

# Frequency Response of Servomechanisms with Non-linear Friction

By

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In this paper, the effect of non-linear friction, i.e. static and coulomb friction, on frequency response of the servomechanism, is considered in order to get the knowledge for synthesis of the servomechanism. The effect of non-linear friction does not depend on the absolute value of friction torque, but on the ratio of the static friction torque to the amplitude of sinusoidal input torque  $F/A$ .

If the amplitude of sinusoidal input torque is not so large as the static friction torque, the output speed of the servomechanism has a dead zone in the low frequency range, and whether the output speed has a dead zone or not depends on the frequency of the sinusoidal input torque, the ratio  $F/A$ , and the other system parameters. The limiting condition of the output speed with a dead zone is considered.

When the input torque varies sinusoidally, the waveforms of output speed of the servomechanism in which the non-linear friction characteristic is contained, are strictly analysed. From the results of the above analyses, the effect of non-linear friction on the frequency response of the servomechanism is discussed through the method of comparing the first harmonic component of the output speed with input torque. The magnitude of gain is less than that of the linear first order system and converges to a certain value determined by the characteristic of the non-linear friction in the low frequency range. The shape of gain curves is like the linear first order system and the curves approach the  $-20$  dB/decade line in the high frequency range. The break point frequency becomes a little larger than that of the linear first order system as  $F/A$  becomes larger. On the other hand, the phase angle delays more than that of the linear first order system in the low frequency range and less than that in the high frequency range. Moreover, the phase characteristic curves converge to a certain angle larger than  $-90^\circ$ . So, in the high frequency range, the non-linear friction acts so as to stabilize the performance of the whole system.

## 1. Introduction

In a practical servomechanism, the static and the coulomb friction (which together we shall call non-linear friction) have a great influence on its performance. Nevertheless, few have undertaken the investigation except J. Tou

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and P. M. Schultheiss,<sup>1)</sup> and M. Y. Silberberg,<sup>2)</sup> who have studied it through the concept of describing function. Their results, though they are partially successful in explaining the hunting phenomena of the servomechanism with non-linear friction, do not enable us to estimate its relative stability or to acquire general knowledge for its synthesis, which can only be done by analyzing precisely the contribution of the non-linear friction.

## 2. Speed Waveform of the Servomechanism for Sinusoidal Input Torque

The block diagram of a servomechanism, taking into consideration the non-linear friction, is generally represented as in Fig. 1, where  $N$  represents the non-linear friction characteristic. In order to make clear the effect of the non-linear friction upon the response of the servomechanism, we shall study Fig. 2 which is enclosed within the broken line of Fig. 1. For the sake of simplicity we shall assume the non-linear characteristic to be as shown in Fig. 3, where  $\pm F$  is the static friction torque required to initiate a motion

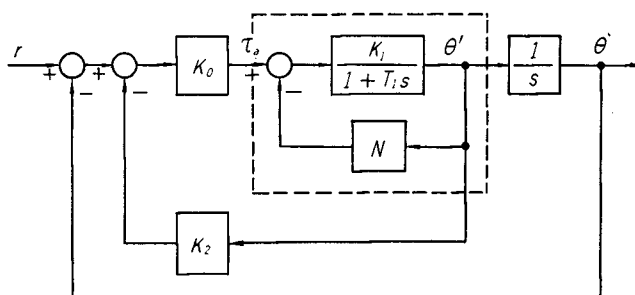


Fig. 1. Block diagram of a servomechanism with non-linear friction.

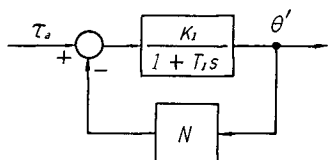


Fig. 2. Block diagram of the part containing non-linear friction.

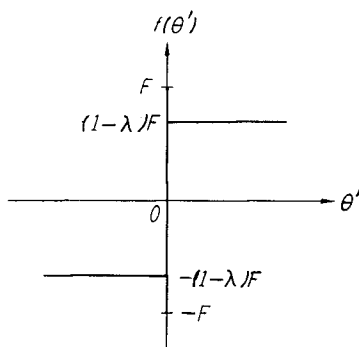


Fig. 3. Non-linear friction characteristic.

from the stationary state, and  $\pm(1-\lambda)F$  is the coulomb friction torque which is constant independently of the speed of motion and changes its polarity according as the speed is positive or negative.

Now in Fig. 4, if we regard the input torque  $\tau_a(t)$  as a sinusoid of amplitude  $A$ , the speed  $\theta'(t)$  remains stationary untill the input torque reaches  $|F|$ . The speed begins to change only after the torque exceeds  $|F|$ , overcoming the static friction torque. It sometimes happens, therefore, that, if the amplitude  $A$  of the sinusoidal input torque is not large enough compared with the static friction torque  $|F|$ , there appears a dead zone in the speed waveform.

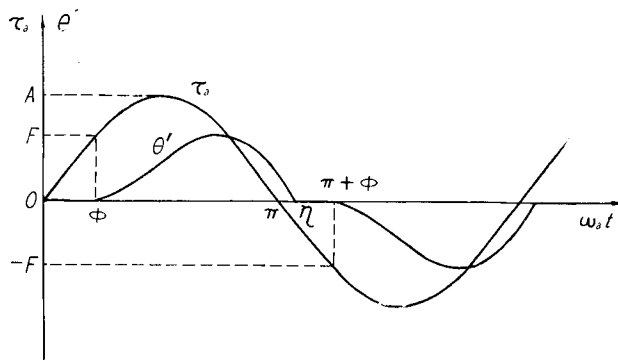


Fig. 4. Waveforms of input torque and output speed in the case where there occurs a dead zone in the speed.

Once the speed thus falls down to zero, the conditions for initiating the motion being the same as the beginning, it begins to move only when the input torque overcomes the static friction torque, so that the same speed waveform, only differing in sign, repeats itself for the sinusoidal input torque at every half-cycle, as shown in Fig. 4. Here the phase angle of the input torque at the moment of the speed starting from the dead zone is given by

$$\phi = \sin^{-1} F/A. \quad (1)$$

If we represent by  $\eta$  the phase angle of the input torque at the next zero intercept of the speed.

$$\eta < \pi + \phi,$$

if, as in Fig. 4, there appears a dead zone in the speed; but sometimes it happens that

$$\eta \geq \pi + \phi$$

according to the amplitude of the input torque and the parameters of the

servomechanism, so that no dead zone occurs just as in a linear system. We shall proceed to analyze the frequency response of the speed in each case.

### 3. Frequency Response of the Speed

#### 3.1. The case with dead zone

If we put the input torque in Fig. 2 as

$$\tau_a(t) = A \sin \omega_a t, \quad (2)$$

the speed  $\theta'(t)$  satisfies the differential equation

$$T_1 \theta'' + \theta' = K_1 \{A \sin \omega_a t - f(\theta')\}, \quad ' \equiv d/dt \quad (3)$$

where  $f(\theta')$  is, as shown in Fig. 3,

$$\left. \begin{aligned} f(\theta') &\leq |F|, & \theta'(t) &= 0, \\ f(\theta') &= (1-\lambda)F, & \theta'(t) &> 0, \\ f(\theta') &= -(1-\lambda)F, & \theta'(t) &< 0. \end{aligned} \right\} \quad (4)$$

By putting

$$\left. \begin{aligned} 1/T_1 &= 2\zeta\omega_0, & \omega_0 t &= \tau, \\ K_1/\omega_0 &= K_1', & \omega_a/\omega_0 &= \omega \end{aligned} \right\} \quad (5)$$

and making Eq. (3) dimensionless, we obtain

$$\ddot{\theta} + 2\zeta\dot{\theta} = 2\zeta K_1' \{A \sin \omega\tau - f(\dot{\theta})\}, \quad \cdot \equiv d/d\tau \quad (6)$$

In this case the block diagram can be transformed as in Fig. 5 for  $p=s/\omega_0$ .

If there occurs a dead zone, the same speed waveform differing only in sign, repeats itself at every half-cycle. Now take the positive half-cycle. During the time between the starting and the next zero intercept of the speed, Eq. (6) becomes

$$\ddot{\theta} + 2\zeta\dot{\theta} = 2\zeta K_1' \{A \sin \omega\tau - (1-\lambda)F\}. \quad (7)$$

By applying to this equation the initial condition  $\dot{\theta}(\tau)=0$  for  $\omega\tau=\phi$ , we obtain the solution

$$\left. \begin{aligned} \dot{\theta}(\tau) &= \frac{K_1' A}{\sqrt{1+(\omega/2\zeta)^2}} \left[ \sin(\omega\tau - \phi) \right. \\ &\quad \left. + \{K' - \sin(\phi - \psi)\} \exp\{-2\zeta(\omega\tau - \phi)/\omega\} - K' \right], \end{aligned} \right\} \quad (8)$$

where

$$\psi = \tan^{-1} \omega/2\zeta, \quad K' = \sqrt{1+(\omega/2\zeta)^2} (1-\lambda)F/A,$$

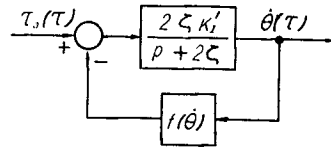


Fig. 5. Normalized block diagram.

The speed waveform of the positive half-cycle is obtained from Eq. (8) for the period in which  $\dot{\theta}(\tau)$  starts from zero and again falls down to zero. The speed for the negative half-cycle is likewise obtained by changing sign. Thus, by detecting the fundamental harmonic component of  $\dot{\theta}(\tau)$  and comparing it with the input torque, we have the approximate frequency response of the system with due regard to the effect of non-linear friction.

The fundamental component  $\dot{\theta}_1(\tau)$  of  $\dot{\theta}(\tau)$ , obtained from Eq. (8) is

$$\begin{aligned}\dot{\theta}_1(\tau) &= \frac{K_1' A}{\sqrt{1 + (\omega/2\zeta)^2}} (a_1 \sin \omega\tau + b_1 \cos \omega\tau) \\ &= \frac{K_1' A}{\sqrt{1 + (\omega/2\zeta)^2}} \sqrt{a_1^2 + b_1^2} \sin (\omega\tau + \tan^{-1} b_1/a_1),\end{aligned}\quad (9)$$

where

$$\left. \begin{aligned}\frac{\pi}{2} a_1 &= \frac{1}{4} \left\{ 2(\eta - \phi) \cos \phi - \sin (2\eta - \psi) + \sin (2\phi - \psi) \right\} \\ &\quad - \frac{K' - \sin (\phi - \psi)}{\sqrt{1 + (2\zeta/\omega)^2}} \left[ \sin (\eta + \psi) \exp \left\{ -\frac{2\zeta}{\omega} (\eta - \phi) \right\} \right. \\ &\quad \left. - \sin (\phi + \psi) \right] + K' (\cos \eta - \cos \phi), \\ \frac{\pi}{2} b_1 &= \frac{1}{4} \left\{ -2(\eta - \phi) \sin \phi - \cos (2\eta - \psi) + \cos (2\phi - \psi) \right\} \\ &\quad - \frac{K' - \sin (\phi - \psi)}{\sqrt{1 + (2\zeta/\omega)^2}} \left[ \cos (\eta + \psi) \exp \left\{ -\frac{2\zeta}{\omega} (\eta - \phi) \right\} \right. \\ &\quad \left. - \cos (\phi + \psi) \right] - K' (\sin \eta - \sin \phi).\end{aligned}\right\} \quad (10)$$

As for  $\eta$  in Eq. (10), by solving

$$\sin (\omega\tau - \psi) + \left\{ K' - \sin (\phi - \psi) \right\} \exp \left\{ -2\zeta(\omega\tau - \phi)/\omega \right\} - K' = 0, \quad (11)$$

we could obtain  $\eta (= \omega\tau)$  such that  $\dot{\theta}(\tau) = 0$ . Figures 6 and 7 are examples of the frequency response of the fundamental component  $\dot{\theta}_1(\tau)$  to the input torque. In these calculations, the non-linear friction characteristics are taken as  $F/A = 0.8$  and  $0.4$  respectively and  $\lambda = 0.25$ . Also, to facilitate the comparison with a linear system, the amplitude of the fundamental component of the speed is represented as  $|\dot{\theta}_1(\tau)/K_1' A|$ . In the case of Fig. 6, the speed waveform will possess a dead zone over the whole frequency range, enabling us to compute the frequency response by the above method, while in the case of Fig. 7

$$\eta = \pi + \phi \quad (12)$$

as  $\omega = 0.86, 1.29$  and  $1.72$  for  $\zeta = 0.4, 0.6$  and  $0.8$ , respectively, so that the frequency response to the input torque with angular frequency less than these

values will be obtained by the same method, though for a larger angular frequency, since  $\eta > \pi + \phi$ , another method of analysis, which will be described in the next paragraph, should be applied. The limiting condition satisfying Eq. (12) is represented by a broken line in Fig. 7.

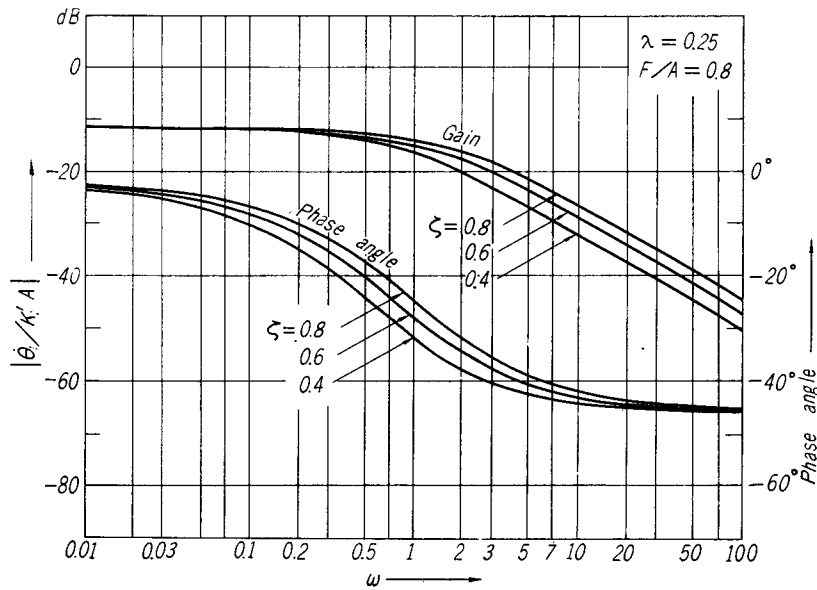


Fig. 6. Frequency response of the speed ( $F/A$  : constant) (1).

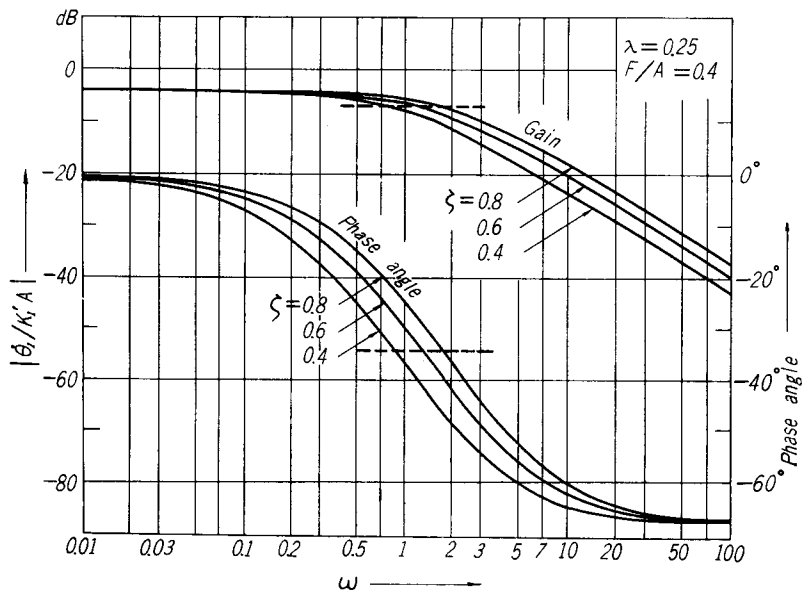


Fig. 7. Frequency response of the speed ( $F/A$  : constant) (2).

### 3.2. The case without dead zone

In the case where there occurs a dead zone in the speed, the phase angle  $\eta$  gradually approaches  $\pi + \phi$  as the angular frequency  $\omega$  of the input torque increased, and finally it becomes  $\eta \geq \pi + \phi$ . Then, the absolute value of the input torque at the phase angle  $\eta$  overcoming the static friction torque  $|F|$ , the speed starts for the next half-cycle without any dead zone. Under this condition, only the coulomb friction torque is acting as a non-linear friction torque which is  $\pm(1-\lambda)F$  according to the sign of the speed. We shall, therefore, transform the block diagram of Fig. 5 to the form shown in Fig. 8 and therefrom analyze the speed waveform in the steady-state in order to obtain the frequency response.

Now, if we put the input torque in Fig. 8 as

$$\tau_a(\tau) = A \sin(\omega\tau + \xi), \quad (13)$$

in the steady-state after a long elapse of time

$$\left. \begin{aligned} x(\tau) &= \frac{K'_1 A}{\sqrt{1 + (\omega/2\zeta)^2}} \sin(\omega\tau + \xi - \psi), \\ \psi &= \tan^{-1} \omega/2\zeta, \end{aligned} \right\} \quad (14)$$

where

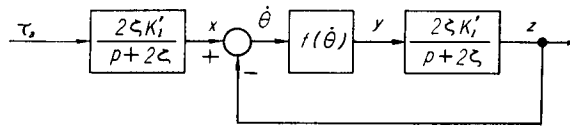


Fig. 8. Transformed block diagram.

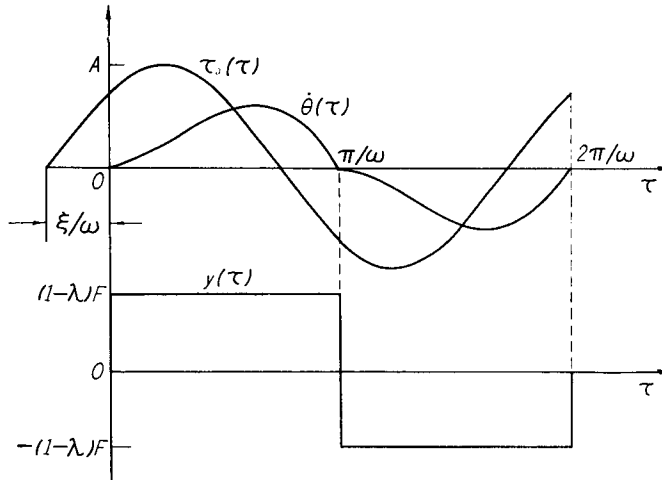


Fig. 9. Waveforms of input torque and output speed in the case where there occurs no dead zone in the speed.

which is to be applied as the input of the closed loop with non-linear friction in Fig. 8. The speed waveform in this steady-state is shown in Fig. 9. The output  $y(\tau)$  of  $f(\dot{\theta})$  has a periodic rectangular waveform.

Now if we apply the Laplace transformation to  $y(\tau)$ , we get

$$X(p) = \frac{(1-\lambda)F}{p} \frac{1 - \exp\left(-\frac{\pi}{\omega}p\right)}{1 + \exp\left(-\frac{\pi}{\omega}p\right)}. \quad (15)$$

The Laplace transform, therefore, of  $z(\tau)$  in Fig. 8, will be

$$Z(p) = \frac{2\zeta K_1'}{p+2\zeta} \frac{(1-\lambda)F}{p} \frac{1 - \exp\left(-\frac{\pi}{\omega}p\right)}{1 + \exp\left(-\frac{\pi}{\omega}p\right)}. \quad (16)$$

By applying the inverse Laplace transformation, we obtain the function  $z(\tau)$  in time domain, which in the steady-state is:

$$z(\tau) = K_1'(1-\lambda)F \left\{ 1 - \left( 1 + \tanh \frac{\pi}{2} \frac{2\zeta}{\omega} \right) \exp(-2\zeta\tau) \right\}, \quad 0 \leq \tau \leq \pi/\omega. \quad (17)$$

From Eqs. (14) and (17), we obtain

$$\begin{aligned} \dot{\theta}(\tau) &= x(\tau) - z(\tau) \\ &= \frac{K_1'A}{\sqrt{1+(\omega/2\zeta)^2}} \left[ \sin(\omega\tau + \xi - \psi) - \sqrt{1+(\omega/2\zeta)^2} (1-\lambda) \frac{F}{A} \right. \\ &\quad \left. \times \left\{ 1 - \left( 1 + \tanh \frac{\pi}{2} \frac{2\zeta}{\omega} \right) \exp(-2\zeta\tau) \right\} \right], \quad \left. \begin{aligned} &0 \leq \tau \leq \pi/\omega. \end{aligned} \right\} \quad (18) \end{aligned}$$

While in the steady-state, in which there occurs no dead zone in the speed waveform,  $\dot{\theta}(\tau)$  should be zero at  $\tau=0, \pi/\omega, \dots$  as shown in Fig. 9. That is:

$$\begin{aligned} \dot{\theta}(0) &= \frac{K_1'A}{\sqrt{1+(\omega/2\zeta)^2}} \left\{ \sin(\xi - \psi) + \sqrt{1+(\omega/2\zeta)^2} (1-\lambda) \frac{F}{A} \right. \\ &\quad \left. \times \tanh \frac{\pi}{2} \frac{2\zeta}{\omega} \right\} = 0. \quad (19) \end{aligned}$$

We obtain from this equation

$$\xi = \psi - \sin^{-1} \left\{ \sqrt{1+(\omega/2\zeta)^2} (1-\lambda) \frac{F}{A} \tanh \frac{\pi}{2} \frac{2\zeta}{\omega} \right\}. \quad (20)$$

The change of  $\xi$  with  $\omega$ , obtained from Eq. (20) by giving suitable values to  $F/A$ ,  $\lambda$  and  $\zeta$ , is as shown in Fig. 10. The limit within which there occurs a dead zone in the speed is represented by a broken line. Thus, having known  $\xi$  from Eq. (20), we can obtain by means of it the fundamental component  $\dot{\theta}_1(\tau)$  of the speed waveform from Eq. (18). That is:

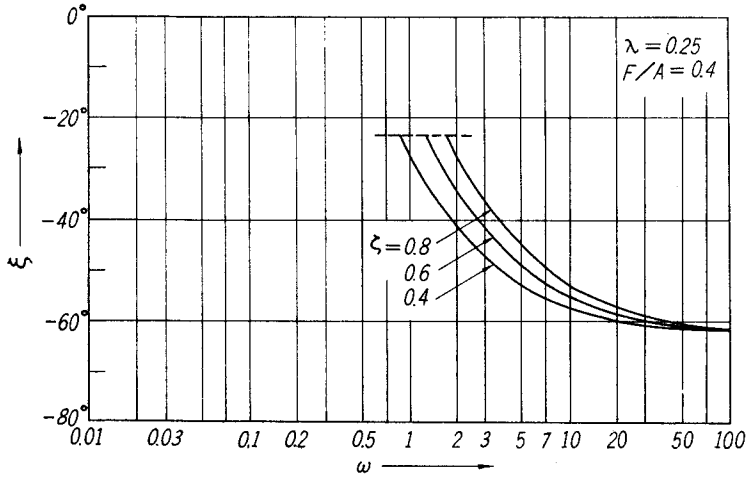


Fig. 10. Phase angle  $\xi$  in the case where there occurs no dead zone in the speed.

$$\dot{\theta}_1(\tau) = \frac{K_1' A}{\sqrt{1 + (\omega/2\zeta)^2}} \sqrt{a_1^2 + b_1^2} \sin \left( \omega\tau + \tan^{-1} \frac{b_1}{a_1} \right),$$

where

$$\left. \begin{aligned} \frac{\pi}{2} a_1 &= \frac{\pi}{2} \cos \psi - 2(1-\lambda) \frac{F}{A} \left\{ \sqrt{1 + (\omega/2\zeta)^2} \cos \xi - \frac{\omega}{2\zeta} \sin (\xi + \psi) \right\}, \\ \frac{\pi}{2} b_1 &= -\frac{\pi}{2} \sin \psi + 2(1-\lambda) \frac{F}{A} \left\{ \sqrt{1 + (\omega/2\zeta)^2} \sin \xi + \frac{\omega}{2\zeta} \cos (\xi + \psi) \right\}. \end{aligned} \right\} \quad (21)$$

The frequency response of the fundamental component  $\dot{\theta}_1(\tau)$  of the speed for  $F/A=0.4$  and  $\lambda=0.25$ , obtained by means of Eqs. (20) and (21) is given in Fig. 7.

#### 4. The Limit Within which There Occurs a Dead Zone in the Speed

Equation (8) in the preceding paragraph gives the speed waveform in the case where there occurs a dead zone and the limit within which there occurs a dead zone in the speed is given by  $\dot{\theta}(\tau)=0$  at  $\omega\tau=\pi+\phi$ . There we obtain from Eq. (8)

$$\sin (\pi + \phi - \psi) + \left\{ K' - \sin (\phi - \psi) \right\} \exp (-2\zeta\pi/\omega) - K' = 0.$$

Therefore

$$\begin{aligned} \sin (\phi - \psi) &= -K' \tanh \frac{\pi}{2} \frac{2\zeta}{\omega} \\ &= -\sqrt{1 + (\omega/2\zeta)^2} (1-\lambda) \frac{F}{A} \tanh \frac{\pi}{2} \frac{2\zeta}{\omega}. \end{aligned} \quad (22)$$

This gives the limiting condition for causing a dead zone in the speed. By substituting  $\phi$  and  $\psi$  of Eqs. (1) and (8) into Eq. (22), we get

$$\sin \left( \sin^{-1} \frac{F}{A} - \tan^{-1} \frac{\omega}{2\zeta} \right) = -\sqrt{1 + (\omega/2\zeta)^2} (1-\lambda) \frac{F}{A} \tanh \frac{\pi}{2} \frac{2\zeta}{\omega}. \quad (23)$$

Putting for simplicity

$$u = \omega/2\zeta, \quad v = F/A, \quad (24)$$

Eq. (23) is transformed into

$$\frac{\sqrt{1-v^2}}{v} = \frac{1}{u} + \frac{1-\lambda}{\sin(\tan^{-1} u)} \sqrt{1+u^2} \tanh \frac{\pi}{2u}. \quad (25)$$

The left-hand side of this equation is a function solely of  $v$  and its right-hand side is a function solely of  $u$ . We get from Eq. (25)

$$v = \left[ 1 + \left\{ \frac{1}{u} + \frac{1-\lambda}{\sin(\tan^{-1} u)} \sqrt{1+u^2} \tanh \frac{\pi}{2u} \right\}^2 \right]^{-1/2}. \quad (26)$$

This equation gives the relation between  $u$  and  $v$ , that is, between  $\omega/2\zeta$  and  $F/A$  at the limit within which there occurs a dead zone in the speed. In Fig. 11 is shown the relation between  $\omega$  and  $F/A$  obtained by giving appropriate values for  $\lambda$  and  $\zeta$  in Eq. (26). It is noticed in this figure that there occurs a dead zone in the speed on the left side of the curves, whereas it does not on the right. It is to be observed from this figure that, if the frequency of

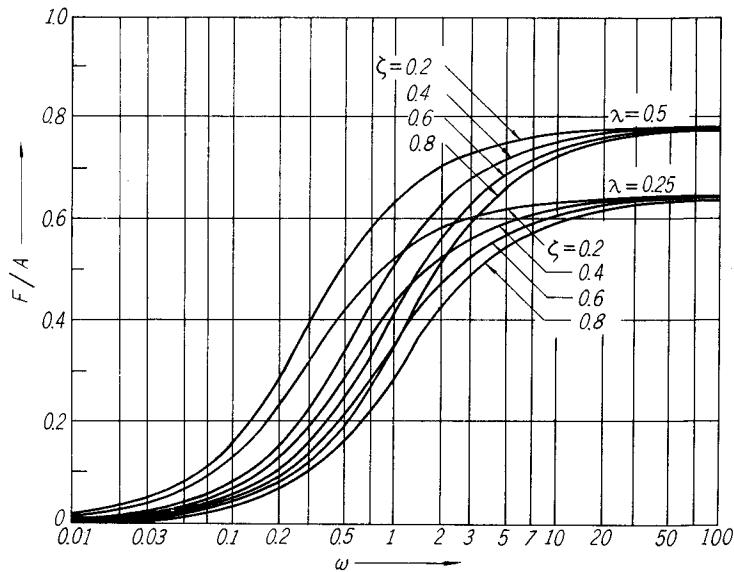


Fig. 11. The limit within which there occurs a dead zone in the speed.

the input torque remains the same, a dead zone is more likely to occur in the speed as the ratio  $F/A$  of the static friction torque to the amplitude of the input torque becomes larger, and that, even if the frequency of the input torque and the value of  $F/A$  remain the same, a dead zone is more likely to occur in the speed as the ratio of the coulomb friction torque to the static friction torque becomes larger, that is, as the value of  $\lambda$  becomes smaller, and that even if the value of  $F/A$ ,  $\lambda$  and the frequency of the input torque remain the same, a dead zone is more likely to occur in the speed as the value of  $\zeta$  becomes larger. It is also evident from the figure that, if the frequency of the input torque is low, a dead zone will appear even if  $F/A$  is fairly small, while with a high frequency, it will not appear unless  $F/A$  is fairly large. Further, it is clearly seen from the figure that in the case shown in Fig. 6 there will occur a dead zone over the whole frequency range, whereas in the case shown in Fig. 7 there will be a boundary frequency represented by a broken line.

### 5. The Effect of Non-Linear Friction on Frequency Response of the Speed

After we have theoretically obtained in the two previous paragraphs the frequency response of the speed of the servomechanism with due regard to non-linear friction characteristics, we proceed to consider briefly the effect of

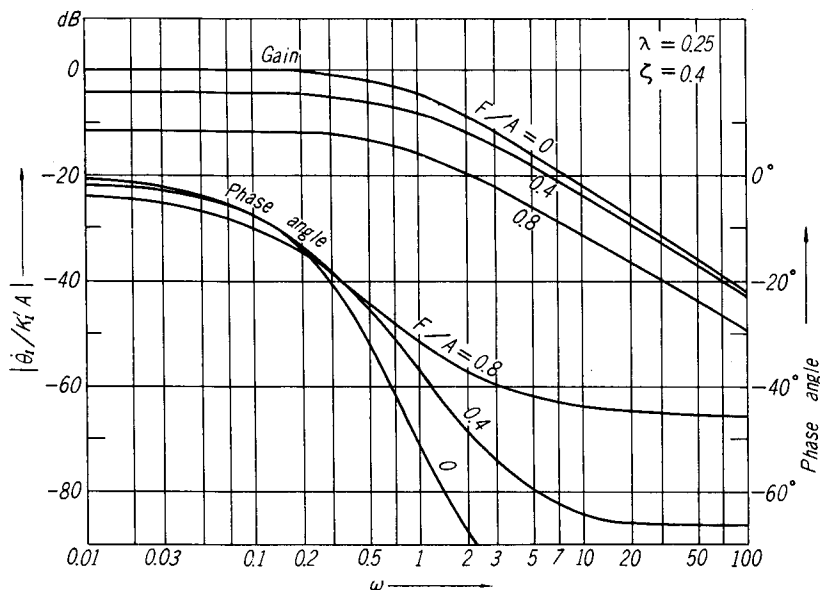


Fig. 12. Frequency response of the speed ( $\zeta$  : constant) (1).

the non-linear friction on the frequency response of the servomechanism by summing up these results.

Figures 12-14 show examples of the frequency response of the speed derived from Figs. 6 and 7 with the values of  $\lambda$  and  $\zeta$  fixed. The response

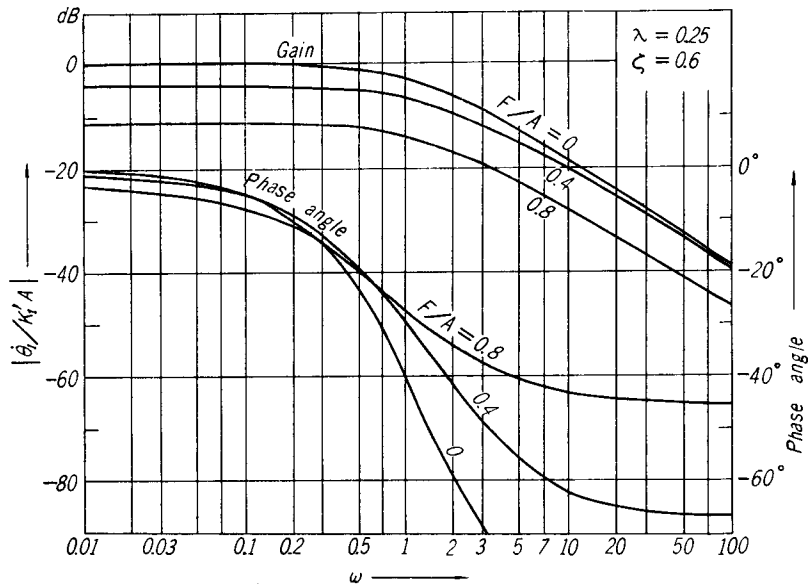


Fig. 13. Frequency response of the speed ( $\zeta$  : constant) (2).

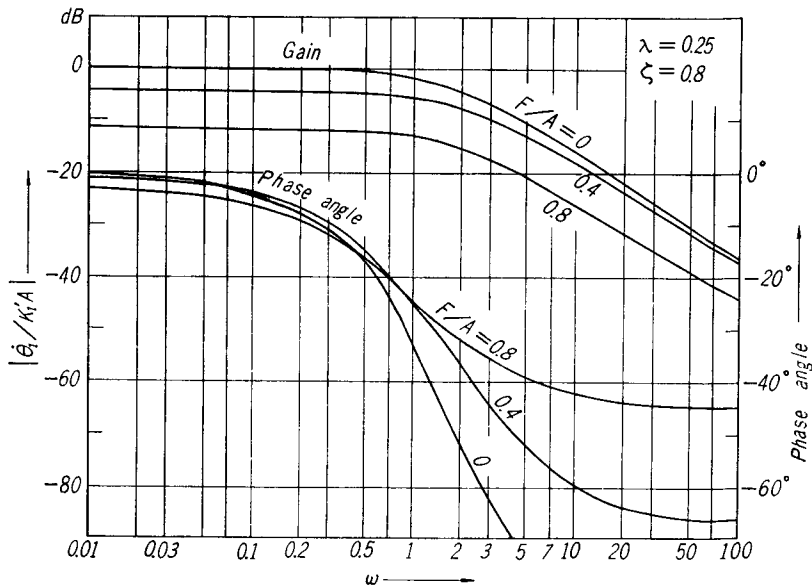


Fig. 14. Frequency response of the speed ( $\zeta$  : constant) (3).

of the linear first order system is shown side by side for reference.

As seen from these figures, if there is the non-linear friction the gain diminishes compared with the linear system as the value of  $F/A$  increases, and comes to a certain value determined by  $F/A$  in a low frequency range. If we consider here the gain at sufficiently low frequency as the steady-state gain, we have its relation to  $F/A$  in the curves of Fig. 15, from which we can infer the effect of the non-linear friction upon the steady-state gain. In a high frequency range, the gain curve, as with the linear system, approximates the  $-20$  dB/decade line, but the corner frequency becomes somewhat larger than in the linear system as the value of  $F/A$  increases. This is equivalent to saying that the time constant diminishes when we approximate the system containing the

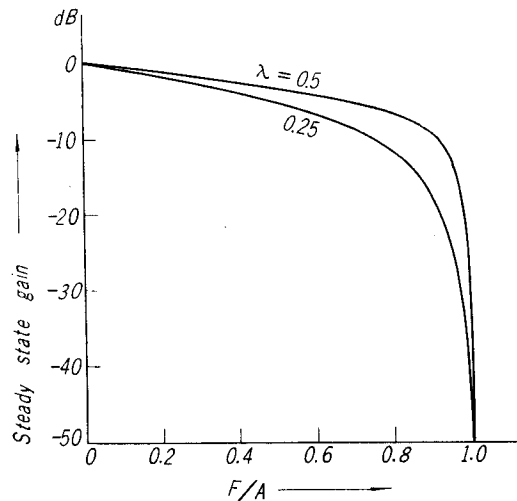


Fig. 15. Gain characteristics in steady-state.

non-linear friction by a linear system. In short, the gain of the system decreases due to the non-linear friction, and the decrease is great in lower frequency range, and, unless the value of  $F/A$  is quite near 1, the gain curve has much the same form as the linear first order system.

Further, if the frequency is low, the phase lag increases compared with the linear first order system, on account of the non-linear friction, and conversely, if the frequency becomes high, the phase lag becomes smaller than the linear system and converges to an angle less than  $90^\circ$ .

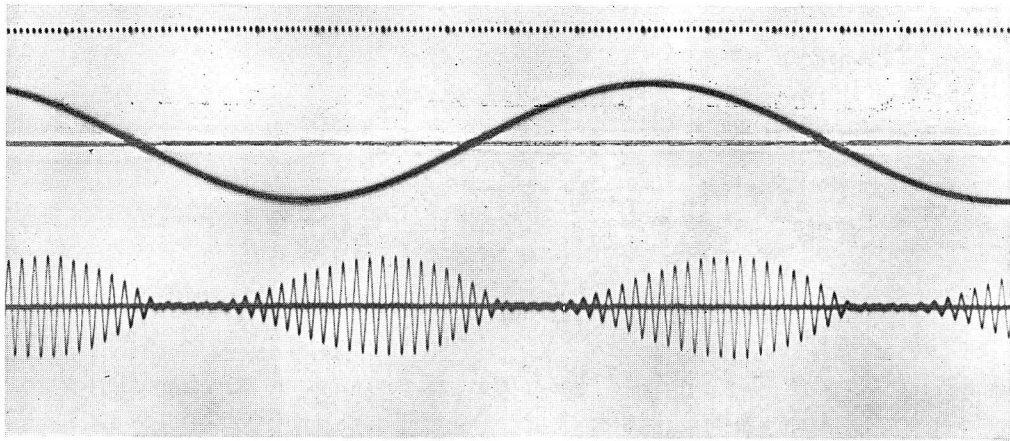
This angle to which the phase lag curve converges is smaller as the value of  $F/A$  is larger, and with the diminution of  $F/A$  approaches as the phase lag curve of the linear system, till it is near  $90^\circ$ . Thus looked at from the view

point of the phase lag curve, the non-linear friction diminishes the phase angle in a high frequency range if the input torque is of small amplitude, thus stabilizing the behaviour of the servomechanism. The relative stability of the whole servomechanism can be estimated from the gain margin and the phase margin by synthesizing the frequency response of that part which contains the above mentioned non-linear friction and that of the other linear parts.

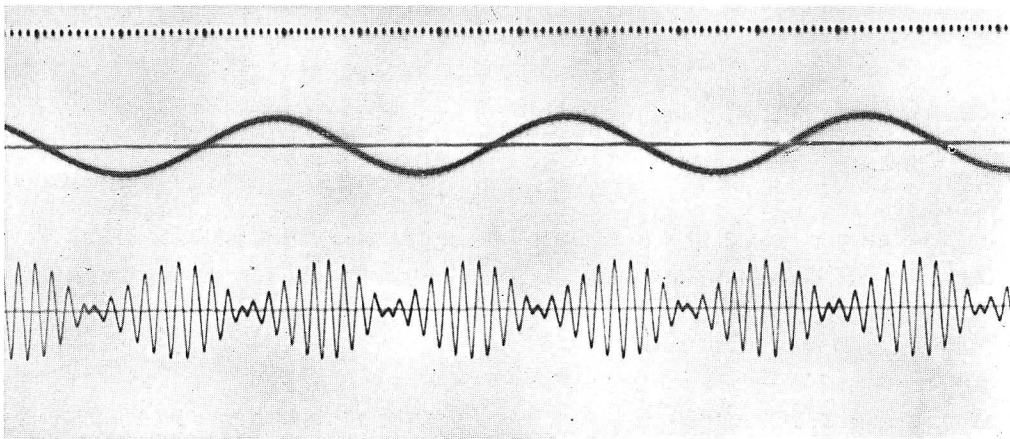
## 6. Measurement of the Speed Waveform and the Frequency Response of the Servomechanism

Here will be briefly described the results obtained by actually measuring the speed waveforms and the frequency responses of a small servomechanism with an a-c two phase servomotor.

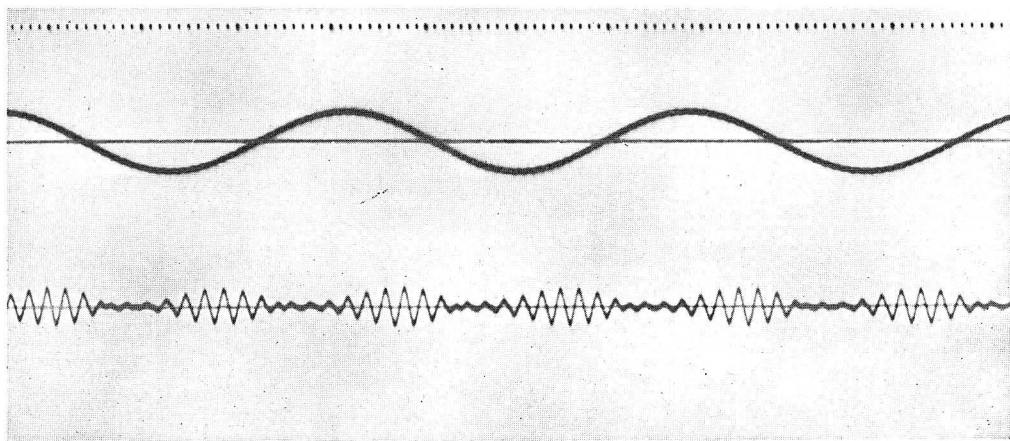
The non-linear friction torque has been changed by adjusting the pressure of the spring around the disc fixed upon the axis of the servomechanism. It is difficult to obtain by experiment the relation of input torque to output speed changing with time. However, as the attenuation and the phase lag at the electrical parts, that is, at the servo-amplifier and the servomotor, are considered to be small enough compared with those at the mechanical parts, we have assumed that the developed torque of the servomotor is to be change sinusoidally without attenuation and phase lag in the case where a sinusoidal voltage is applied to the input terminal of the servo-amplifier. The output speed was inferred from the output voltage of the a-c induction-generator tachometer directly connected to the servomotor. In this way, the static friction torque is given equivalently as the d-c input voltage  $F_v$  of the servo-amplifier at the moment the servomotor begins to rotate, so that it is possible from the  $F_v$  and the amplitude  $A_v$  of the sinusoidal input voltage to know the equivalent value of  $F/A$  mentioned in previous paragraphs. Figure 16 is an example of the oscillogram showing the waveforms of the input voltage and the output speed for changing values of the angular frequency  $\omega_a$  of the input voltage and of the amplitude  $A_v$  with  $F_v$  constant (0.2 V). Even if  $F_v/A_v$  be the same, a dead zone appears in the speed waveform if  $\omega_a$  is small, and even if  $\omega_a$  be the same, a dead zone appears in the speed waveform if  $F_v/A_v$  is large. This result agrees in its trend with the theoretical result in Fig. 11. Figure 17 is an example of the phase characteristic obtained from the speed waveform measured for changing  $F_v/A_v$ , which agrees fairly well in its trend with the theoretical results in Figs. 12-14.



(a)  $F_v/A_v=0.5$ ,  $\omega_a=7$



(b)  $F_v/A_v=0.5$ ,  $\omega_a=20$



(c)  $F_v/A_v=0.7$ ,  $\omega_a=20$

Fig. 16. Waveforms of the input voltage and the speed ( $F_v=0.2V$ ).

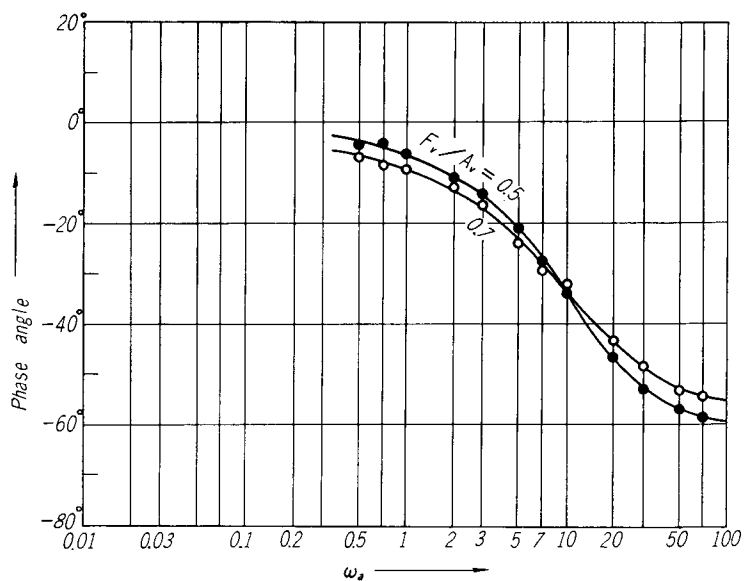


Fig. 17. Phase characteristics of the speed (experiment).

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