

TRAVELING WAVES FOR DISCRETE QUASILINEAR MONOSTABLE DYNAMICS

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1. INTRODUCTION

We are concerned with traveling waves for the infinite ODE system

$$(1) \quad \dot{u}_j = g(u_{j+1}) + g(u_{j-1}) - 2g(u_j) + f(u_j), \quad j \in \mathbb{Z},$$

where g is increasing and f monostable: $f(0) = f(1) = 0$ and $f > 0$ in $(0, 1)$.

This equation is a discrete version of the quasilinear parabolic equation

$$(2) \quad u_t = (g(u))_{xx} + f(u).$$

When $g(u) = u$ and $f(u) = u(1 - u)$, this equation is known as the Fisher's equation or Kolmogorov, Petrovsky and Piskunov (KPP) equation, and has been extensively studied. The discrete version (1) came directly from many biological models (cf. the book of Shorrocks & Swinglad (1990)).

A solution $\{u_j(\cdot)\}_{j \in \mathbb{Z}}$ of (1) is called a *traveling wave of speed c* if $u_j(1/c) = u_{j-1}(0)$ for all $j \in \mathbb{Z}$.

We look for traveling waves connecting the steady states 1 and 0. If we define $U \in C^1(\mathbb{R})$ by $U(j - ct) = u_j(t)$ for all $j \in \mathbb{Z}$ and $t \in [0, 1/c)$, then (c, U) satisfy

$$(3) \quad \begin{cases} cU' + D_2[g(U)] + f(U) = 0 & \text{on } \mathbb{R}, \\ U(-\infty) = 1, \quad U(\infty) = 0, \quad 0 \leq U(\cdot) \leq 1 & \text{on } \mathbb{R}. \end{cases}$$

Here

$$D_2[\phi](x) := \phi(x+1) + \phi(x-1) - 2\phi(x).$$

There has been constant interest in traveling waves for (2); see, for example, Aronson and Weinberger [1975, 1980], Fife and McLeod [1977], De Pablo and Vazquez [1991], Ebert and Saarloo [2000], Hamel and Nadirashvili [2002], and especially the references therein.

For the discrete version (1), there has been growing interest in the last decade; see Zinner [1991, 1992], Chow, Mallet-Paret, and Shen [1998], Bates, Chen, and Chmaj [2002], Mallet-Paret [1999a, 1999b] (for bistable $f = u(1 - u)(u - a)$), Weinberger [1982], Zinner, Harris, and Hudson [1993], Wu and Zou [1997], Fu, Guo, and Shieh [2002], and the references therein.

We are interested in the existence, uniqueness, and asymptotic stability of traveling waves for (1) when f is monostable.

2. EXISTENCE

Definition 1. A non-constant continuous function ψ from $\mathbb{R}$ to $(0, 1]$ is called a *super-solution* for a wave speed c if $\psi(-\infty) = 1$ and for some $\mu \geq \frac{1}{c} \|f' - 2g'\|_{L^\infty([0,1])}$,

$$(4) \quad \psi(x) \geq T[\psi](x), \quad \forall x \in \mathbb{R}.$$

where

$$T[\psi](x) := \frac{1}{c} \int_0^\infty e^{-\mu s} \{D_2[g(\psi)] + f(\psi) + c\mu\psi\}(x+s) ds.$$

A sufficient condition for (4) for a Lipschitz continuous function ψ is the differential inequality

$$(5) \quad -c\psi' - D_2[g(\psi)] - f(\psi) \geq 0 \quad \text{on } \mathbb{R},$$

from which (4) follows by integration with an integrating factor $e^{-\mu x}$. Sub-solution can be defined similarly.

Earlier existence result can be found in the papers of Zinner-Harris-Hudson (1993), Wu-Zou (1997), Fu-Guo-Shieh (2002), Chen-Guo (2002).

The following two classical methods were used:

1. **sub-super-solution method** (with monotone iteration method)
2. **homotopy method** (with degree theory)

The sub-super-solution method iterates (4) (with equal sign) from a sub-super-solution set. The homotopy method in Z-H-H [1993] uses a degree theory and relies only on the existence of a super-solution. Though it is not difficult to construct sub-solutions, the results obtained from these two methods can be very different, since the sub-super-solution method requires super-solutions be bigger than sub-solutions.

We proposed a **new approach**. Under the following assumption:

(A) f and g are Lipschitz continuous on $[0, 1]$, g is strictly increasing, and $f(0) = f(1) = g(0) = 0 < f(s) \quad \forall s \in (0, 1)$,

we have the following existence result of traveling waves

Theorem 1. Assume (A). Then the following hold:

- (i) there exists $c_{\min} > 0$ such that (3) admits a solution if and only if $c \geq c_{\min}$;
- (ii) every solution of (3) satisfies $0 < U(\cdot) < 1$ on $\mathbb{R}$;
- (iii) (Monotonicity) for each $c \geq c_{\min}$, (3) admits a solution U satisfying $U' < 0$ on $\mathbb{R}$;
- (iv) if there exists a super-solution for a wave speed c^* , then $c_{\min} \leq c^*$. (Existence of super-solution $\Rightarrow$ existence of a solution.)

- Assertion (i) is analogous to the continuum case (2).
- Assertion (ii), however, is different from the continuum case, since for $g = u^m$ ($m > 1$), there are traveling waves to (2) satisfying $U(\cdot) = 0$ on $[0, \infty)$ (see, Aronson-Weinberger (1975), Aronson (1980), De Pablo-Vazquez (1991)).
- Assertion (iv) gives upper bounds for $c_{\min}$.

We use a key idea, with significant simplification, from Zinner-Harris-Hudson [1993] who approximate traveling waves by solutions of an initial boundary value problem for (1). The use of a degree theory is replaced by a monotone iteration method (of Wu-Zou [1997] and recently of Fu-Guo-Shieh [2002] and Chen-Guo [2002]) and the construction of a sub-super-solution set is avoided.

Outline of the idea:

We consider, for each $n > 0$, the problem for the equation

$$cU'(x) + D_2[g(U)](x) + f(U(x)) = 0 \quad \forall x \in [0, n],$$

with the “boundary” condition

$$U(x) = 1 \quad \text{for } x < 0, \quad U(x) = \varepsilon \quad \text{for all } x \geq n$$

where $\varepsilon \in [0, 1]$ is a number to be chosen such that $U(n/2)$ is away from 0 and 1, say, $1/2$.

- The freedom of ε is introduced so that the problem becomes much easier and a degree theory is avoided.

The problem can be written in its integral form as

$$(6) \quad U(x) = T^n[U](x) \quad \forall x \in [0, n],$$

where

$$\begin{aligned} T^n[U](x) &:= \frac{1}{c} \int_0^{n-x} e^{-\mu s} \{ D_2[g(U)] + f(U) + c\mu U \}(x+s) ds \\ &\quad + \int_{n-x}^{\infty} \mu \varepsilon e^{-\mu s} ds, \\ \mu &:= \frac{1}{c} \|f' - 2g'\|_{L^\infty([0,1])}. \end{aligned}$$

- Here μ is introduced to (i) make the integral convergent, and (ii) ensure that T^n is monotonic in U . Hence the problem (6) has a unique solution $U^{\varepsilon, n} \in C^1([0, n]) \cap C^0([0, \infty))$ for each $\varepsilon \in [0, 1]$ and $n > 0$.

Then we extract a useful convergent subsequence from $\{U^{\varepsilon, n}(n/2 + \cdot)\}$ by constructing a super-solution. The existence can be derived by passing to the limit in (6). $\square$

The following gives lower bounds.

Theorem 2. Assume (A) and let $c_{\min}$ be as in Theorem 1. Then

$$c_{\min} \geq \max_{0 < a < b < 1} \frac{\min\{\min_{a \leq s \leq b} \{[g(1) - g(s)]f(s)\}, [g(b) - g(a)]^2\}}{\int_0^1 [g(1) - g(s)] ds}.$$

Also, if $g'(0)$ exists and is positive, and if $\underline{m} := \liminf_{u \searrow 0} \frac{f(u)}{u}$, then

$$c_{\min} \geq c_* := \min_{r > 0} \frac{g'(0)[e^r + e^{-r} - 2] + \underline{m}}{r}.$$

When g is linear and $f(u) \leq \bar{m}u$ for all $u \in [0, 1]$, a super-solution is given by $\psi = \min\{1, e^{-rx}\}$, where $cr = g'(0)[e^r + e^{-r} - 2] + \bar{m}$. Hence, if $\bar{m} = \underline{m}$, then $c_{\min} = c_*$ (the optimal result in Zinner-Harris-Hudson [1993]).

Proof of the first estimate in Theorem 2.

Suppose (c, U) solves (3). We may assume that $U' < 0$ on $\mathbb{R}$. Integrating

$$cU' + D_2[g(U)] + f(U) = 0$$

$$g(U)\{cU' + D^2[g(U)] + f(U)\} = 0$$

over $[-M, M]$ and sending $M \rightarrow \infty$, one obtains the identities

$$c = \int_{\mathbb{R}} f(U) dx,$$

$$\int_{\mathbb{R}} f(U)g(U) dx = c \int_0^1 g(s) ds + \int_{\mathbb{R}} \{g(U(x-1)) - g(U(x))\}^2 dx.$$

Fix $a, b \in (0, 1)$ with $a < b$ and let x_0 be such that $U(x_0) = b$.

Then $U(\cdot - 1) \geq b$ in $[x_0, x_0 + 1]$ so that

$$\begin{aligned} & c \int_0^1 [g(1) - g(s)] ds \\ &= \int_{\mathbb{R}} (g(1) - g(U))f(U) dx + \int_{\mathbb{R}} \{g(U(x-1)) - g(U(x))\}^2 dx \\ &\geq \int_{x_0}^{x_0+1} \{(g(1) - g(U))f(U) + [g(b) - g(U)]^2\} dx \\ &\geq \min \left\{ \min_{a \leq s \leq b} \{[g(1) - g(s)]f(s)\}, [g(b) - g(a)]^2 \right\}, \end{aligned}$$

by dividing $[x_0, x_0 + 1]$ into sets $\{x \in [x_0, x_0 + 1] \mid U(x) \geq a\}$ and $\{x \in [x_0, x_0 + 1] \mid U(x) < a\}$. The estimate thus follows by first taking the maximum over a and b and then the infimum over c . $\square$

3. UNIQUENESS

It seems that uniqueness of traveling waves for discrete monostable dynamics is largely open.

(cf. J. Carr & A. Chmaj, a recent preprint for nonlocal case.)

For this, we assume

(A) f and g are Lipschitz continuous on $[0, 1]$, g is strictly increasing, and $f(0) = f(1) = g(0) = 0 < f(s) \quad \forall s \in (0, 1)$,

(B) f and g are differentiable at 0 and 1, and $f'(1) < 0 < f'(0)$.

In general, we can only establish uniqueness for those solutions which satisfy

$$(7) \quad \exists \lambda < 0 \quad \ni \quad \lim_{x \rightarrow \infty} \frac{U'(x)}{U(x)} = \lambda.$$

Theorem 3. Assume (A) and (B). Then the following holds:

(i) Any solution U of (3) is monotonic, i.e., $U'(\cdot) < 0$ on $\mathbb{R}$.

(ii) Any solution of (3) satisfies, for σ the unique positive root of $c\sigma + g'(1)[e^\sigma + e^{-\sigma} - 2] + f'(1) = 0$,

$$(8) \quad \lim_{x \rightarrow -\infty} \frac{U'(x)}{U(x) - 1} = \sigma.$$

(iii) Let (c, U_1) and (c, U_2) be two solutions of (3) and (7). Then $U_1(\cdot) = U_2(\cdot + \xi)$ on $\mathbb{R}$ for some $\xi \in \mathbb{R}$; i.e., traveling waves are unique up to a translation.

(iv) Suppose $g'(0) > 0$. Then any solution U of (3) satisfies (7) with λ a root of the characteristic equation $c\lambda + g'(0)[e^\lambda + e^{-\lambda} - 2] + f'(0) = 0$. If $c > c_{\min}$, then λ is the larger (less negative) root.

The proof is based on techniques one of the authors used in [Chen(1997)] for dealing with bistable nonlocal equations.

As a consequence of Theorem 3 (iii) and (iv), we have

Theorem 4. Assume (A), (B), and $g'(0) > 0$. Then solutions of (3) are unique up to a translation.

Remark 1. When $g'(0) = 0$, we are unable to show (7). Indeed, we show that $(c, U) = (1, e^{-\epsilon^*})$ is a traveling wave for some (f, g) satisfying (A) and (B). Note that $\lim_{x \rightarrow \infty} |U'(x)|/U(x) = \infty$.

Indeed, we have the following alternatives.

Theorem 5. Assume (A), (B) and $g'(0) = 0$. Let (c, U) be a solution of (3). Then

$$(9) \quad \text{either } \lim_{x \rightarrow \infty} \frac{U'(x)}{U(x)} = -\frac{f'(0)}{c} \quad \text{or} \quad \lim_{x \rightarrow \infty} \frac{U(x)}{U(x-1)} = 0.$$

Key steps for the proof of uniqueness:

- We first study the linearization of (3) near $U = 0$ and 1.

This can be put in a general form

$$(10) \quad a u'(x) + u(x+1) + u(x-1) + b u(x) = 0 \quad \forall x \in \mathbb{R}$$

where a and b are constants. For our application, we shall be interested only in positive solutions.

Equation (10) has exact solutions of the form $u = e^{\lambda x}$ if λ is a root of the *characteristic equation*

$$P(a, b, \lambda) := \lambda a + e^\lambda + e^{-\lambda} + b = 0.$$

As $P(a, b, \cdot)$ is convex, there are at most two real characteristic values (roots).

For a positive solution u , we can define $r = u'/u$. Then r satisfies

$$(11) \quad a r'(x) + e^{\int_x^{x+1} r(y) dy} + e^{\int_x^{x-1} r(y) dy} + b = 0 \quad \forall x \in \mathbb{R}.$$

Assume $a \neq 0$ and let $P(a, b, \lambda) := a\lambda + e^\lambda + e^{-\lambda} + b$.

(i) If $P(a, b, \cdot) = 0$ has no real roots, then (11) has no solution.

(ii) If $P(a, b, \cdot) = 0$ has only one root Λ^* , then (11) has only the trivial solution $r(\cdot) \equiv \Lambda^*$.

(iii) If $P(a, b, \cdot) = 0$ has two real roots $\{\Lambda_1, \Lambda_2\}$ with $\Lambda_1 < \Lambda_2$, then all solutions to (11) are given by

$$r(x) = \frac{\theta \Lambda_1 e^{\Lambda_1 x} + (1 - \theta) \Lambda_2 e^{\Lambda_2 x}}{\theta e^{\Lambda_1 x} + (1 - \theta) e^{\Lambda_2 x}}, \quad \theta \in [0, 1].$$

In particular, all non-constant solutions of (11) are strictly increasing.

- Next, we study the asymptotic behaviors of solution U of (3) as $x \rightarrow \pm\infty$. That is to establish (7) and

(8).

- We have the following *Strong Comparison Principle*:

Let U_1 and U_2 be two solutions of (3) satisfying $U_1 \geq U_2$ on $\mathbb{R}$. Then either $U_1 \equiv U_2$ or $U_1 > U_2$ on $\mathbb{R}$.

- There exists $q_0 \in (0, 1)$ (depending only on c, f, g) such that for any solution (c, U) of (3) and any $q \in (0, q_0]$,

$$D_2[g(U + qU)] + f(U + qU) - (1 + q)\{D_2[g(U)] + f(U)\} < 0$$

on $\{x \mid U(x) > 1 - q_0\}$.

- Suppose (c, U_1) and (c, U_2) solve (3) and there exists a constant $q \in (0, q_0]$ such that $(1 + q)U_1(\cdot + \ell_0 q) \geq U_2(\cdot)$ on $\mathbb{R}$, where $\ell_0 = \ell_0(U_1)$. Then $U_1(\cdot) \geq U_2(\cdot)$ on $\mathbb{R}$.

- Finally, let (c, U_1) and (c, U_2) be two solutions of (3) and (7). Then $\lim_{x \rightarrow \infty} \frac{U_1(x)}{U_2(x)}$ exists. $\square$

4. ASYMPTOTIC STABILITY

We now study the asymptotic stability of traveling waves for

$$(12) \quad \dot{u}_j = [g(u_{j+1}) + g(u_{j-1}) - 2g(u_j)] + f(u_j), \quad j \in \mathbb{Z}.$$

More convenient than (12) is to consider its continuum version

$$(13) \quad u_t(x, t) = D_2[g(u(\cdot, t))](x) + f(u(x, t)) \quad \forall x \in \mathbb{R}, t > 0,$$

where D_2 is a linear operator from $C(\mathbb{R})$ to $C(\mathbb{R})$ defined by

$$D_2[\phi](x) := \phi(x+1) + \phi(x-1) - 2\phi(x).$$

Note that if $u(j, 0) = u_j(0)$ for all $j \in \mathbb{Z}$, then $u_j(t) = u(j, t)$ for all j and t .

We shall assume the following:

(A1) $f, g \in C^{1+\alpha}([0, 1])$ for some $\alpha \in (0, 1]$, g strictly increasing, and $g(0) = f(0) = f(1) = 0 < f(u)$ $\forall u \in (0, 1)$.

(A2) There exists $M_g \in [0, \infty)$ such that

$$|g(u) - g'(0)u| \leq M_g u^{1+\alpha} \quad \forall u \in [0, 1].$$

(A3) There exist constants $M_f^- > 0$ and $M_f^+ \in \mathbb{R}$ such that

$$-M_f^- u^{1+\alpha} \leq f(u) - f'(0)u \leq M_f^+ u^{1+\alpha} \quad \forall u \in [0, 1].$$

(A4) $f'(0) > 0$ and $f'(1) < 0$.

We denote by $\Lambda_1(c)$ the larger (less negative) root of the equation

$$c\lambda + g'(0)[e^\lambda + e^{-\lambda} - 2] + f'(0) = 0.$$

Note that $\Lambda_1(c) \uparrow$ as $c \uparrow$. Here we consider the case when $c > c_{\min}$.

Theorem 6. Assume that f and g satisfy (A1)–(A4). Let u be the solution of (13) with initial value $u(\cdot, 0) = u_o(\cdot)$ satisfying

(U0) $u_o \in C(\mathbb{R} \rightarrow [0, 1])$, $\liminf_{x \rightarrow -\infty} u_o(x) > 0$ and $\lim_{x \rightarrow \infty} u_o(x)e^{-\lambda x} = 1$ for some $\lambda \in (\Lambda_1(c_{\min}), 0)$.

Then

$$(14) \quad \lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \frac{u(x, t)}{U(x - ct)} - 1 \right| = 0$$

where (c, U) is the traveling wave satisfies

$$(15) \quad \lim_{\xi \rightarrow \infty} U(\xi)e^{-\lambda \xi} = 1$$

with $\lambda = \Lambda_1(c)$.

Key steps for the proof of stability:

- We first study the initial value problem

$$(16) \quad \begin{cases} u_t = \mathcal{L}[u] := D_2[g(u)] + f(u) & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_o(x) & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

We have the existence, uniqueness, and (strong) comparison principle for this problem.

- Next, by constructing the sub-super-solution for (3) we can sandwich solutions of (13) accurately for large x .

- The following function is a super/sub solution of (13):

$$u^\pm(x, t) := (1 \pm q)U(x - ct \pm \ell q), \quad q := \epsilon e^{-\eta t}, \quad (x, t) \in \mathbb{R} \times [0, \infty),$$

which sandwich accurately the solution of (13) for large $-x$. □

5. SUB-SUPER-SOLUTION METHOD

Lemma 1. Let f and g satisfy (A1). Then for each fixed $c > 0$, (3) admits a solution U if and only if there exist functions $\phi^+, \phi^- \in C(\mathbb{R} \rightarrow [0, 1])$ with the following properties:

- (i) $\phi^- \not\equiv 0$, $\lim_{\xi \rightarrow \infty} \phi^-(\xi) = 0$, and $-c(\phi^-)' - D_2[g(\phi^-)] - f(\phi^-) \leq 0$ on $\mathbb{R}$;
- (ii) ϕ^+ is non-increasing, $\lim_{\xi \rightarrow \infty} \phi^+(\xi) = 0$, and $-c(\phi^+)' - D_2[g(\phi^+)] - f(\phi^+) \geq 0$ on $\mathbb{R}$;
- (iii) $\phi^- \leq \phi^+$ on $\mathbb{R}$.

Here

$$(\phi^-)'(\xi) := \limsup_{h \rightarrow 0} \frac{\phi^-(\xi + h) - \phi^-(\xi)}{h}; \quad (\phi^+)' \leftrightarrow \liminf$$

We are seeking those TW (c, U) satisfying

$$(17) \quad \lim_{\xi \rightarrow \infty} U(\xi)e^{-\lambda\xi} = 1$$

for some $\lambda < 0$. Since

$$\lim_{\xi \rightarrow \infty} [-cU'(\xi)e^{-\lambda\xi}] = f'(0) + g'(0)[e^\lambda + e^{-\lambda} - 2]$$

and

$$c\lambda U(\xi) + c\lambda \int_{\xi}^{\infty} U'(s)ds = 0,$$

we obtain that

$$(18) \quad 0 = \Phi(c, \lambda) := c\lambda + \{f'(0) + g'(0)[e^\lambda + e^{-\lambda} - 2]\}.$$

Define

$$c^* := \inf_{\lambda < 0} C(\lambda), \quad C(\lambda) := -\{f'(0) + g'(0)[e^\lambda + e^{-\lambda} - 2]\}/\lambda.$$

Then

- $g'(0) = 0 \Rightarrow c^* = 0$, and $\Lambda_1(c) := -f'(0)/c$, $\Lambda_2(c) = -\infty$ are two solutions of (18) for each $c > 0$.
- $g'(0) > 0 \Rightarrow c^* > 0$, and $-\infty < \Lambda_2(c) < \Lambda_1(c) < 0$ are two solutions of (18) for each $c > c^*$.

Lemma 2. Assume (A1)–(A3). Let $c > c^*$ be any number. Then for every $\beta \in (1, \min\{1 + \alpha, \frac{\Lambda_2(c)}{\Lambda_1(c)}\})$, there exists $K_1(c, \beta) \geq 1$ such that for each $k \geq K_1(c, \beta)$,

$$(19) \quad \phi(\xi) := \max\{0, e^{\Lambda_1(c)\xi} - k e^{\beta\Lambda_1(c)\xi}\} \quad \forall \xi \in \mathbb{R}$$

is a sub-solution of (3) with speed c .

To construct super-solutions, we shall use the solution to the ode problem

$$(20) \quad \varphi' = \left(\Lambda_1 - m \min\{1, (k\varphi)^\alpha\} \right) \varphi \quad \text{on } \mathbb{R}, \quad \lim_{\xi \rightarrow \infty} \varphi(\xi)e^{-\Lambda_1\xi} = 1,$$

where $\Lambda_1 = \Lambda_1(c)$, $k \geq 1$ is arbitrary and $m \geq 0$ is some constant. Note that

$$\varphi(\xi) \geq e^{\Lambda_1\xi} \quad \forall \xi \in \mathbb{R}.$$

Lemma 3. Assume (A1)–(A3). There is a constant $\bar{c} \geq c^*$ such that for every $c \geq \bar{c}$ and every $k \geq 1$, the function $\phi := \min\{1, \varphi\}$, where φ is the solution of (20), is a super-solution of (3) with speed c .

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