

Nonlinear Operators and Convergence Theorems in Optimization

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Abstract. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . A mapping $U : C \rightarrow H$ is called extended hybrid if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha(1+\gamma)\|Ux - Uy\|^2 + (1 - \alpha(1+\gamma))\|x - Uy\|^2 \\ & \leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ & \quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned}$$

for all $x, y \in C$. In this article, we first deal with fundamental properties for extended hybrid mappings in a Hilbert space. Then we deal with weak and strong convergence theorems for these nonlinear mappings in a Hilbert space.

1 Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . A mapping $T : C \rightarrow H$ is called generalized hybrid [11] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \quad (1.1)$$

for all $x, y \in C$. We call such a mapping an (α, β) -generalized hybrid mapping. Kocourek, Takahashi and Yao [11] proved a fixed point theorem for such mappings in a Hilbert space. Furthermore, they proved a nonlinear mean convergence theorem of Baillon's type [2] in a Hilbert space. Notice that the class of the mappings above covers several classes of well-known mappings. For example, an (α, β) -generalized hybrid mapping T is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is also nonspreading [12, 13] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Furthermore, it is hybrid [28] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

The classes of nonexpansive mappings, nonspreading mappings and hybrid mappings are deduced from the equilibrium problem in optimization; see [6] and [28]. Putting $x = u$ with $u = Tu$ in (1.1), we have that for any $y \in C$,

$$\alpha\|u - Ty\|^2 + (1 - \alpha)\|u - Ty\|^2 \leq \beta\|u - y\|^2 + (1 - \beta)\|u - y\|^2$$

and hence $\|u - Ty\| \leq \|u - y\|$. This means that an (α, β) -generalized hybrid mapping with a fixed point is quasi-nonexpansive. Recently, Hojo, Takahashi and Yao [8] defined the following class of nonlinear mappings which contains the class of generalized hybrid mappings. A mapping $U : C \rightarrow H$ is called extended hybrid if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \\ & \leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ & \quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned} \quad (1.2)$$

for all $x, y \in C$. We note that an extended hybrid mapping is not quasi-nonexpansive generally.

In this article, we first deal with fundamental properties for extended hybrid mappings in a Hilbert space. Then we deal with weak and strong convergence theorems for these nonlinear mappings in a Hilbert space.

2 Preliminaries

Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. From [27], we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \quad (2.1)$$

Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \quad (2.2)$$

From [18], a Hilbert space H satisfies Opial's condition, i.e., for a sequence $\{x_n\}$ of H such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|. \quad (2.3)$$

Let C be a nonempty closed convex subset of H and let $T : C \rightarrow H$ be a mapping. We denote by $F(T)$ be the set of fixed points of T . A mapping $T : C \rightarrow H$ with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [10]. Since a generalized hybrid mapping T defined in Introduction is quasi-nonexpansive, $F(T)$ is closed and convex.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a Banach

limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞ . If μ is a Banach limit on l^∞ , then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \rightarrow a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [24]. Using Banach limits, Kocourek, Takahashi and Yao [11] proved the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 2.1 ([11]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \rightarrow C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the metric projection of H onto C . It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for all $x \in H$ and $u \in C$; see [27] for more details. We also know the following lemma.

Lemma 2.2 ([30]). *Let F be a nonempty closed convex subset of a Hilbert space H , let P be the metric projection of H onto F and let $\{x_n\}$ be a sequence in H such that $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in F$ and $n \in \mathbb{N}$. Then $\{P x_n\}$ converges strongly.*

3 New Class of Extended Hybrid Mappings

Let H be a real Hilbert space and let C be a nonempty subset of H . A mapping $U : C \rightarrow H$ is called extended hybrid [8] if there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \\ & \leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ & \quad - (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned} \quad (3.1)$$

for all $x, y \in C$ and such a mapping U is called (α, β, γ) -extended hybrid. In [8], the authors derived a relation between the class of generalized hybrid mappings and the class of extended hybrid mappings in a Hilbert space.

Theorem 3.1 ([8]). *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let T and U be mappings of C into H such that $U = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \rightarrow H$ is an (α, β) -generalized hybrid mapping if and only if $U : C \rightarrow H$ is an (α, β, γ) -extended hybrid mapping. In this case, $F(T) = F(U)$.*

A mapping $U : C \rightarrow H$ is called a widely strict pseudo-contraction if there exists a real number $k \in \mathbb{R}$ with $k < 1$ such that

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called a widely k -strict pseudo-contraction. A widely k -strict pseudo-contraction [5] is a strict pseudo-contraction if $0 \leq k < 1$. It is also nonexpansive if $k = 0$. Conversely, if $T : C \rightarrow H$ is a nonexpansive mapping, then for any $n \in \mathbb{N}$,

$$U = \frac{1}{1+n}T + \frac{n}{1+n}I$$

is a widely $(-n)$ -strict pseudo-contraction. The following result is in [32]:

Proposition 3.2 ([32]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let $\alpha > 0$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = 2\alpha U + (1 - 2\alpha)I$. Then, the following are equivalent:*

(a) A is an α -inverse-strongly monotone mapping, i.e.,

$$\alpha \|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay \rangle, \quad \forall x, y \in C;$$

(b) U is a widely $(1 - 2\alpha)$ -strict pseudo-contraction, i.e.,

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + (1 - 2\alpha)\|(I - U)x - (I - U)y\|^2, \quad \forall x, y \in C;$$

(c) U is a $(1, 0, 2\alpha - 1)$ -extended hybrid mapping, i.e.,

$$\begin{aligned} & 2\alpha \|Ux - Uy\|^2 + (1 - 2\alpha)\|x - Uy\|^2 \\ & \leq (2\alpha - 1)\|Ux - y\|^2 + 2(1 - \alpha)\|x - y\|^2 \\ & \quad - (2\alpha - 1)\|x - Ux\|^2 - (2\alpha - 1)\|y - Uy\|^2, \quad \forall x, y \in C; \end{aligned}$$

(d) T is a nonexpansive mapping.

In this case, $Z(A) = F(U) = F(T)$, where $Z(A) = \{u \in C : Au = 0\}$.

Let $\alpha > 0$ and let $A : C \rightarrow H$ be α -inverse-strongly monotone. Then for any $\beta \in \mathbb{R}$ with $0 < \beta \leq 2\alpha$, A is $\frac{\beta}{2}$ -inverse-strongly monotone. Thus

$$T = I - \beta A = I - \beta(I - U) = \beta U + (1 - \beta)I$$

is nonexpansive. Using Proposition 3.2, we can get the following result:

Proposition 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let k be a real number with $k < 1$ and let A, U and T be mappings of C into H such that $U = I - A$ and $T = (1 - k)U + kI$. Then, the following are equivalent:*

(a) A is a $\frac{1-k}{2}$ -inverse-strongly monotone mapping;

(b) U is a widely k -strict pseudo-contraction;

(c) U is a $(1, 0, -k)$ -extended hybrid mapping;

(d) T is a nonexpansive mapping.

In this case, $Z(A) = F(U) = F(T)$.

Let $k < 1$ and let U be a widely k -strict pseudo-contraction. Then for any $t \in \mathbb{R}$ with $k \leq t < 1$, U is a widely t -strict pseudo-contraction. Thus

$$T = (1 - t)U + tI$$

is nonexpansive. We also have the following important result [31] for extended hybrid mappings in a Hilbert space.

Theorem 3.4 ([32]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let α, β, γ be real numbers and let $U : C \rightarrow H$ be an (α, β, γ) -extended hybrid mapping with $1 + \gamma > 0$. Then, $I - U$ is demiclosed, i.e., $x_n \rightarrow z$ and $x_n - Ux_n \rightarrow 0$ imply $z \in F(U)$.*

Using Theorem 3.5, we have the following result for k -strict pseudo-contractions obtained by Marino and Xu [15]; see also [1].

Corollary 3.5 (Marino and Xu [15]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let k be a real number with $0 \leq k < 1$ and $U : C \rightarrow H$ be a k -strict pseudo-contraction. Then, $I - U$ is demiclosed, i.e., $x_n \rightarrow z$ and $x_n - Ux_n \rightarrow 0$ imply $z \in F(U)$.*

4 Weak Convergence Theorems

Motivated by Propositions 3.2 and 3.3, we are interested in weak and strong convergence theorems for extended hybrid mappings in a Hilbert space. In this section, we first state the following weak convergence theorem of Baillon's type [2] by using Lemma 2.2.

Theorem 4.1 ([8]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let α, β and γ be real numbers with $0 \leq -\gamma < 1$. Let $S : C \rightarrow C$ be an (α, β, γ) -extended hybrid mapping with $F(S) \neq \emptyset$ and let P be the metric projection of H onto $F(S)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=1}^n ((1 + \gamma)S - \gamma I)^k x$$

converges weakly to $z \in F(S)$, where $z = \lim_{n \rightarrow \infty} PT^n x$ and $T = (1 + \gamma)S - \gamma I$.

The following weak convergence theorem was proved by Takahashi, Wong and Yao [31].

Theorem 4.2 ([31]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C . Let α, β and γ be real numbers. Let $U : C \rightarrow H$ be an (α, β, γ) -extended hybrid mapping such that $1 + \gamma > 0$ and $F(U) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = P_C \{ \alpha_n x_n + (1 - \alpha_n)((1 + \gamma)Ux_n - \gamma x_n) \}, \quad n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges weakly to an element v of $F(U)$, where $v = \lim_{n \rightarrow \infty} P_{F(U)} x_n$ and $P_{F(U)}$ is the metric projection of H onto $F(U)$.

As direct consequences of Theorem 4.2, we obtain the following results.

Corollary 4.3. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C . Let γ be a real number with $1 + \gamma > 0$ and let $U : C \rightarrow H$ be an $(2, 1, \gamma)$ -extended hybrid mapping, i.e.,*

$$\begin{aligned} & 2(1 + \gamma)\|Ux - Uy\|^2 - (1 + 2\gamma)\|x - Uy\|^2 \\ & \leq (1 + 2\gamma)\|Ux - y\|^2 - 2\gamma\|x - y\|^2 \\ & \quad - \gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \end{aligned}$$

for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C\{\alpha_n x_n + (1 - \alpha_n)((1 + \gamma)Ux_n - \gamma x_n)\}, \quad n \in \mathbb{N}.$$

If $F(U) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element v of $F(U)$, where $v = \lim_{n \rightarrow \infty} P_{F(U)}x_n$ and $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Corollary 4.4. Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C . Let γ be a real number with $1 + \gamma > 0$ and let $U : C \rightarrow H$ be an $(\frac{3}{2}, \frac{1}{2}, \gamma)$ -extended hybrid mapping, i.e.,

$$\begin{aligned} 3(1 + \gamma)\|Ux - Uy\|^2 - (1 + 3\gamma)\|x - Uy\|^2 \\ \leq (1 + 3\gamma)\|Ux - y\|^2 + (1 - 3\gamma)\|x - y\|^2 \\ - 2\gamma\|x - Ux\|^2 - 2\gamma\|y - Uy\|^2 \end{aligned}$$

for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)((1 + \gamma)Ux_n - \gamma x_n)), \quad n \in \mathbb{N}.$$

If $F(U) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element v of $F(U)$, where $v = \lim_{n \rightarrow \infty} P_{F(U)}x_n$ and $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Taking $\gamma = -\frac{1}{2}$ in Corollaries 4.3 and 4.4, we obtain two mappings such that

$$2\|Ux - Uy\|^2 \leq 2\|x - y\|^2 + \|x - Ux\|^2 + \|y - Uy\|^2$$

and

$$\begin{aligned} 3\|Ux - Uy\|^2 + \|x - Uy\|^2 + \|y - Ux\|^2 \\ \leq 5\|x - y\|^2 + 2\|x - Ux\|^2 + 2\|y - Uy\|^2 \end{aligned}$$

for all $x, y \in C$, respectively. We can apply Corollaries 4.3 and 4.4 for such mappings and then obtain weak convergence theorems in a Hilbert space.

5 Strong Convergence Theorems

Using an idea of mean convergence, we can prove the following strong convergence theorem [31] of Halpern's type for extended hybrid mappings in a Hilbert space.

Theorem 5.1 ([31]). Let C be a nonempty closed convex subset of a real Hilbert space H and let α, β and k be real numbers. Let $U : C \rightarrow C$ be an $(\alpha, \beta, -k)$ -extended hybrid mapping such that $0 \leq k < 1$ and $F(U) \neq \emptyset$ and let P be the metric projection of H onto $F(U)$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C$, $u \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + (1 - \alpha_n)z_n, \\ z_n = \frac{1}{n} \sum_{m=1}^n ((1 - k)U + kI)^m x_n \end{cases}$$

for all $n = 1, 2, \dots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to Pu .

Using the hybrid method by Nakajo and Takahashi [17], we can prove the following strong convergence theorem for extended hybrid non-self mappings in a Hilbert space. The method of the proof is due to Nakajo and Takahashi [17] and Marino and Xu [15].

Theorem 5.2 ([31]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let α, β and k be real numbers and let $U : C \rightarrow H$ be an $(\alpha, \beta, -k)$ -extended hybrid mapping such that $k < 1$ and $F(U) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)\{(1 - k)Ux_n + kx_n\}, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)^2 \alpha_n (1 - \alpha_n) \|x_n - Ux_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = R_{F(U)}x$, where $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Using Theorem 5.2, we can prove the following theorem obtained by Marino and Xu [15].

Theorem 5.3 (Marino and Xu [15]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let k be a real number with $0 \leq k < 1$ and let $U : C \rightarrow C$ be a k -strict pseudo contraction such that $F(U) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Ux_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (\beta_n - k)(1 - \beta_n) \|x_n - Ux_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\beta_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = R_{F(U)}x$, where $P_{F(U)}$ is the metric projection of H onto $F(U)$.

Proof. We first know that a $(1, 0, -k)$ -extended hybrid mapping with $0 \leq k < 1$ is a k -strict pseudo contraction. We also have that for any $n \in \mathbb{N}$,

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n)Ux_n \\ &= \frac{\beta_n - k}{1 - k} x_n + (1 - \frac{\beta_n - k}{1 - k})\{(1 - k)Ux_n + kx_n\}. \end{aligned}$$

Putting $\alpha_n = \frac{\beta_n - k}{1 - k}$, we have from $1 > \beta_n$ that $1 - k > \beta_n - k$ and hence $1 > \frac{\beta_n - k}{1 - k} = \alpha_n$. Furthermore, we have that for any $n \in \mathbb{N}$ and $z \in C$,

$$\begin{aligned} \|y_n - z\|^2 &\leq \|x_n - z\|^2 - (\beta_n - k)(1 - \beta_n) \|x_n - Ux_n\|^2 \\ &\iff \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)\alpha_n(1 - k)(1 - \alpha_n) \|x_n - Ux_n\|^2 \\ &\iff \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)^2 \alpha_n(1 - \alpha_n) \|x_n - Ux_n\|^2. \end{aligned}$$

From Theorem 5.2, we have the desired result. $\square$

Next, we prove a strong convergence theorem by the shrinking projection method [29] for extended hybrid non-self mappings in a Hilbert space.

Theorem 5.4 ([31]). Let H be a Hilbert space and let C be a nonempty closed convex subset of H . Let α, β and k be real numbers and let $U : C \rightarrow H$ be an $(\alpha, \beta, -k)$ -extended hybrid mapping such that $k < 1$ and $F(U) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)\{(1 - k)Ux_n + kx_n\}, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)^2 \alpha_n (1 - \alpha_n) \|Ux_n - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} , and $\{\alpha_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(U)} x$, where $P_{F(U)}$ is the metric projection of H onto $F(U)$.

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