

Classical Boussinesq Equation

広 大 工      広 田   良 吾      (Ryogo Hirota)

Abstract.

The classical Boussinesq equation

$$u_t = [(1 + u)v + v_{xx}]_x$$

$$v_t = (u + \frac{1}{2} v^2)_x$$

describing shallow water wave, is a  $pq = c$  reduction of a first-modified KP equation

$$(D_1^2 + D_2) \tau \cdot \tau' = 0 ,$$

$$(D_1^3 - 3D_1 D_2 - 4D_3) \tau \cdot \tau' = 0 .$$

We find a coupled equation in the text book, (Whitham: Linear and Non-linear Waves Equation (13. 101) p.465)

$$\eta_t + (1 + \alpha\eta)w_x - \frac{1}{6} \beta w_{xxx} + o(\alpha\beta, \beta^2) = 0 ,$$

$$w_t + \alpha w w_x + \eta_x - \frac{1}{2} \beta w_{xxt} + o(\alpha\beta, \beta^2) = 0 .$$

Let

$$w = u + \frac{1}{2} \beta u_{xx} + o(\alpha\beta, \beta^2) .$$

Then

$$\eta_t + [(1 + \alpha\eta)u]_x + \frac{1}{3} \beta u_{xxx} + o(\alpha\beta, \beta^2) = 0 ,$$

$$u_t + [\eta + \frac{\alpha}{2} u^2]_x + o(\alpha\beta, \beta^2) = 0 .$$

cf. S. Kawamoto: J. Phys. Soc. Jpn. 53 (1984) 2922.

E.V. Krishnan: J. Phys. Soc. Jpn. 51 (1982) 2391.

Let  $u = w_x$  . Then we have

$$\begin{aligned} w_{tt} + (1 + \alpha)w_x w_{xt} + \alpha w_t w_{xx} - w_{xx} \\ + \frac{3\alpha}{2} (w_x)^2 w_{xx} - \frac{1}{3} \beta w_{xxxx} = o(\alpha\beta, \beta^2) , \end{aligned}$$

which is Kaup's higher order water wave equation.

cf. D.J. Kaup: Prog. Theor. Phys. 54 (1975) 396.

On the other hand, we have the Jaulent-Miodek eq.

$$u_t = \frac{1}{4} v_{3x} - uv_x - \frac{1}{2} u_x v ,$$

$$v_t = -u_x - \frac{3}{2} v v_x ,$$

which is the integrability conditions of the following spectral problem

$$\left[ \frac{\partial^2}{\partial x^2} + (\lambda^2 - \lambda v - u) \right] \psi = 0 ,$$

$$\psi_t = \left[ \frac{1}{4} v_x - \frac{1}{2} (u^2 + 2\lambda) \frac{\partial}{\partial x} \right] \psi .$$

M. Joulent and I. Miodek: Lett. Math. Phys. 1 (1976) 243.

The Joulent-Miodek equation is transformed into the classical Boussinesq equation

$$\phi_\tau = [(1 + \phi)v + v_{\xi\xi}]_\xi ,$$

$$v_\tau = \left( \phi + \frac{1}{2} v^2 \right)_\xi ,$$

through the transformation

$$\phi = u + \frac{1}{2} v^2 - 1 ,$$

$$\tau = -\frac{1}{2} t , \quad \xi = \frac{1}{2} x .$$

Sato and Mori have studied a hierarchy of the classical Boussinesq equations

$$w_{t1} = (v_{xx} + 2vw)_x ,$$

$$v_{t1} = (w + v^2)_x ,$$

$$w_{t2} = (w_{xx} + 3vv_{xx} + \frac{3}{2} v_x^2 + \frac{3}{2} v^2 + 3v^2 w)_x ,$$

$$v_{t2} = (v_{xx} + 3vw + v^3)_x .$$

They called it Higher order Nonlinear Schrödinger equations.

M. Sato and Y. Mori: *RIMS Kôkyûroku* 388 (1980) 183.

Ito has studied a symmetries and conservation laws of the classical Boussinesq eq. and found a recursion operator for symmetries.

M. Ito: *Physics Lett.* 104 A (1984) 248.

The classical Boussinesq eq.

$$u_t = [(1 + u)v + v_{xx}]_x ,$$

$$v_t = (u + \frac{1}{2} v^2)_x ,$$

is transformed into the bilinear form

$$(D_\tau - D_x^2) f \cdot g = 0 ,$$

$$[D_x D_\tau - (1 + u_0) D_x - D_x^3] f \cdot g = 0 ,$$

where  $D_\tau = D_t - v_0 D_x$  through the transformation

$$v = v_0 + 2\phi_x ,$$

$$u = u_0 + 2\rho_{xx} ,$$

$$\phi = \log(f/g) ,$$

$$\rho = \log(fg) .$$

Hierarchies of K-P equation and first modified K-P equation are obtained by Jimbo and Miwa;

K-P equation

$$(D_1^4 + 3D_2^2 - 4D_1D_3) \tau \cdot \tau = 0 ,$$

$$(D_1^3D_2 + 2D_2D_3 - 3D_1D_4) \tau \cdot \tau = 0 ,$$

...

First modified K-P equation

$$(D_1^2 + D_2) \tau \cdot \tau' = 0 ,$$

$$(D_1^3 - 3D_1D_2 - 4D_3) \tau \cdot \tau' = 0 ,$$

...

M. Jimbo and T. Miwa: "Solitons and Infinite Dimensional Lie Algebras", RIMS - 439, March (1983).

N-Soliton solution to the first modified K-P equation is well-known. We

write it for  $N = 2$

$$\begin{aligned}\tau &= 1 + q_1 e^{\eta_1} + q_2 e^{\eta_2} + q_1 q_2 a_{12} e^{\eta_1 + \eta_2}, \\ \tau' &= 1 + p_1 e^{\eta_1} + p_2 e^{\eta_2} + p_1 p_2 a_{12} e^{\eta_1 + \eta_2},\end{aligned}$$

where  $\eta_i = (p_i - q_i)x_1 + (p_i^2 - q_i^2)x_2 + (p_i^3 - q_i^3)x_3 + \dots$ ,  
for  $i = 1, 2, \dots$

$$a_{12} = \frac{(p_1 - q_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)}.$$

where  $p_i$  and  $q_i$  for  $i = 1, 2$ , are free parameters.

Now we impose the following conditions on  $p_i$  and  $q_i$

$$p_i q_i = c \quad \text{for all } i,$$

which we call "pq = c reduction" of first modified K-P equation. Then,  
we find that the following equation holds

$$(D_1^3 + 3cD_1 - D_3) \tau \cdot \tau' = 0.$$

Substituting the above relation into the first modified K-P equation,  
we have

$$(D_1^2 + D_2) \tau \cdot \tau' = 0,$$

$$(D_1^3 + D_1 D_2 + 4cD_1) \tau \cdot \tau' = 0,$$

which becomes the bilinear form of the classical Boussinesq equation by the following identification

$$x_1 = x, \quad x_2 = -\tau, \quad 1 + u_0 = 4c.$$

Hence, two-soliton solution to the classical Boussinesq equation is readily constructed by the two-soliton solution of the first modified K-P equation.

All we do is to calculate the factor  $a_{12}$  which describes a phase shift of a soliton after colliding another soliton, using the condition  $p_i q_i = c$ ,

$$\begin{aligned} a_{12} &= \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)} \\ &= \frac{c(p_1 - p_2)^2}{(p_1 p_2 - c)^2} \end{aligned}$$

and the dispersion relation which relate the imaginary frequency

$\Omega_i \equiv p_i^2 - q_i^2$  to the imaginary wave number  $P_i \equiv p_i - q_i$ :

$$\begin{aligned} \Omega_i^2 &= (p_i^2 - q_i^2)^2 \\ &= (p_i - q_i)^2 [(p_i - q_i)^2 + 4p_i q_i] \\ &= P_i^2 [P_i^2 + 4c]. \end{aligned}$$

We rewrite "pq = c reduction" of the modified K-P equation

$$(iD_t + D_x^2) g \cdot f = 0,$$

$$(iD_x D_t + D_x^3 - 4\rho_0^2 D_x)g \cdot f = 0 ,$$

where we put

$$x_1 = x , \quad x_2 = -it , \quad c = -\rho_0^2 , \quad \tau = g , \quad \tau' = f .$$

We also rewrite the two-soliton solution

$$g = 1 + (q_1/p_1)e^{\eta_1} + (q_2/p_2)e^{\eta_2} + (q_1 q_2 / p_1 p_2) a_{12} e^{\eta_1 + \eta_2} ,$$

$$f = 1 + e^{\eta_1} + e^{\eta_2} + a_{12} e^{\eta_1 + \eta_2} ,$$

where  $\eta_i = P_i x - \Omega_i t$  ,

$$q_i/p_i = - (p_i^2 + i\Omega_i) / (p_i^2 - i\Omega_i) ,$$

$$\Omega_i^2 = P_i^2 (4\rho_0^2 - P_i^2) .$$

We note that  $f$  and  $g$  constitute two-dark solitons of the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0$$

$$\psi = \rho_0 \exp(-2i\rho_0^2 t) (g/f) .$$

In fact, the bilinear equation for real  $f$

$$(iD_t + D_x^2)g \cdot f = 0 ,$$

$$(iD_t D_x + D_x^3 - 4\rho_0^2 D_x)g \cdot f = 0$$



is transformed into

$$(iD_t + D_x^2)g \cdot f = 0 ,$$

$$D_x^2 f f - 2\rho_0^2 f^2 = c_0 g^* g$$

where  $c_0$  is determined by the boundary condition on  $f$  and  $g$  at  $|x| = \infty$ . Now we consider the case  $\rho_0 = 0$  or "pq = 0 reduction" of the modified K-P equation. We have for  $q_1 = c/p_1$ ,  $p_2 = c/q_2$

$$f = 1 + \frac{c}{p_1} e^{\eta_1} + q_2 e^{\eta_2} + \frac{c}{p_1} q_2 \frac{(c - p_1 q_2)^2}{c(p_1 - q_2)^2} e^{\eta_1 + \eta_2} ,$$

$$g = 1 + p_1 e^{\eta_1} + \frac{c}{q_2} e^{\eta_2} + \frac{c}{q_2} \frac{(c - p_1 q_2)^2}{c(p_1 - q_2)^2} e^{\eta_1 + \eta_2} ,$$

where

$$\eta_1 = (p_1 - \frac{c}{p_1})x - i(p_1^2 - \frac{c^2}{p_1^2})t + \eta_1^0 ,$$

$$\eta_2 = (\frac{c}{q_2} - q_2)x - i(\frac{c^2}{q_2^2} - q_2^2)t + \eta_2^0 ,$$

which becomes, in the limit  $c = 0$ ,

$$f = 1 + q_2 e^{\eta_2} + \frac{p_1 q_2^3}{(p_1 - q_2)^2} e^{\eta_1 + \eta_2} ,$$

$$g = 1 + p_1 e^{\eta_1} + \frac{p_1^3 q_2}{(p_1 - q_2)^2} e^{\eta_1 + \eta_2} ,$$

where

$$\eta_1 = p_1 x - i p_1^2 t + \eta_1^0 ,$$

$$\eta_2 = -q_2 x + i q_2^2 t + \eta_2^0 .$$

Let  $q_2 = -p_1^*$  ,  $\eta_2^0 = \eta_1^{0*} + i\pi$  . Then

$$f = 1 + p_1^* e^{\eta_1^*} + \frac{p_1 p_1^{*3}}{(p_1 + p_1^*)^2} e^{\eta_1 + \eta_1^*} , \quad g = f^* .$$

Accordingly  $f$  and  $g$  are solutions to the bilinear equation

$$(iD_t + D_x^2) g \cdot f = 0 ,$$

$$(iD_x D_t + D_x^3) g \cdot f = 0 ,$$

but they cannot be a solution to the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0$$

any more because  $f$  is no longer real.

Polynomial solutions to

$$(iD_t + D_x^2) f^* f = 0$$

$$(iD_x D_t + D_x^3) f^* f = 0$$

have been found (A. Nakamura and R. Hirota, submitted to J. phys. Soc. Jpn. (1984) ). They are

$$f_0 = 1 , \quad g_0 = 1 ,$$

$$f_1 = x^2 + 2it , \quad g_1 = f_1^* ,$$

$$f_2 = (x^6 - 36x^2t^2) + i(6x^4t + 72t^3), \quad g_2 = f_2^*,$$

...

Rational solutions to the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0$$

are constructed using  $f_n$  and  $f_n^*$  as follows.

$$\psi = \frac{D_x f_n^* \cdot f_{n+1}^*}{f_n f_{n+1}^* + f_n^* f_{n+1}}.$$

We have found the following reductions.

Reductions of K-P equation

$$p^2 - q^2 = c(p - q), \quad \dots \text{KdV equation}$$

$$p^3 - q^3 = c(p - q), \quad \dots \text{Boussinesq equation}$$

$$p^4 - q^4 = 0, \quad \dots \text{Coupled KdV equation}$$

Reductions of modified K-P equation

$$p^2 - q^2 = c(p - q), \quad \dots \text{modified KdV equation}$$

$$p^3 - q^3 = c(p - q), \quad \dots \text{modified Boussinesq equation}$$

$$p \cdot q = c, \quad \dots \text{Classical Boussinesq equation}$$