

QUASI-FREE FINITE GROUP ACTIONS ON THE CUNTZ ALGEBRA $\mathcal{O}_\infty$

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1. INTRODUCTION

The Cuntz algebra $\mathcal{O}_n$, for $n = 2, 3, \dots, \infty$, is the universal C^* -algebra generated by isometries $\{S_i\}_{i=1}^n$ with mutually orthogonal ranges, satisfying $\sum_{i=1}^n S_i S_i^* = I$ if n is finite. Let $\mathcal{H}_n$ be the closed linear span of $\{S_i\}_{i=1}^n$, which has a Hilbert space structure with inner product $T^*S = \langle S, T \rangle I$. An action α of a group G on $\mathcal{O}_n$ is said to be *quasi-free* if $\alpha_g(\mathcal{H}_n) = \mathcal{H}_n$ for all $g \in G$. There is a one-to-one correspondence between the quasi-free actions of G and the unitary representation of G in $\mathcal{H}_n$. For a quasi-free action α , we denote by $(\pi_\alpha, \mathcal{H}_n)$ the corresponding unitary representation of G . We assume that G is finite in what follows.

A priori, the conjugacy class of a quasi-free action α depends on the unitary equivalence class of the unitary representation $(\pi_\alpha, \mathcal{H}_n)$. Indeed, it really does when n is finite (see, for example, [2], [3]). However, when $n = \infty$, the pair $(\mathcal{O}_\infty, \alpha)$ is equivariantly KK -equivalent to the pair $(\mathbb{C}, \text{id})$, and there is no way to differentiate quasi-free actions as far as K -theory is concerned. In fact, P. Goldstein announced in his preprint [1] (ten years old by now) that any two non-trivial quasi-free $\mathbb{Z}_2$ -actions are mutually conjugate. His idea is to develop a $\mathbb{Z}_2$ -equivariant version of Lin-Phillips' argument, which gives a uniqueness theorem for unital homomorphisms from $\mathcal{O}_\infty$ to purely infinite C^* -algebras up to approximately unitary equivalence (see [7, Section 7.2]). However, his proof is based on Evans-Su's unpublished work, which is not available for some reason.

The aim of this short note is to announce that any two faithful quasi-free actions are indeed mutually conjugate for every finite group G . Our strategy is basically the same as Goldstein's. However, while Goldstein compares two quasi-free actions directly using an equivariant version of Elliott's intertwining argument, we use a model action splitting argument, which in fact works for every outer action on the Kirchberg algebras.

2. MAIN RESULTS

A Kirchberg algebra is a purely infinite, simple, nuclear, separable C^* -algebra. The reader is referred to [7] for basic properties and classification results for Kirchberg algebras. The Cuntz algebras are typical examples of Kirchberg algebras. We denote by $\mathbb{K}$ the set of compact operators on a separable infinite dimensional Hilbert space.

We fix a finite group G . By a G - C^* -algebra (A, α) , we mean a C^* -algebra A with a fixed G -action α . A G -homomorphism φ from a G - C^* -algebra (A, α) into another G - C^* -algebra (B, β) is a homomorphism from A into B intertwining the two G -actions

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α and β . We denote by $\text{Hom}_G(A, B)$ the set of G -homomorphisms from (A, α) into (B, β) . Two G -homomorphisms $\varphi, \psi \in \text{Hom}_G(A, B)$ are said to be G -approximately unitarily equivalent if there exists a sequence of unitaries $\{u_n\}$ in the fixed point algebra B^G such that

$$\lim_{n \rightarrow \infty} \|\varphi(x) - \text{Ad}u_n \circ \psi(x)\|, \quad \forall x \in A.$$

When A and B are separable, an equivariant version of Elliott's intertwining argument implies the following: if there exist $\varphi \in \text{Hom}_G(A, B)$ and $\psi \in \text{Hom}_G(B, A)$ such that $\psi \circ \varphi$ is G -approximately unitarily equivalent to $\text{id}_{(A, \alpha)}$ and $\varphi \circ \psi$ is G -approximately unitarily equivalent to $\text{id}_{(B, \beta)}$, then the two actions α and β are conjugate.

Let β be a faithful outer action of G on a simple C^* -algebra B , and let $\{\lambda_g\}$ be the implementing unitary representation of G in the crossed product $B \rtimes_\beta G$. Let

$$e_\beta = \frac{1}{\#G} \sum_{g \in G} \lambda_g,$$

which is a projection in $B \rtimes_\beta G$. Then the homomorphism

$$\Phi_\beta : B^G \ni x \mapsto xe_\beta \in B \rtimes_\beta G$$

induces the isomorphism $K_*(\Phi_\beta) : K_*(B^G) \rightarrow K_*(B \rtimes_\beta G)$.

Let $\hat{G}$ be the unitary dual of G . The dual coaction $\hat{\beta}$ of the action β induces an action of the representation ring $\mathbb{Z}\hat{G}$ on $K_*(B \rtimes_\beta G)$. For a finite dimensional unitary representation (π, H_π) of G , we denote by $K_*(\hat{\beta}_\pi)$ the corresponding endomorphism in $\text{End}(K_*(B \rtimes_\beta G))$.

The following theorem is an equivariant version of Rørdam's result (cf. [7, Theorem 5.1.2]):

Theorem 2.1. *Let α be a quasi-free action of G on $\mathcal{O}_n$ with finite n , and let (B, β) be a G - C^* -algebra. We assume that B is unital purely infinite simple, and β is outer. For two unital G -homomorphisms $\varphi, \psi \in \text{Hom}_G(\mathcal{O}_n, B)$, let*

$$u = \sum_{i=1}^n \psi(S_i) \varphi(S_i)^*,$$

which is a unitary in B^G . Then the following conditions are equivalent.

- (1) *The K_1 -class $K_1(\Phi_\beta)([u])$ is in the image of $K_1(\hat{\beta}_{\pi_\alpha}) - 1$.*
- (2) *The G -homomorphisms φ and ψ are G -approximately unitarily equivalent.*
- (3) *$[\varphi] = [\psi]$ in the equivariant KK -group $KK_G(\mathcal{O}_n, B)$.*

The equivalence of (1) and (2) follows from the Rohlin property of the shift of the UHF algebra M_{n^∞} restricted to the fixed point algebra of a product type G -action.

Let $\mathcal{T}_n$ be the Cuntz-Toeplitz algebra, which is the universal C^* -algebra generated by isometries $\{T_i\}_{i=1}^n$ with mutually orthogonal ranges. Then there exists a homomorphism from $\mathcal{T}_n$ to $\mathcal{O}_n$ sending T_i to S_i for $i = 1, 2, \dots, n$, which gives a short exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{T}_n \longrightarrow \mathcal{O}_n \longrightarrow 0.$$

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Quasi-free actions on $\mathcal{T}_n$ are defined in the same way as in the case of the Cuntz algebras, and this short exact sequence is actually a semi-splitting short exact sequence of G - C^* -algebras. Therefore we get the following 6-term exact sequence of equivariant KK -groups:

$$\begin{array}{ccccc} KK_G(\mathcal{O}_n, B) & \longrightarrow & KK_G(\mathcal{T}_n, B) & \longrightarrow & KK_G(\mathbb{K}, B) \\ \delta \uparrow & & & & \downarrow \\ KK_G^1(\mathbb{K}, B) & \longleftarrow & KK_G^1(\mathcal{T}_n, B) & \longleftarrow & KK_G^1(\mathcal{O}_n, B) \end{array}$$

Let $\tilde{\alpha}$ be the quasi-free action of G on $\mathcal{T}_n$ that is a lift of α . Since $(\mathcal{T}_n, \tilde{\alpha})$ is equivariantly KK -equivalent to $(\mathbb{C}, \text{id})$, we get an exact sequence

$$\begin{array}{ccccc} KK_G(\mathcal{O}_n, B) & \longrightarrow & K_0(B \rtimes_{\beta} G) & \xrightarrow{1-K_0(\hat{\beta}_{\pi_{\alpha}})} & K_0(B \rtimes_{\beta} G) \\ \delta \uparrow & & & & \downarrow \\ K_1(B \rtimes_{\beta} G) & \xleftarrow{1-K_1(\hat{\beta}_{\pi_{\alpha}})} & K_1(B \rtimes_{\beta} G) & \longleftarrow & KK_G^1(\mathcal{O}_n, B) \end{array}$$

A tedious computation shows $\delta(K_1(\Phi_{\beta})([u])) = [\varphi] - [\psi]$, which shows equivalence of (1) and (3). This argument also shows that there exists a short exact sequence

$$0 \rightarrow \text{Coker}(1 - K_{1-*}(\hat{\beta}_{\pi_{\alpha}})) \rightarrow KK_G^*(\mathcal{O}_n, B) \rightarrow \text{Ker}(1 - K_*(\hat{\beta}_{\pi_{\alpha}})) \rightarrow 0.$$

As in [7, Proposition 7.2.5], Theorem 2.1 implies

Theorem 2.2. *Let α be a quasi-free action of G on $\mathcal{O}_\infty$, and let (B, β) be a unital G - C^* -algebra. We assume that B is purely infinite simple, and β is outer. Then any two unital G -homomorphisms in $\text{Hom}_G(\mathcal{O}_\infty, B)$ are G -approximately unitarily equivalent.*

Thanks to Theorem 2.2, we get a G -equivariant version of Kirchberg-Phillips' $\mathcal{O}_\infty$ theorem (cf. [7, Theorem 7.2.6]).

Theorem 2.3. *Let B be a Kirchberg algebra, and let β be an outer action of G on B . Let $\{\gamma^{(i)}\}_{i=1}^\infty$ be any sequence of quasi-free actions of G on $\mathcal{O}_\infty$. Then (B, β) is conjugate to*

$$(B \otimes \bigotimes_{i=1}^\infty \mathcal{O}_\infty, \beta \otimes \bigotimes_{i=1}^\infty \gamma^{(i)}).$$

Applying Theorem 2.3 to $B = \mathcal{O}_\infty$ with a faithful quasi-free action β , we obtain

Corollary 2.4. *Any two faithful quasi-free G -actions on $\mathcal{O}_\infty$ are mutually conjugate.*

3. APPROXIMATELY REPRESENTABLE ACTIONS

An action α of G on a unital separable C^* -algebra A is said to be approximately representable if there exists a sequence of unitaries $\{u(g)_n\}_{n=1}^\infty$ in A for each $g \in G$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u(g)_n x u(g)_n^* - \alpha_g(x)\| &= 0, \quad \forall x \in A, \forall g \in G, \\ \lim_{n \rightarrow \infty} \|u_n(g) u(h)_n - u(gh)_n\| &= 0, \quad \forall g, h \in G, \end{aligned}$$

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$$\lim_{n \rightarrow \infty} \|\alpha_g(u(h)_n) - u(ghg^{-1})_n\| = 0, \quad \forall g, h \in G.$$

When G is abelian, an action α is approximately representable if and only if its dual action has the Rohlin property. When G is a cyclic group of prime power order, approximately representable quasi-free actions on $\mathcal{O}_n$ with finite n are completely characterized in [3], and there exist quasi-free actions that are not approximately representable.

Theorem 3.1. *Every quasi-free G -action on $\mathcal{O}_\infty$ is approximately representable.*

It does not seem to me that one could show Theorem 3.1 directly from the definition of quasi-free actions. Our proof uses the intertwining argument between two model actions; one is obviously quasi-free, and the other is an infinite tensor product action, which is obviously approximately representable.

4. EQUIVARIANT RØRDAM GROUP

Let A and B be simple C^* -algebras. Following Rørdam [6], we denote by $H(A, B)$ the set of approximately unitary equivalence classes of non-zero homomorphisms from $A \otimes \mathbb{K}$ into $B \otimes \mathbb{K}$. Choosing two isometries S_1 and S_2 satisfying the $\mathcal{O}_2$ relation in the multiplier algebra of $B \otimes \mathbb{K}$, we can define the direct sum $[\varphi] \oplus [\psi]$ of two classes $[\varphi]$ and $[\psi]$ in $H(A, B)$ to be the class of the homomorphism

$$A \otimes \mathbb{K} \ni x \mapsto S_1 \varphi(x) S_1^* + S_2 \psi(x) S_2^* \in B \otimes \mathbb{K}.$$

This makes $H(A, B)$ a semigroup. When A is a Kirchberg algebra, Rørdam semigroup $H(A, B)$ is in fact a group, which is isomorphic to $KL(A, B)$, a certain quotient of $KK(A, B)$ (see [6]).

When (A, α) (resp. (B, β)) is a unital G - C^* -algebra, we equip $A \otimes \mathbb{K}$ (resp. $B \otimes \mathbb{K}$) with a G - C^* -algebra structure by the diagonal action $\alpha_g \otimes \text{Adu}(g)$ (resp. $\beta_g \otimes \text{Adu}(g)$), where $u(g)$ is a countable direct sum of the regular representation of G . Then one can introduce an equivariant version $H_G(A, B)$ of Rørdam's semigroup $H(A, B)$ in an obvious way.

Theorem 4.1. *Let (A, α) and (B, β) be G - C^* -algebras with outer actions α and β . We assume that A and B are Kirchberg algebras. Then $H_G(A, B)$ is a group.*

The proof uses Theorem 2.3 and the fact that $(A \otimes \mathcal{O}_2, \alpha \otimes \text{id})$ is conjugate to $(\mathcal{O}_\infty \otimes \mathcal{O}_2, \gamma \otimes \text{id})$ with a quasi-free action γ (see [2]).

Note that there are two natural homomorphisms

$$\mu : H_G(A, B) \rightarrow H(A, B),$$

$$\nu : H_G(A, B) \rightarrow H(A \rtimes_\alpha G, B \rtimes_\beta G).$$

Theorem 4.2. *Let the notation be as above.*

- (1) *If β has the Rohlin property, then μ is injective.*
- (2) *If β is approximately representable, then ν is injective.*

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Moreover, when the assumption of (1) (resp. (2)) is satisfied, the image of μ (resp. ν) can be completely characterized, so that $H_G(A, B)$ is computable if the algebras involved satisfy UCT.

We conjecture that if G -approximately unitary equivalence in the definition of $H_G(A, B)$ is replaced by G -asymptotically unitary equivalence, one would get the equivariant KK -group $KK_G(A, B)$, as it is the case for trivial G (see [5]).

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