_{45} = q,$$

$$q_{23} = q_{41} = p,$$

then the system of differential equations

$$\frac{da_i}{dt} = \sum_{k=1}^5 p_{ik} a_k, \quad i = 1, 2, \dots, 5,$$

may be transformed into the system § 4 (1) and hence into § 4 (4). (For the case $n=4$, we may treat similarly.) Serret-Frenets' formulas (1) are a special type of kinematical equations for which

$$p_{12} = \frac{1}{R_1}, \quad p_{13} = p_{14} = p_{15} = 0,$$

$$p_{23} = \frac{1}{R_2}, \quad p_{24} = p_{25} = 0,$$

$$p_{34} = \frac{1}{R_3}, \quad p_{35} = 0,$$

$$p_{45} = \frac{1}{R_4}.$$

Hence, by the above theory, this system of equations may be transformed into § 4 (4). Thus Prof. Darboux's method¹ admits theoretically of a similar extension²

6. Line Element on Hypersphere.

If ds_n denote the line element on the hypersphere of n dimensions, then for the hypersphere

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 = 1, \quad (1)$$

$$ds_5^2 = d\alpha^2 + d\beta^2 + d\gamma^2 + d\delta^2 + d\varepsilon^2. \quad (2)$$

On the other hand, from the equations § 4 (2)

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -(1-\varepsilon)^2 xy^{-1} = 1-\varepsilon^2,$$

or dividing by $(1-\varepsilon)$, and rearranging we have

$$\left. \begin{aligned} \varepsilon &= (x+y)(x-y)^{-1} \\ 1-\varepsilon &= -2y(x-y)^{-1} = -2(x-y)^{-1}y. \end{aligned} \right\} \quad (2)$$

Now $d\alpha + d\beta i_1 + d\gamma i_2 + d\delta i_3 = (1-\varepsilon) dx - x d\varepsilon$

$$= -2 dx (x-y)^{-1} y - x d\varepsilon,$$

also $d\alpha - d\beta i_1 - d\gamma i_2 - d\delta i_3 = (1-\varepsilon) y^{-1} dy y^{-1} + y^{-1} d\varepsilon;$

¹ *Loc. cit.*

² Mr. Eiesland treated the same problem of integration of kinematical equations, from another point of view, in *Amer. J. Math.*, 20 (1898); 28 (1906).

³ When the product of two quaternions is equal to a scalar, then the factors are permutable.

$$\text{hence } d\alpha^2 + d\beta^2 + d\gamma^2 + d\delta^2 = 4 dx (x-y)^{-1} dy (x-y)^{-1} - xy^{-1} d\epsilon^2 \\ + 2 (xy^{-1} dy - dx)(x-y)^{-1} d\epsilon.$$

$$\text{But } d\epsilon (x-y) = (\epsilon-1) \left(\frac{1+\epsilon}{\epsilon-1} dy - dx \right) = (\epsilon-1)(xy^{-1} dy - dx),$$

$$\text{hence } xy^{-1} dy - dx = (x-y) \frac{d\epsilon}{\epsilon-1}.$$

$$\text{Therefore } d\alpha^2 + d\beta^2 + d\gamma^2 + d\delta^2 = 4 dx (x-y)^{-1} dy (x-y)^{-1} \\ - \frac{1+\epsilon}{\epsilon-1} d\epsilon^2 + 2 \frac{d\epsilon^2}{\epsilon-1},$$

$$\text{or } ds_6^2 = d\alpha^2 + d\beta^2 + d\gamma^2 + d\delta^2 + d\epsilon^2 = 4 dx (x-y)^{-1} dy (x-y)^{-1}, \quad (4)$$

which is quite analogical with

$$ds_3^2 = d\alpha^2 + d\beta^2 + d\gamma^2 = \frac{4 dx dy^1}{(x-y)^2}$$

7. Parametric Representation.

By aid of x and y , the five variables $\alpha, \beta, \gamma, \delta, \epsilon$ can be represented as in the case of three variables. Now add the equations

$$\left. \begin{aligned} \alpha + \beta i_1 + \gamma i_2 + \delta i_3 &= (1-\epsilon) x, \\ \alpha - \beta i_1 - \gamma i_2 - \delta i_3 &= -(1-\epsilon) y^{-1}. \end{aligned} \right\} \quad (1)$$

Then we have

$$2\alpha = (x-y^{-1})(1-\epsilon),$$

by § 6 (3)

$$= -2(x-y^{-1})y(xy-y)^{-1}.$$

Hence

$$\alpha = (1-xy)(x-y)^{-1}. \quad (2)$$

Next from (1)

$$\alpha i_1 - \beta - \gamma i_3 + \delta i_2 = (1-\epsilon) x i_1,$$

$$\alpha i_1 + \beta - \gamma i_3 + \delta i_2 = -(1-\epsilon) i_1 y^{-1}.$$

Adding we have

$$2\beta = -(1-\epsilon)(x i_1 + i_1 y^{-1}), \\ = 2(x i_1 + i_1 y^{-1})y(xy-y)^{-1}.$$

Hence

$$\beta = i_1(1-i_1 x i_1 y)(x-y)^{-1}. \quad (3)$$

γ and δ may be obtained in the same way; hence, adding the first formula § 6 (3), we have

¹ Darboux: *Loc. cit.* p. 30.

$$\left. \begin{aligned} \alpha &= (1 - xy)(x - y)^{-1}, \\ \beta &= i_1 (1 - i_1 x i_1 y)(x - y)^{-1}, \\ \gamma &= i_2 (1 - i_2 x i_2 y)(x - y)^{-1}, \\ \delta &= i_3 (1 - i_3 x i_3 y)(x - y)^{-1}, \\ \varepsilon &= (x + y)(x - y)^{-1}. \end{aligned} \right\} \quad (4)$$

For the hypersphere of four dimensions, we have to put $\varepsilon = 0$, and hence $x = -y$,

$$\left. \begin{aligned} \alpha &= \frac{1}{2} (x^{-1} + x), \\ \beta &= \frac{1}{2} (i_1 x^{-1} - x i_1), \\ \gamma &= \frac{1}{2} (i_2 x^{-1} - x i_2), \\ \delta &= \frac{1}{2} (i_3 x^{-1} - x i_3); \end{aligned} \right\} \quad (5)$$

for the sphere we have to put $i_2 = i_3 = 0$, $i_1 \equiv i$,

$$\left. \begin{aligned} \alpha &= (1 - xy)(x - y)^{-1}, \\ \beta &= i (1 + xy)(x - y)^{-1}, \\ \varepsilon &= (x + y)(x - y)^{-1}; \end{aligned} \right\} \quad (6)$$

for the circle, $i_2 = i_3 = \varepsilon = 0$, $i_1 \equiv i$ and $x = -y$,

$$\left. \begin{aligned} \alpha &= \frac{1}{2} (x^{-1} + x), \\ \beta &= \frac{i}{2} (x^{-1} - x). \end{aligned} \right\} \quad (7)$$

8. Transformation of Line-element.

The square of the line element ds^2 does not change by the substitution

$$x = Ax_1 B + C, \quad y = Ay_1 B + C, \quad (1)$$

where A, B, C are constant quaternions. For

$$\begin{aligned} ds^2 &= 4 dx (x - y)^{-1} dy (x - y)^{-1} \\ &= x A dx_1 B \{A (x_1 - y_1) B\}^{-1} A dy_1 B \{A (x_1 - y_1) B\}^{-1}, \end{aligned}$$

$$= 4 A dx_1 (x_1 - y_1)^{-1} dy_1 (x_1 - y_1)^{-1} A_1.$$

Since ds^2 is a scalar

$$ds^2 = 4 dx_1 (x_1 - y_1)^{-1} dy_1 (x_1 - y_1)^{-1}.$$

Next ds^2 does not change by the substitutions

$$x = x_1^{-1}, \quad y = y_1^{-1}. \quad (2)$$

For since $(x - y)^{-1} = x_1 (y_1 - x_1)^{-1} y_1 = y_1 (y_1 - x_1)^{-1} x_1$,

$$ds^2 = dx_1 (x_1 - y_1)^{-1} dy_1 (x_1 - y_1)^{-1}.$$

Lastly ds^2 does not change by the substitutions

$$\left. \begin{aligned} x &= (Ax_1 + B)(Cx_1 + D)^{-1}, \\ y &= (Ay_1 + B)(Cy_1 + D)^{-1} \end{aligned} \right\} \quad (3)$$

where A, B, C, D are constant quaternions.

$$\begin{aligned} \text{Since } x &= \{AC^{-1}(Cx_1 + D) + B - AC^{-1}D\}(Cx_1 + D)^{-1} \\ &= AC^{-1} + (B - AC^{-1}D)(Cx_1 + D)^{-1}. \end{aligned}$$

$$\text{Putting } x_2 \equiv Cx_1 + D, \quad x_3 \equiv x_2^{-1},$$

$$\text{then } x \equiv AC^{-1} + (B - AC^{-1}D)x_3,$$

in the same way

$$y_2 \equiv Cy_1 + D, \quad y_3 \equiv y_2^{-1},$$

$$\text{then } y = AC^{-1} + (B - AC^{-1}D)y_3.$$

Therefore by (1) and (2), we may prove the invariance of ds^2 . We remark that since

$$\begin{aligned} ds^2 &= 4 dx (x - y)^{-1} dy (x - y)^{-1}, \\ &= 4 dy (x - y)^{-1} dx (x - y)^{-1}, \end{aligned}$$

ds^2 is also invariant by the transformations

$$\left. \begin{aligned} x &= (Ay_1 + B)(Cy_1 + D)^{-1}, \\ y &= (Ax_1 + B)(Cx_1 + D)^{-1}. \end{aligned} \right\} \quad (4)$$

9. Meaning of the Parameter x .

When $(\omega, \xi, \eta, \zeta)$ denote the coordinates of the projection of a point on the hypersphere

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 = 1,$$

from its pole on the equatorial hyperspace of four dimensions, then by analogy, the equations of projection are

$$\frac{\alpha}{\omega} = \frac{\beta}{\xi} = \frac{\gamma}{\eta} = \frac{\delta}{\zeta} = 1 - \varepsilon,$$

whence by easy calculations

$$\alpha = \frac{2\omega}{\omega^2 + \xi^2 + \eta^2 + \zeta^2 + 1},$$

$$\beta = \frac{2\xi}{\omega^2 + \xi^2 + \eta^2 + \zeta^2 + 1},$$

$$\gamma = \frac{2\eta}{\omega^2 + \xi^2 + \eta^2 + \zeta^2 + 1},$$

$$\delta = \frac{2\zeta}{\omega^2 + \xi^2 + \eta^2 + \zeta^2 + 1},$$

$$\varepsilon = \frac{\omega^2 + \xi^2 + \eta^2 + \zeta^2 - 1}{\omega^2 + \xi^2 + \eta^2 + \zeta^2 + 1}.$$

Now
$$\alpha + \beta i_1 + \gamma i_2 + \delta i_3 = \frac{2(\omega + \xi i_1 + \eta i_2 + \zeta i_3)}{\omega^2 + \xi^2 + \eta^2 + \zeta^2 + 1},$$

$$= (1 - \varepsilon) x,$$

hence
$$= \frac{2}{\omega^2 + \xi^2 + \eta^2 + \zeta^2 + 1} x.$$

Therefore
$$x = \omega + \xi i_1 + \eta i_2 + \zeta i_3.$$

For a sphere, x is the complex variable on the Gauss' plane.
