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A CONTROL PROBLEM FOR A THERMOELASTIC SYSTEM IN SHAPE MEMORY MATERIALS

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Abstract

The control problem for a two- or three-dimensional model of the nonlinear thermoelastic material is considered. The Fréchet differentiability of the general goal functional with respect to the mechanical and thermal controls is proved. The mathematical description may represent, among others, the shape memory materials.

1 Introduction

The main objective of the paper consists in proving the existence and characterizing the control laws for optimization problems concerning fairly general nonlinear thermoelastic evolution systems. The main representative of such systems describes the behaviour of shape memory materials (SMM) and its study was the primary motivation of this work.

The shape memory materials have a peculiar property that their free energy functions posess, depending on temperature, variable number of stable minima in terms of strain. Above certain temperature there is only one minimum, corresponding to the strain-free state, and below it the minima occur also for several nonzero strains.

Thus, at a temperature below critical, an external force may cause shift of the state from the strain-free configuration to another stable shape, and the subsequent heating causes the appearence of elastic forces striving to restore the initial configuration. This property, known as shape memory effect, is a consequence of structural phase transitions between low-temperature martensitic phases and high-temperature austenitic phase. It is used in many applications, see e.g. [4],[8].

As we see, the choice of control variables is natural, namely the intensity and location of external heat sources and forces. The goal functional should refer to a desired evolution of a structure made of SMM. Therefore it can depend in particular on the variable configuration (displacement) and strain, which in turn is related to the material phases, as well as temperature distribution.

The generality of the problem statement is due to the fact that the system expresses balance laws of linear momentum and energy with constitutive relations characteristic for a broad class of materials. In particular, we admit governing free energy function corresponding to several types of SMM models, like those proposed in [3],[17].

The thermodynamical background of such thermoelastic systems, the existence and uniqueness of solutions as well as their stability with respect to data have been addressed in the previous papers [11],[12],[13],[14]. Here we study the differentiability properties of these solutions with respect to control variables. Furthermore, we prove an existence result for optimal control problem and formulate the neccessary optimality conditions. We note that our analysis of the differentiability properties is based on the technique developed in [13] for the global in time existence.

Similar control problems, but for special kinds of 2–D systems, have been treated in [5],[6],[17].

2 State equations

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be a bounded domain with a smooth boundary $\partial\Omega$ occupied by an elastic body in a reference configuration. Let also $I = (0,T)$, $Q_t = (0,t) \times \Omega$, $\Omega_t = \{t\} \times \Omega$, $S_t = (0,t) \times \partial\Omega$, and $\mathbf{n}$ stands for the unit outward normal to $\partial\Omega$.
Let $\mathbf{u} : Q_T \to \mathbb{R}^n$ be the displacement vector, and $\theta : Q_T \to \mathbb{R}_+$ the absolute temperature.
We denote by $\boldsymbol{\epsilon} = (\epsilon_{ij})$, with $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i/j} + u_{j/i})$, the linearized strain tensor, and by $\boldsymbol{\epsilon}_t = \boldsymbol{\epsilon}(\mathbf{u}_t)$ the strain rate tensor.
Throughout the paper we use the notation $f_{/i} = \partial f/\partial x_i$, $f_t = \partial f/\partial t$.

The state equations to be considered express balances of linear momentum and energy which, under simplifying assumption of constant material density $\rho \equiv 1$, are given by

$$\mathbf{u}_{tt} - \nu \mathbf{Q}\mathbf{u}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{u} = \nabla \cdot F_{/\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta) + \mathbf{b}, \tag{2.1}$$

$$c(\boldsymbol{\epsilon}, \theta)\theta_t - k\Delta\theta = \theta F_{/\theta\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \theta) : \boldsymbol{\epsilon}_t + \nu(\mathbf{A}\boldsymbol{\epsilon}_t) : \boldsymbol{\epsilon}_t + g \quad \text{in} \quad Q_T, \tag{2.2}$$

with initial

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}_t(0,\mathbf{x}) = \mathbf{u}_1(\mathbf{x}), \tag{2.3}$$

$$\theta(0,\mathbf{x}) = \theta_0(\mathbf{x}) \quad \text{in} \quad \Omega, \tag{2.4}$$

and boundary conditions

$$\mathbf{u} = 0, \quad \mathbf{Q}\mathbf{u} = 0, \tag{2.5}$$

$$\nabla\theta \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T, \tag{2.6}$$

where

$$c(\boldsymbol{\epsilon}, \theta) = c_v - \theta F_{/\theta\theta}(\boldsymbol{\epsilon}, \theta). \tag{2.7}$$

We shall refer to (2.1)–(2.7) as problem (P).
The quantities in (P) have the following meaning: $F(\epsilon,\theta)$ – elastic energy, $c(\epsilon,\theta)$ – specific heat coefficient: The positive constants c_v, k, ν and κ correspond to thermal specific heat, heat conductivity, viscosity and interface energy.
The vector $\mathbf{b}$ is a distributed external force and g is a distributed heat source which represent possible mechanical and thermal controls.
The linear map

$$\mathbf{u} \mapsto \mathbf{A}\epsilon(\mathbf{u}) = \lambda \operatorname{trace} \epsilon(\mathbf{u})\,\mathbf{I} + 2\mu\epsilon(\mathbf{u}), \tag{2.8}$$

where $\lambda, \mu > 0$ are Lamé constants and $\mathbf{I} = (\delta_{ij})$ is the unit matrix, represents Hooke's law for the homogeneous isotropic material. Here $\mathbf{A} = (A_{ijkl})$ with

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

is the fourth order elasticity tensor.
The second order differential operator $\mathbf{Q}$ defined by

$$\mathbf{u} \mapsto \mathbf{Q}\mathbf{u} = \nabla \cdot (\mathbf{A}\epsilon(\mathbf{u})), \tag{2.9}$$

is known as operator of linearized elasticity. By (2.8) it admits the representation

$$\mathbf{Q}\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}). \tag{2.10}$$

In the divergence $\nabla\cdot$ we use the convention of the contraction over the last index, i.e.

$$\nabla\cdot(\mathbf{A}\epsilon(\mathbf{u})) = \partial_j(\,A_{ijkl}\epsilon_{kl}(\mathbf{u})\,) = A_{ijkl}\partial_j\epsilon_{kl}(\mathbf{u}) = \mathbf{A}\nabla\epsilon(\mathbf{u}).$$

Moreover, the summation convention is used, and the following notation: for vectors $\mathbf{a} = (a_i)$, $\mathbf{b} = (b_i)$ and tensors $\mathbf{B} = (B_{ij})$, $\mathbf{C} = (C_{ij})$, $\mathbf{A} = (A_{ijkl})$ we write $\mathbf{a}\cdot\mathbf{b} = a_ib_i$, $\mathbf{B}:\mathbf{C} = B_{ij}C_{ij}$, $\mathbf{aB} = a_iB_{ij}$, $\mathbf{Ba} = B_{ij}a_j$, $\mathbf{BA} = B_{ij}A_{ijkl}$, etc.
To problem (P) corresponds the free energy functional of the Ginzburg–Landau form

$$f(\epsilon,\nabla\epsilon,\theta) = -c_v\theta\log\theta + F(\epsilon(\mathbf{u}),\theta) + \frac{\kappa}{8}\mid\mathbf{Q}\mathbf{u}\mid^2 \tag{2.11}$$

with the subsequent terms representing thermal energy, elastic energy and interfacial energy.
The main characteristic feature of (2.11) as a model of shape memory materials is the nonlinearity in ϵ: $F(\epsilon,\theta)$ is a multiple–well in ϵ with the shape changing qualitatively with θ. The second characteristic feature is the presence of higher order term with coefficient κ which accounts for interaction effects on phase interfaces. Terms of this type are known in the so called multiscale approach to modelling of phase transitions. The particular form of κ–term in (2.11) can be interpreted as a resultant of mechanical forces acting on a layer element of interface.

A typical example of the elastic energy is the Falk–Konopka model [3] in the form of sixth order polynomial in terms of ϵ_{ij}:

$$F(\epsilon,\theta) = \sum_{i=1}^{3} F_i^2(\theta)J_i^2(\epsilon) + \sum_{i=1}^{5} F_i^4(\theta)J_i^4(\epsilon) + \sum_{i=1}^{2} F_i^6(\theta)J_i^6(\epsilon), \tag{2.12}$$

where $J_i^k(\epsilon)$, $i = 1, \dots, i^k$, are k-th order crystallographical invariants, that is appropriate combinations of the strain tensor components ϵ_{ij}, and $F_i^k(\theta)$ are corresponding temperature-dependent coefficients.

The form (2.12) represents a generalization of the well known 1–D Landau–Devonshire energy proposed for shape memory alloys by Falk [2],

$$F(\epsilon,\theta) = \alpha_1(\theta - \theta_c)\epsilon^2 - \alpha_2\epsilon^4 + \alpha_3\epsilon^6, \tag{2.13}$$

where $\alpha_i > 0$ are constant parameters, and $\theta_c > 0$ is a critical temperature.
Our formulation (2.1)–(2.7) constitutes an analog of the 1–D dynamical Falk's model [2].

The problem (P) is studied under several conditions concerning data and constitutive functions. We assume that

(D) the boundary $\partial\Omega$ is of class C^2.

Further assumptions concern the elastic energy:

(FE–1) Structure: $F(\epsilon,\theta)$ is of class C^3 on $S^2 \times [0,\infty)$, where S^2 denotes the set of symmetric tensors of second order in $\mathbb{R}^n$. We assume the splitting

$$F(\epsilon,\theta) = F_1(\epsilon,\theta) + F_2(\epsilon),$$

where $F_1(\epsilon,\theta)$ is linear in θ over certain interval $[0,\theta_1)$ and satisfies (FE–2) for large values of θ.

(FE–2) Growth conditions: Let ϵ_1 and θ_1 be certain constants. There exists a constant Λ such that for $|\epsilon| \geq \epsilon_1$ and $\theta \geq \theta_1$ the following conditions are satisfied:

$$|F_{1/\epsilon\epsilon}(\epsilon,\theta)| \leq \Lambda + \Lambda\theta^r|\epsilon|^{q-1}, \qquad |F_{2/\epsilon\epsilon}(\epsilon)| \leq \Lambda + \Lambda|\epsilon|^{\bar{q}-1},$$
$$|F_{1/\epsilon\theta}(\epsilon,\theta)| \leq \Lambda + \Lambda\theta^{r-1}|\epsilon|^{q}, \qquad |F_{1/\theta\theta}(\epsilon,\theta)| \leq \Lambda + \Lambda\theta^{r-2}|\epsilon|^{q+1},$$

where

$$0 < r < \frac{2}{p_n}, \qquad 1 < q \leq q_n\left(\frac{1}{p_n} - \frac{r}{2}\right), \qquad 1 < \bar{q} \leq \frac{q_n}{p_n},$$

$p_n = n+2$, and q_n is the Sobolev exponent for which the imbedding of $W_2^1(\Omega)$ into $L_{q_n}(\Omega)$ is continuous, that is $q_n = 2n/(n-2)$ for $n \geq 3$ and q_n is any finite number for $n = 2$.

We note that the above conditions imply the following growth of $F(\epsilon,\theta)$:

$$|F_1(\epsilon,\theta)| \leq \Lambda + \Lambda\theta^r|\epsilon|^{q+1}, \qquad |F_2(\epsilon)| \leq \Lambda + \Lambda|\epsilon|^{\bar{q}+1}.$$

(FE–3) Concavity with respect to θ (thermal stability):

$$F_{1/\theta\theta}(\epsilon,\theta) \leq 0 \quad \text{for} \quad (\epsilon,\theta) \in S^2 \times [0,\infty).$$

This implies the lower bound for the specific heat coefficient

$$0 < c_v \leq c(\epsilon,\theta) \quad \text{for} \quad (\epsilon,\theta) \in S^2 \times [0,\infty).$$

(FE–4) Lower bound for the internal energy:

$$-\Lambda \le (F_1(\epsilon,\theta) - \theta F_{1/\theta}(\epsilon,\theta)) + F_2(\epsilon) \quad \text{for} \quad (\epsilon,\theta) \in S^2 \times [0,\infty).$$

The most restrictive is the assumption on θ–growth exponent $r < 1/2$ and the assumption on ϵ–growth exponent $\bar{q} \le 6/5$ in 3–D case.
In 2–D case the latter is not active, since q and $\bar{q}$ are then any large numbers. Hence our assumptions admit the form of sixth order polynomial (2.12) only in 2–D case. In 3–D case they require the growth with respect to ϵ close to quadratic. The temperature dependence is restricted to quadratic terms $F_i^2(\theta)$ (as in 1–D model (2.13)).
The growth condition on r is needed both in 2– and 3–D case.
We are looking for the solution in the anisotropic Sobolev space

$$V(p) = \{\, (\mathbf{u},\theta) \in \mathbf{W}_p^{4,2}(Q_T) \times W_p^{2,1}(Q_T) \,\},$$

with a parameter p related to the L_p–integrability. The assumptions on the initial data and the source terms correspond to this space.

(BV–p) Let $\delta > 0$, $p > 1$, $p_1 = p + \delta$. The initial conditions satisfy the inclusions

$$\mathbf{u}_0 \in \mathbf{W}_p^{4-2/p}(\Omega), \quad \mathbf{u}_1 \in \mathbf{W}_p^{2-2/p}(\Omega),$$

$$0 \le \theta_0 \in W_{p_1}^{2-2/p_1}(\Omega),$$

and the compatibility relations. The source terms satisfy

$$\mathbf{b} \in \mathbf{L}_p(Q_T), \quad g \in L_{p_1}(Q_T), \quad g \ge 0 \ \text{a.e.}$$

Further on Λ denotes a generic constant, depending in general on the data of the problem, domain Ω and the time horizon T.
In [13] there has been proved the existence result:

Theorem 2.1 *Existence.*
Under assumptions (D), (FE–1) – (FE–4), (BV–p) and $0 < \sqrt{\kappa} < \nu$ *there exists for*

$$p \ge p_n$$

a solution $(\mathbf{u},\theta) \in V(p)$ *to problem (P) for any* $T > 0$. *Moreover, the following a priori estimates hold,*

$$\| \mathbf{u} \|_{\mathbf{W}_p^{4,2}(Q_T)} \le \Lambda, \quad \| \theta \|_{W_p^{2,1}(Q_T)} \le \Lambda, \tag{2.14}$$

with a constant Λ *dependent on the data of the problem,* Ω *and time* T.

The condition between κ and ν is needed for parabolic decomposition of elasticity equation (2.1).
This theorem has several consequences concerning regularity of the solution:

Corollary 2.1 *For a solution to problem (P) the following holds: the functions* $\mathbf{u}$, $\nabla\mathbf{u}$, $\nabla^2\mathbf{u}$, $\mathbf{u}_t$, θ *are continuous in* Q_T, *and*

$$|\mathbf{u}|,\ |\nabla\mathbf{u}|,\ |\nabla^2\mathbf{u}|,\ |\mathbf{u}_t| \le \Lambda, \qquad 0 \le \theta \le \Lambda \quad \text{in } Q_T,$$

$$\| \nabla^3\mathbf{u} \|_{\mathbf{L}_p(Q_T)},\ \| \nabla\mathbf{u}_t \|_{\mathbf{L}_p(Q_T)},\ \| \nabla\theta \|_{\mathbf{L}_p(Q_T)} \le \Lambda \qquad \text{for } p_n \le p < \infty,$$

$$c_v \le c(\epsilon,\theta) \le c_{max} = c_{max}(\Lambda).$$

The proof of the solution uniqueness requires an additional regularity which holds provided $p > p_n$. Moreover, stronger assumptions on $F(\epsilon, \theta)$ and g have to be imposed.

(FE–5) The function $F_1(\epsilon, \theta)$ is of class C^4 on the set $S^2 \times [0, \infty)$, and the heat source satisfies

$$g \in L_\infty(Q_T) \quad \text{and} \quad g \geq 0 \ \ a.e.$$

The uniqueness result proved in [13] states:

Theorem 2.2 *Uniqueness.*
Let the assumptions of Theorem 2.1 and (FE–5) be satisfied, and $p > p_n$. Then the solution to the problem (P) is unique for any $T > 0$.

Throughout the rest of the paper we postulate that the assumptions required for the uniqueness result are satisfied. Then the solution has an additional continuity property.

Corollary 2.2 *For a solution to problem (P) the following holds in case $p > p_n$:*
$\nabla^3\mathbf{u}, \nabla\mathbf{u}_t, \nabla\theta$ *are continuous in Q_T and satisfy the bounds*

$$|\nabla^3\mathbf{u}|,\ |\nabla\mathbf{u}_t|,\ |\nabla\theta| \ \leq\ \Lambda.$$

In [14] we have proved also the stability of solutions $(\mathbf{u}, \theta)$ of problem (P) with respect to control parameters $(\mathbf{b}, g)$. Let $(\mathbf{u}^1, \theta^1)$ and $(\mathbf{u}^2, \theta^2)$ be the solutions corresponding to $(\mathbf{b}^1, g^1)$ and $(\mathbf{b}^2, g^2)$, respectively. We have the following

Theorem 2.3 *Stability.*
Under the assumptions of Theorem 2.2 the solutions (u^i, θ^i) corresponding to the right–hand sides $(\mathbf{b}^i, g^i)$, $i = 1, 2$, satisfy the inequality

$$||(\mathbf{u}^2 - \mathbf{u}^1, \theta^2 - \theta^1)||_{V(p)} \leq \Lambda(||\mathbf{b}^2 - \mathbf{b}^1||_{\mathbf{L}_p(Q_T)} + ||g^2 - g^1||_{L_p(Q_T)}) \tag{2.15}$$

for any finite $p > p_n$ and $T > 0$, where Λ is a constant dependent on the data of the problem, Ω and time T.

Both the existence and stability proofs are based on the parabolic decomposition (see [17]) of the problem (P). The same decomposition is used here for the proof of the differentiability result. Chosing numbers α, β so that

$$\alpha + \beta = \nu, \qquad \alpha\beta = \frac{\kappa}{4}, \tag{2.16}$$

the system (2.1) with initial conditions (2.3) and boundary conditions (2.5) is equivalent to the following two sets of BVP's for a vector field $\mathbf{w}$:

$$\begin{aligned} \mathbf{w}_t - \beta\mathbf{Q}\mathbf{w} &= \nabla\cdot F_{/\epsilon}(\epsilon, \theta) + \mathbf{b}, \quad \text{in } Q_T, \\ \mathbf{w}(0, \mathbf{x}) &= \mathbf{u}_1(\mathbf{x}) - \alpha\mathbf{Q}\mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \\ \mathbf{w} &= 0 \quad \text{on } S_T, \end{aligned} \tag{2.17}$$

and the displacement $\mathbf{u}$:

$$\begin{aligned} \mathbf{u}_t - \alpha\mathbf{Q}\mathbf{u} &= \mathbf{w}, \quad \text{in } Q_T, \\ \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \\ \mathbf{u} &= 0 \quad \text{on } S_T, \end{aligned} \tag{2.18}$$

The condition between parameters κ and ν, required by Theorem 2.1, assures that $\Re\alpha, \Re\beta > 0$.

3 Differentiability

Let us consider two control pairs $(\mathbf{b}^i, g^i) \in \mathbf{L}_p(Q_T) \times L_{p_1}(Q_T)$, $g^i \geq 0$ a.e. in Q_T, $i = 1, 2$, such that

$$\mathbf{b}^2 = \mathbf{b}^1 + \tau\boldsymbol{\phi}, \qquad g^2 = g^1 + \tau\psi. \tag{3.1}$$

We assume that

$$g^i \leq g_{max}, \quad 0 \leq \tau \leq \tau_0, \tag{3.2}$$

where g_{max}, τ_0 are given constants.

Let $(\mathbf{u}^i, \theta^i) \in V(p)$, $p > p_n$, be unique solutions of problem (P) corresponding to $(\mathbf{b}^i, g^i)$. According to Theorem 2.3, we have the following stability estimate

$$\|(\mathbf{u}^2 - \mathbf{u}^1, \theta^2 - \theta^1)\|_{V(p)} \leq \Lambda(\|\tau\boldsymbol{\phi}\|_{\mathbf{L}_p(Q_T)} + \|\tau\psi\|_{L_p(Q_T)}) \leq \Lambda\tau \tag{3.3}$$

for $p > p_n$. Consequently, by the imbedding theorem, similar bound holds pointwise in Q_T for the differences $\mathbf{u}^2 - \mathbf{u}^1$, $\theta^2 - \theta^1$, $\boldsymbol{\nabla}(\mathbf{u}_t^2 - \mathbf{u}_t^1)$, $\boldsymbol{\nabla}^i(\mathbf{u}^2 - \mathbf{u}^1)$, $i = 1, 2, 3$, and $\boldsymbol{\nabla}(\theta^2 - \theta^1)$. Our goal is to find a pair $(\mathbf{v}, \eta) \in V(p)$ such that

$$\mathbf{u}^2 = \mathbf{u}^1 + \tau\mathbf{v} + o(\tau), \qquad \theta^2 = \theta^1 + \tau\eta + o(\tau)$$

in the sense of the space $V(p)$.
Let us rewrite problem (P) in the following form:

$$\mathbf{u}_{tt} - \nu\mathbf{Q}\mathbf{u}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{u} = \boldsymbol{\nabla} \cdot F_{/\epsilon}(\epsilon, \theta) + \mathbf{b}, \tag{3.4}$$

$$\theta_t - k\gamma(\epsilon, \theta)\Delta\theta = G(\epsilon, \epsilon_t, \theta) + \gamma(\epsilon, \theta)g \quad \text{in} \quad Q_T, \tag{3.5}$$

with boundary and initial conditions

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{u}_t(0, \mathbf{x}) = \mathbf{u}_1(\mathbf{x}), \quad \theta(0, \mathbf{x}) = \theta_0(\mathbf{x}) \quad \text{in} \quad \Omega, \tag{3.6}$$

$$\mathbf{u} = \mathbf{Q}\mathbf{u} = 0, \quad \boldsymbol{\nabla}\theta \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T, \tag{3.7}$$

where

$$\gamma(\epsilon, \theta) = \frac{1}{c(\epsilon, \theta)}, \qquad G(\epsilon, \epsilon_t, \theta) = \gamma(\epsilon, \theta)[\theta F_{/\theta\epsilon}(\epsilon, \theta) : \epsilon_t + \nu(\mathbf{A}\epsilon_t) : \epsilon_t].$$

Using formal approximation by Taylor series we obtain the following system of equations for the pair $(\mathbf{v}, \eta)$:

$$\mathbf{v}_{tt} - \nu\mathbf{Q}\mathbf{v}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{v} = \boldsymbol{\nabla} \cdot (F^1_{/\epsilon\epsilon}\epsilon(\mathbf{v}) + F^1_{/\epsilon\theta}\eta) + \boldsymbol{\phi}, \tag{3.8}$$

$$\eta_t - k\gamma^1\Delta\eta = \mathbf{H}_1 : \epsilon(\mathbf{v}) + \mathbf{H}_2 : \epsilon(\mathbf{v}_t) + H_3\eta + \gamma^1\psi \quad \text{in} \quad Q_T, \tag{3.9}$$

with initial and boundary conditions

$$\mathbf{v}(0, \mathbf{x}) = 0, \quad \mathbf{v}_t(0, \mathbf{x}) = 0, \quad \eta(0, \mathbf{x}) = 0 \quad \text{in} \quad \Omega, \tag{3.10}$$

$$\mathbf{v} = \mathbf{Q}\mathbf{v} = 0, \quad \boldsymbol{\nabla}\eta \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T, \tag{3.11}$$

$$\mathbf{H}_1 = G^1_{/\epsilon} + k\gamma^1_{/\epsilon}\Delta\theta^1 + g^1\gamma^1_{/\epsilon}, \quad \mathbf{H}_2 = G^1_{/\epsilon_t}, \quad H_3 = G^1_{/\theta} + k\gamma^1_{/\theta}\Delta\theta^1 + g^1\gamma^1_{/\theta}.$$

The superscript $(\cdot)^i$ means that the quantity is evaluated at $(\mathbf{u}^i, \theta^i)$, for $i = 1, 2$. We note that due to the regularity properties of the solutions $(\mathbf{u}^i, \theta^i)$ there holds: $\mathbf{H}_1 \in \mathbf{L}_{p_1}(Q_T)$, $H_3 \in L_{p_1}(Q_T)$, and $\mathbf{H}_2$ is continuous in Q_T. By similar arguments as in Theorem 2.3, we can claim that there exists the unique solution $(\mathbf{v}, \eta) \in V(p)$ to the problem (3.8)–(3.11) for any $T > 0$.
We shall prove here the following differentiability result:

Theorem 3.1 *Let the assumptions of Theorem 2.2 hold and the data $(\mathbf{b}^i, g^i)$ satisfy (3.1), (3.2). Then the solutions $(\mathbf{u}^i, \theta^i)$ of (3.4)–(3.7) and $(\mathbf{v}, \eta)$ of (3.8)–(3.11) fulfil the following relation*

$$\|(\mathbf{u}^2 - \mathbf{u}^1 - \tau\mathbf{v}, \theta^2 - \theta^1 - \tau\eta)\|_{V(p)} \le \Lambda\tau^2 \tag{3.12}$$

for any $p > p_n$, where Λ is a constant dependent on the data of the problem (in particular L_∞-norm of g), Ω and time T. Hence

$$\lim_{\tau\to 0+} \frac{1}{\tau} \|(\mathbf{u}^2 - \mathbf{u}^1 - \tau\mathbf{v}, \theta^2 - \theta^1 - \tau\eta)\|_{V(p)} = 0, \tag{3.13}$$

what means that the pair $(\mathbf{v}, \eta)$ constitutes a Gateaux derivative of the solution with respect to the parameters $(\mathbf{b}, g)$. In fact, this convergence is uniform with respect to the norms of ϕ, ψ, that is $(\mathbf{v}, \eta)$ defines a Fréchet derivative.

Proof. Let us define functions

$$\mathbf{z} = \mathbf{u}^2 - \mathbf{u}^1 - \tau\mathbf{v}, \qquad \varphi = \theta^2 - \theta^1 - \tau\eta. \tag{3.14}$$

Due to their construction, they satisfy the following BVP:

$$\mathbf{z}_{tt} - \nu\mathbf{Q}\mathbf{z}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{z} = \boldsymbol{\nabla}\cdot(F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi + F^{1,2}_{/\epsilon}) \quad \text{in} \quad Q_T, \tag{3.15}$$

$$\begin{aligned}\varphi_t - k\gamma^1\Delta\varphi &= (G^1_{/\epsilon} + g^1\gamma^1_{/\epsilon}) : \epsilon(\mathbf{z}) + G^1_{/\epsilon_t} : \epsilon(\mathbf{z}_t) + (G^1_{/\theta} + g^1\gamma^1_{/\theta})\varphi \\ &\quad + G^{1,2} + g^1\gamma^{1,2} + \tau\psi(\gamma^2 - \gamma^1) \\ &\quad + k(\gamma^1_{/\epsilon} : \epsilon(\mathbf{z}) + \gamma^1_{/\theta}\varphi)\Delta\theta^1 \\ &\quad + k\gamma^{1,2}\Delta\theta^1 \\ &\quad + k(\gamma^2 - \gamma^1)\Delta(\theta^2 - \theta^1) \\ &=: R_1 + R_2 + R_3 + R_4 + R_5 \quad \text{in} \quad Q_T,\end{aligned} \tag{3.16}$$

with conditions

$$\mathbf{z}(0, \mathbf{x}) = 0, \quad \mathbf{z}_t(0, \mathbf{x}) = 0, \quad \varphi(0, \mathbf{x}) = 0 \quad \text{in} \quad \Omega, \tag{3.17}$$

$$\mathbf{z} = \mathbf{Q}\mathbf{z} = 0, \quad \boldsymbol{\nabla}\varphi\cdot\mathbf{n} = 0 \quad \text{on} \quad S_T, \tag{3.18}$$

where

$$\begin{aligned}F^{1,2}_{/\epsilon} &= F^2_{/\epsilon} - F^1_{/\epsilon} - F^1_{/\epsilon\epsilon}(\epsilon^2 - \epsilon^1) - F^1_{/\epsilon\theta}(\theta^2 - \theta^1), \\ G^{1,2} &= G^2 - G^1 - G^1_{/\epsilon} : (\epsilon^2 - \epsilon^1) - G^1_{/\epsilon_t} : (\epsilon^2_t - \epsilon^1_t) - G^1_{/\theta}(\theta^2 - \theta^1), \\ \gamma^{1,2} &= \gamma^2 - \gamma^1 - \gamma^1_{/\epsilon} : (\epsilon^2 - \epsilon^1) - \gamma^1_{/\theta}(\theta^2 - \theta^1).\end{aligned}$$

In view of the known regularity of solutions $(\mathbf{u}^i, \theta^i)$, there exists the unique solution $(\mathbf{z}, \varphi) \in V(p)$ to the problem (3.15)–(3.18) for any $p > p_n$.
We shall show that

$$\|(\mathbf{z}, \varphi)\|_{V(p)} \le \Lambda \tau^2. \tag{3.19}$$

The assumptions concerning the function $F(\epsilon, \theta)$ and the regularity of solutions $(\mathbf{u}^i, \theta^i) \in V(p)$ allow us to obtain immediately the following bounds:

$$|F^{1,2}_{/\epsilon}|,\ |\gamma^{1,2}| \le \Lambda(|\epsilon^2 - \epsilon^1|^2 + |\theta^2 - \theta^1|^2), \tag{3.20}$$

$$|G^{1,2}| \le \Lambda(|\epsilon^2 - \epsilon^1|^2 + |\epsilon^2_t - \epsilon^1_t|^2 + |\theta^2 - \theta^1|^2), \tag{3.21}$$

$$|\gamma^2 - \gamma^1| \le \Lambda(|\epsilon^2 - \epsilon^1| + |\theta^2 - \theta^1|). \tag{3.22}$$

The reasoning will follow closely the arguments of Theorem 2.3 in [14]. We start from energy estimates for $\mathbf{z}$. Multiplying the equation (3.15) by $\mathbf{z}_t$ and integrating over Q_t yields

$$\int_{Q_t} \left(\frac{1}{2}\frac{d}{dt}|\mathbf{z}_t|^2 + \frac{\kappa}{8}\frac{d}{dt}|\mathbf{Q}\mathbf{z}|^2 \right) dx dt' + \nu \int_{Q_t} (\mathbf{A}\epsilon(\mathbf{z}_t)) : \epsilon(\mathbf{z}_t)\, dx dt'$$
$$= - \int_{Q_t} (F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi) : \epsilon(\mathbf{z}_t)\, dx dt' - \int_{Q_t} F^{1,2}_{/\epsilon} : \epsilon(\mathbf{z}_t)\, dx dt'. \tag{3.23}$$

From this, after estimating the right–hand side and applying Gronwall's inequality together with estimate (3.20) on $F^{1,2}_{/\epsilon}$, we get

$$\|\mathbf{z}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\epsilon(\mathbf{z})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{Q}\mathbf{z}\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\epsilon(\mathbf{z}_t)\|_{\mathbf{L}_2(Q_T)} \le$$
$$\le \Lambda\left(\|\varphi\|_{L_2(Q_T)} + \tau^2\right), \tag{3.24}$$

where in the last inequality we have applied stability estimate (3.3). Hence, by the elipticity property of Q,

$$\|\mathbf{z}\|_{L_\infty(0,T;\mathbf{W}^3_2(\Omega))} \le \Lambda\left(\|\varphi\|_{L_2(Q_T)} + \tau^2\right). \tag{3.25}$$

In order to obtain energy estimates for φ we multiply equation (3.16) by φ and integrate over Q_t to get

$$\frac{1}{2}\int_{\Omega_t} \varphi^2\, dx + \int_{Q_t} k\gamma^1 |\nabla\varphi|^2\, dx dt' = - \int_{Q_t} k\varphi(\nabla\varphi \cdot \nabla\gamma^1)\, dx dt' + \sum_{i=1}^{5} \int_{Q_t} R_i\varphi\, dx dt'. \tag{3.26}$$

For the first term on the right–hand side we have, due to continuity of $\nabla\gamma^1$,

$$\left| \int_{Q_t} k\varphi(\nabla\varphi \cdot \nabla\gamma^1)\, dx dt' \right| \le \Lambda\delta_1 \int_{Q_t} |\nabla\varphi|^2\, dx dt' + \Lambda\delta_1^{-1} \int_{Q_t} \varphi^2\, dx dt'. \tag{3.27}$$

For terms containing R_i we get, using (3.20)–(3.22), (3.24), the following inequalities:

$$\left|\int_{Q_t} R_1\varphi\, dx dt'\right|,\ \left|\int_{Q_t} R_2\varphi\, dx dt'\right| \le \Lambda \int_{Q_t} (\varphi^2 + \tau^4)\, dx dt',$$

$$\left|\int_{Q_t} R_3\varphi\, dx dt'\right| \le \Lambda\delta_2 \int_{Q_t} |\nabla\varphi|^2\, dx dt' + \Lambda(1 + \delta_2^{-1}) \int_{Q_t} (\varphi^2 + \tau^4)\, dx dt',$$

$$\left|\int_{Q_t} R_4\varphi\, dx dt'\right| \le \Lambda\delta_3 \int_{Q_t} |\nabla\varphi|^2\, dx dt' + \Lambda(1 + \delta_3^{-1}) \int_{Q_t} (\varphi^2 + \tau^4)\, dx dt',$$

$$\left|\int_{Q_t} R_5\varphi\, dx dt'\right| \le \Lambda\delta_4 \int_{Q_t} |\nabla\varphi|^2\, dx dt' + \Lambda(1 + \delta_4^{-1}) \int_{Q_t} (\varphi^2 + \tau^4)\, dx dt'.$$

In the process we have used the bounds for gradients of γ and the stability estimate. As a result, after suitable choice of δ_i, we get from (3.26)

$$\int_{\Omega_t} \varphi^2\,dx + \int_{Q_t} |\nabla\varphi|^2\,dxdt' \le \Lambda \int_{Q_t} (\varphi^2 + \tau^4)\,dxdt', \tag{3.28}$$

and, applying Gronwall's inequality,

$$\|\varphi\|_{L_\infty(0,T;L_2(\Omega))} + \|\nabla\varphi\|_{\mathbf{L}_2(Q_T)} \le \Lambda\tau^2. \tag{3.29}$$

Substituting (3.29) into (3.24) yields

$$\|\mathbf{z}_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\epsilon(\mathbf{z})\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\mathbf{Qz}\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\epsilon(\mathbf{z}_t)\|_{\mathbf{L}_2(Q_T)} \le \Lambda\tau^2. \tag{3.30}$$

Hence, the classical imbeddings and parabolic estimates [1] imply the following bounds:

$$\|\varphi\|_{L_{2p_n/n}(Q_T)} + \|\epsilon(\mathbf{z})\|_{L_\infty(0,T;\mathbf{L}_{q_n}(\Omega))} \le \Lambda\tau^2. \tag{3.31}$$

In order to obtain still more refined estimates we employ the parabolic decomposition of the system (3.15) into BVP's:

$$\begin{aligned} \mathbf{w}_t - \beta\mathbf{Qw} &= \nabla\cdot(F^1_{/\epsilon\epsilon} : \epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi + F^{1,2}_{/\epsilon}) \quad \text{in} \quad Q_T,\\ \mathbf{w}(0,\mathbf{x}) &= 0 \quad \text{in} \quad \Omega,\\ \mathbf{w} &= 0 \quad \text{on} \quad S_T, \end{aligned} \tag{3.32}$$

$$\begin{aligned} \mathbf{z}_t - \alpha\mathbf{Qz} &= \mathbf{w} \quad \text{in} \quad Q_T,\\ \mathbf{z}(0,\mathbf{x}) &= 0 \quad \text{in} \quad \Omega,\\ \mathbf{z} &= 0 \quad \text{on} \quad S_T. \end{aligned} \tag{3.33}$$

Using (3.20) and the stability estimate we get

$$\int_{Q_t} |F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi + F^{1,2}_{/\epsilon}|^p\,dxdt' \le \Lambda(\|\epsilon(\mathbf{z})\|^p_{\mathbf{L}_p(Q_T)} + \|\varphi\|^p_{L_p(Q_T)} + \tau^{2p}). \tag{3.34}$$

Therefore, thanks to the regularity theory of parabolic systems (see [13]),

$$\|\nabla\mathbf{z}\|_{\mathbf{W}^{2,1}_p(Q_T)} + \|\nabla\mathbf{w}\|_{\mathbf{L}_p(Q_T)} \le \Lambda(\|\epsilon(\mathbf{z})\|_{\mathbf{L}_p(Q_T)} + \|\varphi\|_{L_p(Q_T)} + \tau^2). \tag{3.35}$$

Consequently, because of (3.31),

$$\|\epsilon(\mathbf{z})\|_{\mathbf{W}^{2,1}_p(Q_T)} \le \Lambda\tau^2 \qquad \text{for} \;\; p \le \frac{2p_n}{n}. \tag{3.36}$$

As a result, since $p_n/2 \le 2p_n/n$,

$$\|\nabla\epsilon(\mathbf{z})\|_{\mathbf{L}_{p_n}(Q_T)},\ \|\epsilon(\mathbf{z})\|_{\mathbf{L}_p(Q_T)} \le \Lambda\tau^2 \qquad \text{for} \;\; p \ge \frac{p_n}{2}. \tag{3.37}$$

In the next step we improve the bounds for the function φ. Let us write (3.16) in the form

$$c^1\varphi_t - k\Delta\varphi = \sum_{i=1}^5 R_i^\star \qquad \text{where} \;\; R_i^\star = c^1 R_i, \tag{3.38}$$

and assess the right–hand side in $L_2(Q_T)$–norm.
Using (3.20)–(3.22), the stability estimate and (3.31), (3.36), (3.37) yield

$$\|R_1^\star\|_{L_2(Q_T)} + \|R_2^\star\|_{L_2(Q_T)} + \|R_3^\star\|_{L_2(Q_T)} + \|R_4^\star\|_{L_2(Q_T)} + \|R_5^\star\|_{L_2(Q_T)} \leq \Lambda\tau^2,$$

Therefore, by the classical parabolic theory [7],

$$\|\varphi\|_{W_2^{2,1}(Q_T)} \leq \Lambda\tau^2, \tag{3.39}$$

and by the appropriate imbedding,

$$\|\varphi\|_{L_p(Q_T)} \leq \Lambda\tau^2 \qquad \text{for} \quad 2 \leq p \leq \frac{q_n p_n}{n}. \tag{3.40}$$

Now we can limit the right–hand side of (3.38) in the stronger $L_{p_n/2}(Q_T)$–norm:

$$\|R_1^\star\|_{L_{p_n/2}(Q_T)} + \|R_3^\star\|_{L_{p_n/2}(Q_T)} + \|R_4^\star\|_{L_{p_n/2}(Q_T)} + \|R_5^\star\|_{L_{p_n/2}(Q_T)} \leq \Lambda\tau^2,$$

$$\|R_2^\star\|_{L_p(Q_T)} \leq \Lambda\tau^2 \qquad \text{for} \quad p \geq p_n.$$

Hence

$$\|\varphi\|_{W^{2,1}_{p_n/2}(Q_T)} \leq \Lambda\tau^2, \tag{3.41}$$

and

$$\|\nabla\varphi\|_{L_{p_n}(Q_T)}, \ \ \|\varphi\|_{L_p(Q_T)} \leq \Lambda\tau^2 \qquad \text{for} \quad p \geq \frac{p_n}{2}. \tag{3.42}$$

We return now to the decomposed system (3.32), (3.33). By the regularity of solutions the following estimates hold:

$$|\nabla \cdot (F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi)| \leq \Lambda(|\epsilon(\mathbf{z})| + |\nabla\epsilon(\mathbf{z})| + |\varphi| + |\nabla\varphi|),$$

$$|\nabla \cdot F^{1,2}_{/\epsilon}| \leq \Lambda(|\epsilon^2 - \epsilon^1|^2 + |\nabla(\epsilon^2 - \epsilon^1)|^2 + |\theta^2 - \theta^2|^2 + |\nabla(\theta^2 - \theta^1)|^2).$$

Therefore,

$$\|\nabla \cdot (F^1_{/\epsilon\epsilon}\epsilon(\mathbf{z}) + F^1_{/\epsilon\theta}\varphi)\|_{\mathbf{L}_p(Q_T)} \leq \Lambda\tau^2 \qquad \text{for} \quad p = p_n, \tag{3.43}$$

$$\|\nabla \cdot F^{1,2}_{/\epsilon}\|_{\mathbf{L}_p(Q_T)} \leq \Lambda\tau^2 \qquad \text{for} \quad p \geq p_n. \tag{3.44}$$

Applying the Solonnikov theory of parabolic systems [15],[16] to (3.32) and (3.33) yields

$$\|\mathbf{w}\|_{\mathbf{W}^{2,1}_{p_n}(Q_T)} \leq \Lambda\tau^2 \quad \Longrightarrow \quad \|\mathbf{z}\|_{\mathbf{W}^{4,2}_{p_n}(Q_T)} \leq \Lambda\tau^2. \tag{3.45}$$

Finally, by imbedding,

$$\|\nabla^2\epsilon(\mathbf{z})\|_{\mathbf{L}_p(Q_T)} + \|\epsilon(\mathbf{z}_t)\|_{\mathbf{L}_p(Q_T)} \leq \Lambda\tau^2 \qquad \text{for} \quad p \geq p_n. \tag{3.46}$$

With this estimate we can bound the right–hand side of φ–equation in $L_p(Q_T)$–norm for any $p \geq p_n$.
Indeed, we may directly improve the bounds for $R_1^\star, R_2^\star$,

$$\|R_1^\star\|_{L_p(Q_T)} + \|R_2^\star\|_{L_p(Q_T)} \leq \Lambda\tau^2 \qquad \text{for} \quad p \geq p_n.$$

In analysis of $R_3^\star$, $R_4^\star$ and $R_5^\star$ we make use of the fact that $\Delta\theta^i \in W_{p_1}^{2,1}(Q_T)$, due to assumption (BV–p). Hence,

$$\|R_3^\star\|_{L_p(Q_T)} \le \Lambda \left(\int_{Q_T} |\Delta\theta^1|^{p_1}\, dxdt \right)^{\frac{1}{p_1}} \left(\int_{Q_T} |\gamma^1_{/\epsilon} : \epsilon(\mathbf{z}) + \gamma^1_{/\theta}\varphi|^{\frac{pp_1}{p_1-p}}\, dxdt \right)^{\frac{p_1-p}{pp_1}} \le \Lambda\tau^2,$$

and similarly for $R_4^\star$ and $R_5^\star$.
In consequence,

$$\|\varphi\|_{W_p^{2,1}(Q_T)} + \|\nabla\varphi\|_{\mathbf{L}_p(Q_T)} \le \Lambda\tau^2 \qquad \text{for} \quad p \ge p_n. \tag{3.47}$$

Hence the estimate (3.43) holds also for $p \ge p_n$. As a result,

$$\|\mathbf{w}\|_{\mathbf{W}_p^{2,1}(Q_T)} \le \Lambda\tau^2 \quad \text{and} \quad \|\mathbf{z}\|_{\mathbf{W}_p^{4,2}(Q_T)} \le \Lambda\tau^2 \quad \text{for} \quad p \ge p_n.$$

This completes the proof. □

4 Optimal control problem

Let us denote the control space by

$$\mathcal{U} = \mathbf{L}_p(Q_T) \times L_p(Q_T) \qquad \text{for} \quad p > p_n,$$

and assume that g is subject to the additional pointwise constraint, i.e.

$$\mathbf{f} = (\mathbf{b}, g) \in \mathcal{U}_{ad} = \{\ (\mathbf{b}, g) \in \mathcal{U} \mid 0 \le g \le g_{max} \ \text{a.e. in} \quad Q_T\ \}.$$

Let $\mathcal{S}$ denotes the solution operator, i.e. the map from the admissible set $\mathcal{U}_{ad}$ into $V(p) = \mathbf{W}_p^{4,2}(Q_T) \times W_p^{2,1}(Q_T)$, defined by

$$\mathcal{S}(\mathbf{f}) = (\mathbf{u}, \theta), \tag{4.1}$$

where $(\mathbf{u}, \theta)$ is the solution of (P) corresponding to $\mathbf{f} = (\mathbf{b}, g)$. From Theorem 2.3 it follows that the map $\mathcal{S}$ is Lipschitz continuous.
Thanks to the a priori estimates in Theorem 2.1 it is easy to prove the following continuity property.

Lemma 4.1 *Under assumptions of Theorem 2.2 the map $\mathcal{S}$ is continuous from $\mathcal{U}_{ad}$ (weak) into $V(p)$ (weak).*

By virtue of the stability estimate (3.3), the result of Theorem 3.1 can be easily reformulated in terms of $\mathcal{S}$ in the following way:
Let

$$\mathbf{f}, \mathbf{f} + \delta\mathbf{f} \ \in \mathcal{U}_{ad}, \quad \mathbf{f} = (\mathbf{b}, g), \ \delta\mathbf{f} = (\phi, \psi), \tag{4.2}$$

and $\mathcal{S}(\mathbf{f})$, $\mathcal{S}(\mathbf{f} + \delta\mathbf{f})$ be the corresponding solutions of (P). Then

$$\|\mathcal{S}(\mathbf{f} + \delta\mathbf{f}) - \mathcal{S}(\mathbf{f}) - \mathcal{S}'(\mathbf{f})\,\delta\mathbf{f}\|_{V(p)} \le \Lambda\|\delta\mathbf{f}\|^2_{\mathcal{U}}, \tag{4.3}$$

where $\mathcal{S}'(\mathbf{f}) : \mathcal{U}_{ad} \to V(p)$ is a linear operator, and $(\mathbf{v}, \eta) = \mathcal{S}'(\mathbf{f})\,\delta\mathbf{f}$ is the solution of the problem (3.8)–(3.11), where the coefficients $\mathbf{F}_{/\epsilon\epsilon}, \mathbf{F}_{/\epsilon\theta}, \gamma, \mathbf{H}_1, \mathbf{H}_2, H_3$ are evaluated at $(\mathbf{u}, \theta) = \mathcal{S}(\mathbf{f})$. The operator $\mathcal{S}'(\mathbf{f})$ is the Fréchet derivative of $\mathcal{S}$.
We consider the following cost functional

$$J[\mathbf{u}, \theta; \mathbf{f}] = \frac{1}{2}\int_{Q_T} \Phi(|\mathbf{u}-\bar{\mathbf{u}}|^2, |\epsilon(\mathbf{u}-\bar{\mathbf{u}})|^2, |\nabla(\theta-\bar{\theta})|^2)\, dxdt + \frac{\rho}{2}\int_{Q_T} (|\mathbf{b}|^{2s} + g^{2s})\, dxdt, \quad (4.4)$$

where the function $\Phi(s_1, s_2, s_3)$ is assumed to be of a class $C^1(\bar{\mathbb{R}}^3_+)$, Lipschitz continuous, and the regularizing weight coefficient ρ is positive. Moreover, $s \in \mathcal{N}$ and $2s > p_n$. The functions $\bar{\mathbf{u}}, \bar{\theta}$ are given reference solutions satisfying initial and boundary conditions of problem (P).
The following holds

Theorem 4.1 *There exists an optimal control $\hat{\mathbf{f}} \in \mathcal{U}_{ad}$ minimizing the cost functional (4.4) of the problem (P), i.e.*

$$J[\hat{\mathbf{u}}, \hat{\theta}; \hat{\mathbf{f}}] = \inf_{\mathbf{f}\in\mathcal{U}_{ad}} J[\mathbf{u}, \theta; \mathbf{f}], \quad (4.5)$$

where $(\hat{\mathbf{u}}, \hat{\theta}) = \mathcal{S}(\hat{\mathbf{f}})$ *and* $(\mathbf{u}, \theta) = \mathcal{S}(\mathbf{f})$.

Proof. The proof follows by standard arguments. Let $(\mathbf{u}^n, \theta^n; \mathbf{f}^n)$, $(\mathbf{u}^n, \theta^n) = \mathcal{S}(\mathbf{f}^n)$ be a minimizing sequence for the functional J. Since $J[\mathbf{u}^n, \theta^n; \mathbf{f}^n] \le \Lambda$, thanks to the positivity of ρ we have

$$\|\mathbf{f}^n\|_{\mathcal{U}} \le \Lambda.$$

Due to the Lemma 4.1 we can select a subsequence of $\{\mathbf{f}^n\}$ and $\{(\mathbf{u}^n, \theta^n)\}$, denoted by the same index n, such that $\mathbf{f}^n \to \mathbf{f}$ weakly in $\mathcal{U}$, and

$$(\mathbf{u}^n, \theta^n) = \mathcal{S}(\mathbf{f}^n) \to (\mathbf{u}, \theta) = \mathcal{S}(\mathbf{f}) \quad \text{weakly in} \quad V(p).$$

By the weak l.s.c. of $J[\mathbf{u}, \theta; \mathbf{f}]$,

$$\liminf_{n\to\infty} J[\mathbf{u}^n, \theta^n; \mathbf{f}^n] \ge J[\mathbf{u}, \theta; \mathbf{f}].$$

Thus $\hat{\mathbf{f}} := \mathbf{f}$ is an optimal control for (P). □

5 Necessary optimality conditions

We turn now to the neccessary optimality conditions which have to be satisfied by any optimal control $\mathbf{f}$. The variation of the goal functional (4.4) is given by

$$\begin{aligned}
\delta J &= \frac{d}{d\tau} J[\mathcal{S}(\mathbf{f} + \tau\,\delta\mathbf{f}; \mathbf{f} + \tau\,\delta\mathbf{f}]|_{\tau=0} \\
&= \int_{Q_T} [\Phi_{/s_1}(\mathbf{u} - \bar{\mathbf{u}}) \cdot \mathbf{v} + \Phi_{/s_2}\epsilon(\mathbf{u} - \bar{\mathbf{u}}) : \epsilon(\mathbf{v}) + \Phi_{/s_3}\nabla(\theta - \bar{\theta}) \cdot \nabla\eta]\, dxdt \\
&+ \rho s \int_{Q_T} (\mathbf{b}^{2s-1} \cdot \boldsymbol{\phi} + g^{2s-1}\psi)\, dxdt.
\end{aligned}$$

Performing integration by parts gives

$$\delta J = \int_{Q_T} (\mathbf{\Phi}_1 \cdot \mathbf{v} + \Phi_2 \eta)\, dxdt + \rho s \int_{Q_T} (\mathbf{b}^{2s-1} \cdot \boldsymbol{\phi} + g^{2s-1}\psi)\, dxdt, \tag{5.1}$$

where

$$\mathbf{\Phi}_1 = \Phi_{/s_1}(\mathbf{u} - \bar{\mathbf{u}}) - \nabla \cdot [\Phi_{/s_2} \boldsymbol{\epsilon}(\mathbf{u} - \bar{\mathbf{u}})],$$
$$\Phi_2 = -\nabla \cdot [\Phi_{/s_3} \nabla(\theta - \bar{\theta})].$$

We note that by the regularity properties of solution $(\mathbf{u}, \theta) \in V(p)$ the function $\mathbf{\Phi}_1$ is continuous in Q_T, and $\Phi_2 \in L_p(Q_T)$.

In order to derive the adjoint equations it is advantageous to rewrite (3.8)–(3.11) as a first order system, introducing an artificial variable $\mathbf{z}$:

$$\mathbf{v}_t = \mathbf{z}, \tag{5.2}$$
$$\mathbf{z}_t = \nu \mathbf{Q}\mathbf{z} - \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{v} + \nabla \cdot (F_{/\epsilon\epsilon}\boldsymbol{\epsilon}(\mathbf{v}) + F_{/\epsilon\theta}\eta) + \boldsymbol{\phi}, \tag{5.3}$$
$$\eta_t = k\gamma \Delta \eta + \mathbf{H}_1 : \boldsymbol{\epsilon}(\mathbf{v}) + \mathbf{H}_2 : \boldsymbol{\epsilon}(\mathbf{z}) + H_3 \eta + \gamma \psi \quad \text{in} \quad Q_T, \tag{5.4}$$

with initial and boundary conditions

$$\mathbf{v} = \mathbf{z} = 0, \quad \eta = 0 \quad \text{on} \quad \{0\} \times \Omega, \tag{5.5}$$
$$\mathbf{v} = \mathbf{z} = \mathbf{Q}\mathbf{v} = 0, \quad \nabla \eta \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T. \tag{5.6}$$

Denoting the adjoint variables by $\mathbf{p}, \mathbf{r}, q$ we may formally write down the adjoint system as

$$\mathbf{p}_t = \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{r} - \nabla \cdot (F_{/\epsilon\epsilon}\boldsymbol{\epsilon}(\mathbf{r}) + \mathbf{H}_1 q) - \mathbf{\Phi}_1, \tag{5.7}$$
$$\mathbf{r}_t = -\mathbf{p} - \nu \mathbf{Q}\mathbf{r} + \nabla \cdot (\mathbf{H}_2 q), \tag{5.8}$$
$$q_t = F_{/\epsilon\theta} : \boldsymbol{\epsilon}(\mathbf{r}) - \nabla \cdot [\nabla(k\gamma q)] - H_3 q - \Phi_2 \quad \text{in} \quad Q_T, \tag{5.9}$$

with terminal and boundary conditions

$$\mathbf{p} = \mathbf{r} = 0, \quad q = 0 \quad \text{on} \quad \{T\} \times \Omega, \tag{5.10}$$
$$\mathbf{r} = \mathbf{Q}\mathbf{r} = 0, \quad \nabla(k\gamma q) \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T. \tag{5.11}$$

The first order adjoint system (5.7)–(5.11) is equivalent to

$$\mathbf{r}_{tt} + \nu \mathbf{Q}\mathbf{r}_t + \frac{\kappa}{4}\mathbf{Q}\mathbf{Q}\mathbf{r} = \nabla \cdot (F_{/\epsilon\epsilon}\boldsymbol{\epsilon}(\mathbf{r})) - \mathbf{H}_1 q + (\mathbf{H}_2 q)_t) + \mathbf{\Phi}_1, \tag{5.12}$$
$$q_t + \nabla \cdot [\nabla(k\gamma q)] = F_{/\epsilon\theta} : \boldsymbol{\epsilon}(\mathbf{r}) - H_3 q - \Phi_2 \quad \text{in} \quad Q_T, \tag{5.13}$$

with terminal and boundary conditions

$$\mathbf{r}(T, \mathbf{x}) = 0, \quad \mathbf{r}_t(T, \mathbf{x}) = 0, \quad q(T, \mathbf{x}) = 0 \quad \text{in} \quad \Omega, \tag{5.14}$$
$$\mathbf{r} = \mathbf{Q}\mathbf{r} = 0, \quad \nabla(k\gamma q) \cdot \mathbf{n} = 0 \quad \text{on} \quad S_T. \tag{5.15}$$

Multiplying equations (5.2)–(5.4) correspondingly by $\mathbf{p}, \mathbf{r}, q$ and integrating over Q_T gives, after several integration by parts and use of boundary conditions, the following identity

$$\int_{Q_T} (\mathbf{p} \cdot \mathbf{v}_t + \mathbf{r} \cdot \mathbf{z}_t + q\eta_t)\, dxdt =$$
$$= -\int_{Q_T} (\mathbf{v} \cdot \mathbf{p}_t + \mathbf{z} \cdot \mathbf{r}_t + \eta q_t)\, dxdt - \int_{Q_T} (\mathbf{\Phi}_1 \cdot \mathbf{v} + \Phi_2 \eta)\, dxdt + \int_{Q_T} (\boldsymbol{\phi} \cdot \mathbf{r} + \gamma\psi q)\, dxdt. \tag{5.16}$$

Hence, from conditions (5.5), (5.10) it follows that

$$\int_{Q_T} (\mathbf{\Phi}_1 \cdot \mathbf{v} + \Phi_2 \eta)\, dxdt = \int_{Q_T} (\boldsymbol{\phi} \cdot \mathbf{r} + \gamma\psi q)\, dxdt. \tag{5.17}$$

This identity corresponds to the definition of the solution $(\mathbf{r}, q)$ of the adjoint system (5.7)–(5.11) in the transposition method sense of Lions, Magenes [9].

As common in the control theory, despite the lower regularity of the solution $(\mathbf{r}, q)$, identity (5.17) allows to formulate the first order optimality condition.

Actually, according to (5.1), the first variation of the cost functional has the representation

$$\delta J = \int_{Q_T} (\boldsymbol{\phi} \cdot \mathbf{r} + \gamma\psi q)\, dxdt + \rho s \int_{Q_T} (\boldsymbol{\phi} \cdot \mathbf{b}^{2s-1} + \psi g^{2s-1})\, dxdt. \tag{5.18}$$

Concluding, we get the following characterization of optimality conditions

Theorem 5.1 *Let* $\mathbf{f} = (\mathbf{b}, g) \in \mathcal{U}_{ad}$ *be an optimal control for problem (P). If* $(\mathbf{u}, \theta) = \mathcal{S}(\mathbf{f})$ *is the corresponding solution of (P) and* $(\mathbf{r}, q)$ *the corresponding solution of the adjoint system (5.12)–(5.13), then they satisfy the first order optimality condition*

$$\int_{Q_T} [(\mathbf{r} + \rho s \mathbf{b}^{2s-1}) \cdot (\bar{\mathbf{b}} - \mathbf{b}) + (\gamma q + \rho s g^{2s-1})(\bar{g} - g)]\, dxdt \geq 0$$

for all $(\bar{\mathbf{b}}, \bar{g}) \in \mathcal{U}_{ad}$.

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