DEVELOPMENT OF FUNDAMENTAL THEORY ON UNSTEADY OPEN CHANNEL FLOWS

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ABSTRACT

The unsteady open channel flows are considered as one of the most important research topics in hydraulic engineering and river engineering fields. Although many researchers have studied flood flows, dam-break flows, tsunamis, etc. from various points of view over one century, there are still unresolved problems to make clear from theoretical points of view. Thus, this study was conducted to obtain some insights advancing the classical theory of unsteady open channel flows.

The shallow water equations are applied to explain the basic features of the hydrograph propagation in sub-critical unsteady flows and to reproduce the unsteady flood waves in super-critical flows. For simplicity, the shallow water equations with constant friction coefficient are used as the governing equations. In order to clarify the basic features of the propagation of the hydrographs in sub-critical flows, the approximate analytical solutions are derived by applying the wave front expansion method in a moving coordinate system (ξ, τ). The analytical solutions with wide range of parameters $\left(T_0, Fr_0, x, h_{0peak}\right)$ are in good agreement with the numerical results by means of Method of Characteristics (MOC) for linearized equations, pointing out the validity of the analytical solutions. The non-linear analytical solutions with up to second order terms on time are also considered and compared with the numerical results obtained using TVD-Lax-Wendroff scheme.

Then, the analytical solutions of linearized shallow water equations for super-critical flows are derived by using the perturbation method and MOC. The analytical solutions of flood waves for super-critical flows are calculated under the water depth and velocity hydrographs given at the upstream end. The analytical solutions are verified through the comparison with the numerical results. It is proved that two upstream boundary hydrographs can be reproduced using one depth hydrograph through the theoretical considerations of the analytical solutions, indicating the possibility of inversion analysis.

Finally, the approximated solution of the dam-break flow is derived considering the bottom shear stresses. The approximate solution is compared to the numerical results, indicating the necessity of the further improvement of theoretical analysis.

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CHAPTER 1

INTRODUCTION

Open channel flows with a water surface in natural rivers and man-made channels are basically unsteady in time and non-uniform in space, producing the wide variety of flow patterns. Since the unsteady open channel flows such as flood flows, dam break flows, tsunamis, etc. are closely related to natural disasters, the flows are considered one of the most important research topics in hydraulic engineering and river engineering fields and have been well studied from the practical and theoretical points of view by a lot of researchers over one century [1~5].

Speaking of the basic theory on unsteady open channel flows, the shallow water equations, so called *Saint-Venant* equations, have been commonly applied to clarify the basic features of flood flows, dam break flows, etc. Although it seems that all the basic characteristics have been made clear theoretically on the basic flows, I found some unresolved problems which are attributed to the non-linearity of the basic equations with the inertia and bottom friction terms.

In this thesis, a few basic flows are re-examined from theoretical points of view. In Chapter 2, the propagation of a hydrograph in sub-critical uniform flows is studied without assuming that the non-dimensional parameter defined in 2.2 is small. Although the classical flood wave theories such as *Hayami*'s diffusion theory were formulated under the assumption that the inertia effect is small, the effect of the inertia term is considered in Chapter 2. The analytical solutions of the linearized equations for hydrograph propagation are derived applying the wave front expansion method and then verified through comparison with numerical results.

In Chapter 3, the propagation of a hydrograph in super-critical uniform flows is studied applying the method of characteristics to derive the analytical solutions of the linearized equations. Based on the theoretical findings in this chapter, it is pointed out that two boundary hydrographs of depth and velocity at the upstream end can be reproduced using one depth hydrograph at one site in the downstream.

In Chapter 4, a new analytical solution for a dam break flow is derived without assuming the balance between the bottom friction and pressure forces. The approximate solutions are expressed as the depth and velocity distributions between the positive and negative wave front positions in a moving coordinate system. Since the results are not in good agreement with simulated results, the mathematical framework is represented to improve the accuracy of the approximate solutions in the future.

Finally, Chapter 5 summarizes all the results and conclusions of this study suggesting the necessity of further researches to improve the performance.

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CHAPTER 2

APPLICATION OF THE WAVE FRONT EXPANSION METHOD TO THE HYDROGRAPH PROPAGATION IN SUBCRITICAL OPEN CHANNEL FLOWS

2.1. Objectives and Previous Research

2.1.1. Objectives

Since the prediction of flood waves in rivers and man-made channels is considered as one of the most important research subjects in hydraulics and river engineering fields, a lot of researches have been conducted in view of various points from several decades ago.

The shallow water equations have been commonly applied to clarify the basic features of the propagation of hydrographs in the upstream and downstream directions theoretically and to develop the computational method to predict the actual flood flows.

In chapter 2, the previous theoretical works on the hydrograph propagation (*Hayami* (1951), *Hayashi* (1953), *Lighthill* and *Whitham* (1955), *Inoue* (1987)) are summarized making clear the limitation and drawbacks. The previous flood wave theories have been dealing with the waves with large wave length or large time period in base flows with low Froude numbers, which are well characterized by the non -dimensional parameters such as the spatial scale parameter σ and the Froude number Fr_0.

In this case, it was shown that the theories based on the extended kinematic wave model can be conveniently applied to such waves and the propagation of flood waves are well reproduced by the previous theories. The non-linear *Hayami*'s theory with the diffusion term is considered as a representative theory among the previous theories.

However, it is clear that when the spatial scale parameter and the Froude number become relatively large, the inertia terms of the equation of motion cannot be neglected and the extended kinematic wave model loses the applicability. It seems that the progress to extend the previous theories for unsteady flows with the wide range of non-dimensional parameters has not been seen for several decades.

Thus, this research is conducted to intend to develop the fundamental theory on the wave propagation applicable to the unsteady open channel flows with wide range of parameters.

2.1.2. Summary of the Previous Theoretical Studies for Flood Waves

The previous theoretical studies on the hydrograph propagation in rivers or channels are summarized in this section to make clear the objectives of this study. The shallow water equations have been commonly applied to analyse the unsteady open channel flows denoted as Eq. (2.1.1) and (2.1.2). Eq. (2.1.1) indicates the continuity equation and Eq. (2.1.2) the momentum equation, respectively.

$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=0 \tag{2.1.1}$$

$$\frac{\partial q}{\partial t}+\frac{\partial uq}{\partial x}+gh\frac{\partial h}{\partial x}=gh\left(i_0-i_f\right) \tag{2.1.2}$$

where, x = spatial coordinate,

t = time,

h = water depth,

q = flow rate per unit width,

u = depth averaged velocity ($\equiv q/h$),

g = gravity acceleration,

i_0 = bed gradient,

i_f = energy gradient due to friction ($i_f = n^2q^2/h^{10/3}$; n = Manning's roughness coefficient).

Let's consider the hydrograph propagation shown in Fig.2.1.1. Introducing the non-dimensional variables lead the non-dimensional form of basic equations with two non-dimensional parameters λ and Fr_0.

$$t' = \frac{t}{T_0}\ ,\ x' = \frac{x}{L}\ ,\ h' = \frac{h}{h_0}\ ,\ q' = \frac{q}{h_0 u_0}$$

where, T_0 : the time scale of the depth hydrograph observed at the upstream end, L : the spatial scale of waves, h_0 : the depth of base flow and u_0 : the velocity of base flow. L is considered to be evaluated as $L \equiv u_0 T_0$.

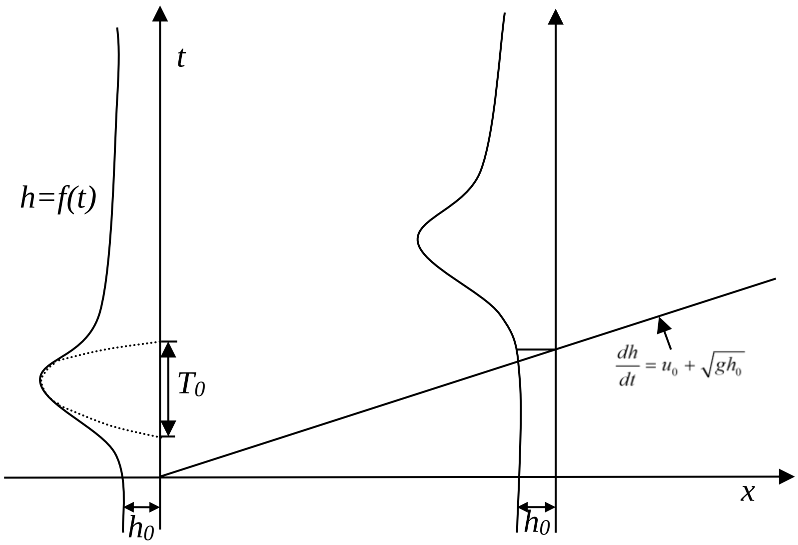


Figure.2.1.1. Propagation of flood wave form

Non-dimensional form of Eq. (2.1.1) and (2.1.2) can be expressed as the following equations;

$$\frac{\partial h'}{\partial t'} + \frac{\partial q'}{\partial x'} = 0 \tag{2.1.3}$$

$$\lambda Fr_0^{\,2}\left(\frac{\partial q'}{\partial t'} + 2\frac{q'}{h'}\frac{\partial q'}{\partial x'} - \frac{q'^2}{h'^2}\frac{\partial h'}{\partial x'} + \frac{h'}{Fr_0^{\,2}}\frac{\partial h'}{\partial x'}\right) = h'\left(1 - \frac{q'^2}{h'^{10/3}}\right) \tag{2.1.4}$$

in which, $\lambda = \frac{h_0}{Li_0}, Fr_0 = \frac{u_0}{\sqrt{gh_0}}$.

2.1.2.1. Wave Propagation near the Front

Based on the Method of Characteristics, it is known that the hydrograph near a wave front propagates with the speed of the dynamic wave $u_0 + \sqrt{gh_0}$.

Lighthill and *Whitham* (1955) proposed to derive the disturbances of depth and velocity near a front using the series expansion with respect to τ defined as the following equation.

$$\tau = t - \frac{x}{u_0 + \sqrt{gh_0}} \tag{2.1.5}$$

The depth and velocity distributions are expressed as Eq. (2.1.6) and (2.1.7).

$$h = h_0 + \tau h_1(x) + \tau^2 h_2(x) + ... \tag{2.1.6}$$

$$u = u_0 + \tau u_1(x) + \tau^2 u_2(x) + ... \quad ; \tau \geq 0 \tag{2.1.7}$$

Substituting Eq. (2.1.6) and (2.1.7) into the basic equations, the relations for h_1, h_2, u_1 and u_2 were derived. The depth distribution calculated using the wave front expansion is shown as the broken line in Fig.2.1.2.

2.1.2.2. Hydrograph around the Peak and Its Propagation

Hayashi (1953) and *Inou*e (1987) considered to extend the kinematic wave model introducing the non-dimensional parameter $\sigma \equiv \lambda Fr_0{}^2$. When σ is very small, the terms on the left side of Eq. (2.1.4) can be neglected. The model with Eq. (2.1.3) and Eq. (2.1.4) without the inertia terms is referred to as Kinematic Wave Model.

Although the Kinematic Wave Model (KWM) can be considered as the simplest model to predict the propagation of flood waves, there are a few fatal drawbacks such that the peak value of a hydrograph does not attenuate in the downstream direction, the discontinuity called the kinematic shock is inevitably generated, etc. In order to resolve these problems, the first order KWM was improved including the effect of the inertia terms in Eq. (2.1.4).

Inoue (1987) applied the perturbation method to Eq. (2.1.3) and (2.1.4). In this method, h' and q' are denoted as the following equations.

$$h' = h_1' + \sigma h_2' + ... \tag{2.1.8}$$

$$q' = q_1' + \sigma q_2' + ... \tag{2.1.9}$$

Substituting these equations into Eq. (2.1.3) and (2.1.4) leads to 0th order and 1st order equations of σ as follows.

$$\left.\begin{aligned}&\sigma^0 : \frac{\partial h_1}{\partial t} + \frac{\partial q_1}{\partial x} = 0\\ &h_1 - \frac{q_1^{\,2}}{h_1^{\,7/3}} = 0\end{aligned}\right\} \tag{2.1.10}$$

$$\left.\begin{aligned}&\sigma^1 : \frac{\partial h_2}{\partial t} + \frac{\partial q_2}{\partial x} = 0\\ &\frac{1}{h_1}\frac{\partial q_1}{\partial t} + 2\frac{q_1}{h_1^{\,2}}\frac{\partial q_1}{\partial x} + \left(\frac{1}{Fr_0^{\,2}} - \frac{q_1^{\,2}}{h_1^{\,2}}\right)\frac{\partial h_1}{\partial x} = \frac{10}{3}\frac{h_2}{h_1} - 2\frac{q_2}{q_1}\end{aligned}\right\} \tag{2.1.11}$$

Eq. (2.1.10) indicates the KWM and Eq. (2.1.11) includes the effect of the inertia terms.

Inoue (1987) derived the analytical solutions of Eq. (2.1.10) and (2.1.11). Eq. (2.1.10) was firstly solved under the following boundary condition at the upstream boundary $x = 0$.

$$h_1(0,t) = f(t) \quad \text{at} \quad x = 0 \tag{2.1.12}$$

The solution of Eq. (2.1.10) given in Eq. (2.1.13) was substituted into Eq. (2.1.11) and then the second order solution was derived as Eq. (2.1.14).

$$h_1(x,t) = f\left(t - \frac{x}{\omega_1}\right) \ ; \ \omega_1 = \frac{5}{3}h_1^{\,2/3} \tag{2.1.13}$$

$$\begin{aligned}h = h_1 + \sigma h_2 = f\Bigg[1 + \sigma\frac{9f}{10\left(2\dot{f}x - 3\omega_1 f\right)}&\left\{\dot{f}\left(\frac{3}{Fr_0^{\,2}} - \frac{14}{25}\frac{\omega_1^{\,2}}{f}\right)\ln\left(1 - \frac{2\dot{f}x}{3\omega_1 f}\right)\right.\\ &\left.-\left(\frac{1}{Fr_0^{\,2}} - \frac{4}{25}\frac{\omega_1^{\,2}}{f}\right)\left(\frac{5f^2 - 3f\ddot{f}}{2\dot{f}x - 3\omega_1 f}\right)x\right\}\Bigg]\end{aligned} \tag{2.1.14}$$

where, $\dot{f} = {df}/{dt}$.

Substituting $\dot{f} = 0$ into Eq. (2.1.14), the peak value of depth $h_p(x)$ is given as Eq. (2.1.15).

$$h_p(x) = f_p\left[1 - \sigma\frac{27}{250}\frac{\ddot{f}_p}{f_p}x\left(\frac{1}{Fr_0^{\,2} f_p^{\,1/3}} - \frac{4}{9}\right)\right] \tag{2.1.15}$$

2.1.2.3. *Hayami*'s Diffusion Theory

If λ is not small and $Fr_0(<1)$ is close to 1 in Eq. (2.1.4), the fourth term on the left side cannot be neglected. *Hayami* (1951) proposed to include the fourth term in Eq. (2.1.4). This model denoted as Eq. (2.1.16) using the dimensional form is called the non-linear *Hayami*'s diffusion model.

$$\frac{\partial h}{\partial t} + \omega\frac{\partial h}{\partial x} = \frac{1}{2}\frac{q}{\left(i_0 - \dfrac{\partial h}{\partial x}\right)}\frac{\partial^2 h}{\partial x^2}\ ,\ \omega = \frac{5}{3}\frac{q}{h} \tag{2.1.16}$$

The depth distributions calculated using the wave front expansion Eq. (2.1.6) and (2.1.7), the extended KWM from Eq. (2.1.14) and the diffusion model from Eq. (2.1.16) with $\lambda = 0.23$ and $Fr_0 = 0.44$ were shown by *Hosoda* (1992) in Fig. 2.1.2. The results from the diffusion model seem to reproduce the basic features of the flood wave propagation well qualitatively.

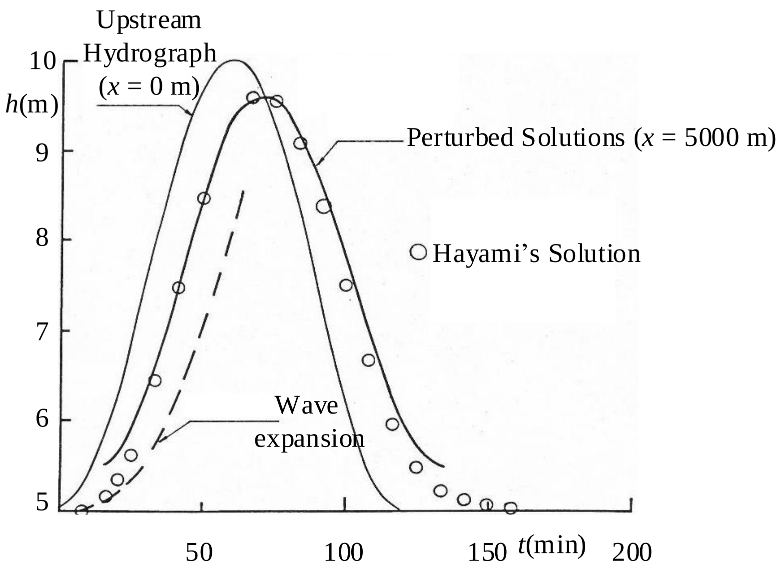


Figure.2.1.2. Calculation results of flood waveform (*Hosoda* (1992)).
($i_0 = 1/1000, n = 0.03, \lambda = 0.23, Fr_0 = 0.44$)

2.1.2.4. Solution of the Linearized Dynamic Wave Equation for Large *t*

Assuming that the deviations of depth and velocity from the uniform flow depth and velocity are small, the linearzed equation of the shallow water equations in dimensional form is derived as Eq. (2.1.17).

$$\frac{\partial^2\eta}{\partial t^2}+2u_0\frac{\partial^2\eta}{\partial t\partial x}-\left(gh_0-u_0{}^2\right)\frac{\partial^2\eta}{\partial x^2}+2\lambda\left(\frac{\partial\eta}{\partial t}+\frac{5}{3}u_0\frac{\partial\eta}{\partial x}\right)=0 \tag{2.1.17}$$

Applying the Laplace transformation, *Lighthill* and *Whitham* (1955) derived an approximate solution of Eq. (2.1.17) for large *t* under the following initial and boundary conditions.

Initial condition, $t=0$: $\eta=0$, $\dfrac{\partial\eta}{\partial t}=0$.

Boundary condition, $x=0$: $\eta=f\left(t\right)$.

The approximate solution for the maximum depth of η at x and its time of occurrence are described as the following equations.

$$\eta_{\max}\sim\left(\frac{\lambda_1}{2\pi t}\right)^{\frac{1}{2}}\frac{5Fr_0}{3\left(1-\left(4/9\right)Fr_0{}^2\right)}\int_0^\infty f\left(t\right)dt \tag{2.1.18}$$

$$\lambda_1=\frac{gi_0}{u_0} \tag{2.1.19}$$

$$x=\frac{5}{3}u_0t_{\max}$$

2.2. Application of the Wave Front Expansion to the Propagation of Hydrographs

2.2.1. Mathematical Formulation

In this study, for simplicity the shallow water equations with constant friction coefficient are used as the governing equations. In Chapter 2, the unsteady flows are assumed to be in the subcritical flow condition.

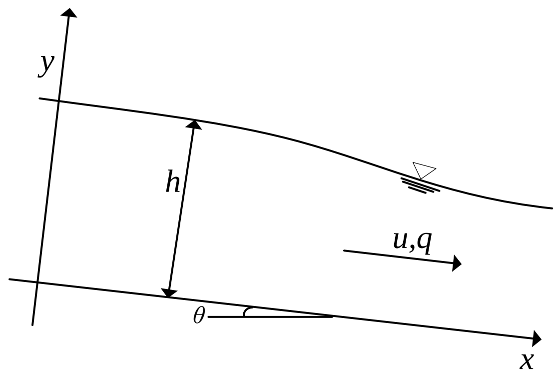


Figure.2.2.1. Coordinate system

$$\frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}=0 \tag{2.2.1}$$

$$\frac{\partial q}{\partial t}+\frac{\partial uq}{\partial x}+g\cos\theta\frac{\partial}{\partial x}\left(\frac{h^2}{2}\right)=gh\sin\theta-c_f u^2 \tag{2.2.2}$$

where, c_f = constant friction coefficient.

Introducing the non-dimensional variables with, the non-dimensional forms of Eq. (2.2.1) and (2.2.2) can be rewritten as follow:

$$\frac{\partial h'}{\partial t'}+\frac{\partial h'u'}{\partial x'}=0 \tag{2.2.3}$$

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+\frac{1}{Fr_0^{\ 2}}\frac{\partial h}{\partial x'}'=c_f(1-\frac{u'^2}{h'}) \tag{2.2.4}$$

where, $x'=x/h_0, u'=u/u_0, t'=tu_0/h_0, h'=h/h_0$ and $gh_0\sin\theta=c_f u_0^{\ 2}$.

Hereafter, for simplicity, “ ' ” indicating non-dimensional variables is omitted. Applying the wave front expansion, the propagation of a hydrograph shown in Fig.2.1.1 is investigated theoretically. Since the wave front in Fig.2.1.1 propagates with the speed of $dx/dt=1+1/Fr_0$, h and u seem to be approximated as Eq. (2.2.5) and (2.2.6).

$$h=1+h_1(\xi)\tau+h_2(\xi)\tau^2+h_3(\xi)\tau^3+h_4(\xi)\tau^4+h_5(\xi)\tau^5+h_6(\xi)\tau^6+..... \tag{2.2.5}$$

$$u=1+u_1(\xi)\tau+u_2(\xi)\tau^2+u_3(\xi)\tau^3+u_4(\xi)\tau^4+u_5(\xi)\tau^5+u_6(\xi)\tau^6+..... \tag{2.2.6}$$

Eq. (2.2.3) and (2.2.4) can be transformed from (x,t) system into the moving coordinate system (ξ,τ) with the relation of $\xi = x$ and $\tau = t - x \Big/ \left(\frac{Fr_0 + 1}{Fr_0} \right)$. Using the relation of $\frac{\partial}{\partial x} = \partial \left({-Fr_0} \big/ {Fr_0 + 1} \right) \Big/ \partial \tau + \frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}$, the non-dimensional equations in (ξ,τ) plane are given by

$$\frac{\partial h}{\partial \tau} + u\left(-F_1 \frac{\partial h}{\partial \tau} + \frac{\partial h}{\partial \xi} \right) + h\left(-F_1 \frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial \xi} \right) = 0 \tag{2.2.7}$$

$$\frac{\partial u}{\partial \tau} + u\left(-F_1 \frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial \xi} \right) + \frac{1}{Fr_0^{\ 2}}\left(-F_1 \frac{\partial h}{\partial \tau} + \frac{\partial h}{\partial \xi} \right) = c_f \left(1 - \frac{u^2}{h} \right) \tag{2.2.8}$$

where, $F_1 = \frac{Fr_0}{Fr_0 + 1}$.

2.2.2. Derivation of the Analytical Solution

In order to obtain the approximate solutions, the solutions are assumed to be defined as the power series expansion. Substituting Eq. (2.2.5) and (2.2.6) into Eq. (2.2.7) and (2.2.8), the resulting equations can be given by

$$\begin{aligned}
& h_1 - F_1 h_1 - F_1 u_1 + \tau \left\{ 2h_2 - 2F_1 h_2 + \frac{dh_1}{d\xi} - F_1 h_1 u_1 - 2F_1 u_2 + \frac{du_1}{d\xi} - F_1 h_1 u_1 \right\} \\
& \quad + \tau^2 \left\{ 3h_3 - 3F_1 h_3 + \frac{dh_2}{d\xi} - 2F_1 h_2 u_1 + u_1 \frac{dh_1}{d\xi} - F_1 h_1 u_2 \right. \\
& \quad \left. -3F_1 u_3 + \frac{du_2}{d\xi} - 2F_1 h_1 u_2 + h_1 \frac{du_1}{d\xi} - F_1 h_2 u_1 \right\} \\
& \quad + \tau^3 \left\{ 4h_4 - 4F_1 h_4 + \frac{dh_3}{d\xi} - 3F_1 h_3 u_1 + u_1 \frac{dh_2}{d\xi} - 2F_1 h_2 u_2 + u_2 \frac{dh_1}{d\xi} - F_1 h_1 u_3 \right. \\
& \quad \left. -4F_1 u_4 + \frac{du_3}{d\xi} - 3F_1 h_1 u_3 + h_1 \frac{du_2}{d\xi} - 2F_1 h_2 u_2 + h_2 \frac{du_1}{d\xi} - F_1 h_3 u_1 \right\} \\
& \quad + \tau^4 \left\{ 5h_5 - 5F_1 h_5 + \frac{dh_4}{d\xi} - 4F_1 h_4 u_1 + u_1 \frac{dh_3}{d\xi} - 3F_1 h_3 u_2 + u_2 \frac{dh_2}{d\xi} - 2F_1 h_2 u_3 \right. \\
& \quad + u_3 \frac{dh_1}{d\xi} - F_1 h_1 u_4 - 5F_1 u_5 + \frac{du_4}{d\xi} - 4F_1 h_1 u_4 + h_1 \frac{du_3}{d\xi} \\
& \quad \left. -3F_1 h_2 u_3 + h_2 \frac{du_2}{d\xi} - 2F_1 h_3 u_2 + h_3 \frac{du_1}{d\xi} - F_1 h_4 u_1 \right\}
\end{aligned}$$

$$+\tau^5\left\{6h_6-6F_1h_6+\frac{dh_5}{d\xi}-5F_1h_5u_1+u_1\frac{dh_4}{d\xi}-4F_1h_4u_2+u_2\frac{dh_3}{d\xi}-3F_1h_3u_3\right.$$

$$+u_3\frac{dh_2}{d\xi}-2F_1h_2u_4+u_4\frac{dh_1}{d\xi}-F_1h_1u_5-6F_1u_6+\frac{du_5}{d\xi}-5F_1h_1u_5+h_1\frac{du_4}{d\xi}$$

$$\left.-4F_1h_2u_4+h_2\frac{dh_3}{d\xi}-3F_1h_3u_3+h_3\frac{du_2}{d\xi}-2F_1h_4u_2+h_4\frac{du_1}{d\xi}-F_1h_5u_1\right\}=0 \qquad (2.2.9)$$

$$u_1-F_1u_1-\frac{F_1}{Fr_0^{\ 2}}h_1+\tau\left\{2u_2+h_1u_1-2F_1u_2+\frac{du_1}{d\xi}-F_1h_1u_1-F_1u_1^{\ 2}-2\frac{F_1h_2}{Fr_0^{\ 2}}+\frac{1}{Fr_0^{\ 2}}\frac{dh_1}{d\xi}-\frac{F_1h_1^{\ 2}}{Fr_0^{\ 2}}\right\}$$

$$+\tau^2\left\{3u_3+2h_1u_2+h_2u_1-3F_1u_3+\frac{du_2}{d\xi}-2F_1h_1u_2+h_1\frac{du_1}{d\xi}-F_1h_2u_1-2F_1u_1u_2+u_1\frac{du_1}{d\xi}\right.$$

$$\left.-F_1h_1u_1^{\ 2}-F_1u_1u_2-3\frac{F_1}{Fr_0^{\ 2}}h_3+\frac{1}{Fr_0^{\ 2}}\frac{dh_2}{d\xi}-2\frac{F_1h_1h_2}{Fr_0^{\ 2}}+\frac{h_1}{Fr_0^{\ 2}}\frac{dh_1}{d\xi}-\frac{F_1h_1h_2}{Fr_0^{\ 2}}\right\}$$

$$+\tau^3\left\{4u_4+3h_1u_3+2h_2u_2+h_3u_1-4F_1u_4+\frac{du_3}{d\xi}-3F_1h_1u_3+h_1\frac{du_2}{d\xi}-2F_1h_2u_2\right.$$

$$+h_2\frac{du_1}{d\xi}-F_1h_3u_1-3F_1u_1u_3+u_1\frac{du_2}{d\xi}-2F_1h_1u_1u_2+h_1u_1\frac{du_1}{d\xi}-F_1h_2u_1^{\ 2}-2F_1u_2^{\ 2}$$

$$+u_2\frac{du_1}{d\xi}-F_1h_1u_1u_2-F_1u_1u_3-4\frac{F_1}{Fr_0^{\ 2}}h_4+\frac{1}{Fr_0^{\ 2}}\frac{dh_3}{d\xi}-3\frac{F_1}{Fr_0^{\ 2}}h_1h_3+\frac{1}{Fr_0^{\ 2}}h_1\frac{dh_2}{d\xi}$$

$$\left.-2\frac{F_1}{Fr_0^{\ 2}}h_2^{\ 2}+\frac{1}{Fr_0^{\ 2}}h_2\frac{dh_1}{d\xi}-\frac{F_1}{Fr_0^{\ 2}}h_1h_3\right\}$$

$$+\tau^4\left\{5u_5+4h_1u_4+3h_2u_3+2h_3u_2+h_4u_1-5F_1u_5+\frac{du_4}{d\xi}-4F_1h_1u_4+h_1\frac{du_3}{d\xi}\right.$$

$$-3F_1h_2u_3+h_2\frac{du_2}{d\xi}-2F_1h_3u_2+h_3\frac{du_1}{d\xi}-F_1h_4u_1-4F_1u_1u_4+u_1\frac{du_3}{d\xi}-3F_1h_1u_1u_3$$

$$+h_1u_1\frac{du_2}{d\xi}-2F_1h_2u_1u_2+h_2u_1\frac{du_1}{d\xi}-F_1h_3u_1^{\ 2}-3F_1u_2u_3+u_2\frac{du_2}{d\xi}-2F_1h_1u_2^{\ 2}$$

$$+h_1u_2\frac{du_1}{d\xi}-F_1h_1u_1u_2-2F_1u_2u_3+u_3\frac{du_1}{d\xi}-F_1h_1u_1u_3-F_1u_1u_4-5\frac{F_1}{Fr_0^{\ 2}}h_5$$

$$+\frac{1}{Fr_0^{\ 2}}\frac{dh_4}{d\xi}-4\frac{F_1}{Fr_0^{\ 2}}h_1h_4+\frac{1}{Fr_0^{\ 2}}h_1\frac{dh_3}{d\xi}-3\frac{F_1}{Fr_0^{\ 2}}h_2h_3+\frac{1}{Fr_0^{\ 2}}h_2\frac{dh_2}{d\xi}$$

$$-2\frac{F_1}{Fr_0^{\ 2}}h_2h_3+\frac{1}{Fr_0^{\ 2}}h_3\frac{dh_1}{d\xi}-\frac{F_1}{Fr_0^{\ 2}}h_1h_4\Bigg\}$$

$$+\tau^5\Bigg\{6u_6+5h_1u_5+4h_2u_4+3h_3u_3+2h_4u_2+h_5u_1-6F_1u_6+\frac{du_5}{d\xi}-5F_1h_1u_5+h_1\frac{du_4}{d\xi}$$

$$-4F_1h_2u_4+h_2\frac{du_3}{d\xi}-3F_1h_3u_3+h_3\frac{du_2}{d\xi}-2F_1h_4u_2+h_4\frac{du_1}{d\xi}-F_1h_5u_1-5F_1u_1u_5$$

$$+u_1\frac{du_4}{d\xi}-4F_1h_1u_1u_4+h_1u_1\frac{du_3}{d\xi}-3F_1h_2u_1u_3+h_2u_1\frac{du_2}{d\xi}-2F_1h_3u_1u_2+h_3u_1\frac{du_1}{d\xi}$$

$$-F_1h_4u_1^{\ 2}-4F_1u_2u_4+u_2\frac{du_3}{d\xi}-3F_1h_1u_2u_3+h_1u_2\frac{du_2}{d\xi}-2F_1h_2u_2^{\ 2}+h_2u_2\frac{du_1}{d\xi}$$

$$-F_1h_3u_1u_2-3F_1u_3^{\ 2}+u_3\frac{du_2}{d\xi}-2F_1h_1u_2u_3+h_1u_3\frac{du_1}{d\xi}-F_1h_2u_1u_3-2F_1u_2u_4+u_4\frac{du_1}{d\xi}$$

$$-F_1h_1u_1u_4-F_1u_1u_5-\frac{6F_1}{Fr_0^{\ 2}}h_5+\frac{1}{Fr_0^{\ 2}}\frac{dh_5}{d\xi}-\frac{5F_1}{Fr_0^{\ 2}}h_1h_5+\frac{1}{Fr_0^{\ 2}}h_1\frac{dh_4}{d\xi}-\frac{4F_1}{Fr_0^{\ 2}}h_2h_4$$

$$+\frac{1}{Fr_0^{\ 2}}h_2\frac{dh_3}{d\xi}-3\frac{F_1}{Fr_0^{\ 2}}h_3^{\ 2}+\frac{1}{Fr_0^{\ 2}}h_3\frac{dh_2}{d\xi}-2\frac{F_1}{Fr_0^{\ 2}}h_2h_4+\frac{1}{Fr_0^{\ 2}}h_4\frac{dh_1}{d\xi}-\frac{F_1}{Fr_0^{\ 2}}h_1h_5\Bigg\}$$

$$=c_f(1)-c_f(1)+\tau\{c_fh_1-2c_fu_1\}+\tau^2\{c_fh_2-c_f(2u_2+u_1^{\ 2})\}$$

$$+\tau^3\{c_fh_3-c_f(2u_3+2u_1u_2)\}+\tau^4\{c_fh_4-c_f(2u_4+2u_1u_3+u_2^2)\}$$

$$+\tau^5\{c_fh_5-c_f(2u_5+2u_1u_4+u_2u_3)\} \tag{2.2.10}$$

The zeroth order terms in Eq. (2.2.9) and (2.2.10) are given by

$$\tau^0:\ \ (1-F_1)h_1-F_1u_1=0 \tag{2.2.11}$$

$$-\frac{F_1}{Fr_0^{\ 2}}h_1+(1-F_1)u_1=0 \tag{2.2.12}$$

Eq. (2.2.11) and (2.2.12) indicate the relationship of the h_1 and u_1

$$h_1=Fr_0u_1 \tag{2.2.13}$$

The first order terms of the Eq. (2.2.9) and (2.2.10) are written as

$$\tau^1: \ 2(1-F_1)h_2 - 2F_1u_2 + \frac{dh_1}{d\xi} + \frac{du_1}{d\xi} - 2F_1h_1u_1 = 0 \tag{2.2.14}$$

$$-\frac{2F_1}{Fr_0^{\ 2}}h_2 + 2(1-F_1)u_2 + \frac{1}{Fr_0^{\ 2}}\frac{dh_1}{d\xi} + \frac{du_1}{d\xi} + (1-F_1)h_1u_1$$

$$-F_1u_1^{\ 2} - \frac{F_1h_1^{\ 2}}{Fr_0^{\ 2}} = c_f\left(h_1 - 2u_1\right) \tag{2.2.15}$$

From the above Eq. (2.2.14) and (2.2.15), the equations relevant to h_1, h_2 and u_2 can be derived as follow:

$$-\frac{dh_1}{d\xi} + \frac{3Fr_0}{2\left(Fr_0+1\right)^2}h_1^{\ 2} + \frac{F_1\left(Fr_0-2\right)}{2}c_f h_1 = 0 \tag{2.2.16}$$

$$h_2 - Fr_0u_2 = \frac{1}{4}h_1^{\ 2} - \frac{\left(Fr_0+1\right)\left(Fr_0-2\right)}{4}c_f h_1 \tag{2.2.17}$$

Similarly, the second order, third order, fourth order and fifth order terms of the non-linear equations in the moving coordinate system are given by

$$\tau^2: \ 3(1-F_1)h_3 - 3F_1u_3 + \frac{dh_2}{d\xi} + \frac{du_2}{d\xi} + u_1\frac{dh_1}{d\xi} + h_1\frac{du_1}{d\xi} - 3F_1h_2u_1 - 3F_1h_1u_2 = 0 \tag{2.2.18}$$

$$-\frac{3F_1}{Fr_0^{\ 2}}h_3 + 3(1-F_1)u_3 + \frac{1}{Fr_0^{\ 2}}\frac{dh_2}{d\xi} + \frac{du_2}{d\xi} + \frac{1}{Fr_0^{\ 2}}h_1\frac{dh_1}{d\xi} + h_1\frac{du_1}{d\xi} + u_1\frac{du_1}{d\xi} - F_1h_1u_1^{\ 2}$$

$$-3F_1u_1u_2 + 2(1-F_1)h_1u_2 + (1-F_1)h_2u_1 - \frac{3F_1}{Fr_0^{\ 2}}h_1h_2 = c_f\left(h_2 - 2u_2 - u_1^{\ 2}\right) \tag{2.2.19}$$

$$\tau^3: \ 4(1-F_1)h_4 - 4F_1u_4 + \frac{dh_3}{d\xi} + u_1\frac{dh_2}{d\xi} + u_2\frac{dh_1}{d\xi} + \frac{du_3}{d\xi} + h_1\frac{du_2}{d\xi} + h_2\frac{du_1}{d\xi}$$

$$-4F_1h_3u_1 - 4F_1h_2u_2 - 4F_1h_1u_3 = 0 \tag{2.2.20}$$

$$-\frac{4F_1}{Fr_0^{\ 2}}h_4 + 4(1-F_1)u_4 + \frac{1}{Fr_0^{\ 2}}\frac{dh_3}{d\xi} + \frac{h_1}{Fr_0^{\ 2}}\frac{dh_2}{d\xi} + \frac{h_2}{Fr_0^{\ 2}}\frac{dh_1}{d\xi} + \frac{du_3}{d\xi} + h_1\frac{du_2}{d\xi} + u_1\frac{du_2}{d\xi}$$

$$+h_2\frac{du_1}{d\xi} + u_2\frac{du_1}{d\xi} + h_1u_1\frac{du_1}{d\xi} + 3(1-F_1)h_1u_3 + 2(1-F_1)h_2u_2 + (1-F_1)h_3u_1$$

$$-\frac{4F_1}{Fr_0^{\ 2}}h_1h_3 - \frac{2F_1}{Fr_0^{\ 2}}h_2^{\ 2} - 3F_1h_1u_1u_2 - F_1h_2u_1^{\ 2} - 2F_1u_2^{\ 2} - 4F_1u_1u_3$$

$$= c_f\left(h_3 - 2u_3 - 2u_1u_2\right) \tag{2.2.21}$$

$$\tau^4:\ 5(1-F_1)h_5+\frac{dh_4}{d\xi}-5F_1h_4u_1-5F_1h_3u_2-5F_1h_2u_3-5F_1h_1u_4+u_1\frac{dh_3}{d\xi}+u_2\frac{dh_2}{d\xi}$$

$$+u_3\frac{dh_1}{d\xi}-5F_1u_5+\frac{du_4}{d\xi}+h_1\frac{du_3}{d\xi}+h_2\frac{du_2}{d\xi}+h_3\frac{du_1}{d\xi}=0 \tag{2.2.22}$$

$$5(1-F_1)u_5+4(1-F_1)h_1u_4+3(1-F_1)h_2u_3+2(1-F_1)h_3u_2+(1-F_1)h_4u_1+\frac{du_4}{d\xi}$$

$$+h_1\frac{du_3}{d\xi}+h_2\frac{du_2}{d\xi}+h_3\frac{du_1}{d\xi}-5F_1u_1u_4+u_1\frac{du_3}{d\xi}-4F_1h_1u_1u_3+h_1u_1\frac{du_2}{d\xi}$$

$$-3F_1h_2u_1u_2+h_2u_1\frac{du_1}{d\xi}-F_1h_3u_1{}^2-5F_1u_2u_3+u_2\frac{du_2}{d\xi}-2F_1h_1u_2{}^2+h_1u_2\frac{du_1}{d\xi}$$

$$+u_3\frac{du_1}{d\xi}-\frac{5F_1}{Fr_0{}^2}h_5+\frac{1}{Fr_0{}^2}\frac{dh_4}{d\xi}-\frac{5F_1}{Fr_0{}^2}h_1h_4+\frac{1}{Fr_0{}^2}h_1\frac{dh_3}{d\xi}-\frac{5F_1}{Fr_0{}^2}h_2h_3$$

$$+\frac{1}{Fr_0{}^2}h_2\frac{dh_2}{d\xi}+\frac{1}{Fr_0{}^2}h_3\frac{dh_1}{d\xi}=c_f\left(h_4-2u_4-2u_1u_3-u_2{}^2\right) \tag{2.2.23}$$

$$\tau^5:\ 6(1-F_1)h_6+\frac{dh_5}{d\xi}-6F_1h_5u_1-6F_1h_4u_2-6F_1h_3u_3-6F_1h_2u_4-6F_1h_1u_5+u_1\frac{dh_4}{d\xi}$$

$$+u_2\frac{dh_3}{d\xi}+u_3\frac{dh_2}{d\xi}+u_4\frac{dh_1}{d\xi}-6F_1u_6+\frac{du_5}{d\xi}+h_1\frac{du_4}{d\xi}$$

$$+h_2\frac{du_3}{d\xi}+h_3\frac{du_2}{d\xi}+h_4\frac{du_1}{d\xi}=0 \tag{2.2.24}$$

$$6(1-F_1)u_6+5(1-F_1)h_1u_5+4(1-F_1)h_2u_4+3(1-F_1)h_3u_3+2(1-F_1)h_4u_2$$

$$+(1-F_1)h_5u_1+\frac{du_5}{d\xi}+h_1\frac{du_4}{d\xi}+h_2\frac{du_3}{d\xi}+h_3\frac{du_2}{d\xi}+h_4\frac{du_1}{d\xi}-6F_1u_1u_5+u_1\frac{du_4}{d\xi}$$

$$-5F_1h_1u_1u_4+h_1u_1\frac{du_3}{d\xi}-4F_1h_2u_1u_3+h_2u_1\frac{du_2}{d\xi}-3F_1h_3u_1u_2+h_3u_1\frac{du_1}{d\xi}-F_1h_4u_1^2$$

$$-6F_1u_2u_4-5F_1h_1u_2u_3+u_2\frac{du_3}{d\xi}-2F_1h_2u_2^2+h_1u_2\frac{du_2}{d\xi}+h_2u_2\frac{du_1}{d\xi}-3F_1u_3^2$$

$$+u_3\frac{du_2}{d\xi}+h_1u_3\frac{du_1}{d\xi}+u_4\frac{du_1}{d\xi}-\frac{6F_1}{Fr_0^2}h_6+\frac{1}{Fr_0^2}\frac{dh_5}{d\xi}-\frac{6F_1}{Fr_0^2}h_1h_5$$

$$+\frac{1}{Fr_0^2}h_1\frac{dh_4}{d\xi}-\frac{6F_1}{Fr_0^2}h_2h_4+\frac{1}{Fr_0^2}h_2\frac{dh_3}{d\xi}-\frac{3F_1}{Fr_0^2}h_3^2+\frac{1}{Fr_0^2}h_3\frac{dh_2}{d\xi}+\frac{1}{Fr_0^2}h_4\frac{dh_1}{d\xi}$$

$$=c_f\left(h_5-2u_5-2u_1u_4-2u_2u_3\right) \tag{2.2.25}$$

From the equations described above, the relationships for each order on water depth and velocity ($h_1, u_1, h_2, u_2, h_3, u_3, h_4, u_4, h_5$ and u_5) can be derived. These equations are expressed below.

$$-\frac{1}{Fr_0}\frac{dh_2}{d\xi}-\frac{du_2}{d\xi}+2F_1\left(1-F_1\right)h_2u_1+F_1\left(1-F_1\right)h_1u_2+\left\{-\left(1-F_1\right)u_1-\frac{F_1}{Fr_0^{\ 2}}h_1\right\}\frac{dh_1}{d\xi}$$
$$+\left\{-h_1-F_1u_1\right\}\frac{du_1}{d\xi}+3F_1^{\ 2}u_1u_2+F_1^{\ 2}h_1u_1^2+\frac{3F_1^{\ 2}}{Fr_0^{\ 2}}h_1h_2+F_1c_fh_2$$
$$-2F_1c_fu_2-F_1c_fu_1^2=0 \tag{2.2.26}$$

$$\frac{3}{Fr_0}h_3-3u_3=-\frac{\left(Fr_0^2-1\right)}{Fr_0^2}\frac{dh_2}{d\xi}+\left(2F_1+1\right)h_2u_1+\left(F_1+2\right)h_1u_2-\frac{\left(Fr_0-2\right)}{Fr_0^2}h_1\frac{dh_1}{d\xi}$$
$$-3F_1u_1u_2-F_1h_1u_1^2-\frac{3F_1}{Fr_0^2}h_1h_2-c_fh_2+2c_fu_2+c_fu_1^2 \tag{2.2.27}$$

$$-\frac{1}{Fr_0}\frac{dh_3}{d\xi}-\frac{du_3}{d\xi}-\frac{2\left(1-F_1\right)}{Fr_0}h_1\frac{dh_2}{d\xi}-\frac{\left(Fr_0+2\right)}{\left(Fr_0+1\right)}h_1\frac{du_2}{d\xi}+\left\{-\left(1-F_1\right)u_2-\frac{2F_1}{Fr_0^{\ 2}}h_2\right\}\frac{dh_1}{d\xi}$$
$$+\left\{-h_2-F_1h_1u_1-F_1u_2\right\}\frac{du_1}{d\xi}+3F_1\left(1-F_1\right)h_3u_1+2F_1\left(1-F_1\right)h_2u_2$$
$$+F_1\left(1-F_1\right)h_1u_3+4F_1^{\ 2}u_1u_3+3F_1^{\ 2}h_1u_1u_2+2F_1^{\ 2}h_1u_2^2+F_1^{\ 2}h_2u_1^{\ 2}$$
$$+\frac{4F_1^{\ 2}}{Fr_0^{\ 2}}h_1h_3+\frac{2F_1^{\ 2}}{Fr_0^{\ 2}}h_2^{\ 2}+F_1c_fh_3-2F_1c_fu_3-2F_1c_fu_1u_2=0 \tag{2.2.28}$$

$$\frac{4}{Fr_0}h_4-4u_4=-\frac{\left(Fr_0^2-1\right)}{Fr_0^2}\frac{dh_3}{d\xi}+\left(3F_1+1\right)h_3u_1+\left(2F_1+2\right)h_2u_2+\left(F_1+3\right)h_1u_3$$
$$-\frac{\left(Fr_0-1\right)}{Fr_0^2}h_1\frac{dh_2}{d\xi}+u_1\frac{du_2}{d\xi}+\left\{-u_2+\frac{1}{Fr_0^2}h_2\right\}\frac{dh_1}{d\xi}+\left\{h_1u_1+u_2\right\}\frac{du_1}{d\xi}$$
$$-4F_1u_1u_3-3F_1h_1u_1u_2-F_1h_2u_1^2-2F_1u_2^2-\frac{4F_1}{Fr_0^2}h_1h_3$$
$$-\frac{2F_1}{Fr_0^2}h_2^{\ 2}-c_fh_3+2c_fu_3+c_fu_1u_2 \tag{2.2.29}$$

$$-\frac{1}{Fr_0}\frac{dh_4}{d\xi}-\frac{du_4}{d\xi}-\frac{2\left(1-F_1\right)}{Fr_0}h_1\frac{dh_3}{d\xi}-\frac{\left(Fr_0+2\right)}{\left(Fr_0+1\right)}h_1\frac{du_3}{d\xi}+\left\{-\left(1-F_1\right)u_2-\frac{F_1}{Fr_0^{\ 2}}h_2\right\}\frac{dh_2}{d\xi}$$

$$+\left\{-h_2-F_1h_1u_1-F_1u_2\right\}\frac{du_2}{d\xi}+\left\{-\left(1-F_1\right)u_3-\frac{F_1}{Fr_0{}^2}h_3\right\}\frac{dh_1}{d\xi}$$

$$+\left\{-h_3-F_1\left(h_2u_1-h_1u_2-u_3\right)\right\}\frac{du_1}{d\xi}+4F_1\left(1-F_1\right)h_4u_1+3F_1\left(1-F_1\right)h_3u_2$$

$$+2F_1\left(1-F_1\right)h_2u_3+F_1\left(1-F_1\right)h_1u_4+5F_1^2u_1u_4+4F_1{}^2h_1u_1u_3+3F_1{}^2h_2u_1u_2$$

$$+5F_1{}^2u_2u_3+2F_1{}^2h_1u_2^2+F_1{}^2h_3u_1{}^2+\frac{5F_1{}^2}{Fr_0{}^2}h_1h_4+\frac{5F_1{}^2}{Fr_0{}^2}h_2h_3+F_1c_fh_4$$

$$-2F_1c_fu_4-2F_1c_fu_1u_3-F_1c_fu_2{}^2=0 \tag{2.2.30}$$

$$\frac{5}{Fr_0}h_5-5u_5=-\frac{\left(Fr_0^2-1\right)}{Fr_0^2}\frac{dh_4}{d\xi}+\left(4F_1+1\right)h_4u_1+\left(3F_1+2\right)h_3u_2+\left(2F_1+3\right)h_2u_3$$

$$+\left(F_1+4\right)h_1u_4-\frac{\left(Fr_0-1\right)}{Fr_0^2}h_1\frac{dh_3}{d\xi}+u_1\frac{du_3}{d\xi}+\left\{-u_2+\frac{1}{Fr_0^2}h_2\right\}\frac{dh_2}{d\xi}$$

$$+\left\{h_1u_1+u_2\right\}\frac{du_2}{d\xi}+\left\{-u_3+\frac{1}{Fr_0{}^2}h_3\right\}\frac{dh_1}{d\xi}+\left\{h_2u_1+h_1u_2+u_3\right\}\frac{du_1}{d\xi}$$

$$-5F_1u_1u_4-4F_1h_1u_1u_3-3F_1h_2u_1u_2-F_1h_3u_1^2-2F_1h_1u_2^2-5F_1u_2u_3$$

$$-\frac{5F_1}{Fr_0^2}h_1h_4-\frac{5F_1}{Fr_0^2}h_2h_3-c_fh_4+2c_fu_4+2c_fu_1u_3+c_fu_2{}^2 \tag{2.2.31}$$

$$-\frac{1}{Fr_0}\frac{dh_5}{d\xi}-\frac{du_5}{d\xi}-\frac{2\left(1-F_1\right)}{Fr_0}h_1\frac{dh_4}{d\xi}-\frac{\left(Fr_0+2\right)}{\left(Fr_0+1\right)}h_1\frac{du_4}{d\xi}+\left\{-\left(1-F_1\right)u_2-\frac{F_1}{Fr_0{}^2}h_2\right\}\frac{dh_3}{d\xi}$$

$$+\left\{-\left(1-F_1\right)h_2-F_1\left(h_2+h_1u_1+u_2\right)\right\}\frac{du_3}{d\xi}+\left\{-\left(1-F_1\right)u_3-\frac{F_1}{Fr_0{}^2}h_3\right\}\frac{dh_2}{d\xi}$$

$$+\left\{-\left(1-F_1\right)h_3-F_1\left(h_3+h_2u_1+h_1u_2+u_3\right)\right\}\frac{du_2}{d\xi}+\left\{-\left(1-F_1\right)u_4-\frac{F_1}{Fr_0{}^2}h_4\right\}\frac{dh_1}{d\xi}$$

$$+\left\{-\left(1-F_1\right)h_4-F_1\left(h_4+h_3u_1+h_2u_2+h_1u_3+u_4\right)\right\}\frac{du_1}{d\xi}+5F_1\left(1-F_1\right)h_5u_1$$

$$+4F_1\left(1-F_1\right)h_4u_2+3F_1\left(1-F_1\right)h_3u_3+2F_1\left(1-F_1\right)h_2u_4+F_1\left(1-F_1\right)h_1u_5$$

$$+6F_1{}^2u_1u_5+5F_1{}^2h_1u_1u_4+4F_1{}^2h_2u_1u_3+3F_1{}^2h_3u_1u_2+F_1{}^2h_4u_1{}^2+5F_1{}^2h_1u_2u_3$$

$$+6F_1{}^2u_1u_4+2F_1{}^2h_2u_2^2+3F_1{}^2u_3{}^2+\frac{6F_1{}^2}{Fr_0{}^2}h_1h_5+\frac{6F_1{}^2}{Fr_0{}^2}h_2h_4+\frac{3F_1{}^2}{Fr_0{}^2}h_3{}^2+F_1c_fh_5$$

$$-2F_1c_fu_5-2F_1c_fu_1u_4-2F_1c_fu_2u_3=0 \tag{2.2.32}$$

2.2.3. Derivation of the Solution of Linearized Equations

In order to derive the approximate solutions, only the linear terms in the above equations are firstly considered. The linearized equation of Eq. (2.2.16) becomes

$$\frac{dh_1}{d\xi}-\frac{1}{2}F_1\left(Fr_0-2\right)c_fh_1=0 \tag{2.2.33}$$

Rearranging and integrating this equation, the first order linear approximate solution can be calculated as Eq. (2.2.36).

$$\int_0^\xi \frac{dh_1}{h_1}=\int_0^\xi \frac{1}{2}F_1\left(Fr_0-2\right)c_fd\xi \tag{2.2.34}$$

$$\ln h_1\left(\xi\right)-\ln h_1\left(0\right)=\frac{1}{2}F_1\left(Fr_0-2\right)c_f\xi=\frac{1}{2}Y_1\xi \tag{2.2.35}$$

in which, $Y_1=F_1\left(Fr_0-2\right)c_f$.

$$h_1\left(\xi\right)=h_1\left(0\right)\exp\left(\frac{1}{2}Y_1\xi\right) \tag{2.2.36}$$

Substituting Eq. (2.2.36) into Eq. (2.2.13) gives the first order solution for velocity.

$$u_1\left(\xi\right)=\frac{1}{Fr_0}h_1\left(0\right)\exp\left(\frac{1}{2}Y_1\xi\right) \tag{2.2.37}$$

The linear equations of Eq. (2.2.17) and (2.2.26) can also be written as follows;

$$h_2-Fr_0u_2=-\frac{\left(Fr_0+1\right)\left(Fr_0-2\right)}{4}c_fh_1 \tag{2.2.38}$$

$$-\frac{1}{Fr_0}\frac{dh_2}{d\xi}-\frac{du_2}{d\xi}=-F_1c_fh_2+2F_1c_fu_2 \tag{2.2.39}$$

Rearranging these two equations and substituting the solution of $h_1\left(\xi\right)$ given by Eq. (2.2.36), the linear solution for h_2 and u_2 can be derived.

$$u_2=\frac{1}{Fr_0}h_2+\frac{\left(Fr_0+1\right)\left(Fr_0-2\right)}{4Fr_0}c_fh_1 \tag{2.2.38'}$$

$$\frac{1}{Fr_0}\frac{dh_2}{d\xi}+\frac{1}{Fr_0}\frac{dh_2}{d\xi}+\frac{\left(Fr_0-2\right)\left(Fr_0-1\right)}{4Fr_0}c_f\frac{dh_1}{d\xi}=F_1c_fh_2$$

$$-2F_1 c_f\left(\frac{1}{Fr_0}h_2+\frac{(Fr_0+1)(Fr_0-2)}{4Fr_0}c_f h_1\right) \quad (2.2.39')$$

$$\frac{dh_2}{d\xi}-\frac{1}{2}F_1(Fr_0-2)c_f h_2=-\frac{1}{16}Fr_0(Fr_0-2)^2 c_f^2 h_1-\frac{1}{4}Fr_0(Fr_0-2)c_f^2 h_1 \quad (2.2.39'')$$

Integrating the equation above, the analytical solutions of second order can be derived as follows:

$$h_2(\xi)=H_{2S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+H_{2G}\exp\left(\frac{1}{2}Y_1\xi\right) \quad (2.2.40)$$

$$u_2(\xi)=\frac{1}{Fr_0}\left\{H_{2S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+H_{2G}\exp\left(\frac{1}{2}Y_1\xi\right)\right\}$$

$$+\frac{1}{4Fr_0}(Fr_0-2)(Fr_0+1)c_f h_1(0)\exp\left(\frac{1}{2}Y_1\xi\right) \quad (2.2.41)$$

where, $H_{2S_1}=-\frac{1}{16}Fr_0\left(Fr_0^2-4\right)c_f{}^2 h_1(0)$ and $H_{2G}=h_2(0)$.

Similarity, the linear equations of Eq. (2.2.27) to (2.2.32) can be expressed as

$$\frac{3}{Fr_0}h_3-3u_3=-\frac{\left(Fr_0^2-1\right)}{Fr_0^2}\frac{dh_2}{d\xi}-c_f h_2+2c_f u_2 \quad (2.2.42)$$

$$-\frac{1}{Fr_0}\frac{dh_3}{d\xi}-\frac{du_3}{d\xi}=F_1 c_f h_3-2F_1 c_f u_3 \quad (2.2.43)$$

$$\frac{4}{Fr_0}h_4-4u_4=-\frac{\left(Fr_0^2-1\right)}{Fr_0^2}\frac{dh_3}{d\xi}-c_f\left(h_3-2u_3\right) \quad (2.2.44)$$

$$-\frac{1}{Fr_0}\frac{dh_4}{d\xi}-\frac{du_4}{d\xi}=-F_1 c_f h_4+2F_1 c_f u_4 \quad (2.2.45)$$

$$\frac{5}{Fr_0}h_5-5u_5=-\frac{\left(Fr_0^2-1\right)}{Fr_0^2}\frac{dh_4}{d\xi}-c_f\left(h_4-2u_4\right) \quad (2.2.46)$$

$$-\frac{1}{Fr_0}\frac{dh_5}{d\xi}-\frac{du_5}{d\xi}=-F_1 c_f h_5+2F_1 c_f u_5 \quad (2.2.47)$$

Applying the same procedures, the higher order analytical solutions can be derived by using Eq. (2.2.42) to Eq. (2.2.47). The solutions can be expressed as:

$$h_3(\xi)=H_{3S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+H_{3S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right)+H_{3G}\exp\left(\frac{1}{2}Y_1\xi\right) \quad (2.2.48)$$

where, $H_{3S_1} = -\dfrac{1}{6Fr_0}\left(Fr_0^{\,2}-1\right)C_{3,1} - \dfrac{1}{6}Fr_0 c_f C_{3,3}$,

$$H_{3S_2} = \frac{1}{2}\left\{-\frac{1}{6Fr_0}\left(Fr_0^{\,2}-1\right)C_{3,2} - \frac{1}{6}Fr_0 c_f C_{3,4}\right\} \text{ and } H_{3G} = h_3(0),$$

$$C_{3,1} = \left(\frac{1}{4}Y_1^2 + F_1 c_f Y_1\right)H_{2G} + \left(Y_1 + 2F_1 c_f\right)H_{2S_1},$$

$$C_{3,2} = \left(\frac{1}{4}Y_1^2 + F_1 c_f Y_1\right)H_{2S_1},$$

$$C_{3,3} = \frac{\left(Fr_0 - 2\right)}{Fr_0}\left\{H_{2S_1} + \left(\frac{1}{2}Y_1 + 2F_1 c_f\right)H_{2G} - \frac{1}{2}\left(Fr_0 + 1\right)c_f\left(\frac{1}{2}Y_1 + 2F_1 c_f\right)h_1(0)\right\},$$

$$C_{3,4} = \frac{\left(Fr_0 - 2\right)}{Fr_0}\left\{\left(\frac{1}{2}Y_1 + 2F_1 c_f\right)H_{2S_1}\right\}.$$

$$u_3(\xi) = U_{3S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right) + U_{3S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right) + u_3(0)\exp\left(\frac{1}{2}Y_1\xi\right) \tag{2.2.49}$$

where, $U_{3S_1}(\xi) = \dfrac{1}{Fr_0}H_{3S_1} + \dfrac{1}{6Fr_0^{\,2}}\left(Fr_0^{\,2}-1\right)Y_1 H_{2S_1} + \dfrac{1}{3Fr_0}\left(Fr_0 - 2\right)c_f H_{2S_1}$.

$$U_{3S_2}(\xi) = \frac{1}{Fr_0}H_{3S_2},$$

$$u_3(0) = \frac{1}{Fr_0}h_3(0) + \frac{1}{6Fr_0^{\,2}}\left(Fr_0^{\,2}-1\right)Y_1 h_2(0) + \frac{1}{3Fr_0}\left(Fr_0 - 2\right)c_f h_2(0)$$

$$+\frac{1}{3Fr_0^{\,2}}\left(Fr_0^{\,2}-1\right)H_{2S_1} - \frac{1}{6Fr_0}\left(Fr_0 - 2\right)\left(Fr_0 + 1\right)c_f^{\,2}h_1(0).$$

$$h_4(\xi) = H_{4S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right) + H_{4S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right) + H_{4S_3}\xi^3\exp\left(\frac{1}{2}Y_1\xi\right)$$

$$+H_{4G}\exp\left(\frac{1}{2}Y_1\xi\right) \tag{2.2.50}$$

where, $H_{4S_1} = -\dfrac{1}{8Fr_0}\left(Fr_0^{\,2}-1\right)C_{4,1} - \dfrac{1}{8}Fr_0 c_f C_{4,4}$,

$$H_{4S_2} = \frac{1}{2}\left\{-\frac{1}{8Fr_0}\left(Fr_0^{\,2}-1\right)C_{4,2} - \frac{1}{8}Fr_0 c_f C_{4,5}\right\},$$

$$H_{4S_3} = \frac{1}{3}\left\{ -\frac{1}{8Fr_0}\left(Fr_0^{\ 2}-1\right)C_{4,3} - \frac{1}{8}Fr_0 c_f C_{4,6} \right\},$$

$$H_{4G} = h_4(0),$$

$$C_{4,1} = \left(\frac{1}{4}Y_1^2 + F_1 c_f Y_1\right)H_{3G} + \left(Y_1 + 2F_1 c_f\right)H_{3S_1} + 2H_{3S_2},$$

$$C_{4,2} = \left(\frac{1}{4}Y_1^2 + F_1 c_f Y_1\right)H_{3S_1} + \left(2Y_1 + 4F_1 c_f\right)H_{3S_2},$$

$$C_{4,3} = \left(\frac{1}{4}Y_1^2 + F_1 c_f Y_1\right)H_{3S_2},$$

$$C_{4,4} = \frac{\left(Fr_0 - 2\right)}{Fr_0}H_{3S_1} + \left\langle -\frac{1}{3Fr_0^{\ 2}}\left(Fr_0^{\ 2}-1\right)\left(Y_1 + 4F_1 c_f\right)\right.$$

$$\left. -\frac{1}{3Fr_0}\left(Fr_0 - 2\right)\left(Fr_0 + 1\right)c_f \right\rangle H_{2S_1} + \frac{\left(Fr_0 - 2\right)}{Fr_0}\left(\frac{1}{2}Y_1 + 2F_1 c_f\right)H_{3G}$$

$$-\frac{1}{3Fr_0}\left(Fr_0 - 2\right)\left(Fr_0 + 1\right)c_f\left(\frac{1}{2}Y_1 + 2F_1 c_f\right)H_{2G}$$

$$+\frac{1}{3Fr_0}\left(Fr_0 - 2\right)\left(Fr_0 + 1\right)c_f^{\ 2}\left(\frac{1}{2}Y_1 + 2F_1 c_f\right)h_1(0),$$

$$C_{4,5} = \frac{\left(Fr_0 - 2\right)}{Fr_0}\left\{ 2H_{3S_2} + \left(\frac{1}{2}Y_1 + 2F_1 c_f\right)H_{3S_1}\right.$$

$$\left. -\frac{1}{3}\left(Fr_0 + 1\right)c_f\left(\frac{1}{2}Y_1 + 2F_1 c_f\right)H_{2S_1}\right\},$$

$$C_{4,6} = \frac{\left(Fr_0 - 2\right)}{Fr_0}\left\{\left(\frac{1}{2}Y_1 + 2F_1 c_f\right)H_{3S_2}\right\}.$$

$$u_4(\xi) = U_{4S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right) + U_{4S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right) + U_{4S_3}\xi^3\exp\left(\frac{1}{2}Y_1\xi\right)$$

$$+u_4(0)\exp\left(\frac{1}{2}Y_1\xi\right) \tag{2.2.51}$$

where, $U_{4S_1}(\xi) = \dfrac{1}{Fr_0}H_{4S_1} + \dfrac{1}{4Fr_0^{\ 2}}\left(Fr_0^{\ 2}-1\right)\left(2H_{3S_2} + \dfrac{1}{2}Y_1 H_{3S_1}\right)$

$$+\frac{1}{4Fr_0}\left(Fr_0 - 2\right)c_f H_{3S_1} - \frac{1}{3Fr_0}\left(Fr_0 - 2\right)\left(Fr_0 + 1\right)c_f H_{2S_1},$$

$$U_{4S_2}(\xi)=\frac{1}{Fr_0}H_{4S_2}+\frac{1}{8Fr_0^{\ 2}}\left(Fr_0^{\ 2}-1\right)Y_1H_{3S_2}+\frac{1}{4Fr_0}\left(Fr_0-2\right)c_fH_{3S_2},$$

$$U_{4S_3}(\xi)=\frac{1}{Fr_0}H_{4S_3},$$

$$u_4(0)=\frac{1}{Fr_0}h_4(0)+\frac{1}{4Fr_0^{\ 2}}\left(Fr_0^{\ 2}-1\right)\left(H_{3S_1}+\frac{1}{2}Y_1h_3(0)\right)$$

$$+\frac{1}{4}c_f\left(-\frac{2}{3Fr_0^{\ 2}}\left(Fr_0^{\ 2}-1\right)H_{2S_1}+\frac{\left(Fr_0-2\right)}{Fr_0}H_{3G}\right.$$

$$\left.-\frac{1}{3Fr_0}\left(Fr_0-2\right)\left(Fr_0+1\right)c_fH_{2G}+\frac{1}{3Fr_0}\left(Fr_0-2\right)\left(Fr_0+1\right)c_f^{\ 2}h_1(0)\right).$$

$$h_5(\xi)=H_{5S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+H_{5S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right)+H_{5S_3}\xi^3\exp\left(\frac{1}{2}Y_1\xi\right)$$

$$+H_{5S_4}\xi^4\exp\left(\frac{1}{2}Y_1\xi\right)+H_{5G}\exp\left(\frac{1}{2}Y_1\xi\right) \tag{2.2.52}$$

where, $H_{5S_1}=-\frac{1}{10Fr_0}\left(Fr_0^{\ 2}-1\right)C_{5,1}-\frac{1}{10}Fr_0c_fC_{5,5}$,

$$H_{5S_2}=\frac{1}{2}\left\{-\frac{1}{10Fr_0}\left(Fr_0^{\ 2}-1\right)C_{5,2}-\frac{1}{10}Fr_0c_fC_{5,6}\right\},$$

$$H_{5S_3}=\frac{1}{3}\left\{-\frac{1}{10Fr_0}\left(Fr_0^{\ 2}-1\right)C_{5,3}-\frac{1}{10}Fr_0c_fC_{5,7}\right\},$$

$$H_{5S_4}=\frac{1}{4}\left\{-\frac{1}{10Fr_0}\left(Fr_0^{\ 2}-1\right)C_{5,4}-\frac{1}{10}Fr_0c_fC_{5,8}\right\},$$

$$H_{5G}=h_5(0),$$

$$C_{5,1}=\left(\frac{1}{4}Y_1^2+F_1c_fY_1\right)H_{4G}+\left(Y_1+2F_1c_f\right)H_{4S_1}+2H_{4S_2},$$

$$C_{5,2}=\left(\frac{1}{4}Y_1^2+F_1c_fY_1\right)H_{4S_1}+\left(2Y_1+4F_1c_f\right)H_{4S_2}+6H_{4S_3},$$

$$C_{5,3}=\left(\frac{1}{4}Y_1^2+F_1c_fY_1\right)H_{4S_2}+\left(3Y_1+6F_1c_f\right)H_{4S_3},$$

$$C_{5,4}=\left(\frac{1}{4}Y_1^2+F_1c_fY_1\right)H_{4S_3},$$

$$C_{5,5}=\left(\frac{1}{2}Y_1+2F_1c_f\right)H_{4G}-\left(Y_1+4F_1c_f\right)u_4\left(0\right)+H_{4S_1}-2U_{4S_1},$$

$$C_{5,6}=\left(\frac{1}{2}Y_1+2F_1c_f\right)H_{4S_1}-\left(Y_1+4F_1c_f\right)U_{4S_1}+2H_{4S_2}-4U_{4S_2},$$

$$C_{5,7}=\left(\frac{1}{2}Y_1+2F_1c_f\right)H_{4S_2}-\left(Y_1+4F_1c_f\right)U_{4S_2}+3H_{4S_3}-6U_{4S_3},$$

$$C_{5,8}=\left(\frac{1}{2}Y_1+2F_1c_f\right)H_{4S_3}-\left(Y_1+4F_1c_f\right)U_{4S_3},$$

$$u_5\left(\xi\right)=U_{5S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+U_{5S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right)+U_{5S_3}\xi^3\exp\left(\frac{1}{2}Y_1\xi\right)$$
$$+U_{5S_4}\xi^4\exp\left(\frac{1}{2}Y_1\xi\right)+u_5\left(0\right)\exp\left(\frac{1}{2}Y_1\xi\right) \qquad (2.2.53)$$

where, $U_{5S_1}\left(\xi\right)=\frac{1}{Fr_0}H_{5S_1}+\frac{1}{5Fr_0^{\,2}}\left(Fr_0^{\,2}-1\right)\left(2H_{4S_2}+\frac{1}{2}Y_1H_{4S_1}\right)+\frac{1}{5}c_f\left(H_{4S_1}-2U_{4S_1}\right),$

$$U_{5S_2}\left(\xi\right)=\frac{1}{Fr_0}H_{5S_2}+\frac{1}{5Fr_0^{\,2}}\left(Fr_0^{\,2}-1\right)\left(3H_{4S_3}+\frac{1}{2}Y_1H_{4S_2}\right)+\frac{1}{5}c_f\left(H_{4S_2}-2U_{4S_2}\right),$$

$$U_{5S_3}\left(\xi\right)=\frac{1}{Fr_0}H_{5S_3}+\frac{1}{5Fr_0^{\,2}}\left(Fr_0^{\,2}-1\right)\left(\frac{1}{2}Y_1H_{4S_3}\right)+\frac{1}{5}c_f\left(H_{4S_3}-2U_{4S_3}\right),$$

$$U_{5S_4}\left(\xi\right)=\frac{1}{Fr_0}H_{4S_4},$$

$$u_5\left(0\right)=\frac{1}{Fr_0}h_5\left(0\right)+\frac{1}{5Fr_0^{\,2}}\left(Fr_0^{\,2}-1\right)\left(H_{4S_1}+\frac{1}{2}Y_1h_4\left(0\right)\right)$$
$$+\frac{1}{4}c_f\left(h_4\left(0\right)-2u_4\left(0\right)\right).$$

Substituting Eq. (2.2.36), (2.2.40), (2.2.48), (2.2.50) and (2.2.52) into Eq. (2.2.7) gives:

$$h\left(\xi\right)=1.0+\left[h_1\left(0\right)\exp\left(\frac{1}{2}Y_1\xi\right)\right]\tau+\left[H_{2S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+H_{2G}\exp\left(\frac{1}{2}Y_1\xi\right)\right]\tau^2$$
$$+\left[H_{3S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+H_{3S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right)+H_{3G}\exp\left(\frac{1}{2}Y_1\xi\right)\right]\tau^3$$
$$+\left[H_{4S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+H_{4S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right)+H_{4S_3}\xi^3\exp\left(\frac{1}{2}Y_1\xi\right)\right.$$

$$+H_{4G}\exp\left(\frac{1}{2}Y_1\xi\right)\Bigg]\tau^4+\Bigg[H_{5S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+H_{5S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right)$$

$$+H_{5S_3}\xi^3\exp\left(\frac{1}{2}Y_1\xi\right)+H_{5S_4}\xi^4\exp\left(\frac{1}{2}Y_1\xi\right)+H_{5G}\exp\left(\frac{1}{2}Y_1\xi\right)\Bigg]\tau^5 \qquad (2.2.54)$$

Similarly, Eq. (2.2.55) can be obtained by substituting Eq. (2.2.37), (2.2.41), (2.2.49), (2.2.51) and (2.2.53) into Eq. (2.2.8).

$$u(\xi)=1.0+\left[\frac{1}{Fr_0}h_1(0)\exp\left(\frac{1}{2}Y_1\xi\right)\right]\tau+\Bigg[\frac{1}{Fr_0}\left\{H_{2S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+H_{2G}\exp\left(\frac{1}{2}Y_1\xi\right)\right\}$$

$$+\frac{1}{4Fr_0}(Fr_0-2)(Fr_0+1)c_f h_1(0)\exp\left(\frac{1}{2}Y_1\xi\right)\Bigg]\tau^2$$

$$+\left[U_{3S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+U_{3S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right)+u_3(0)\exp\left(\frac{1}{2}Y_1\xi\right)\right]\tau^3$$

$$+\Bigg[U_{4S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+U_{4S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right)+U_{4S_3}\xi^3\exp\left(\frac{1}{2}Y_1\xi\right)$$

$$+u_4(0)\exp\left(\frac{1}{2}Y_1\xi\right)\Bigg]\tau^4+\Bigg[U_{5S_1}\xi\exp\left(\frac{1}{2}Y_1\xi\right)+U_{5S_2}\xi^2\exp\left(\frac{1}{2}Y_1\xi\right)$$

$$+U_{5S_3}\xi^3\exp\left(\frac{1}{2}Y_1\xi\right)+U_{5S_4}\xi^4\exp\left(\frac{1}{2}Y_1\xi\right)+u_5(0)\exp\left(\frac{1}{2}Y_1\xi\right)\Bigg]\tau^5 \qquad (2.2.55)$$

2.2.4. Comparison of Linear Solutions to Numerical Results by MOC

The analytical solutions derived in section 2.2.3 are needed to verify. Therefore, numerical simulations are conducted to compare the calculated results to analytical solutions. In order to get the numerical results of linearized equations, MOC can be applied (*Chaudry* (2008)). Linearized equations of Eq. (2.2.3) and (2.2.4) are expressed as Eq. (2.2.57) and (2.2.58) based on the concept of MOC. The characteristic lines for subcritical flows are expressed in Fig. 2.2.2. The numerical values at the point *R* can be calculated from the known values at points *P* and *Q* under the initial and boundary conditions.

$$h\equiv 1+\delta h \quad , \quad u\equiv 1+\delta u \qquad (2.2.56)$$

$$c_f^{\ 1} \quad \lambda_+ : \frac{D_+\delta h}{Dt}+Fr_0\frac{D_+\delta u}{Dt}=Fr_0(\delta h-2\delta u) \qquad (2.2.57)$$

$$\lambda_- : \frac{D_-\delta h}{Dt} - Fr_0 \frac{D_-\delta u}{Dt} = -Fr_0(\delta h - 2\delta u) \tag{2.2.58}$$

$$\left(\delta h + Fr_0 \delta u\right)_R - \left(\delta h + Fr_0 \delta u\right)_P = \int_{t_P}^{t_R} Fr_0 c_f \left(\delta h - 2\delta u\right) dt \tag{2.2.59}$$

$$\left(\delta h - Fr_0 \delta u\right)_R - \left(\delta h - Fr_0 \delta u\right)_Q = -\int_{t_Q}^{t_R} Fr_0 c_f \left(\delta h - 2\delta u\right) dt \tag{2.2.60}$$

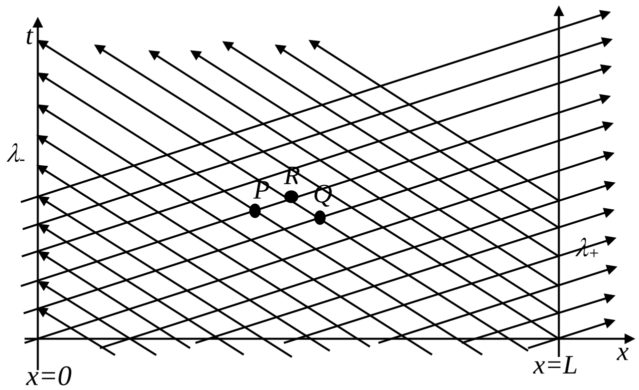


Figure.2.2.2. Expression of characteristic lines for subcritical flow.

The results of the analytical solutions are compared to the numerical simulation results for verification. The comparisons are carried out under the following conditions: the Froude number, 0.2; friction coefficient, 0.0025; the upstream boundary condition; $x = 0$: $h = 1.0 + \alpha_1 t + \alpha_2 t^2$ in which, $\alpha_1 = h_1(0)$ and $\alpha_2 = h_2(0)$, and the initial condition; $t = 0$: $h = h_0 = 1.0$. $u = u_0 = 1.0$. $h = 1.0$ and $u = 1.0$ are given as the downstream boundary conditions.

Fig.2.2.3 represents the water depth hydrographs from analytical solutions and numerical simulation with the method of characteristics for linearized equations (hereafter LinearMOC). The upstream boundary condition was given as a quadratic function of hydrograph with $T_0 = 360$. From the figure, it can be seen that the results from analytical solutions with at least $h_1 \sim h_3$ and numerical results are in good agreement, indicating the validity of the theoretical analysis.

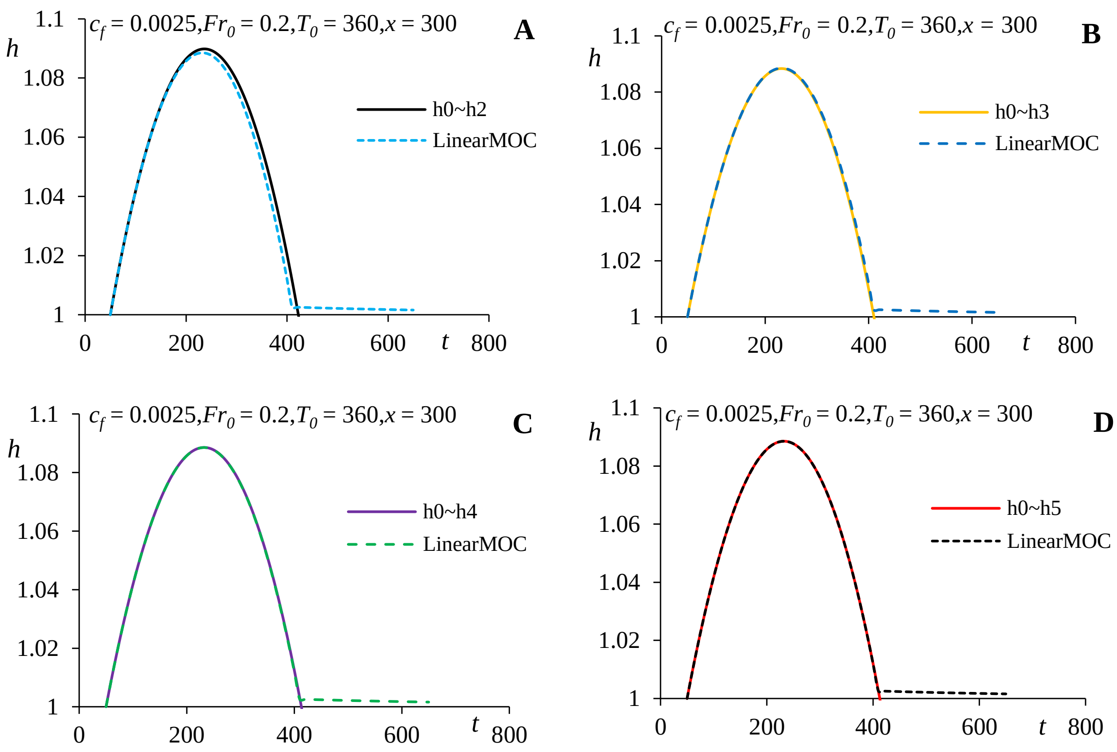
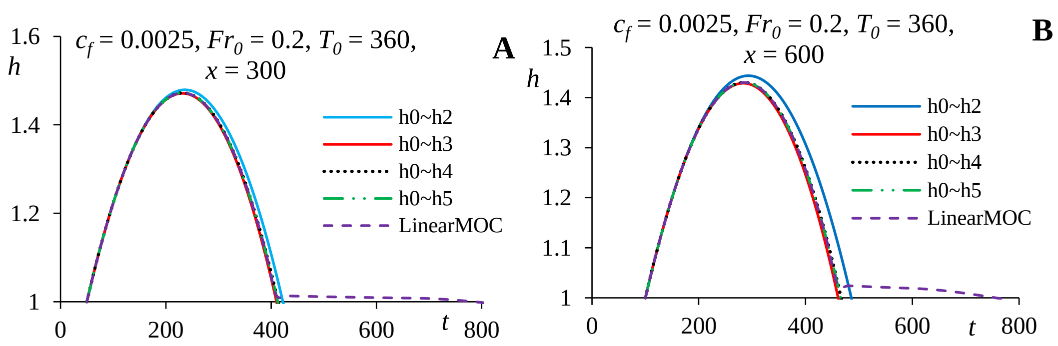


Figure.2.2.3. Comparison of the analytical solutions to numerical results.

In Fig.2.2.3A, the analytical solutions with $h_1 \sim h_2$ are slightly different from the numerical results. The results up to third order term to fifth order term are shown in Fig.2.2.3B, C and D. It can be pointed out that they are completely in agreement. It can be said that the analytical solutions with up to the 3rd order term can be used to consider the propagation of flood waves along the channel.

The water depth hydrographs along the channel at different locations are also shown in Fig.2.2.4. For these hydrographs, the initial conditions: $h(x,0)=1.0$ and $u(x,0)=1.0$ and at the upstream boundary $h_0=1.0$, $h_{peak}=1.5184$ and $T_0=360$ are used.



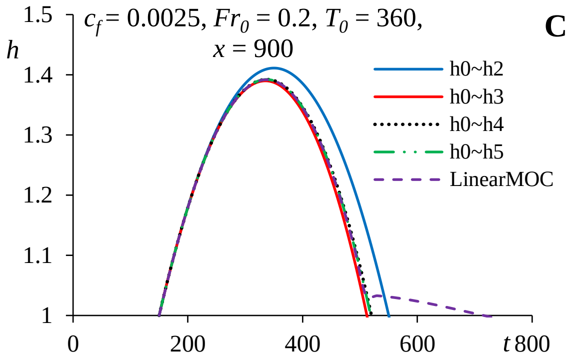


Figure.2.2.4. Propagation of hydrographs along the channel.

In order to verify the analytical solutions with the wide range of parameters, the Froude number, the peak value and time scale of the upstream depth hydrograph, etc are tested. The hydrograph propagations with $T_0 = 360$ and $Fr_0 = 0.5, 0.7$ are shown in Fig. 2.2.5.

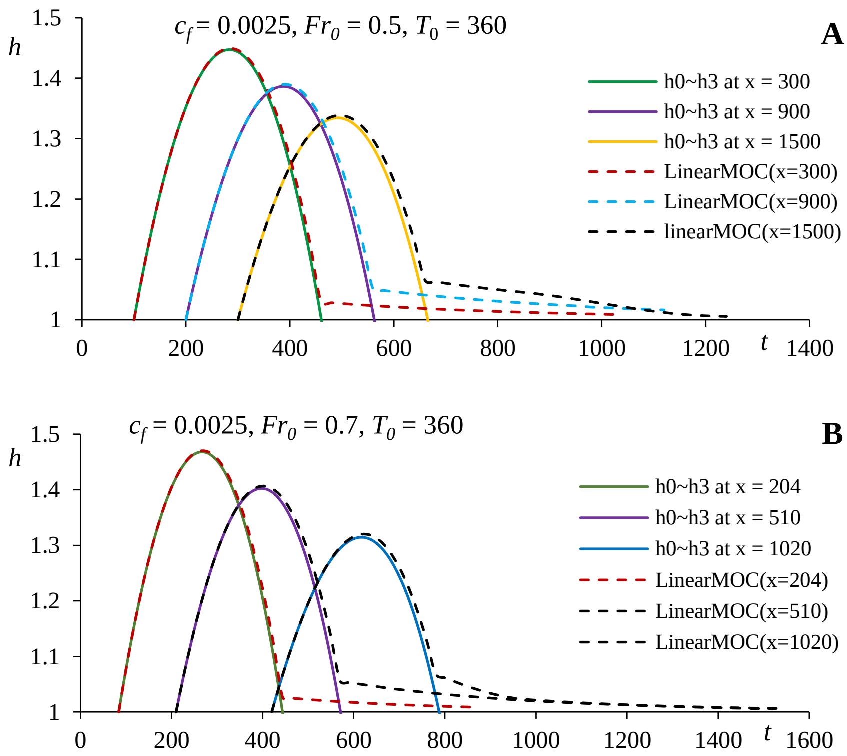
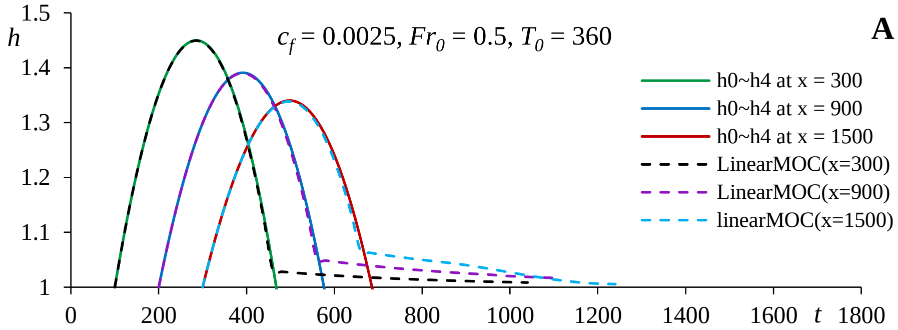


Figure.2.2.5. Hydrograph propagations for $T_0 = 360$ with $Fr_0 = 0.5$ and 0.7.

The analytical solutions summing up to 3rd order terms for large values of the Froude number shown slight differences with the numerical results at the locations far from the upstream end. The results summing up to the 4th order term are also shown in Fig. 2.2.6. In this figure, it can be clearly seen that both the analytical and numerical results are in good agreement.



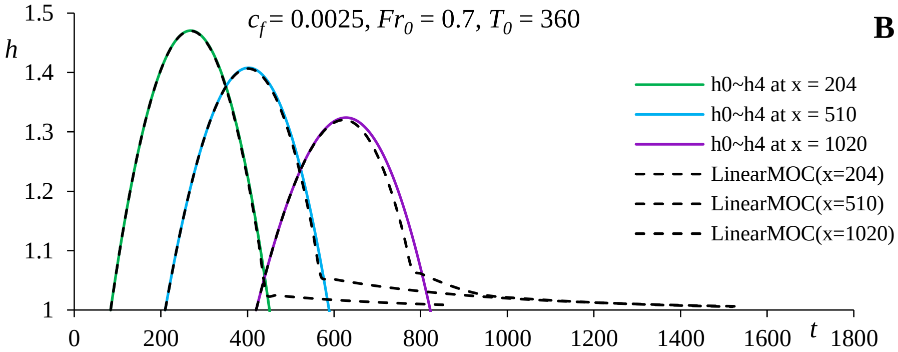
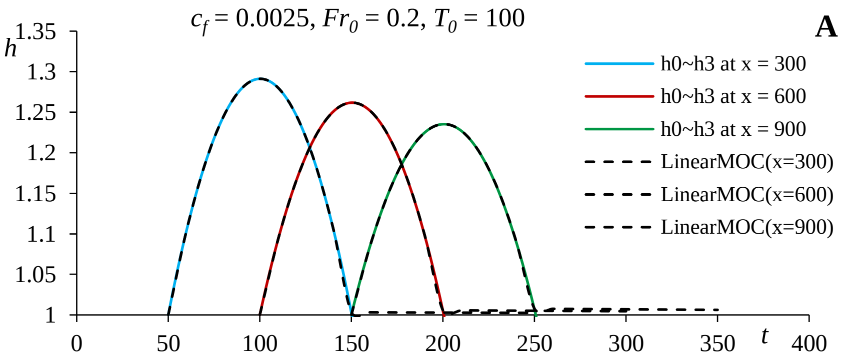


Figure.2.2.6. Propagation of flood waves with higher order terms.

The water depth hydrographs for $T_0 = 100$ and $T_0 = 720$ are shown in Fig. 2.2.7 and Fig. 2.2.8. From Fig.2.2.7, it can be pointed out that the analytical solutions for different Froude numbers are in good agreement with the numerical results. The analytical results for $T_0 = 720$ shown some differences from the numerical results as shown in Fig. 2.2.8. From these figures, it is pointed out that the analytical results for the hydrographs with larger time period differ slightly from the numerical results.



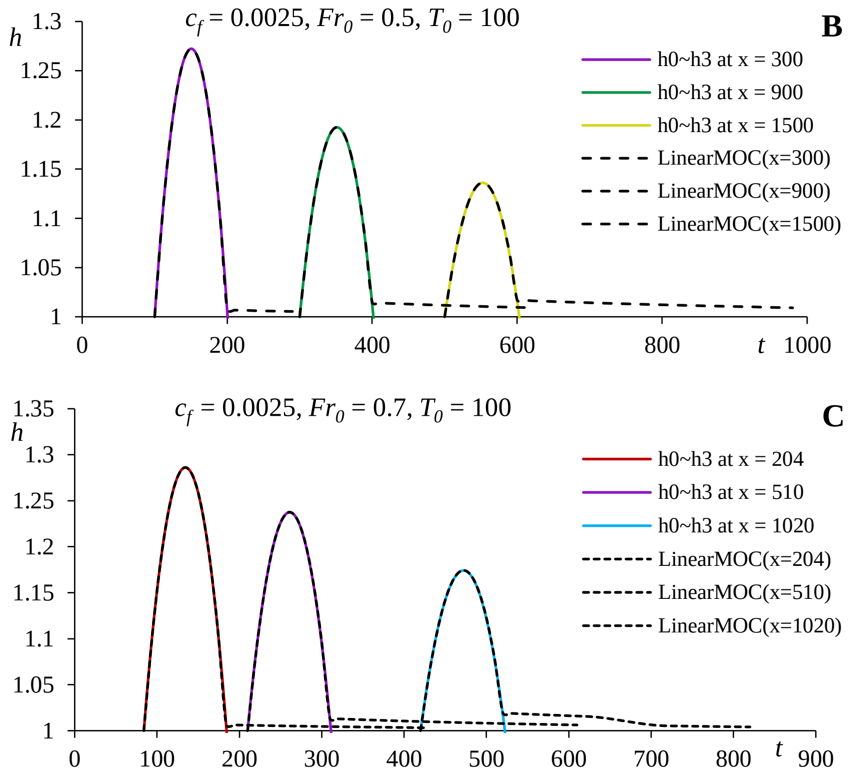


Figure. 2.2.7. Hydrograph propagations for $T_0 = 100$.

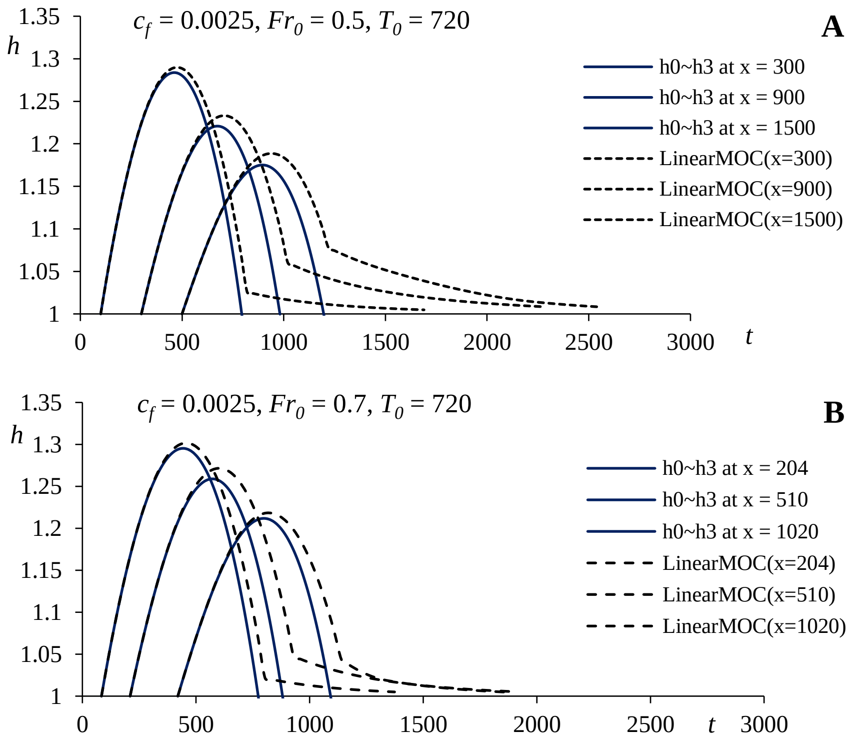


Figure.2.2.8. Hydrographs along the channel for different Fr_0.

2.2.5. Non-linear Analysis

Firstly, the non-linear solutions with 1st and 2nd order terms are derived. In order to derive the non-linear solutions, Eq. (2.2.16) is rearranged and transformed into the following equations.

$$\frac{dh_1}{d\xi_1}\frac{Y_1}{2}=\frac{Y_1}{2}h_1+\frac{P_2}{Y_1/2}h_1^{\,2}$$

in which, $\xi_1=\frac{Y_1}{2}\xi$,

$$P_2=\frac{3F_1}{2\left(Fr_0+1\right)}.$$

$$\frac{dh_1}{d\xi_1}=h_1-P_3h_1^{\,2} \tag{2.2.61}$$

where, $P_3=-\frac{P_2}{Y_1/2}$.

The first order non-linear solution can be calculated by integrating Eq. (2.2.61).

$$\int_0^{\xi_1}\frac{dh_1}{h_1\left(1-P_3h_1\right)}=\int_0^{\xi_1}d\xi_1$$

$$h_1\left(\xi_1\right)=\frac{h_1\left(0\right)\exp\left(\xi_1\right)}{1-P_3h_1\left(0\right)\left(1-\exp\left(\xi_1\right)\right)} \tag{2.2.62}$$

The equations for u_2 and the first order derivative of u_2 can be obtained by rearranging and transforming Eq. (2.2.17).

$$u_2=\frac{1}{Fr_0}h_2-\frac{1}{4}h_1^{\,2}+\frac{\left(Fr_0+1\right)\left(Fr_0-2\right)}{4}c_fh_1 \tag{2.2.63}$$

$$\frac{du_2}{d\xi}=\frac{1}{Fr_0}\frac{dh_2}{d\xi}-\frac{1}{2}h_1\frac{dh_1}{d\xi}+\frac{\left(Fr_0+1\right)\left(Fr_0-2\right)}{4}c_f\frac{dh_1}{d\xi} \tag{2.2.64}$$

Substituting Eq. (2.2.63), Eq. (2.2.64) and Eq. (2.2.13) into Eq. (2.2.26) and rearranging, Eq. (2.2.26) lead to the following equation.

$$\frac{dh_2}{d\xi}=\frac{Y_1}{2}h_2+\frac{9Fr_0}{2\left(Fr_0+1\right)^2}h_1h_2-\frac{1}{16}\left(Fr_0+1\right)\left(Fr_0+2\right)c_fY_1h_1$$

$$+\frac{\left(3Fr_0^2-5Fr_0+4\right)}{16Fr_0\left(Fr_0+1\right)}Y_1h_1^2-\frac{3\left(Fr_0+7\right)F_1}{8\left(Fr_0+1\right)^2}h_1^3 \tag{2.2.65}$$

For simplicity, Eq. (2.2.65) is transformed into Eq. (2.2.66) and Eq. (2.2.67).

$$\frac{dh_2}{d\xi}=R(0,1)h_2+R(1,1)h_1h_2+R(1,0)h_1+R(2,0)h_1^2+R(3,0)h_1^3 \tag{2.2.66}$$

in which, $R(0,1)=\frac{Y_1}{2}$, $R(1,1)=\frac{9Fr_0}{2\left(Fr_0+1\right)^2}$, $R(1,0)=-\frac{1}{16}\left(Fr_0+1\right)\left(Fr_0+2\right)c_fY_1$,

$$R(2,0)=\frac{\left(3Fr_0^2-5Fr_0+4\right)}{16Fr_0\left(Fr_0+1\right)}Y_1,\ R(3,0)=-\frac{3\left(Fr_0+7\right)F_1}{8\left(Fr_0+1\right)^2}h_1^3.$$

$$\frac{dh_2}{d\xi_1}=h_2+R(1,1)'h_1h_2+R(1,0)'h_1+R(2,0)'h_1^2+R(3,0)'h_1^3 \tag{2.2.67}$$

where, $R(1,1)'=\frac{R(1,1)}{Y_1/2}$, $R(1,0)'=\frac{R(1,0)}{Y_1/2}$, $R(2,0)'=\frac{R(2,0)}{Y_1/2}$, $R(3,0)'=\frac{R(3,0)}{Y_1/2}$.

In order to derive the second order non-linear solution, the first and second terms on the right side of Eq. (2.2.67) are firstly considered as described in Eq. (2.2.68).

$$\frac{d\hat{h}_2}{\hat{h}_2}=\left(1+R(1,1)'h_1\right)d\xi_1 \tag{2.2.68}$$

where, $\hat{h}_2$ = tentative solution.

Substituting the solution of $h_1(\xi_1)$ given by Eq. (2.2.62) into Eq. (2.2.68) and integrating it, the second order non-linear solution is calculated as Eq. (2.2.70).

$$\int_0^{\xi_1}\frac{d\hat{h}_2}{\hat{h}_2}=\int_0^{\xi_1}\left(1+R(1,1)'\frac{h_1(0)\exp(\xi_1)}{1-P_3h_1(0)\left(1-\exp(\xi_1)\right)}\right)d\xi_1 \tag{2.2.69}$$

$$\frac{\hat{h}_2(\xi_1)}{h_2(0)}=\left(\frac{c+dZ_1}{c+d}\right)^{e/d}Z_1^{\,f} \tag{2.2.70}$$

in which, $c=1-P_3h_1(0)$, $d=P_3h_1(0)$, $e=R(1,1)'h_1(0)$, $f=1$, $Z_1=\exp(\xi_1)$.

By considering all terms in Eq. (2.2.67), the second order non-linear solution becomes

$$h_2(\xi_1) = C_2(\xi_1) h_2(0) \left(\frac{c + dZ_1}{c + d} \right)^{-3} Z_1 \tag{2.2.71}$$

where, $C_2(\xi_1) = R(1,0)' \dfrac{h_1(0)}{h_2(0)} \left(c^2 \ln Z_1 + 2cdZ_1 + \dfrac{d^2 Z_1^{\,2}}{2} - 2cd - \dfrac{d^2}{2} \right)$

$$+R(2,0)' \frac{h_1(0)^2}{h_2(0)} \frac{1}{2d} \left((c + dZ_1)^2 - (c+d)^2 \right) + R(3,0)' \frac{h_1(0)^3}{h_2(0)} \frac{1}{2} \left(Z_1^{\,2} - 1 \right)$$

Then, the water depth hydrographs for the downstream region can be calculated by using Eq. (2.2.72).

$$h = h_1(\xi_1)\tau + h_2(\xi_1)\tau^2 \tag{2.2.72}$$

Since it is difficult to derive the non-linear solutions with higher order terms, the numerical method is applied to the higher order terms.

In order to verify the analytical solutions given by Eq. (2.2.72), the numerical calculation is also conducted using Eq. (2.2.61) and Eq. (2.2.67) by means of the finite difference method. Theses equations are expressed as Eq. (2.2.73) and Eq. (2.2.74).

$$\frac{h_1^{\,k+1} - h_1^{\,k}}{\Delta\xi_1} = h_1 - P_3 h_1^{\,2}$$

$$h_1^{\,k+1} = h_1^{\,k} + \Delta\xi_1 \left(h_1^{\,k} - P_3 \left(h_1^{\,k} \right)^2 \right) \tag{2.2.73}$$

$$\frac{h_2^{\,k+1} - h_2^{\,k}}{\Delta\xi_1} = h_2 + R(1,1)' h_1 h_2 + R(1,0)' h_1 + R(2,0)' h_1^{\,2} + R(3,0)' h_1^{\,3}$$

$$h_2^{\,k+1} = h_2^{\,k} + \Delta\xi_1 \Big(h_2^{\,k} + R(1,1)' h_1^{\,k} h_2^{\,k} + R(1,0)' h_1^{\,k} + R(2,0)' \left(h_1^{\,k} \right)^2$$

$$+R(3,0)' \left(h_1^{\,k} \right)^3 \Big) \tag{2.2.74}$$

In numerical solutions, the third order term is included in consideration. For the third order term, rearranging Eq. (2.2.27) gives

$$u_3 = \frac{1}{Fr_0} h_3 + \frac{\left(Fr_0^2 - 1 \right)}{3Fr_0^2} \frac{dh_2}{d\xi} + \frac{\left(Fr_0 - 2 \right)}{3Fr_0^2} h_1 \frac{dh_1}{d\xi} + \frac{\left(-2Fr_0 + 1 \right)}{Fr_0 \left(Fr_0 + 1 \right)} h_1 h_2 + \frac{\left(Fr_0 - 2 \right)}{3Fr_0} c_f h_2$$

$$+\frac{1}{4Fr_0} h_1^{\,3} - \left(3Fr_0 + 2 \right)\left(Fr_0 - 1 \right)\left(Fr_0 - 2 \right) c_f h_1^{\,2} - \frac{\left(Fr_0 + 1 \right)\left(Fr_0 - 2 \right)}{6Fr_0} c_f^{\,2} h_1 \tag{2.2.75}$$

$$u_3 = U(1,0,0)h_1 + U(2,0,0)h_1^2 + U(3,0,0)h_1^3 + U(0,1,0)h_2 + U(1,1,0)h_1h_2$$

$$+U(0,0,1)h_3 \tag{2.2.76}$$

in which, $U(1,0,0) = \dfrac{(Fr_0^2-1)}{3Fr_0^2}R(1,0) - \dfrac{(Fr_0-2)(Fr_0+1)}{6Fr_0}c_f^{\,2}$,

$$U(2,0,0) = \frac{(Fr_0^2-1)}{3Fr_0^2}R(2,0) - \frac{(3Fr_0+2)(Fr_0-1)(Fr_0-2)}{12Fr_0^{\,2}}c_f + \frac{(Fr_0-2)}{3Fr_0^{\,2}}R(0,1),$$

$$U(3,0,0) = \frac{(Fr_0^2-1)}{3Fr_0^2}R(3,0) + \frac{1}{4Fr_0} + \frac{(Fr_0-2)}{3Fr_0^{\,2}}P_2,$$

$$U(0,1,0) = \frac{(Fr_0^2-1)}{3Fr_0^2}R(0,1) + \frac{(Fr_0-2)}{3Fr_0}c_f,$$

$$U(1,1,0) = \frac{(Fr_0^2-1)}{3Fr_0^2}R(1,1) + \frac{(-2Fr_0+1)}{Fr_0(Fr_0+1)},$$

$$U(0,0,1) = \frac{1}{Fr_0}.$$

Rearranging and substituting into Eq. (2.2.28), the differential equation of the third order term can be obtained as follows;

$$\frac{dh_3}{d\xi} = -\frac{Fr_0}{2}\{H(1,0,0)h_1 + H(2,0,0)h_1^2 + H(3,0,0)h_1^3 + H(4,0,0)h_1^4$$
$$+H(5,0,0)h_1^5 + H(0,1,0)h_2 + H(0,2,0)h_2^2 + H(1,1,0)h_1h_2$$
$$+H(2,1,0)h_1^2h_2 + H(3,1,0)h_1^3h_2 + H(1,2,0)h_1h_2^2 + H(0,0,1)h_3$$
$$+H(1,0,1)h_1h_3\} \tag{2.2.77}$$

where, $H(1,0,0) = \left\langle \dfrac{2Fr_0}{(Fr_0+1)}c_f + R(0,1)\right\rangle U(1,0,0) + R(1,0)U(0,1,0)$,

$$H(2,0,0) = \left\langle \frac{-5Fr_0}{(Fr_0+1)^2} + P_2\right\rangle U(1,0,0) + \left\langle \frac{2Fr_0}{(Fr_0+1)}c_f + 2R(0,1)\right\rangle U(2,0,0)$$

$$+R(2,0)U(0,1,0)+R(1,0)\left\langle U(1,1,0)+\frac{(Fr_0+4)}{Fr_0(Fr_0+1)}\right\rangle$$

$$+\frac{(Fr_0-2)(Fr_0+4)}{4Fr_0}c_f R(0,1)+\frac{(Fr_0-2)}{2Fr_0}c_f{}^2,$$

$$H(3,0,0)=\left\langle\frac{-5Fr_0}{(Fr_0+1)^2}+2P_2\right\rangle U(2,0,0)+\left\langle\frac{2Fr_0}{(Fr_0+1)}c_f+3R(0,1)\right\rangle U(3,0,0)$$

$$+R(3,0)U(0,1,0)+R(2,0)\left\langle U(1,1,0)+\frac{(Fr_0+4)}{Fr_0(Fr_0+1)}\right\rangle-\frac{1}{2Fr_0}R(0,1)$$

$$+\frac{(Fr_0-2)(Fr_0+4)}{4Fr_0}c_f P_2-\frac{1}{2Fr_0(Fr_0+1)}c_f-\frac{3(Fr_0-2)}{4(Fr_0+1)}c_f$$

$$-\frac{(Fr_0-2)^2}{8}c_f{}^2,$$

$$H(4,0,0)=\left\langle\frac{-5Fr_0}{(Fr_0+1)^2}+3P_2\right\rangle U(3,0,0)+R(3,0)\left\langle U(1,1,0)+\frac{(Fr_0+4)}{Fr_0(Fr_0+1)}\right\rangle$$

$$-\frac{1}{2Fr_0}P_2-\frac{(Fr_0-2)}{4(Fr_0+1)}c_f+\frac{3}{4(Fr_0+1)^2}c_f,$$

$$H(5,0,0)=-\frac{1}{8(Fr_0+1)^2},$$

$$H(0,1,0)=\left\langle\frac{2Fr_0}{(Fr_0+1)}c_f+R(0,1)\right\rangle U(0,1,0),$$

$$H(0,2,0)=-\frac{4}{(Fr_0+1)^2},$$

$$H(1,1,0)=\left\langle\frac{-5Fr_0}{(Fr_0+1)^2}+R(1,1)\right\rangle U(0,1,0)+\left\langle\frac{2Fr_0}{(Fr_0+1)}c_f+2R(0,1)\right\rangle U(1,1,0)$$

$$+\frac{(2Fr_0+9)}{Fr_0(Fr_0+1)}R(0,1)-\frac{(Fr_0-2)}{2(Fr_0+1)}c_f+\frac{2}{Fr_0(Fr_0+1)}c_f,$$

$$H(2,1,0)=\left\langle\frac{-5Fr_0}{(Fr_0+1)^2}+R(1,1)+P_2\right\rangle U(1,1,0)$$

$$-\frac{7}{2\left(Fr_0+1\right)^2}-\frac{\left(Fr_0-2\right)}{\left(Fr_0+1\right)}c_f+\frac{\left(Fr_0+5\right)}{Fr_0\left(Fr_0+1\right)}P_2+\frac{\left(Fr_0+4\right)}{Fr_0\left(Fr_0+1\right)}R(1,1),$$

$$H\left(3,1,0\right)=\frac{1}{\left(Fr_0+1\right)^2},$$

$$H\left(1,2,0\right)=-\frac{2}{\left(Fr_0+1\right)^2},$$

$$H\left(0,0,1\right)=\frac{2Fr_0}{\left(Fr_0+1\right)}c_fU\left(0,0,1\right)-\frac{Fr_0}{\left(Fr_0+1\right)}c_f,$$

$$H\left(1,0,1\right)=\frac{-5Fr_0}{\left(Fr_0+1\right)^2}U\left(0,0,1\right)-\frac{7}{\left(Fr_0+1\right)^2}.$$

For simplicity,

$$\begin{aligned}\frac{dh_3}{d\xi_1}=-\frac{2}{Y_1}\times\frac{Fr_0}{2}\Big\{&H\left(1,0,0\right)h_1+H\left(2,0,0\right)h_1^{\,2}+H\left(3,0,0\right)h_1^{\,3}+H\left(4,0,0\right)h_1^{\,4}\\&+H\left(5,0,0\right)h_1^{\,5}+H\left(0,1,0\right)h_2+H\left(0,2,0\right)h_2^{\,2}+H\left(1,1,0\right)h_1h_2\\&+H\left(2,1,0\right)h_1^{\,2}h_2+H\left(3,1,0\right)h_1^{\,3}h_2+H\left(1,2,0\right)h_1h_2^{\,2}+H\left(0,0,1\right)h_3\\&+H\left(1,0,1\right)h_1h_3\Big\}\end{aligned}\qquad(2.2.78)$$

Using the finite difference method, the third order non-linear solution is calculated.

$$\begin{aligned}h_3^{\,k+1}=h_3^{\,k}-\Delta\xi_1\frac{2}{Y_1}\times\frac{Fr_0}{2}\Big\{&H\left(1,0,0\right)h_1^{\,k}+H\left(2,0,0\right)\left(h_1^{\,k}\right)^2+H\left(3,0,0\right)\left(h_1^{\,k}\right)^3+\\&H\left(4,0,0\right)\left(h_1^{\,k}\right)^4+H\left(5,0,0\right)\left(h_1^{\,k}\right)^5+H\left(0,1,0\right)h_2^{\,k}+H\left(0,2,0\right)\left(h_2^{\,k}\right)^2\\&+H\left(1,1,0\right)h_1^{\,k}h_2^{\,k}+H\left(2,1,0\right)\left(h_1^{\,k}\right)^2h_2^{\,k}+H\left(3,1,0\right)\left(h_1^{\,k}\right)^3h_2^{\,k}+H\left(1,2,0\right)h_1^{\,k}\left(h_2^{\,k}\right)^2\\&+H\left(0,0,1\right)h_3^{\,k}+H\left(1,0,1\right)h_1^{\,k}h_3^{\,k}\Big\}\end{aligned}\qquad(2.2.79)$$

From the equations described above, the water depth hydrographs can be calculated as follows;

$$h=h_1^{\,k+1}\tau+h_2^{\,k+1}\tau^2+h_3^{\,k+1}\tau^3\qquad(2.2.80)$$

The nonlinear solutions from Eq. (2.2.72) and the solutions obtained using numerical calculations by means of finite difference method from Eq. (2.2.80) are shown in Fig. 2.2.9.

These water depth hydrographs for $T_0 = 100$ and $T_0 = 360$ are also compared with the hydrographs from linear analytical solutions. The non-linear analytical solutions are different from linear analytical solutions and it pointed out that the differences are caused by the nonlinear effect. From Fig. 2.2.9, it can be seen that the solutions with numerical calculations show oscillation between the left side and right side of the linear solutions according to the power terms added. It is expected that considering the higher order terms on τ may result in the convergent solutions.

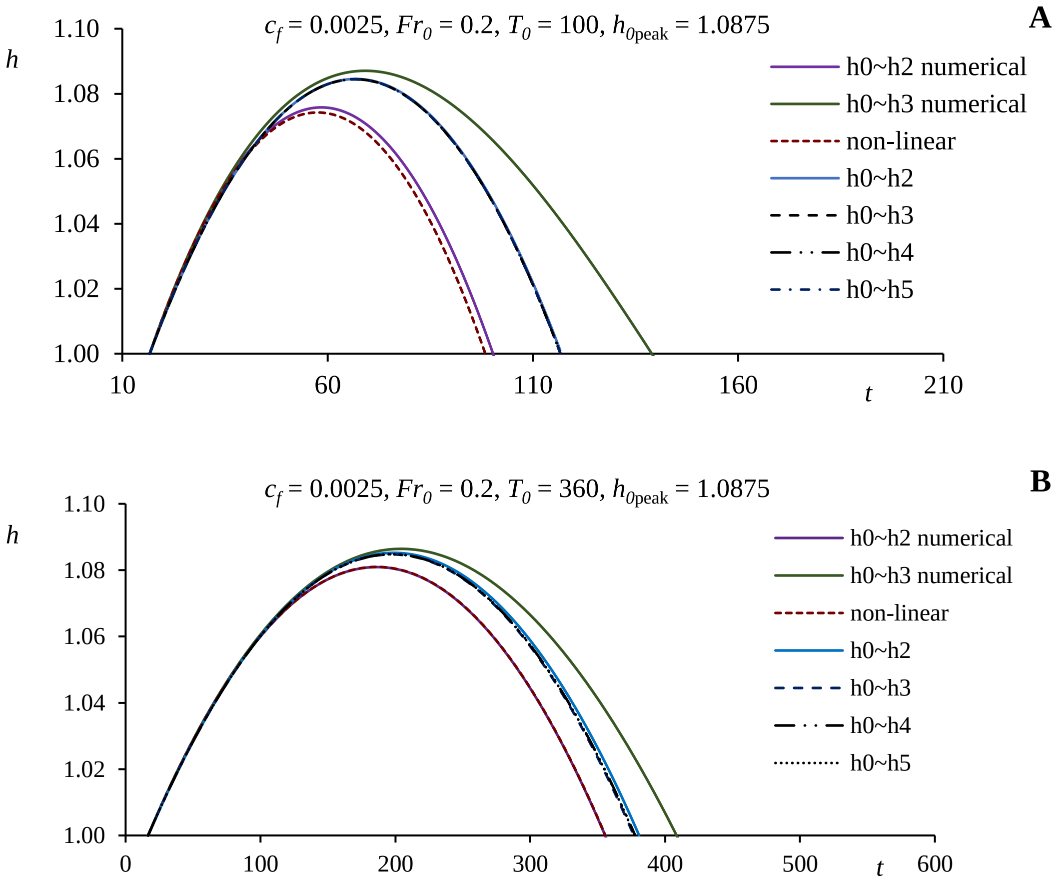


Figure. 2.2.9. Non-linear analytical solutions at x= 100.

2.3. Numerical Simulation of Shallow Water Equations

2.3.1. Calculation of Non-linear Numerical Solution

In order to calculate Eq. (2.2.3) and (2.2.4), directly considering its non-linearity, one of the higher order numerical scheme (second order), the Lax-Wendroff scheme satisfying the TVD condition (TVD-Lax-Wendroff scheme) is applied for simulation. The shallow water equations are transformed into the equations which are satisfied along the characteristic line.

$$\frac{\partial}{\partial t}\left(u \pm \frac{2\sqrt{h}}{Fr_0}\right)+\left(u \pm \frac{\sqrt{h}}{Fr_0}\right)\frac{\partial}{\partial x}\left(u \pm \frac{2\sqrt{h}}{Fr_0}\right)=c_f\left(1-u^2/h\right) \tag{2.3.1}$$

These equations are solved for every h_i^{n+1} and u_i^{n+1} by using the upstream boundary conditions h_1^n and u_1^n and the initial conditions at $t = 0$ (h_i^1, u_i^1) where $i = 1 \sim i_{end}$ indicate the number of grids along the spatial axis and $n = 1 \sim n_{end}$, along the time axis.

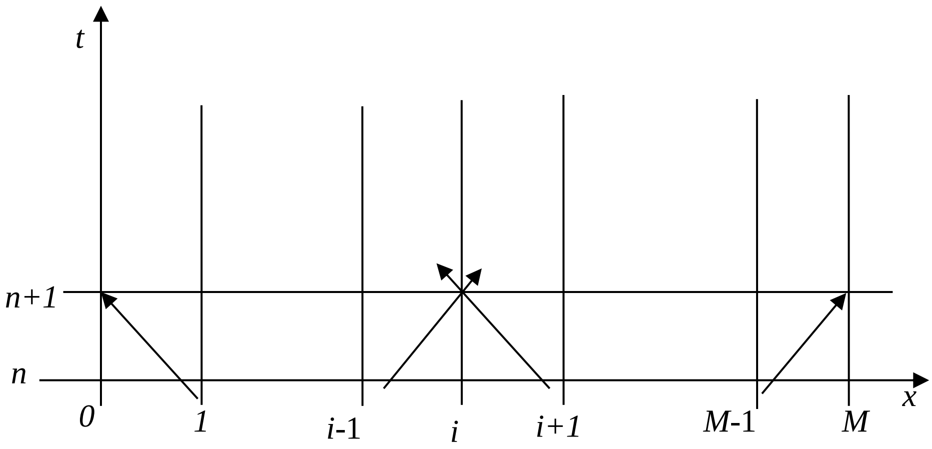


Figure.2.3.1. TVD-Lax-Wendroff Scheme

[TVD-LW Scheme]

$$h_i^{n+1}=\left\{\sqrt{h}_i^{\,n}-\frac{Fr_0}{4}\frac{\Delta t}{\Delta x}\left(FluxD1-FluxD2-FluxU1+FluxU2\right)\right\}^2 \tag{2.3.2}$$

$$u_i^{n+1}=u_i^n-\frac{1}{2}\frac{\Delta t}{\Delta x}\left(FluxD1+FluxD2-FluxU1-FluxU2\right)+\Delta t\times c_f\left(1-\frac{\left(u_i^n\right)^2}{h_i^n}\right) \tag{2.3.3}$$

$$FluxD1=\frac{1}{2}\left\{\left(u+\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\langle\left(u_{i+1}^n+u_i^n\right)+\frac{2}{Fr_0}\left(\sqrt{h_{i+1}^n}+\sqrt{h_i^n}\right)\right\rangle\right.$$

$$-Abs\left(u+\frac{2\sqrt{h}}{Fr_0}\right)_i^n\left\langle\left(u_{i+1}^n-u_i^n\right)+\frac{2}{Fr_0}\left(\sqrt{h_{i+1}^n}-\sqrt{h_i^n}\right)\right\rangle\Bigg\}$$

$$+FLIMIT\times\frac{1}{2}\times Abs\left(u+\frac{2\sqrt{h}}{Fr_0}\right)_i^n\left\{1-Abs\left\langle\frac{\Delta t}{\Delta x}\left(u\pm\frac{2\sqrt{h}}{Fr_0}\right)_i^n\right\rangle\right\}$$

$$\times\left\langle\left(u_{i+1}^n-u_i^n\right)+\frac{2}{Fr_0}\left(\sqrt{h_{i+1}^n}-\sqrt{h_i^n}\right)\right\rangle \tag{2.3.4a}$$

$$FluxU1=\frac{1}{2}\Bigg\{\left(u+\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\langle\left(u_i^n+u_{i-1}^n\right)+\frac{2}{Fr_0}\left(\sqrt{h_i^n}+\sqrt{h_{i-1}^n}\right)\right\rangle$$

$$-Abs\left(u+\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\langle\left(u_i^n-u_{i-1}^n\right)+\frac{2}{Fr_0}\left(\sqrt{h_i^n}-\sqrt{h_{i-1}^n}\right)\right\rangle\Bigg\}$$

$$+FLIMIT\times\frac{1}{2}\times Abs\left(u+\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\{1-Abs\left\langle\frac{\Delta t}{\Delta x}\left(u+\frac{2\sqrt{h}}{Fr_0}\right)_i^n\right\rangle\right\}$$

$$\times\left\langle\left(u_i^n-u_{i-1}^n\right)+\frac{2}{Fr_0}\left(\sqrt{h_i^n}-\sqrt{h_{i-1}^n}\right)\right\rangle \tag{2.3.4b}$$

$$FluxD2=\frac{1}{2}\Bigg\{\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\langle\left(u_{i+1}^n+u_i^n\right)-\frac{2}{Fr_0}\left(\sqrt{h_{i+1}^n}+\sqrt{h_i^n}\right)\right\rangle$$

$$-Abs\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\langle\left(u_{i+1}^n-u_i^n\right)-\frac{2}{Fr_0}\left(\sqrt{h_{i+1}^n}-\sqrt{h_i^n}\right)\right\rangle\Bigg\}$$

$$+FLIMIT\times\frac{1}{2}\times Abs\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\{1-Abs\left\langle\frac{\Delta t}{\Delta x}\left(u-\frac{2\sqrt{h}}{Fr_0}\right)_i^n\right\rangle\right\}$$

$$\times\left\langle\left(u_{i+1}^n-u_i^n\right)-\frac{2}{Fr_0}\left(\sqrt{h_{i+1}^n}-\sqrt{h_i^n}\right)\right\rangle \tag{2.3.4c}$$

$$FluxU2=\frac{1}{2}\Bigg\{\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\langle\left(u_i^n+u_{i-1}^n\right)-\frac{2}{Fr_0}\left(\sqrt{h_i^n}+\sqrt{h_{i-1}^n}\right)\right\rangle$$

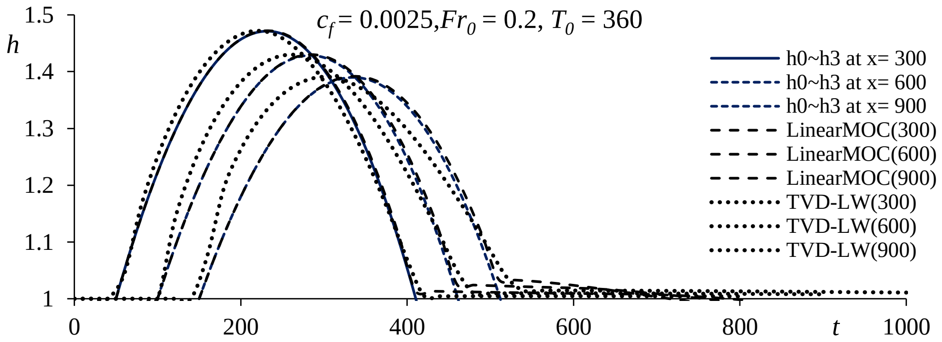
$$-Abs\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\langle\left(u_i^n-u_{i-1}^n\right)-\frac{2}{Fr_0}\left(\sqrt{h_i^n}-\sqrt{h_{i-1}^n}\right)\right\rangle\Bigg\}$$

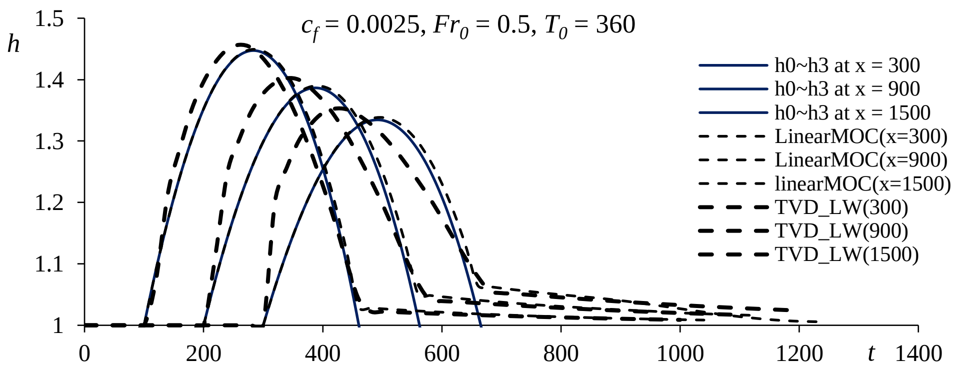
$$+FLIMIT\times\frac{1}{2}\times Abs\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\left\{1-Abs\left\langle\frac{\Delta t}{\Delta x}\left(u-\frac{2\sqrt{h}}{Fr_0}\right)_i^n\right\rangle\right\}$$

$$\times\left\langle\left(u_i^n-u_{i-1}^n\right)-\frac{2}{Fr_0}\left(\sqrt{h_i^n}-\sqrt{h_{i-1}^n}\right)\right\rangle \qquad (2.3.4d)$$

in which, FLIMIT = flux limiter. MINMOD function is used as a limiter function.

Fig. 2.3.2 shows the comparison of the analytical solutions methods to numerical results with $c_f = 0.0025$, $T_0 = 360$ at some locations along the channel. Some differences between the linear solutions and non-linear solutions are detected. Although the analytical solutions and numerical results are in good agreement, the nonlinear results from TVD-LW method are slightly different from analytical solutions and linearMOC results. It is pointed out that these differences are attributed to the effect of nonlinearity.





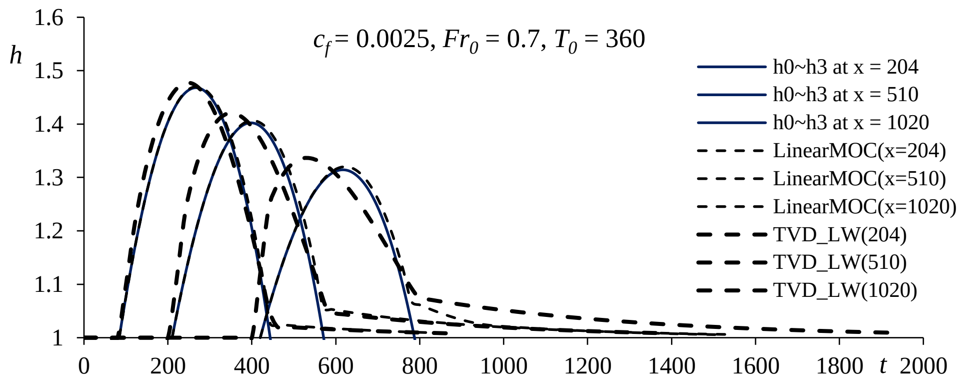


Figure.2.3.2. Comparison of results from different analysis.

Fig.2.3.3. represents the comparison of hydrographs with $T_0 = 720$ and different Froude numbers at different sections of the channel. The analytical solutions with large T_0 are not compatible with the results from linearMOC and nonlinear TVD-LW for downstream region far from upstream boundary.

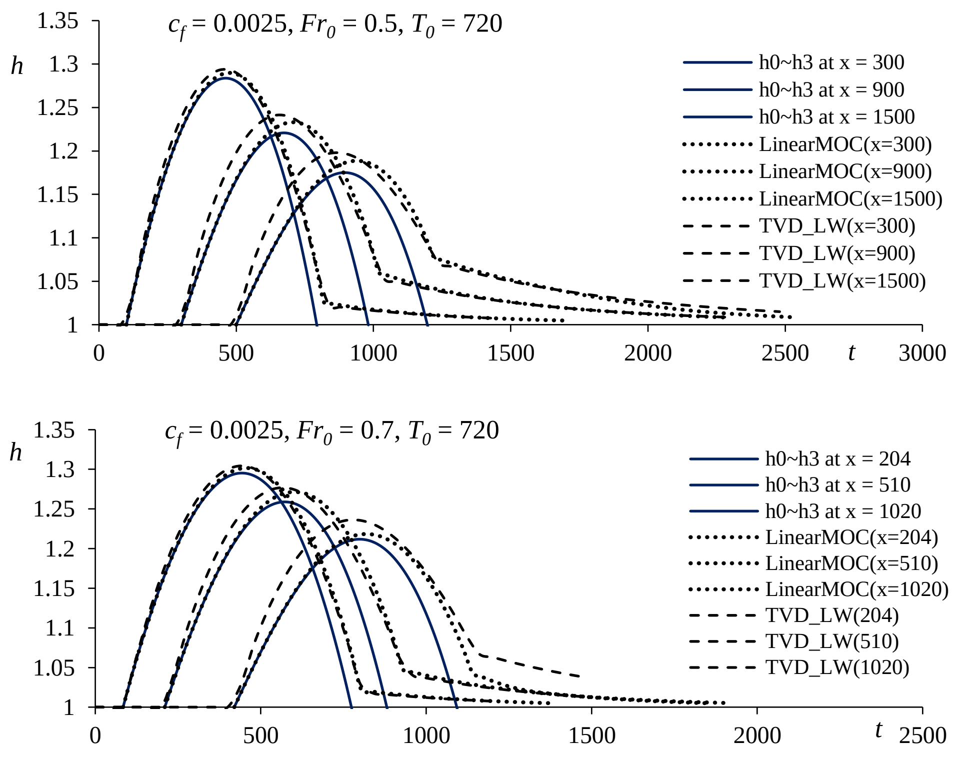


Figure.2.3.3. Comparison of results from different analysis for $T_0 = 720$.

2.4. Summary

The propagation of a hydrograph in sub-critical flows was studied by using shallow water equations without neglecting the inertia terms and the friction term. For simplicity, the linear terms were considered at first and the results were compared with the numerical results from linearMOC method. The linear solutions with up to the fifth order terms of the series expansion series were compatible with the numerical results. It can be pointed out that the solutions can reproduce the basic features of the hydrograph propagation. The new findings are summarized as follows:

1. The analytical solutions of linearized equations can reproduce the hydrograph propagation with small amplitude and time period.
2. The solutions can also be applied for the hydrographs with different Froude numbers in sub-critical flows.
3. The power series expansion with at least up to third order terms should be considered to realize the reasonable accuracy.

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CHAPTER 3

REPRODUCTION OF FLASH FLOOD USING A DEPTH HYDROGRAPH AT ONE SITE BASED ON THE LINEARIZED SHALLOW WATER EQUATIONS

3.1. Objectives

In the classical theory on unsteady open channel flows based on the Method of Characteristics (hereafter MOC), it is well known that one depth-hydrograph or discharge hydrograph at the upstream boundary and another one hydrograph at the downstream boundary of a river section are needed to simulate flood flows in the subcritical flow condition. On the contrary, *Hosoda* et al. (2011) pointed out that it is possible to reproduce flood flows in the subcritical flow condition between the upstream and downstream boundaries of a river course using only one depth hydrograph observed at the middle site of a river course.

In order to prove the possibility of this inversion analysis, *Hosoda* et al. (2011) considered a simple unsteady open channel flow in subcritical flow condition shown in Fig.3.1.1 and Fig.3.1.2. Then they derived the appropriate solutions of the depth hydrograph with the non-linear effect of the shallow water equations under the boundary conditions shown in Fig.3.1.1 and Fig.3.1.2. It was theoretically shown that the two boundary hydrographs can be reproduced using the appropriate solutions at the middle site of a flow domain.

Introducing the small parameter $\varepsilon = h/L$, the depth h and the velocity u were represented as the following Eq. (3.1.1) and (3.1.2).

$$h = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + ... \tag{3.1.1}$$

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + ... \tag{3.1.2}$$

Under the boundary hydrographs given as Eq. (3.1.3) and (3.1.4), substituting Eq. (3.1.1) and (3.1.2) into the shallow flow equations and applying MOC lead to the solutions of *h* and *u* at $x = L/2$:

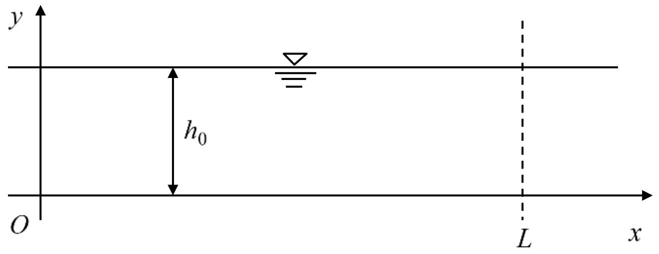


Figure. 3.1.1. Still water

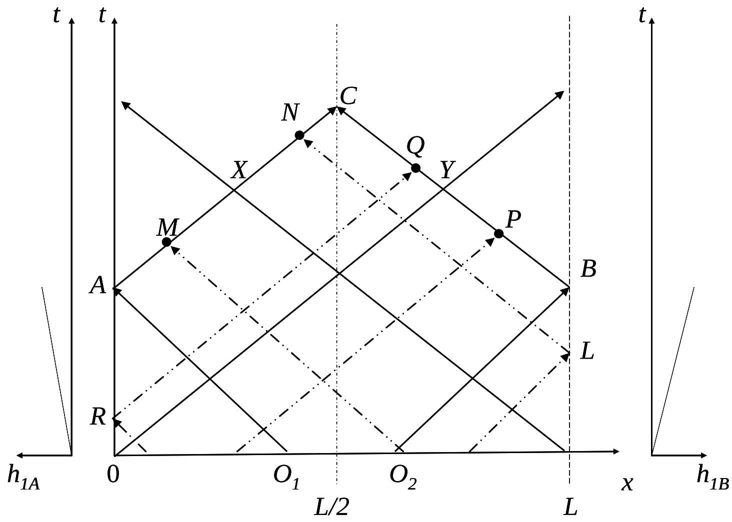


Figure.3.1.2. Networks of characteristic lines.

$$h_{1A} = \alpha_{1u} t_A, \quad \alpha_{1u} : \text{constant} \tag{3.1.3}$$

$$h_{1B} = \alpha_{1d} t_B, \quad \alpha_{1d} : \text{constant} \tag{3.1.4}$$

The value of h_C and the first order derivative of *h* with respect to time at *C* can be denoted as Eq. (3.1.5) and Eq. (3.1.6). It can be pointed out that the coefficients of α_{1u} and α_{1d} can be independently determined by using these values of h_C and $\left.\frac{\partial h}{\partial t}\right|_C$ using Eq.(3.1.5) and Eq. (3.1.6).

$$h\big|_C = h_0 + \varepsilon(\alpha_{1u} + \alpha_{1d})t_A + \varepsilon^2 \left\{ \frac{t_A^{\ 2}}{4h_0}\alpha_{1u}\alpha_{1d} + \frac{3}{4}\frac{Lt_A}{h_0\sqrt{gh_0}}(\alpha_{1u}^{\ 2} + \alpha_{1d}^{\ 2}) \right\} \tag{3.1.5}$$

$$\left.\frac{\partial h}{\partial t}\right|_C = \varepsilon(\alpha_{1u}+\alpha_{1d})+\varepsilon^2\left\{\frac{t_A}{2h_0}\alpha_{1u}\alpha_{1d}+\frac{3}{4}\frac{L}{h_0\sqrt{gh_0}}(\alpha_{1u}{}^2+\alpha_{1d}{}^2)\right\} \tag{3.1.6}$$

$$\varepsilon^4\left(\frac{3Lt_A^4}{4h_0\sqrt{gh_0}}\right)\alpha_{1u}^4+\varepsilon^3 t_A^4\alpha_{1u}^3+\varepsilon^2 t_A^4\left.\frac{\partial h}{\partial t}\right|_C\alpha_{1u}^2+\varepsilon\left\{4h_0t_A^2\left(h_0-h_C+\left.\frac{\partial h}{\partial t}\right|_C t_A\right)\right\}\alpha_{1u}$$

$$+\frac{12Lh_0}{\sqrt{gh_0}}\left(h_0-h_C+\left.\frac{\partial h}{\partial t}\right|_C t_A\right)^2=0 \tag{3.1.7}$$

Eq. (3.1.5) and (3.1.6) also show that α_{1u} and α_{1d} cannot be calculated uniquely without considering the non-linear terms because of Eq. (3.1.8).

$$h_{1C}=(\alpha_{1u}+\alpha_{1d})t_A\ ,\ \frac{\partial h_1}{\partial t_C}=\alpha_{1u}+\alpha_{1d} \tag{3.1.8}$$

These considerations indicate that the upstream and downstream boundary hydrographs can be reproduced if the water depth and the time derivative of depth at one site between the boundaries are known. It can be pointed out that it is possible to determine the unsteady flows during floods in the entire river course by using one depth hydrograph at one site between the upstream and downstream boundaries.

Since the bottom shear stress and bottom slope terms were neglected in the analysis mentioned above, it cannot be concluded yet whether the linearized shallow water equations with these two terms are applicable to the inversion analysis of flood flows in a river section with one hydrograph. In chapter 3, considering supercritical unsteady flows, the reproduction of the depth and velocity hydrographs at the upstream boundary is tested using the linearized shallow water equations using only one depth hydrograph observed in the downstream.

The non-dimensionalized shallow water equations are firstly transformed into the linearized equations considering the derivations of depth and velocity from the uniform flow. The analytical solutions of the linearized equations are derived based on MOC and are verified through the comparison to the results of the numerical simulations. It is finally pointed out through the theoretical considerations of the analytical solutions obtained in chapter 3 that the depth and velocity hydrographs at the upstream boundary for subcritical flows can be reproduced in principle using one depth hydrograph at one site in the downstream region.

3.2. Theoretical Background

3.2.1. Basic Equations

In order to resolve various problems related to unsteady open channel flows in rivers and man-made channels, the shallow water equations, so called the *Saint-Venant* equations, have been utilized in the hydraulic engineering field from old times. The one-dimensional shallow water equations consist of the depth averaged continuity and momentum equations with depth and velocity as unknown variables. Referring to the schematic illustration shown in Fig.3.2.1, the basic equations are denoted as Eq. (3.2.1) and Eq. (3.2.2).

[Continuity Equation]

$$\frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}=0 \tag{3.2.1}$$

[Momentum Equation]

$$\frac{\partial q}{\partial t}+\frac{\partial uq}{\partial x}+g\cos\theta\frac{\partial}{\partial x}\left(\frac{h^2}{2}\right)=gh\sin\theta-c_f u^2 \tag{3.2.2}$$

where, x = the spatial coordinate,

t = the time,

h = the water depth,

u = the depth averaged velocity,

g = the gravity acceleration,

θ = the bed slope

c_f = the friction coefficient.

In this study, c_f is assumed to be constant.

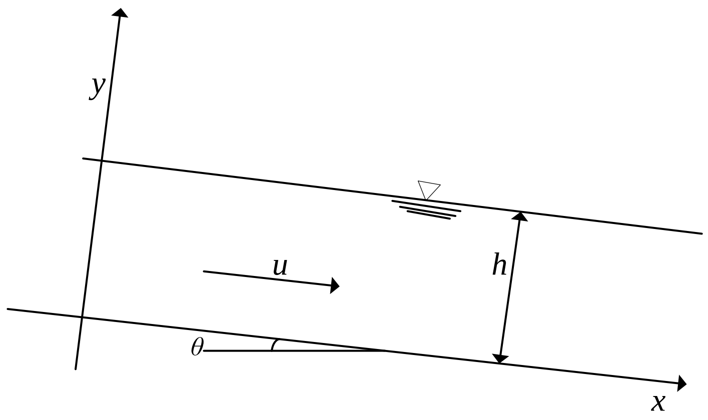


Figure.3.2.1. Description of symbols and coordinate system.

Using Eq. (3.2.1), Eq. (3.2.2) can be converted into the differential form as Eq. (3.2.3).

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x}\cos\theta = g\sin\theta - c_f\frac{u^2}{h} \tag{3.2.3}$$

3.2.2. Formulation based on MOC

Calculating the characteristic lines of Eq. (3.2.1) and Eq. (3.2.3), according to the definition of a characteristic line, it can be shown that the basic equations are classified as Hyperbolic Type. Using a matrix form, Eq. (3.2.1) and Eq. (3.2.2) can be expressed as Eq. (3.2.4).

$$\frac{\partial \boldsymbol{U}}{\partial t} + A\frac{\partial \boldsymbol{U}}{\partial x} = \boldsymbol{B} \tag{3.2.4}$$

where, $\boldsymbol{U}, A$ and $\boldsymbol{B}$ are defined as follows:

$$\boldsymbol{U}=\begin{bmatrix} h \\ u \end{bmatrix},\ A=\begin{bmatrix} u & h \\ g\cos\theta & u \end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix} 0 \\ g\sin\theta - c_f\dfrac{u^2}{h} \end{bmatrix}.$$

Introducing the coordinate transformation from (x,t) system to (ξ,η) system, Eq. (3.2.4) is reduced to Eq. (3.2.5).

$$\left(I\frac{\partial \xi}{\partial t} + A\frac{\partial \xi}{\partial x}\right)\frac{\partial \boldsymbol{U}}{\partial \xi} + \left(I\frac{\partial \eta}{\partial t} + A\frac{\partial \eta}{\partial x}\right)\frac{\partial \boldsymbol{U}}{\partial \eta} = \boldsymbol{B} \tag{3.2.5}$$

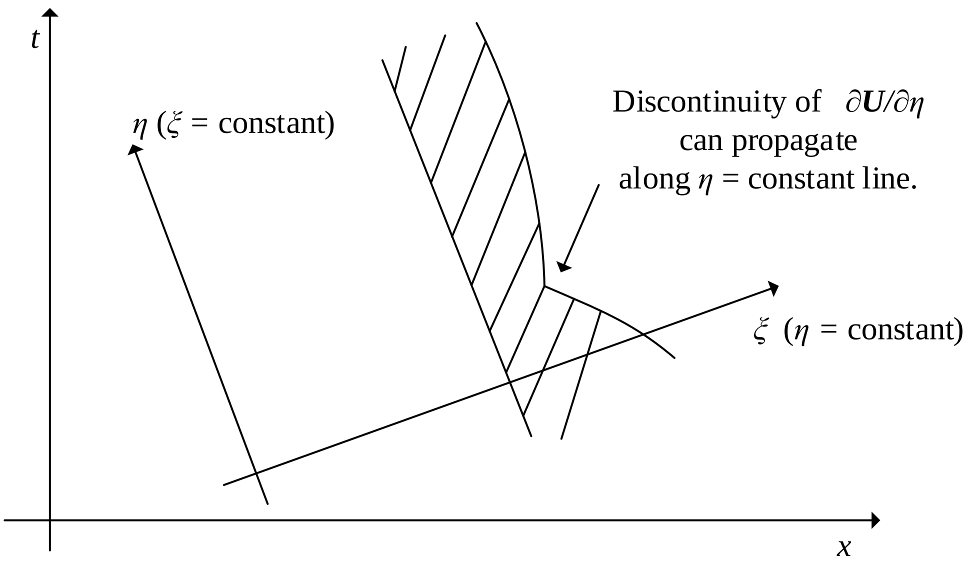


Figure.3.2.2. Illustration of the definition of a characteristic line.

Suppose that $\boldsymbol{U}$ is given along ξ- axis where η is constant. Referring to Fig.3.2.2, a characteristic line is defined as a line where the derivative $\dfrac{\partial \boldsymbol{U}}{\partial \eta}$ cannot be determined uniquely. This definition indicates that the first derivative of $\boldsymbol{U}$ with respect to η, $\dfrac{\partial \boldsymbol{U}}{\partial \eta}$ can be discontinuous along a characteristic line although the values of $\boldsymbol{U}$ are continuous. This definition is mathematically denoted as follows:

$$\det\left(A + I\,\frac{\partial \eta / \partial t}{\partial \eta / \partial x} \right) = 0$$

Since $-\dfrac{\partial \eta / \partial t}{\partial \eta / \partial x}$ is the gradient of a tangential line to a η = constant line, the characteristic lines $\dfrac{dx}{dt}_{\pm}$ are given by Eq. (3.2.6) as the two eigen values of A.

$$\lambda_{\pm} = \frac{dx}{dt}_{\pm} = u \pm \sqrt{gh\cos\theta} \tag{3.2.6}$$

The eigen vectors $\boldsymbol{\mu}_{\pm}$ for $\dfrac{dx}{dt}_{\pm}$ are also given as Eq. (3.2.7a) and Eq. (3.2.7b)

$$\boldsymbol{\mu}_{+} = \left(1, \sqrt{\frac{h}{g\cos\theta}} \right) \tag{3.2.7a}$$

$$\boldsymbol{\mu}_{-} = \left(1, -\sqrt{\frac{h}{g\cos\theta}}\right) \tag{3.2.7b}$$

Multiplying the eigen vectors by Eq. (3.2.4) on the left side, the relations which are satisfied along the characteristic lines can be derived as Eq. (3.2.8) and Eq. (3.2.9).

$$\boldsymbol{\mu}_{\pm}\left(\frac{\partial \boldsymbol{U}}{\partial t} + \lambda_{\pm}\frac{\partial \boldsymbol{U}}{\partial x}\right) = \boldsymbol{\mu}_{\pm}\boldsymbol{B} \tag{3.2.8}$$

$$\frac{\partial h}{\partial t} + \left(u \pm \sqrt{gh\cos\theta}\right)\frac{\partial h}{\partial x} \pm \sqrt{\frac{h}{g\cos\theta}}\left(\frac{\partial u}{\partial t} + \left(u \pm \sqrt{gh\cos\theta}\right)\frac{\partial u}{\partial x}\right)$$

$$= \pm\sqrt{\frac{h}{g\cos\theta}}\left(g\sin\theta - c_f\frac{u^2}{h}\right) \tag{3.2.9}$$

These formulations indicate the computational method how to set the boundary conditions at the upstream and downstream ends.

For subcritical flows, the characteristic lines which reach the upstream boundary give one relation with hydraulic variables in the downstream and the characteristics which reach the downstream boundary provide the information of the upstream region. Thus, one boundary condition as water depth hydrograph or velocity hydrograph at the upstream boundary and another boundary condition at the downstream boundary are required to calculate the hydraulic variables between two boundaries. For supercritical flows, all positive and negative characteristics are emerged from the upstream region giving the information for the entire regions. Therefore, both two boundary conditions at the upstream end are required to produce the hydrographs for the whole region.

3.3. Derivation of Appropriate Solutions for Flash Floods

3.3.1. Theoretical Formulation

In this study, one dimensional shallow water equations for unsteady flow are used as the governing equations. It consists of continuity and momentum equations and in the momentum equation, the friction coefficient is considered as constant. These equations are already given in Eq. (3.2.1) and Eq. (3.2.3).

The equations in the non-dimensional form are represented introducing non-dimensional variables. By using the following relationships, the governing equations in Eq. (3.2.1) and Eq. (3.2.3) can be transformed into Eq. (3.3.2) and Eq. (3.3.3).

$$x' = x/h_0, u' = u/u_0, t' = tu_0/h_0, h' = h/h_0$$

$$gh_0 \sin\theta = c_f {u_0}^2 \tag{3.3.1}$$

where, h_0 is the uniform flow depth and u_0 is the uniform flow velocity of the base flow in super-critical state.

[Non-dimensionalized equations]

$$\frac{\partial h'}{\partial t'} + \frac{\partial h'u'}{\partial x'} = 0 \tag{3.3.2}$$

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + \frac{1}{{Fr_0}^2}\frac{\partial h'}{\partial x'} = c_f(1 - \frac{u'^2}{h'}) \tag{3.3.3}$$

where, Fr_0 is the Froude number in which. $Fr_0 \equiv u_0/\sqrt{g\cos\theta h_0}$

Hereafter, " ' " which indicates non-dimensional variables is omitted.

Based on the perturbation method, h and u can be described as the following with the perturbation parameter ε.

$$h = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + ... \tag{3.3.4}$$

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ... \tag{3.3.5}$$

where, $\varepsilon = h_0/L$ and L = length scale of the study reach.

By substituting the above two equations into the non-dimensional shallow water equations Eq. (3.3.4) and (3.3.5), the equations of the first order on ε are obtained as follows:

$$\varepsilon^1 : \frac{\partial h_1}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{\partial h_1}{\partial x} = 0 \tag{3.3.6a}$$

$$\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x} + \frac{1}{{Fr_0}^2}\frac{\partial h_1}{\partial x} = -c_f(-h_1 + 2u_1) \tag{3.3.6b}$$

Since the super-critical condition is considered, the water depth hydrograph and velocity hydrograph at the upstream boundary are given as power series on time. Introducing

c_f as a small parameter, unknown variables h_1 and u_1 are approximated as Eq. (3.3.7) and Eq. (3.3.8).

$$h_1 = h_{10} + c_f h_{11} + c_f{}^2 h_{12} + \cdots \tag{3.3.7}$$

$$u_1 = u_{10} + c_f u_{11} + c_f{}^2 u_{12} + \cdots \tag{3.3.8}$$

Eq. (3.3.9) to Eq. (3.3.11) can be obtained by substituting h_1 and u_1 equations in Eq. (3.3.6a) and Eq. (3.3.6b) and setting the relations with the same power for c_f to be zero.

$$c_f{}^0 : \frac{\partial h_{10}}{\partial t} + \frac{\partial u_{10}}{\partial x} + \frac{\partial h_{10}}{\partial x} = 0 \tag{3.3.9a}$$

$$\frac{\partial u_{10}}{\partial t} + \frac{\partial u_{10}}{\partial x} + \frac{1}{Fr_0{}^2}\frac{\partial h_{10}}{\partial x} = 0 \tag{3.3.9b}$$

$$c_f{}^1 : \frac{\partial h_{11}}{\partial t} + \frac{\partial u_{11}}{\partial x} + \frac{\partial h_{11}}{\partial x} = 0 \tag{3.3.10a}$$

$$\frac{\partial u_{11}}{\partial t} + \frac{\partial u_{11}}{\partial x} + \frac{1}{Fr_0}\frac{\partial h_{11}}{\partial x} = h_{10} - 2u_{10} \tag{3.3.10b}$$

$$c_f{}^2 : \frac{\partial h_{12}}{\partial t} + \frac{\partial u_{12}}{\partial x} + \frac{\partial h_{12}}{\partial x} = 0 \tag{3.3.11a}$$

$$\frac{\partial u_{12}}{\partial t} + \frac{\partial u_{12}}{\partial x} + \frac{1}{Fr_0}\frac{\partial h_{12}}{\partial x} = h_{11} - 2u_{11} \tag{3.3.11b}$$

3.3.2. Derivation of Analytical Solution

At the upstream boundary, h_{10} and u_{10} are given as the boundary hydrographs as Eq. (3.3.12) and (3.3.13). The quadratic equations are assumed as the boundary hydrographs. h_{11}, u_{11}, h_{12} and u_{12} are set to be zero.

$$h_{10} = \alpha_1 t + \alpha_2 t^2 \tag{3.3.12}$$

$$u_{10} = \beta_1 t + \beta_2 t^2 \tag{3.3.13}$$

These hydrographs with parabolic shape are shown in Fig.3.3.1. In the calculation process, the time period of the water depth hydrograph are considered as the same ($T_E = T_{uE}$) as one of velocity hydrograph.

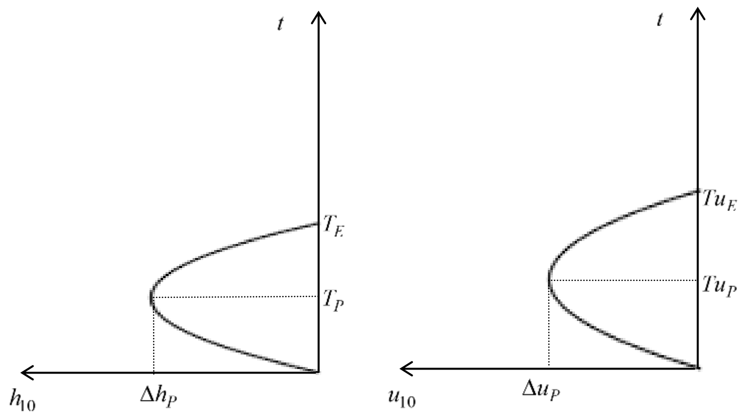


Figure.3.3.1. Upstream boundary hydrographs

The peak values Δh_P and Δu_P are shown in Fig.3.3.1. T_P and Tu_P are the times at which the peak values occur. By using MOC, Eq. (3.3.9) to Eq. (3.3.11) can be transformed into the following equations.

$$c_f^{\ 0} \quad \lambda_+ : \frac{D_+ h_{10}}{Dt} + Fr_0 \frac{D_+ u_{10}}{Dt} = 0 \tag{3.3.14a}$$

$$\lambda_- : \frac{D_- h_{10}}{Dt} - Fr_0 \frac{D_- u_{10}}{Dt} = 0 \tag{3.3.14b}$$

$$c_f^{\ 1} \quad \lambda_+ : \frac{D_+ h_{11}}{Dt} + Fr_0 \frac{D_+ u_{11}}{Dt} = Fr_0 (h_{10} - 2u_{10}) \tag{3.3.15a}$$

$$\lambda_- : \frac{D_- h_{11}}{Dt} - Fr_0 \frac{D_- u_{11}}{Dt} = -Fr_0 (h_{10} - 2u_{10}) \tag{3.3.15b}$$

$$c_f^{\ 2} \quad \lambda_+ : \frac{D_+ h_{12}}{Dt} + Fr_0 \frac{D_+ u_{12}}{Dt} = Fr_0 (h_{11} - 2u_{11}) \tag{3.3.16a}$$

$$\lambda_- : \frac{D_= h_{12}}{Dt} - Fr_0 \frac{D_- u_{12}}{Dt} = -Fr_0 (h_{11} - 2u_{11}) \tag{3.3.16b}$$

For higher order analysis, the same procedure can be applied as the mentioned above. The derivation $D_\pm / Dt$ along the characteristic lines can be defined as following;

$$\frac{D_\pm}{Dt} \equiv \frac{\partial}{\partial t} + \left(1 \pm \frac{1}{Fr_0}\right)\frac{\partial}{\partial x} \tag{3.3.17}$$

Eq. (3.3.14) to Eq. (3.3.16) are applied to derive the solutions for the region between the characteristic lines emerged from O and $t = T_E$.

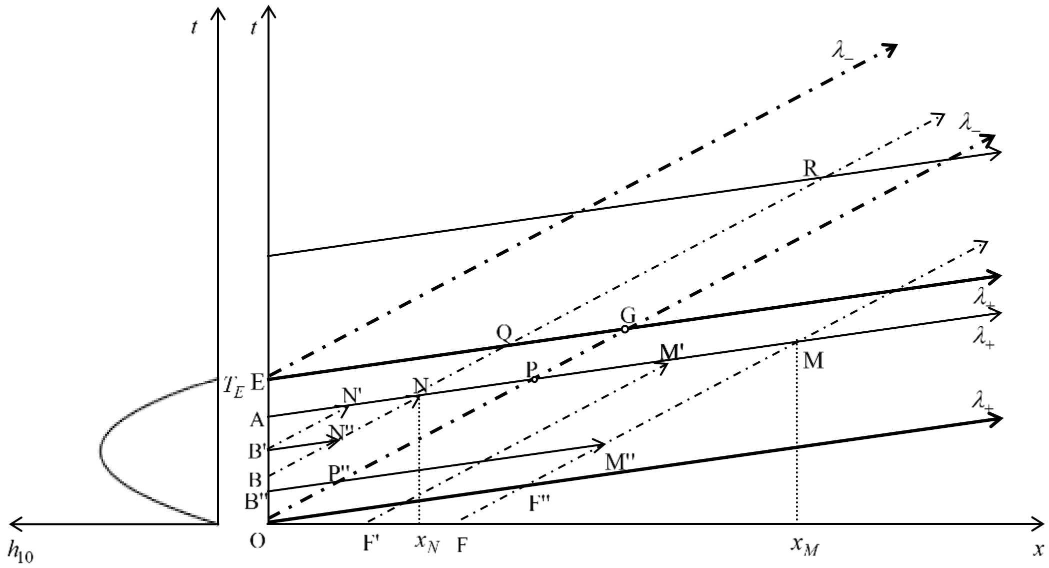


Figure.3.3.2. Illustration of characteristics lines and regions

3.3.3. Derivation of Solutions for Triangle Region (ΔOGE).

Based on MOC, the derivative $D_{\pm}/Dt$ is $1 \pm 1/Fr_0$ throughout the characteristic lines and the values of $h_{10} \pm Fr_0 u_{10}$ are constant along the characteristic lines. The approximate solutions for the triangle region can be obtained using the following procedures.

The values of h_{10A} , u_{10A} , h_{10B} and u_{10B} at the upstream boundary are given as follow:

$$h_{10A} = \alpha_1 t_A + \alpha_2 {t_A}^2 \tag{3.3.18a}$$

$$u_{10A} = \beta_1 t_A + \beta_2 {t_A}^2 \tag{3.3.18b}$$

$$h_{10B} = \alpha_1 t_B + \alpha_2 {t_B}^2 \tag{3.3.19a}$$

$$u_{10B} = \beta_1 t_B + \beta_2 {t_B}^2 \tag{3.3.19b}$$

The solutions for h_{10} and u_{10} at N along the characteristic lines can be obtained by using MOC.

$$h_{10N} + Fr_0 u_{10N} = h_{10A} + Fr_0 u_{10A} \tag{3.3.20a}$$

$$h_{10N} - Fr_0 u_{10N} = h_{10B} - Fr_0 u_{10B} \tag{3.3.20b}$$

$$h_{10N} = \frac{1}{2}\left\{ h_{10_A} + h_{10_B} + Fr_0 (u_{10_A} - u_{10_B}) \right\} \tag{3.3.21a}$$

$$u_{10N}=\frac{1}{2Fr_0}\left\{h_{10_A}-h_{10_B}+Fr_0(u_{10_A}+u_{10_B})\right\} \tag{3.3.21b}$$

$$h_{10N}=\frac{1}{2}\left\{t_A(\alpha_1+Fr_0\beta_1)+t_A{}^2\left(\alpha_2+Fr_0\beta_2\right)+t_B(\alpha_1-Fr_0\beta_1)+t_B{}^2\left(\alpha_2-Fr_0\beta_2\right)\right\} \tag{3.3.22a}$$

$$u_{10N}=\frac{1}{2Fr_0}\left\{t_A(\alpha_1+Fr_0\beta_1)+t_A{}^2\left(\alpha_2+Fr_0\beta_2\right)-t_B(\alpha_1-Fr_0\beta_1)-t_B{}^2\left(\alpha_2-Fr_0\beta_2\right)\right\} \tag{3.3.22b}$$

For h_{11} and u_{11}, h_{10} and u_{10} values are substituted into the integrated equations and the calculation procedures along the lone between A and N are in Eq. (3.3.23)~(3.3.25). The values of h_{11} and u_{11} are set to be zero at the upstream boundary.

$$\int_A^N \frac{D_+}{Dt}(h_{11}+Fr_0u_{11})dt=h_{11_N}+Fr_0u_{11_N}-h_{11_A}-Fr_0u_{11_A}=Fr_0\int_A^N(h_{10N'}-2u_{10N'})dt \tag{3.3.23}$$

$$\begin{aligned}\int_A^N h_{10N'}\,dt=\int_{t_A}^{t_N}\Big\{\ & t_A\frac{1}{2}(\alpha_1+Fr_0\beta_1)+t_A^2\frac{1}{2}(\alpha_2+Fr_0\beta_2)+t_{B'}\frac{1}{2}(\alpha_1-Fr_0\beta_1)\\ &+t_{B'}{}^2\frac{1}{2}(\alpha_2-Fr_0\beta_2)\ \Big\}dt_{N'}\end{aligned} \tag{3.3.24a}$$

$$\begin{aligned}\int_A^N u_{10N'}\,dt=\int_{t_A}^{t_N}\Big\{\ & t_A\frac{1}{2Fr_0}(\alpha_1+Fr_0\beta_1)+t_A^2\frac{1}{2Fr_0}(\alpha_2+Fr_0\beta_2)-t_{B'}\frac{1}{2Fr_0}(\alpha_1-Fr_0\beta_1)\\ &-t_{B'}{}^2\frac{1}{2Fr_0}(\alpha_2-Fr_0\beta_2)\ \Big\}dt_{N'}\end{aligned} \tag{3.3.24b}$$

$$\begin{aligned}h_{11N}+Fr_0u_{11N}=Fr_0\Big\{&(AA_0-2AB_0)(t_N-t_A)+(AA_1-2AB_1)\frac{1}{2}\left(t_N{}^2-t_A{}^2\right)\\ &+(AA_2-2AB_2)\frac{1}{3}(t_N^3-t_A^3)\Big\}\end{aligned} \tag{3.3.25}$$

in which, $AA_0=t_A(I_{1+}+X_1I_{1-})+t_A^2(I_{2+}+X_1^2I_{2-})$, $AA_1=-2X_2I_{1+}-4X_1X_2t_AI_{2-}$,

$AA_2=4X_2{}^2I_{2-}$, $AB_0=t_A\dfrac{1}{Fr_0}(I_{1+}-X_1I_{1-})+t_A^2\dfrac{1}{Fr_0}(I_{2+}-X_1I_{2-})$,

$AB_1=2X_2\dfrac{1}{Fr_0}I_{1-}+4X_1X_2t_A\dfrac{1}{Fr_0}I_{2-}$, $AB_2=-4X_2^2\dfrac{1}{Fr_0}I_{2-}$,

$I_{1+}\equiv\dfrac{1}{2}(\alpha_1+Fr_0\beta_1)$, $\quad I_{1-}\equiv\dfrac{1}{2}(\alpha_1-Fr_0\beta_1)$, $I_{2+}\equiv\dfrac{1}{2}(\alpha_2+Fr_0\beta_2)$, $\quad I_{2-}\equiv\dfrac{1}{2}(\alpha_2-Fr_0\beta_2)$

$X_1\equiv\dfrac{1+\left(1/Fr_0\right)}{1-\left(1/Fr_0\right)}$, $X_2\equiv\dfrac{1/Fr_0}{1-\left(1/Fr_0\right)}$, $t_{B'}=X_1t_A-2X_2t_{N'}$, $t_{B'}=X_3t_B+2X_4t_{N''}$.

Similarly, the calculation procedures from B to N along the negative characteristic line are shown as follow:

$$\int_B^N \frac{D_-}{Dt}(h_{11} - Fr_0 u_{11})dt = h_{11_N} - Fr_0 u_{11_N} - h_{11_B} + Fr_0 u_{11_B} = -Fr_0 \int_B^N (h_{10} - 2u_{10})dt \qquad (3.3.26)$$

$$\int_B^N h_{10N''}dt = \int_{t_B}^{t_N} \{ \ t_{B'} \frac{1}{2}(\alpha_1 + Fr_0\beta_1) + t_{B'}^2 \frac{1}{2}(\alpha_2 + Fr_0\beta_2) + t_B \frac{1}{2}(\alpha_1 - Fr_0\beta_1)$$

$$+ t_B{}^2 \frac{1}{2}(\alpha_2 - Fr_0\beta_2) \ \} dt_{N''} \qquad (3.3.27a)$$

$$\int_B^N u_{10N''}dt = \int_{t_B}^{t_N} \{ \ t_{B'} \frac{1}{2}(\alpha_1 + Fr_0\beta_1) + t_{B'}^2 \frac{1}{2}(\alpha_2 + Fr_0\beta_2) - t_B \frac{1}{2}(\alpha_1 - Fr_0\beta_1)$$

$$- t_B{}^2 \frac{1}{2}(\alpha_2 - Fr_0\beta_2) \ \} dt_{N''} \qquad (3.3.27b)$$

$$h_{11N} - Fr_0 u_{11N} = -Fr_0 \{ (AC_0 - 2AD_0)(t_N - t_B) + (AC_1 - 2AD_1)\frac{1}{2}\left(t_N{}^2 - t_B{}^2\right)$$

$$+ (AC_2 - 2AD_2)\frac{1}{3}\left(t_N{}^3 - t_B{}^3\right) \} \qquad (3.3.28)$$

where, $AC_0 = t_B(I_{1-} + X_3 I_{1+}) + t_B^2(I_{2-} + X_3^2 I_{2+})$, $AC_1 = 2X_4 I_{1+} + 4X_3 X_4 t_B I_{2+}$,

$AC_2 = 4X_4^2 I_{2+}$, $AD_0 = t_B \frac{1}{Fr_0}(-I_{1-} + X_3 I_{1+}) + t_B^2 \frac{1}{Fr_0}(-I_{2-} + X_3^2 I_{2+})$,

$AD_1 = 2X_4 \frac{1}{Fr_0} I_{1+} + 4X_3 X_4 t_B \frac{1}{Fr_0} I_{2+}$, $AD_2 = 4X_4^2 \frac{1}{Fr_0} I_{2+}$,

$X_3 \equiv \frac{1-(1/Fr_0)}{1+(1/Fr_0)}, X_4 \equiv \frac{1/Fr_0}{1+(1/Fr_0)}$.

Thus, the calculation for h_{11} and u_{11} at the intersection of two characteristic lines N are given as Eq. (3.3.29a) and (3.3.29b).

$$h_{11N} = \frac{1}{2} Fr_0 \left[\left\{ (AA_0 - 2AB_0)(t_N - t_A) + (AA_1 - 2AB_1)\frac{1}{2}(t_N^2 - t_A^2) + (AA_2 - 2AB_2)\frac{1}{3}(t_N^3 - t_A^3) \right\} \right.$$

$$- \left\{ (AC_0 - 2AD_0)(t_N - t_B) + \frac{1}{2}(AC_1 - 2AD_1)\left(t_N{}^2 - t_B{}^2\right) \right.$$

$$\left.\left. + \frac{1}{3}(AC_2 - 2AD_2)\left(t_N{}^3 - t_B{}^3\right) \right\} \right] \qquad (3.3.29a)$$

$$u_{11N} = \frac{1}{2} \left[\left\{ (AA_0 - 2AB_0)(t_N - t_A) + (AA_1 - 2AB_1)\frac{1}{2}(t_N^2 - t_A^2) + (AA_2 - 2AB_2)\frac{1}{3}(t_N^3 - t_A^3) \right\} \right.$$

$$+\left\{\left(AC_0-2AD_0\right)\left(t_N-t_B\right)+\frac{1}{2}\left(AC_1-2AD_1\right)\left(t_N{}^2-t_B{}^2\right)\right.$$

$$\left.\left.+\frac{1}{3}\left(AC_2-2AD_2\right)\left(t_N{}^3-t_B{}^3\right)\right\}\right] \tag{3.3.29b}$$

t_A and t_B are replaced by t_N and x_N with the relationship of $t_N=t_A+\frac{x_N}{1+(1/Fr_0)}=t_B+\frac{x_N}{1-(1/Fr_0)}$, resulting in the equations of h_{11N} and u_{11N}

$$h_{11N}=Fr_0\left[-\frac{1}{2}\left(\frac{x_N}{F_1}\right)^2J_1+\frac{1}{2}\left(\frac{x_N}{F_1}\right)^3J_2+\frac{1}{2}\left(\frac{x_N}{F_1}\right)^2X_2F_3I_{1-}-\left(\frac{x_N}{F_1}\right)^3X_2F_3I_{2-}\left(X_1-\frac{2}{3}X_2\right)\right.$$

$$+t_N\left\{\frac{1}{2}\frac{x_N}{F_1}J_1-\left(\frac{x_N}{F_1}\right)^2J_2-\frac{x_N}{F_1}X_2F_3I_{1-}+\left(\frac{x_N}{F_1}\right)^2X_2F_3I_{2-}\left(3X_1-2X_2\right)\right\}$$

$$\left.+t_N^2\left\{\frac{1}{2}\frac{x_N}{F_1}J_2+\frac{x_N}{F_1}X_2F_3I_{2-}\left(-2X_1+2X_2\right)\right\}\right]$$

$$-Fr_0\left[-\frac{1}{2}\left(\frac{x_N}{F_2}\right)^2K_1+\frac{1}{2}\left(\frac{x_N}{F_2}\right)^3K_2-\frac{1}{2}\left(\frac{x_N}{F_2}\right)^2X_4F_4I_{1+}+\left(\frac{x_N}{F_2}\right)^3X_4F_4I_{2+}\left(X_3+\frac{2}{3}X_4\right)\right.$$

$$+t_N\left\{\frac{1}{2}\frac{x_N}{F_2}K_1-\left(\frac{x_N}{F_2}\right)^2K_2+\frac{x_N}{F_2}X_4F_4I_{1+}-\left(\frac{x_N}{F_2}\right)^2X_4F_4I_{2+}\left(3X_3+2X_4\right)\right\}$$

$$\left.+t_N{}^2\left\{\frac{1}{2}\frac{x_N}{F_2}K_2+\frac{x_N}{F_2}X_4F_4I_{2+}\left(2X_3+2X_4\right)\right\}\right] \tag{3.3.30}$$

$$u_{11N}=\frac{1}{Fr_0}\left[\left(\frac{x_N}{F_1}\right)^2J_3-\left(\frac{x_N}{F_1}\right)^3J_4+\left(\frac{x_N}{F_1}\right)^2X_2F_5I_{1-}+\left(\frac{x_N}{F_1}\right)^3X_2F_5I_{2-}\left(-2X_1+\frac{4}{3}X_2\right)\right.$$

$$+t_N\left\{-\frac{x_N}{F_1}J_3+\left(\frac{x_N}{F_1}\right)^2J_4-\frac{x_N}{F_1}2X_2F_5I_{1-}+\left(\frac{x_N}{F_1}\right)^2X_2F_5I_{2-}\left(6X_1-4X_2\right)\right\}$$

$$\left.+t_N^2\left\{-\frac{x_N}{F_1}J_4+4\frac{x_N}{F_1}X_2F_5I_{2-}\left(-X_1+X_2\right)\right\}\right]$$

$$+\frac{1}{Fr_0}\left[\left(\frac{x_N}{F_2}\right)^2K_3-\left(\frac{x_N}{F_2}\right)^3K_4+\left(\frac{x_N}{F_2}\right)^2X_4F_6I_{1+}-\left(\frac{x_N}{F_2}\right)^3X_4F_6I_{2+}\left(2X_3+\frac{4}{3}X_4\right)\right.$$

$$+t_N\left\{-\frac{x_N}{F_2}K_3+2\left(\frac{x_N}{F_2}\right)^2K_4-2\frac{x_N}{F_2}X_4F_6I_{1+}+\left(\frac{x_N}{F_2}\right)^2X_4F_6I_{2+}\left(6X_3+4X_4\right)\right\}$$

$$\left.+t_N{}^2\left\{-\frac{x_N}{F_2}K_4-4\frac{x_N}{F_2}X_4F_6I_{2+}\left(2X_3+2X_4\right)\right\}\right] \tag{3.3.31}$$

where, $F_1 = 1 + \frac{1}{Fr_0}$, $F_2 = 1 - \frac{1}{Fr_0}$, $F_3 = \frac{Fr_0 + 2}{Fr_0}$, $F_4 = \frac{Fr_0 - 2}{Fr_0}$, $F_5 = 1 + \frac{1}{2Fr_0}$, $F_6 = 1 - \frac{1}{2Fr_0}$,

$$J_1 \equiv F_4 I_{1+} + X_1 F_3 I_{1-},\ J_2 \equiv F_4 I_{2+} + X_1^2 F_3 I_{2-},\ J_3 \equiv F_6 I_{1+} - X_1 F_5 I_{1-},$$

$$J_4 \equiv F_6 I_{2+} - X_1^2 F_5 I_{2-},\ K_1 \equiv F_3 I_{1-} + X_3 F_4 I_{1+},\ K_2 \equiv F_3 I_{2-} + X_3^2 F_4 I_{2+},$$

$$K_3 \equiv -F_5 I_{1-} + X_3 F_6 I_{1+},\ K_4 \equiv -F_5 I_{2-} + X_3^2 F_6 I_{2+}.$$

h_{10} and u_{10} are also transformed using the same way.

$$h_{10N} = -I_{1+}\frac{x_N}{F_1} - I_{1-}\frac{x_N}{F_2} + I_{2+}\left(\frac{x_N}{F_1}\right)^2 + I_{2-}\left(\frac{x_N}{F_2}\right)^2 + t_N\left(I_{1+} + I_{1-} - 2I_{2+}\frac{x_N}{F_1} - 2I_{2-}\frac{x_N}{F_2}\right)$$
$$+ t_N^{\ 2}\left(I_{2+} + I_{2-}\right) \tag{3.3.32}$$

$$u_{10N} = +\frac{1}{Fr_0}\left[-I_{1+}\left(\frac{x_N}{F_1}\right) + I_{1-}\left(\frac{x_N}{F_2}\right) + \left(\frac{x_N}{F_1}\right)^2 I_{2+} - \left(\frac{x_N}{F_2}\right)^2\right.$$
$$\left. + t_N\left\{I_{1+} - I_{1-} - 2\left(\frac{x_N}{F_1}\right)I_{2+} + 2\left(\frac{x_N}{F_2}\right)I_{2-}\right\} + t_N^{\ 2}\left(I_{2+} - I_{2-}\right)\right] \tag{3.3.33}$$

Substituting h_{11} and u_{11} into the righthand side of integration equation leads to the solution of h_{12}.

$$h_{12N} = \frac{1}{2}Fr_0\left[\left(Fr_0 - 2\right)\left\{\left(\left(\frac{x_N}{F_1}\right)^3\left(-\frac{1}{4}J_1 + \frac{1}{3}X_2F_3I_{1-}\right) + \left(\frac{x_N}{F_1}\right)^4\left(\frac{1}{4}J_2 + X_2F_3I_{2-}\left(-\frac{2}{3}X_1 + \frac{1}{2}X_2\right)\right)\right)\right.\right.$$
$$+ t_N\left(\left(\frac{x_N}{F_1}\right)^2\left(\frac{1}{4}J_1 - \frac{1}{2}X_2F_3I_{1-}\right) + \left(\frac{x_N}{F_1}\right)^3\left(-\frac{1}{2}J_2 + X_2F_3I_{2-}\left(\frac{5}{3}X_1 - \frac{4}{3}X_2\right)\right)\right)$$
$$\left. - t_N^{\ 2}\left(\left(\frac{x_N}{F_1}\right)^2\left(\frac{1}{4}J_2 + X_2F_3I_{2-}\left(-X_1 + X_2\right)\right)\right)\right\}$$
$$- \left(Fr_0 + 2\right)\left\{\left(\frac{x_N}{F_1}\right)^3\left(-\frac{1}{2}X_1^{\ 2}K_1 + \frac{1}{2}\left(\frac{1}{2} + 2X_2\right)X_1K_1 - \frac{1}{3}\left(1 + 2X_2\right)X_2K_1\right)\right.$$
$$+ \left(\frac{x_N}{F_1}\right)^4\left(\frac{1}{2}X_1^{\ 3}K_2 - \frac{1}{2}\left(\frac{1}{2} + 3X_2\right)X_1^{\ 2}K_2 + \frac{2}{3}\left(1 + 3X_2\right)X_1X_2K_2 - \frac{1}{2}\left(1 + 2X_2\right)X_2^{\ 2}K_2\right)$$
$$+ t_N\left(\left(\frac{x_N}{F_1}\right)^2\left(X_1^{\ 2}K_1 - \frac{3}{2}\left(\frac{1}{2} + 2X_2\right)X_1K_1 + \left(1 + 2X_2\right)X_2K_1\right)\right.$$

$$+\left(\frac{x_N}{F_1}\right)^3\left(-\frac{3}{2}X_1{}^3K_2+2\left(\frac{1}{2}+3X_2\right)X_1{}^2K_2-\frac{8}{3}(1+3X_2)X_1X_2K_2+2(1+2X_2)X_2{}^2K_2\right)\Bigg)$$

$$+t_N{}^2\left(\left(\frac{x_N}{F_1}\right)\left(-\frac{1}{2}X_1{}^2K_1+\left(\frac{1}{2}+2X_2\right)X_1K_1-(1+2X_2)X_2K_1\right)\right.$$

$$+\left(\frac{x_N}{F_1}\right)^2\left(\frac{3}{2}X_1{}^3K_2-\frac{5}{2}\left(\frac{1}{2}+3X_2\right)X_1{}^2K_2+4(1+3X_2)X_1X_2K_2-3(1+2X_2)X_2{}^2K_2\right)\Bigg)$$

$$+t_N{}^3\left(\left(\frac{x_N}{F_1}\right)\left(-\frac{1}{2}X_1{}^3K_2+\left(\frac{1}{2}+3X_2\right)X_1{}^2K_2-(1+3X_2)2X_1X_2K_2+(1+2X_2)2X_2{}^2K_2\right)\right)$$

$$+\frac{1}{2}\left\{\left(\frac{x_N}{F_1}\right)^3\left(-X_1X_2F_4I_{1+}+2X_2{}^2F_4I_{1+}+\frac{1}{3}\left\langle 1-4X_2{}^2\right\rangle X_4F_4I_{1+}\right)\right.$$

$$+\left(\frac{x_N}{F_1}\right)^4\left(2X_1X_2F_4I_{2+}-6X_2{}^2F_4I_{2+}-\frac{2}{3}\left(1-12X_2{}^2\right)X_4F_4I_{2+}+\left(1-4X_2{}^2\right)X_4{}^2F_4I_{2+}\right)$$

$$+t_N\left(\left(\frac{x_N}{F_1}\right)^2\left(2X_1X_2F_4I_{1+}-6X_2{}^2F_4I_{1+}-\left(1-4X_2{}^2\right)X_4F_4I_{1+}\right)\right.$$

$$+\left(\frac{x_N}{F_1}\right)^3\left(\left((-6X_1+24X_2)X_2+\left(\frac{8}{3}\left\langle 1-12X_2{}^2\right\rangle-4\left\langle 1-4X_2{}^2\right\rangle X_4\right)X_4\right)F_4I_{2+}\right)\Bigg)$$

$$+t_N{}^2\left(\left(\frac{x_N}{F_1}\right)\left(\left((-X_1+4X_2)X_2+\left(1-4X_2{}^2\right)X_4\right)F_4I_{1+}\right)\right.$$

$$+\left(\frac{x_N}{F_1}\right)^2\left(\left((6X_1-30X_2)X_2+\left(6\left(1-4X_2{}^2\right)X_4-4\left(1-12X_2{}^2\right)\right)X_4\right)F_4I_{2+}\right)\Bigg)$$

$$+t_N{}^3\left(\left(\frac{x_N}{F_1}\right)\left(\left((-2X_1+12X_2)X_2+\left(2\left(1-12X_2{}^2\right)-4\left(1-4X_2{}^2\right)X_4\right)X_4\right)F_4I_{2+}\right)\right)\Bigg\}$$

$$+\frac{1}{3}\left\{\left(\frac{x_N}{F_1}\right)^4\left(\left((2X_1-6X_2)X_2{}^2+\left(8X_2{}^3-\frac{1}{2}\left(1+8X_2{}^3\right)X_4\right)X_4\right)F_4I_{2+}\right)\right.$$

$$+t_N\left(\left(\frac{x_N}{F_1}\right)^3\left(\left((-6X_1+24X_2)X_2{}^2+\left(-32X_2{}^3+2\left\langle 1+8X_2{}^3\right\rangle X_4\right)X_4\right)F_4I_{2+}\right)\right)$$

$$+t_N{}^2\left(\left(\frac{x_N}{F_1}\right)^2\left(\left((6X_1-30X_2)X_2{}^2+\left(48X_2{}^3-3\left(1+8X_2{}^3\right)X_4\right)X_4\right)F_4I_{2+}\right)\right)$$

$$+t_N{}^3\left(\left(\frac{x_N}{F_1}\right)\left(\left((-2X_1+12X_2)X_2{}^2+\left(-24X_2{}^3+2\left(1+8X_2{}^3\right)X_4\right)X_4\right)F_4I_{2+}\right)\right)\Bigg\}\Bigg\}$$

$$-(Fr_0-2)\Bigg\{\left(\frac{x_N}{F_2}\right)^3\left(-\frac{1}{2}X_3K_1+\frac{1}{2}\left(\frac{1}{2}-2X_4\right)X_3J_1+\frac{1}{3}(1-2X_4)X_4J_1\right)$$

$$+\left(\frac{x_N}{F_2}\right)^4\left(\frac{1}{2}X_3K_2-\frac{1}{2}\left(\frac{1}{2}-3X_4\right)X_3{}^2J_2-\frac{2}{3}(1-3X_4)X_3X_4J_2-\frac{1}{2}(1-2X_4)X_4{}^2J_2\right)$$

$$+t_N\Bigg(\left(\frac{x_N}{F_2}\right)^2\left(X_3K_1+\left(-\frac{3}{2}\left(\frac{1}{2}-2X_4\right)X_3-(1-2X_4)X_4\right)J_1\right)$$

$$+\left(\frac{x_N}{F_2}\right)^3\left(-\frac{3}{2}X_3K_2+2\left(\frac{1}{2}-3X_4\right)X_3{}^2J_2+\frac{8}{3}(1-3X_4)X_3X_4J_2+2(1-2X_4)X_4{}^2J_2\right)\Bigg)$$

$$+t_N{}^2\Bigg(\left(\frac{x_N}{F_2}\right)\left(-\frac{1}{2}X_3K_1+\left(\frac{1}{2}-2X_4\right)X_3J_1+(1-2X_4)X_4J_1\right)$$

$$+\left(\frac{x_N}{F_2}\right)^2\left(\frac{3}{2}X_3K_2-\frac{5}{2}\left(\frac{1}{2}-3X_4\right)X_3{}^2J_2-4(1-3X_4)X_3X_4J_2-3(1-2X_4)X_4{}^2J_2\right)\Bigg)$$

$$+t_N{}^3\left(\left(\frac{x_N}{F_2}\right)\left(-\frac{1}{2}X_3K_2+\left(\frac{1}{2}-3X_4\right)X_3{}^2J_2+2(1-3X_4)X_3X_4J_2+(1-2X_4)2X_4{}^2J_2\right)\right)$$

$$+\frac{1}{2}\Bigg\{\left(\frac{x_N}{F_2}\right)^3\left(X_3X_4F_3I_{1-}+2X_4{}^2F_3I_{1-}-\frac{1}{3}\left\langle 1-4X_4{}^2\right\rangle X_2F_3I_{1-}\right)$$

$$+\left(\frac{x_N}{F_2}\right)^4\left(-2X_3X_4F_3I_{2-}-6X_4{}^2F_3I_{2-}+\frac{2}{3}\left(1-12X_4{}^2\right)X_2F_3I_{2-}+\left(1-4X_4{}^2\right)X_2{}^2F_3I_{2-}\right)$$

$$+t_N\Bigg(\left(\frac{x_N}{F_2}\right)^2\left(-2X_3X_4F_3I_{1-}-6X_4{}^2F_3I_{1-}+\left(1-4X_4{}^2\right)X_2F_3I_{1-}\right)$$

$$+\left(\frac{x_N}{F_2}\right)^3\left(\left((6X_3+24X_4)X_4+\left(-\frac{8}{3}\left\langle 1-12X_4{}^2\right\rangle-4\left\langle 1-4X_4{}^2\right\rangle X_2\right)X_2\right)F_3I_{2-}\right)\Bigg)$$

$$+t_N{}^2\Bigg(\left(\frac{x_N}{F_2}\right)\left(\left((X_3+4X_4)X_4-\left(1-4X_4{}^2\right)X_2\right)F_3I_{1-}\right)$$

$$+\left(\frac{x_N}{F_2}\right)^2\left(\left((-6X_3-30X_4)X_4+\left(6\left(1-4X_4{}^2\right)X_2+4\left(1-12X_4{}^2\right)\right)X_2\right)F_3I_{2-}\right)\Bigg)$$

$$+t_N{}^3\left(\left(\frac{x_N}{F_2}\right)\left(\left((2X_3+12X_4)X_4+\left(-2\left(1-12X_4{}^2\right)-4\left(1-4X_4{}^2\right)X_2\right)X_2\right)F_3I_{2-}\right)\right)\Bigg\}$$

$$+\frac{1}{3}\left\{\left(\frac{x_N}{F_2}\right)^4\left(\left(\left(2X_3+6X_4\right)X_4{}^2+\left(8X_4{}^3-\frac{1}{2}\left(1-8X_4{}^3\right)X_2\right)X_2\right)F_3I_{2-}\right)\right.$$

$$+t_N\left(\left(\frac{x_N}{F_2}\right)^3\left(\left(\left(-6X_3-24X_4\right)X_4{}^2+\left(-32X_4{}^3+2\left\langle 1-8X_4{}^3\right\rangle X_2\right)X_2\right)F_3I_{2-}\right)\right)$$

$$+t_N{}^2\left(\left(\frac{x_N}{F_2}\right)^2\left(\left(\left(6X_3+30X_4\right)X_4{}^2+\left(48X_4{}^3-3\left(1-8X_4{}^3\right)X_2\right)X_2\right)F_3I_{2-}\right)\right)$$

$$\left.+t_N{}^3\left(\left(\frac{x_N}{F_2}\right)\left(\left(\left(-2X_3-12X_4\right)X_4{}^2+\left(-24X_4{}^3+2\left(1-8X_4{}^3\right)X_2\right)X_2\right)F_3I_{2-}\right)\right)\right\}\Bigg\}$$

$$+\left(Fr_0+2\right)\left\{\left(\frac{x_N}{F_2}\right)^3\left(-\frac{1}{4}K_1-\frac{1}{3}X_4F_4I_{1+}\right)+\left(\frac{x_N}{F_2}\right)^4\left(\frac{1}{4}K_2+\left(\frac{2}{3}X_3+\frac{1}{2}X_4\right)X_4F_4I_{2+}\right)\right.$$

$$+t_N\left(\left(\frac{x_N}{F_2}\right)^2\left(\frac{1}{4}K_1+\frac{1}{2}X_4F_4I_{1+}\right)+\left(\frac{x_N}{F_2}\right)^3\left(-\frac{1}{2}K_2+\left(-\frac{5}{3}X_3-\frac{4}{3}X_4\right)X_4F_4I_{2+}\right)\right)$$

$$\left.+t_N{}^2\left(\left(\frac{x_N}{F_2}\right)^2\left(\frac{1}{4}K_1+X_3X_4I_{2+}+X_4{}^2F_4I_{2+}\right)\right)\right\}\Bigg] \tag{3.3.34}$$

Finally, the water depth hydrograph at the downstream locations can be obtained by adding the values of h_{10}, h_{11} and h_{12}.

$$h=1+\Delta h,\quad \Delta h\equiv\varepsilon\left(h_{10}+c_fh_{11}+c_f{}^2h_{12}\right) \tag{3.3.35}$$

As the objective of this research is to study whether the two hydrographs at the upstream boundary can be reproduced using the one observed hydrograph data at the middle site of the river or not, it is necessary to test inversion analysis using the theoretical equations obtained here. In view of the inversion analysis, Eq. (3.3.34) indicates that the two boundary hydrographs of depth and velocity with 4 parameters can be reproduced if the peak value h_P, the time of occurrence t_P of h_P, d^2h_P/dt^2 and d^3h_P/dt^3 are known in ΔOGE region. Since Eq. (3.3.34) includes the terms with $t_N{}^3$, 4 parameters can be calculated using h_P, t_P, d^2h_P/dt^2 and d^3h_P/dt^3.

The calculation results of Eq. (3.3.35) are shown in Fig.3.3.3. In the results of the perturbation equations, a slight difference between Δh and h_{10} can be seen because of the effect of h_{11} and h_{12}. The dotted lines in Fig.3.3.3 connect the water depth (=0) on the positive characteristic line that emerges from the origin and the water depth on the negative characteristic line. The analytical solutions for this time period are derived in next section.

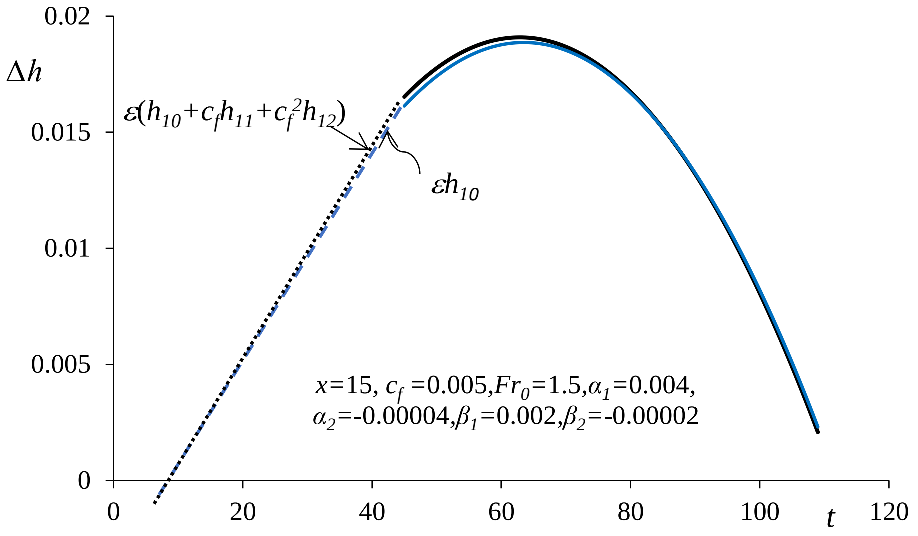


Figure.3.3.3. Resultant hydrographs at x =15 from the u/s boundary in ΔOGE .

3.3.4. Derivation of Analytical Solutions for Downstream Region from ΔOGE Region

The analytical solution for the downstream region from ΔOGE region and the region between the positive characteristic line and the negative characteristic line which emerge from the origin O can be derived by using the solution of ΔOGE region as the boundary conditions.

Firstly, h_{10} and u_{10} at point M is derived by using Eq. (3.3.14a) and Eq. (3.3.14b) from A to M and from F'' to M.

$$h_{10M'} = \frac{1}{2}h_{10A} + \frac{1}{2}Fr_0 u_{10A} = \frac{1}{2}\left(\alpha_1 t_A + \alpha_2 t_A^{\ 2}\right) + \frac{1}{2}Fr_0\left(\beta_1 t_A + \beta_2 t_A^{\ 2}\right) \tag{3.3.36a}$$

$$u_{10M'} = \frac{1}{2Fr_0}h_{10A} + \frac{1}{2}u_{10A} = \frac{1}{2Fr_0}\left(\alpha_1 t_A + \alpha_2 t_A^{\ 2}\right) + \frac{1}{2}\left(\beta_1 t_A + \beta_2 t_A^{\ 2}\right) \tag{3.3.36b}$$

$$h_{10M''} = \frac{1}{2}\left(\alpha_1 t_{B''} + \alpha_2 t_{B''}^{\ 2}\right) + \frac{1}{2}Fr_0\left(\beta_1 t_{B''} + \beta_2 t_{B''}^{\ 2}\right) \tag{3.3.37a}$$

$$u_{10M''} = \frac{1}{2Fr_0}\left(\alpha_1 t_{B''} + \alpha_2 t_{B''}^{\ 2}\right) + \frac{1}{2}\left(\beta_1 t_{B''} + \beta_2 t_{B''}^{\ 2}\right) \tag{3.3.37b}$$

And then, h_{11} and u_{11} can be calculated by substituting h_{10} and u_{10} into the time integration of the right side of Eq. (3.3.15a) and Eq. (3.3.15b). The time integration parts is implemented from point P to M along the positive characteristic line and from point F'' to M along the negative characteristic line as shown in Eq. (3.3.38) and Eq. (3.3.39).

$$\int_P^M \frac{D_+}{Dt}(h_{11}+Fr_0u_{11})dt = h_{11M} + Fr_0u_{11M} - h_{11P} - Fr_0u_{11P} = Fr_0\int_{t_P}^{t_M}(h_{10M'} - 2u_{10M'})dt_{M'}$$

$$= Fr_0\frac{1}{2}\left\{\left(\alpha_1 t_A + \alpha_2 t_A{}^2\right) + Fr_0\left(\beta_1 t_A + \beta_2 t_A{}^2\right)\right\}\left(t_M - t_P\right)$$

$$-\left\{\left(\alpha_1 t_A + \alpha_2 t_A{}^2\right) + Fr_0\left(\beta_1 t_A + \beta_2 t_A{}^2\right)\right\}\left(t_M - t_P\right) \qquad (3.3.38)$$

$$\int_{F''}^M \frac{D_-}{Dt}(h_{11}-Fr_0u_{11})dt = h_{11M} - Fr_0u_{11M} = -Fr_0\int_{F''}^{M}(h_{10M''} - 2u_{10M''})dt_{M''}$$

$$= -Fr_0\frac{1}{2}\left\{\left(\alpha_1 t_{B''} + \alpha_2 t_{B''}{}^2\right) + Fr_0\left(\beta_1 t_{B''} + \beta_2 t_{B''}{}^2\right)\right\}\left(t_M - t_{F''}\right)$$

$$+\left\{\left(\alpha_1 t_{B''} + \alpha_2 t_{B''}{}^2\right) + Fr_0\left(\beta_1 t_{B''} + \beta_2 t_{B''}{}^2\right)\right\}\left(t_M - t_{F''}\right) \qquad (3.3.39)$$

Since $t_{B''}$ is the function of $t_{M''}$, the time integration can be implemented in Eq. (3.3.40) and Eq. (3.3.41).

$$h_{11M} = \frac{1}{2}\left(h_{11P} + Fr_0u_{11P}\right) + \frac{1}{4}\left(Fr_0 - 2\right)\left\{\left(\alpha_1 t_A + \alpha_2 t_A{}^2\right) + Fr_0\left(\beta_1 t_A + \beta_2 t_A{}^2\right)\right\}\left(t_M - t_P\right)$$

$$+\frac{1}{2}Fr_0\left(-BA_0 + 2BB_0\right)\left(t_M - t_{F''}\right) + \frac{1}{4}Fr_0\left(-BA_1 + 2BB_1\right)\left(t_M{}^2 - t_{F''}{}^2\right)$$

$$+\frac{1}{6}Fr_0\left(-BA_2 + 2BB_2\right)\left(t_M{}^3 - t_{F''}{}^3\right) \qquad (3.3.40)$$

$$u_{11M} = \frac{1}{2Fr_0}h_{11P} + \frac{1}{2}u_{11P} + \frac{1}{2Fr_0}\left(Fr_0 - 2\right)\left\{\left(\alpha_1 t_A + \alpha_2 t_A{}^2\right) + Fr_0\left(\beta_1 t_A + \beta_2 t_A{}^2\right)\right\}\frac{1}{2}\left(t_M - t_P\right)$$

$$+\frac{1}{2}\left(BA_0 - 2BB_0\right)\left(t_M - t_{F''}\right) + \frac{1}{2}\left(BA_1 - 2BB_1\right)\frac{1}{2}\left(t_M{}^2 - t_{F''}{}^2\right)$$

$$+\frac{1}{2}\left(-BA_2 + 2BB_2\right)\frac{1}{3}\left(t_M{}^3 - t_{F''}{}^3\right) \qquad (3.3.41)$$

in which, $t_{B''} = 2X_4 t_{M''} - X_5 x_F$, $X_4 \equiv \frac{1}{Fr_0 + 1}$, $X_5 \equiv \frac{Fr_0}{Fr_0 + 1}$, $BA_0 \equiv -I_{1+}X_5x_F + I_{2+}X_5{}^2x_F{}^2$,

$$BA_1 \equiv 2I_{1+}X_4 - 4I_{2+}X_4X_5x_F,\ BA_2 \equiv 4I_{2+}X_4{}^2\ BB_0 \equiv \frac{1}{Fr_0}BA_0, BB_1 \equiv \frac{1}{Fr_0}BA_1, BB_2 \equiv \frac{1}{Fr_0}BA_2.$$

Similarly, by substituting h_{11} and u_{11} into the integration part related to Eq. (3.3.16a) and Eq. (3.3.16b), h_{12} and u_{12} are calculated similarly.

$$h_{12M} + Fr_0 u_{12M} = h_{12P} + Fr_0 u_{12P} + Fr_0 \int_{t_P}^{t_M} (h_{11M'} - 2u_{11M'}) dt_{M'} \tag{3.3.42a}$$

$$h_{12M} - Fr_0 u_{12M} = -Fr_0 \int_{t_{F''}}^{t_M} (h_{11M''} - 2u_{11M''}) dt_{M''} \tag{3.3.42b}$$

$$h_{12M} = \frac{1}{2}\left(h_{12P} + Fr_0 u_{12P} \right) + \frac{Fr_0}{2} \int_P^M (h_{11M'} - 2u_{11M'}) dt_{M'} - \frac{Fr_0}{2} \int_{F''}^M (h_{11M''} - 2u_{11M''}) dt_{M''} \tag{3.3.43}$$

Finally, h_{10M}, h_{11M} and h_{12M} become as the followings by calculating some procedures in the above equations. In this calculation, t_A is the function of t_M and x_M and is defined as $t_A = t_M - X_5 x_M$.

$$h_{10M} = -I_{1+} \frac{x_M}{F_1} + I_{2+} \left(\frac{x_M}{F_1} \right)^2 + t_M \left(I_{1+} - 2I_{2+} \frac{x_M}{F_1} \right) + t_M{}^2 I_{2+} \tag{3.3.44}$$

$$\begin{aligned} h_{11M} = {} & \frac{1}{2}\left(h_{11P} + Fr_0 u_{11P} \right) + \frac{1}{2}\left(Fr_0 - 2 \right)\left(-I_{1+} \frac{x_M}{F_1} + I_{2+} \left(\frac{x_M}{F_1} \right)^2 + t_M \left(I_{1+} - 2I_{2+} \frac{x_M}{F_1} \right) + t_M{}^2 I_{2+} \right)\left(t_M - t_P \right) \\ & - \frac{1}{2}\left(Fr_0 - 2 \right)\Bigg(\left(-X_5 x_F I_{1+} + X_5{}^2 x_F{}^2 I_{2+} \right)\left(t_M - t_F \right) + \left(2X_4 I_{1+} - 4X_4 X_5 x_F I_{2+} \right) \frac{1}{2}\left(t_M{}^2 - t_F{}^2 \right) \\ & + \frac{4}{3} X_4{}^2 I_{2+} \left(t_M{}^3 - t_F{}^3 \right) \Bigg) \end{aligned} \tag{3.3.45}$$

$$\begin{aligned} h_{12M} = {} & \frac{1}{2}\left(h_{12P} + Fr_0 u_{12P} \right) + \frac{1}{4}\left(h_{11P} + Fr_0 u_{11P} \right)\left(Fr_0 - 2 \right)\left(t_M - t_P \right) + \frac{1}{2}\Bigg[\frac{1}{2}\left(Fr_0 - 2 \right)^2 \left(I_{1+} t_A + I_{2+} t_A{}^2 \right) \frac{1}{2}\left(t_M - t_P \right)^2 \\ & + \frac{1}{2}\left(Fr_0{}^2 - 4 \right) \frac{1}{2}\left(Fr_0 + 1 \right) t_A \left(-\frac{4}{3} X_4{}^2 I_{2+} \left(t_M{}^3 - t_P{}^3 \right) + X_4 \left(I_{1+} + 2I_{2+} t_A \right)\left(t_M{}^2 - t_P{}^2 \right) - \left(I_{1+} t_A + I_{2+} t_A{}^2 \right)\left(t_M - t_P \right) \right) \\ & - \frac{1}{2}\left(Fr_0{}^2 - 4 \right) \frac{1}{2}\left(Fr_0 + 1 \right) t_A \left(-\frac{8}{3} X_4{}^2 I_{2+} \left(t_M{}^3 - t_P{}^3 \right) + X_4 \left(I_{1+} + 3I_{2+} t_A \right)\left(t_M{}^2 - t_P{}^2 \right) - \frac{1}{2}\left(I_{1+} t_A + 2I_{2+} t_A{}^2 \right)\left(t_M - t_P \right) \right) \\ & + 2\left(Fr_0{}^2 - 4 \right)\left(-\frac{1}{6} \frac{I_{2+} t_A}{\left(Fr_0 + 1 \right)}\left(t_M{}^3 - t_P{}^3 \right) + \frac{1}{8} I_{2+} t_A{}^2 \left(t_M{}^2 - t_P{}^2 \right) - \frac{1}{24}\left(Fr_0 + 1 \right) I_{2+} t_A{}^3 \left(t_M - t_P \right) \right) \Bigg] \\ & + \frac{1}{2}\Bigg[-\frac{1}{2}\left(Fr_0 - 2 \right)\Bigg\{ \Big(\langle 1 - 2X_4 \rangle \Big(\langle Fr_0 - 2 \rangle 4X_4{}^2 I_{2+} + \langle Fr_0 + 2 \rangle 4X_1{}^2 X_4{}^2 I_{2-} \Big)\Big) \frac{1}{4}\left(t_M{}^4 - \langle X_6 x_F \rangle^4 \right) \\ & + \Big(\langle 1 - 2X_4 \rangle \Big(\langle Fr_0 - 2 \rangle \langle 2X_4 I_{1+} - 4X_4 X_5 x_F I_{2+} \rangle + \langle Fr_0 + 2 \rangle \langle 2X_1 X_4 I_{1-} - 4X_1{}^2 X_4 X_5 x_F I_{2-} \rangle \Big) \\ & + \left\langle -\frac{1}{2} Fr_0 x_F + X_5 x_F \right\rangle \Big(\langle Fr_0 - 2 \rangle 4X_4{}^2 I_{2+} + \langle Fr_0 + 2 \rangle 4X_1{}^2 X_4{}^2 I_{2-} \Big)\Big) \frac{1}{3}\left(t_M{}^3 - \langle X_6 x_F \rangle^3 \right) \\ & + \Big(\langle 1 - 2X_4 \rangle \Big(\langle Fr_0 - 2 \rangle \langle -X_5 x_F I_{1+} + X_5{}^2 x_F{}^2 I_{2+} \rangle + \langle Fr_0 + 2 \rangle \langle -X_1 X_5 x_F I_{1-} + X_1{}^2 X_5{}^2 x_F{}^2 I_{2-} \rangle \Big)\Big) + \left\langle -\frac{1}{2} Fr_0 x_F + X_5 x_F \right\rangle \end{aligned}$$

$$\times\Big(\langle Fr_0-2\rangle\langle 2X_4I_{1+}-4X_4X_5x_FI_{2+}\rangle+\langle Fr_0+2\rangle\langle 2X_1X_4I_{1-}-4X_1{}^2X_4X_5x_FI_{2-}\rangle\Big)\frac{1}{2}\left(t_M{}^2-\langle X_6x_F\rangle^2\right)$$

$$+\left(\left\langle -\frac{1}{2}Fr_0x_F+X_5x_F\right\rangle\Big(\langle Fr_0-2\rangle\langle -X_5x_FI_{1+}+X_5{}^2x_F{}^2I_{2+}\rangle+\langle Fr_0+2\rangle\langle -X_1X_5x_FI_{1-}+X_1{}^2X_5{}^2x_F{}^2I_{2-}\rangle\Big)\right)$$

$$\times\left(t_M-X_6x_F\right)+\left(\frac{1}{2}Y_1\langle Fr_0+2\rangle\langle -8X_1X_2X_4I_{2-}\rangle\right)\frac{1}{4}\left(t_M{}^4-\langle X_6x_F\rangle^4\right)$$

$$+\left(\frac{1}{2}Y_1(Fr_0+2)(-2X_2I_{1-}+4X_1X_2X_5x_FI_{2-})-\frac{1}{2}Y_1Fr_0x_F(Fr_0+2)(-8X_1X_2X_4I_{2-})\right)\frac{1}{3}\left(t_M{}^3-\langle X_6x_F\rangle^3\right)$$

$$+\left(-\frac{1}{2}Fr_0Y_1x_F(Fr_0+2)(-2X_2I_{1-}+4X_1X_2X_5x_FI_{2-})+\frac{1}{8}Y_1Fr_0{}^2x_F{}^2(Fr_0+2)(-8X_1X_2X_4I_{2-})\right)\frac{1}{2}\left(t_M{}^2-\langle X_6x_F\rangle^2\right)$$

$$+\left(\frac{1}{8}Y_1Fr_0{}^2x_F{}^2(Fr_0+2)(-2X_2I_{1-}+4X_1X_2X_5x_FI_{2-})\right)\left(t_M-\langle X_6x_F\rangle\right)+\left(\frac{1}{3}(Fr_0+2)4X_2{}^2I_{2-}Y_2\right)\frac{1}{4}\left(t_M{}^4-\langle X_6x_F\rangle^4\right)$$

$$+\left(-\frac{1}{3}(Fr_0+2)6X_2{}^2I_{2-}Y_2Fr_0x_F\right)\frac{1}{3}\left(t_M{}^3-\langle X_6x_F\rangle^3\right)+\left(\frac{1}{3}(Fr_0+2)3X_2{}^2I_{2-}Y_2Fr_0{}^2x_F{}^2\right)\frac{1}{2}\left(t_M{}^2-\langle X_6x_F\rangle^2\right)$$

$$+\left(-\frac{1}{6}(Fr_0+2)X_2{}^2I_{2-}Y_2Fr_0{}^3x_F{}^3\right)\left(t_M-\langle X_6x_F\rangle\right)\Big\}\Big]$$

$$+\frac{1}{2}\Big[-\frac{1}{2}X_6x_F(Fr_0-2)^2\Big(\langle -X_5x_FI_{1+}+X_5{}^2x_F{}^2I_{2+}\rangle\langle t_M-t_F\rangle+\langle 2X_4I_{1+}-4X_4X_5x_FI_{2+}\rangle\frac{1}{2}\left(t_M{}^2-t_F{}^2\right)$$

$$+\frac{4}{3}X_4{}^2I_{2+}\left(t_M{}^3-t_F{}^3\right)\Big)+\frac{1}{2}\left(Fr_0{}^2-4\right)\left(\langle -X_5x_FI_{1+}+X_5{}^2x_F{}^2I_{2+}\rangle\left(\frac{1}{2}\left(t_M{}^2-t_F{}^2\right)-X_6x_F(t_M-t_F)\right)\right)$$

$$+\frac{1}{4}\left(Fr_0{}^2-4\right)\left(\langle 2X_4I_{1+}-4X_4X_5x_FI_{2+}\rangle\left(\frac{1}{3}\left(t_M{}^3-t_F{}^3\right)-X_6{}^2x_F{}^2(t_M-t_F)\right)\right)$$

$$+\left(\frac{1}{6}\left(Fr_0{}^2-4\right)X_4{}^2I_{2+}\left(\left(t_M{}^4-t_F{}^4\right)-4X_6{}^3x_F{}^3(t_M-t_F)\right)\right)\Big] \tag{3.3.46}$$

The analytical solutions include the terms with t_M^3 and t_M^4 in Eq. (3.3.45) and Eq. (3.3.46). This indicates that there is a possibility of identifying four unknowns at the upstream boundary condition using only one point of water depth hydrograph even in the downstream region.

3.4. Consideration of the Calculation Results.

3.4.1. Numerical Simulation of the Set of Perturbed Linearized Equations

Before considering the calculated results of the analytical solutions, the numerical calculation of the basic equations was carried out in order to verify the theoretical results.

According to MOC, the values of unknown hydraulic variables at *R* can be calculated by using the known values at *P* and *Q*.

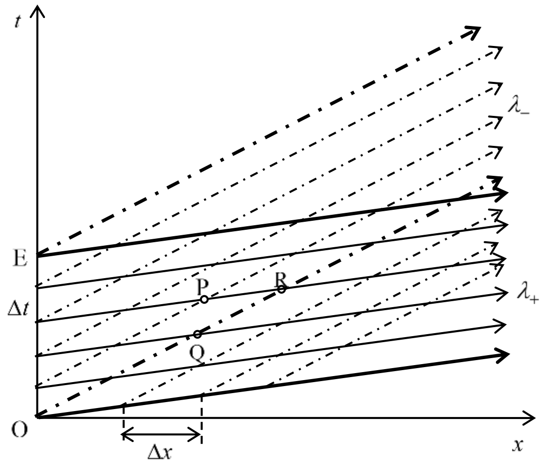


Figure.3.4.1. Characteristic lines for numerical calculation.

The values of hydraulic variables at *R* can be calculated by applying Eq. (3.4.1) with the finite difference equations.

$$c_f^{\ m} \quad \lambda_+ : \frac{D_+ h_{1m}}{Dt} + Fr_0 \frac{D_+ u_{1m}}{Dt} = Fr_0 (h_{1(m-1)} - 2u_{1(m-1)}) \qquad (3.4.1a)$$

$$\lambda_- : \frac{D_- h_{1m}}{Dt} - Fr_0 \frac{D_- u_{1m}}{Dt} = -Fr_0 (h_{1(m-1)} - 2u_{1(m-1)}) \qquad (3.4.1b)$$

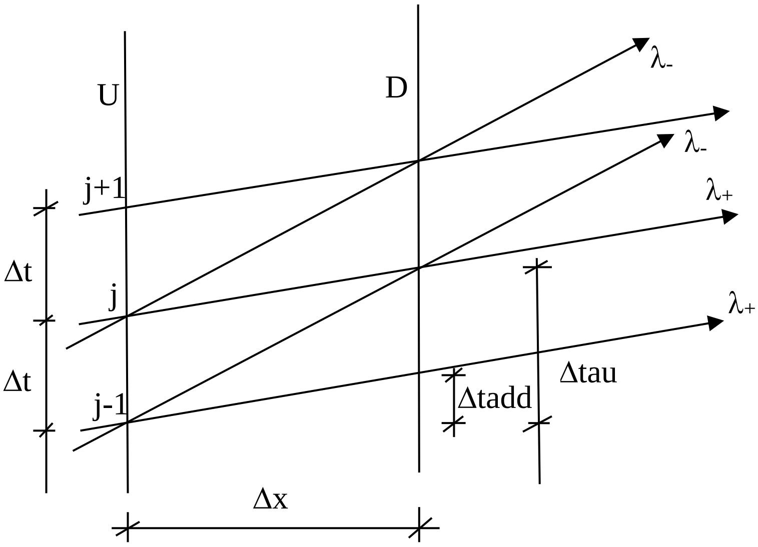


Figure.3.4.2. Illustration of finite difference scheme.

$$h_{1m}\big|_D^j = \frac{1}{2}\left\{\left(h_{1m}\big|_U^j + h_{1m}\big|_U^{j-1}\right) + Fr_0\left(u_{1m}\big|_U^j - u_{1m}\big|_U^{j-1}\right)\right\}$$

$$+\frac{1}{2}Fr_0\frac{1}{2}\Delta tadd(h_{1(m-1)}\Big|_D^j-2u_{1(m-1)}\Big|_D^j+h_{1(m-1)}\Big|_U^j-2u_{1(m-1)}\Big|_U^j)$$

$$-\frac{1}{2}Fr_0\frac{1}{2}\Delta tau(h_{1(m-1)}\Big|_D^j-2u_{1(m-1)}\Big|_D^j+h_{1(m-1)}\Big|_U^{j-1}-2u_{1(m-1)}\Big|_U^{j-1}) \tag{3.4.2}$$

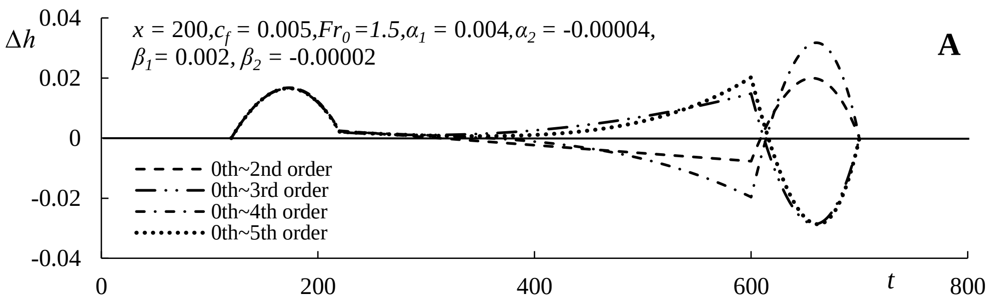
$$u_{1m}\Big|_D^j=\frac{1}{2Fr_0}\left\{\left(h_{1m}\Big|_U^j-h_{1m}\Big|_U^{j-1}\right)+Fr_0\left(u_{1m}\Big|_U^j+u_{1m}\Big|_U^{j-1}\right)\right\}$$

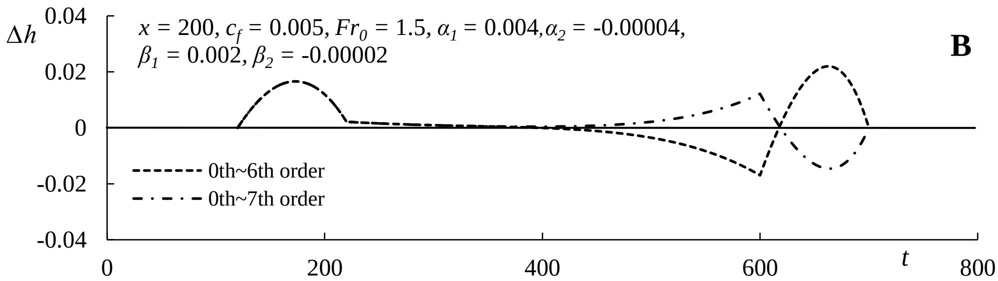
$$+\frac{1}{4}\Delta tadd(h_{1(m-1)}\Big|_D^j-2u_{1(m-1)}\Big|_D^j+h_{1(m-1)}\Big|_U^j-2u_{1(m-1)}\Big|_U^j)$$

$$+\frac{1}{4}\Delta tau(h_{1(m-1)}\Big|_D^j-2u_{1(m-1)}\Big|_D^j+h_{1(m-1)}\Big|_U^{j-1}-2u_{1(m-1)}\Big|_U^{j-1}) \tag{3.4.3}$$

$$h=1+\Delta h,\ \ \Delta h\equiv\varepsilon\left(h_{10}+c_f h_{11}+c_f{}^2h_{12}+......+c_f{}^{m-1}h_{1(m-1)}+c_f{}^m h_{1m}\right) \tag{3.4.4}$$

In this study , the terms up to m=9 are considered for calculation. The resultant hydrographs at the downstream site of the region are shown in Fig.3.4.3 and Fig.3.4.4 for the different friction coefficients. From these figures, it can be seen that the latter part of hydrographs significantly oscillates from positive region to negative region depending on the number of terms included for calculation. However, the hydrographs finally converge with the increase of the terms considered for calculation.





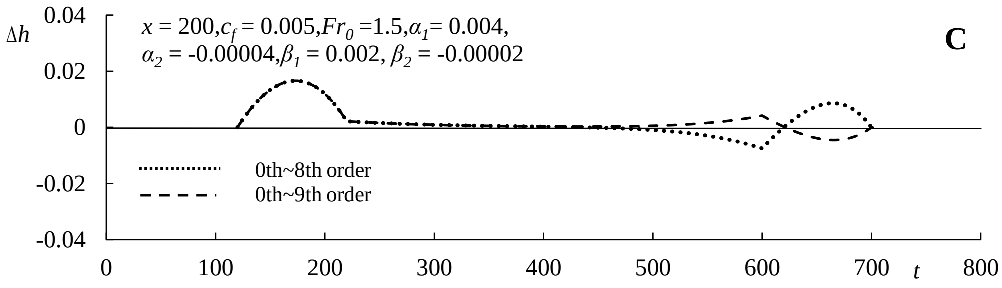


Figure.3.4.3. Hydrographs from numerical simulations of c_f = 0.005.

From Fig.3.4.3, it seems that the latter parts of the hydrographs with the friction coefficient(c_f) 0.005 steadily attenuate when higher order terms are included to calculate the hydrographs. For the friction coefficient 0.0025, the latter part of the hydrograph shown in Fig.3.4.4 almost vanishes with up to the ninth order.

On the other hand, in the former part, there is almost no differences between the hydrographs with up to the second order terms and ones with up to higher order terms. That indicates that considering the second order term is enough to calculate the hydrographs for the former part. The downstream part below the triangle region considering in this study corresponds to the former part of the hydrograph. In this study, only the hydrographs with up to the second order term are considered.

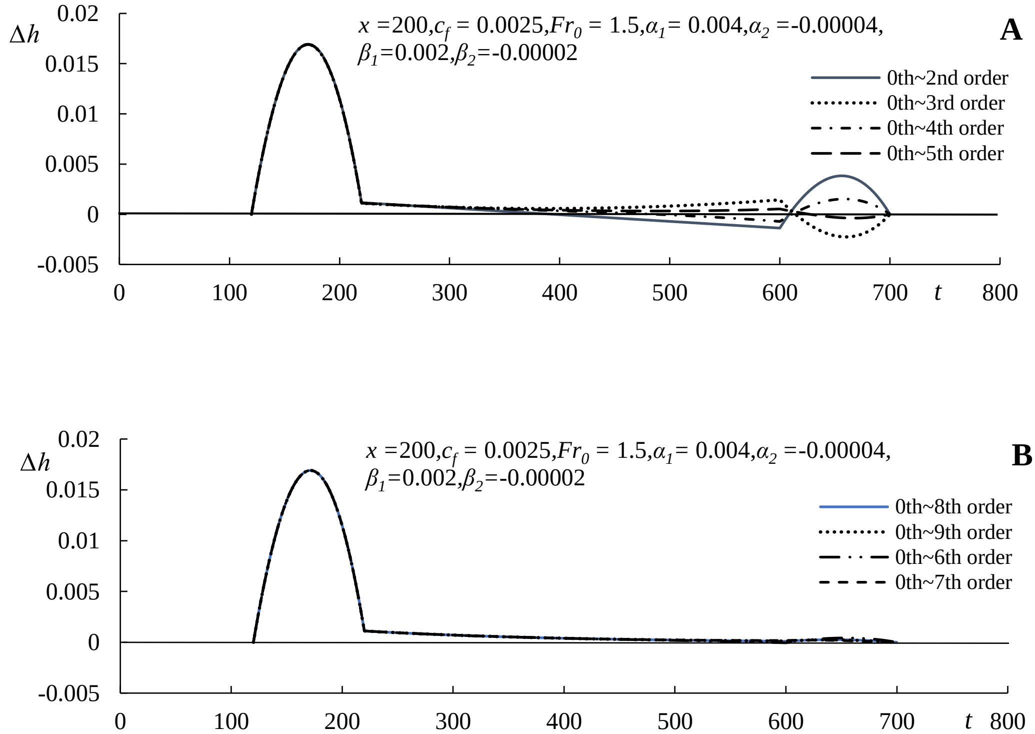


Figure.3.4.4. Resultant hydrographs of numerical simulation with c_f =0.0025.

In order to verify the calculated results of the analytical solutions, the comparisons between the analytical solutions and numerical results are shown in Fig.3.4.5. The hydrographs with solid lines represent the analytical solutions and the dashed lines the numerical results at different locations along the channel course. From these figures, it can be clearly seen that the two results are in good agreement.

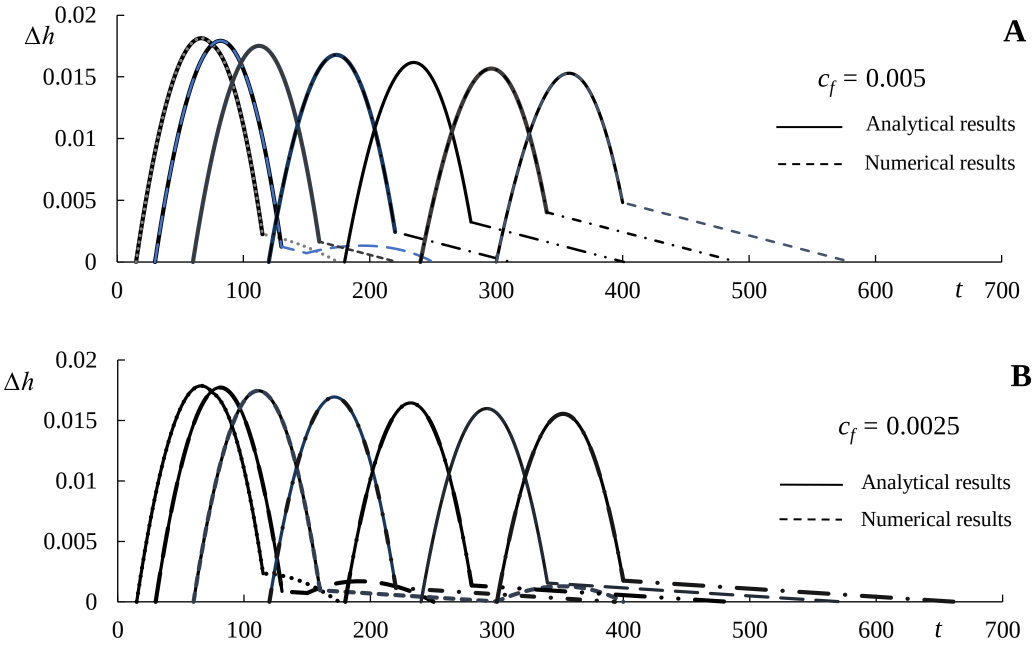


Figure.3.4.5. Comparison between analytical and numerical results.

3.4.2. Numerical Simulation of the Linearized Equations

Eq. (3.3.7) can be transformed into Eq. (3.4.5) and the numerical solutions of these linearized equations can be calculated by using the finite difference method (FDM). The FDM scheme is illustrated in Fig.3.4.2 and Eq. (3.4.6) and Eq. (3.4.7). The calculated results obtained using these two equations are compared to the numerical results of analytical solutions. The comparisons of results from these linarized equations and analytical solutions with friction coefficient of zero which is called *d'Alembert* solution are also shown in Fig.3.4.6.

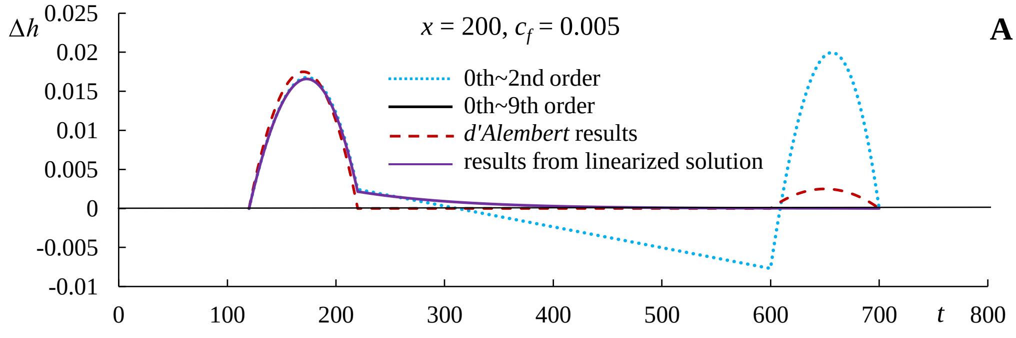
$$\lambda_+ : \frac{D_+ h_1}{Dt} + Fr_0 \frac{D_+ u_1}{Dt} = Fr_0 c_f \left(h_1 - 2u_1 \right) \tag{3.4.5a}$$

$$\lambda_- : \frac{D_- h_1}{Dt} - Fr_0 \frac{D_- u_1}{Dt} = -Fr_0 c_f \left(h_1 - 2u_1 \right) \tag{3.4.5b}$$

$$
\begin{aligned}
h_1\big|_D^j &= \frac{(1+A_1)\times(Fr_0+2A_2)}{\{(1-A_1)(Fr_0+2A_2)+(Fr_0+2A_1)(1+A_2)\}} h_1\big|_U^j \\
&+ \frac{(Fr_0+2A_1)\times(1-A_2)}{\{(1-A_1)(Fr_0+2A_2)+(Fr_0+2A_1)(1+A_2)\}} h_1\big|_U^{j-1} \\
&+ \frac{(Fr_0-2A_1)\times(Fr_0+2A_2)}{\{(1-A_1)(Fr_0+2A_2)+(Fr_0+2A_1)(1+A_2)\}} u_1\big|_U^j \\
&- \frac{(Fr_0+2A_1)\times(Fr_0-2A_2)}{\{(1-A_1)(Fr_0+2A_2)+(Fr_0+2A_1)(1+A_2)\}} u_1\big|_U^{j-1}
\end{aligned} \tag{3.4.6}
$$

$$
\begin{aligned}
u_1\big|_D^j &= \frac{(1+A_1)\times(1+A_2)}{\{(1-A_1)(Fr_0+2A_2)+(Fr_0+2A_1)(1+A_2)\}} h_1\big|_U^j \\
&- \frac{(1-A_1)\times(1-A_2)}{\{(1-A_1)(Fr_0+2A_2)+(Fr_0+2A_1)(1+A_2)\}} h_1\big|_U^{j-1} \\
&+ \frac{(Fr_0-2A_1)\times(1+A_2)}{\{(1-A_1)(Fr_0+2A_2)+(Fr_0+2A_1)(1+A_2)\}} u_1\big|_U^j \\
&+ \frac{(1-A_1)\times(Fr_0-2A_2)}{\{(1-A_1)(Fr_0+2A_2)+(Fr_0+2A_1)(1+A_2)\}} u_1\big|_U^{j-1}
\end{aligned} \tag{3.4.7}
$$

in which, $A_1 = \frac{1}{2} c_f Fr_0 \times \Delta tadd$ and $A_2 = \frac{1}{2} c_f Fr_0 \times \Delta tau$



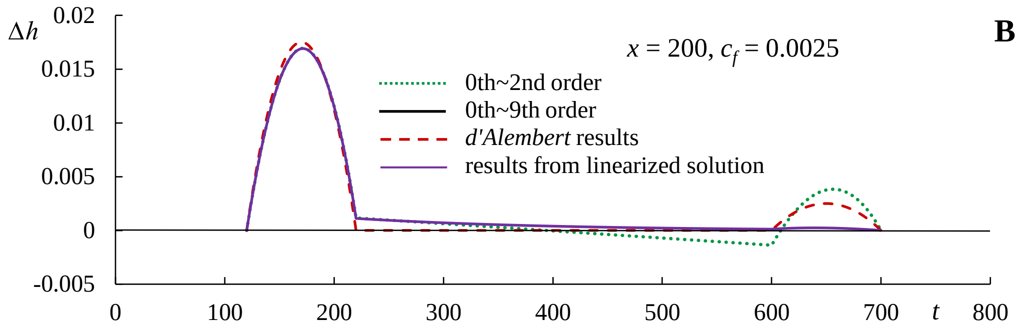


Figure.3.4.6. Comparison of the numerical results.

From Fig.3.4.6, it can be pointed out that the latter parts of hydrographs from the linearized equations are almost the same as the results from the perturbed linearized equations with up to ninth order terms.

3.4.3. Comparison of Numerical Results between Linearized and Shallow Water Equations

Since the shallow water equations are classified as Hyperbolic Type, it is natural to apply MOC in the fixed grid system as a numerical method. Although the first upwind scheme is one of the most robust numerical schemes, the scheme always causes diffusive distribution because of the large numerical diffusion effect. In order to avoid the results with numerical diffusion effect, a higher order numerical scheme with the total variation diminishing conditions is applied to obtain the results with high accuracy and without numerical oscillation. In this study, TVD-Lax-Wendroff scheme is used to obtain reasonable results. The discretized equations of TVD-Lax-Wendroff scheme are denoted as Eq. (3.4.9) ~ Eq. (3.4.11).

$$\frac{\partial}{\partial t}\left(u \pm \frac{2\sqrt{h}}{Fr_0}\right) + \left(u \pm \frac{\sqrt{h}}{Fr_0}\right)\frac{\partial}{\partial x}\left(u \pm \frac{2\sqrt{h}}{Fr_0}\right) = c_f\left(1 - {u^2}/{h}\right) \tag{3.4.8}$$

[TVD-LW Scheme]

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}u_i^n\left(u_i^n - u_{i-1}^n\right) - \frac{\Delta t}{\Delta x}\frac{\sqrt{h_i^n}}{Fr_0}\left(\frac{2\sqrt{h_i^n}}{Fr_0} - \frac{2\sqrt{h_{i-1}^n}}{Fr_0}\right) + \Delta t \times c_f\left(1 - \left(\frac{u^2}{h}\right)_i^n\right)$$

$$+\frac{1}{2}FluxD1 + \frac{1}{2}FluxD2 + \frac{1}{2}FluxU1 + \frac{1}{2}FluxU2 \tag{3.4.9}$$

$$h_i^{n+1} = \left\{\sqrt{h}_i^n - \frac{\Delta t}{\Delta x}u_i^n\left(\sqrt{h}_i^n - \sqrt{h}_{i-1}^n\right) - \frac{\Delta t}{\Delta x}\frac{\sqrt{h}_i^n}{2}\left(u_i^n - u_{i-1}^n\right)\right.$$

$$+\frac{1}{2}FluxD1-\frac{1}{2}FluxD2+\frac{1}{2}FluxU1-\frac{1}{2}FluxU2\Bigg\}^{2} \tag{3.4.10}$$

$$FluxD1=-\frac{1}{2}\frac{\Delta t}{\Delta x}\left(u+\frac{\sqrt{h}}{Fr_0}\right)_i^n\left(1-\frac{\Delta t}{\Delta x}\left(u+\frac{\sqrt{h}}{Fr_0}\right)_i^n\right)\times FLIMIT$$

$$\times\left\{\left(u+\frac{2\sqrt{h}}{Fr_0}\right)_{i+1}^n-\left(u+\frac{2\sqrt{h}}{Fr_0}\right)_i^n\right\} \tag{3.4.11a}$$

$$FluxD2=-\frac{1}{2}\frac{\Delta t}{\Delta x}\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\left(1-\frac{\Delta t}{\Delta x}\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\right)\times FLIMIT$$

$$\times\left\{\left(u-\frac{2\sqrt{h}}{Fr_0}\right)_{i+1}^n-\left(u-\frac{2\sqrt{h}}{Fr_0}\right)_i^n\right\} \tag{3.4.11b}$$

$$FluxU1=\frac{1}{2}\frac{\Delta t}{\Delta x}\left(u+\frac{\sqrt{h}}{Fr_0}\right)_i^n\left(1-\frac{\Delta t}{\Delta x}\left(u+\frac{\sqrt{h}}{Fr_0}\right)_i^n\right)\times FLIMIT$$

$$\times\left\{\left(u+\frac{2\sqrt{h}}{Fr_0}\right)_i^n-\left(u+\frac{2\sqrt{h}}{Fr_0}\right)_{i-1}^n\right\} \tag{3.4.11c}$$

$$FluxU2=\frac{1}{2}\frac{\Delta t}{\Delta x}\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\left(1-\frac{\Delta t}{\Delta x}\left(u-\frac{\sqrt{h}}{Fr_0}\right)_i^n\right)\times FLIMIT$$

$$\times\left\{\left(u-\frac{2\sqrt{h}}{Fr_0}\right)_i^n-\left(u-\frac{2\sqrt{h}}{Fr_0}\right)_{i-1}^n\right\} \tag{3.4.11d}$$

in which, FLIMIT = flux limiter

The calculated results obtained using Eq. (3.4.9) ~ Eq. (3.4.11) are shown in Fig.3.4.7 together with the solutions for the linearized equations discussed in the previous section. The results with TVD-LW are different from the results obtained using the linearized equation. Especially, the temporal gradients of the hydrographs at the front position with TVD-LW is larger than the ones with linearized equation.

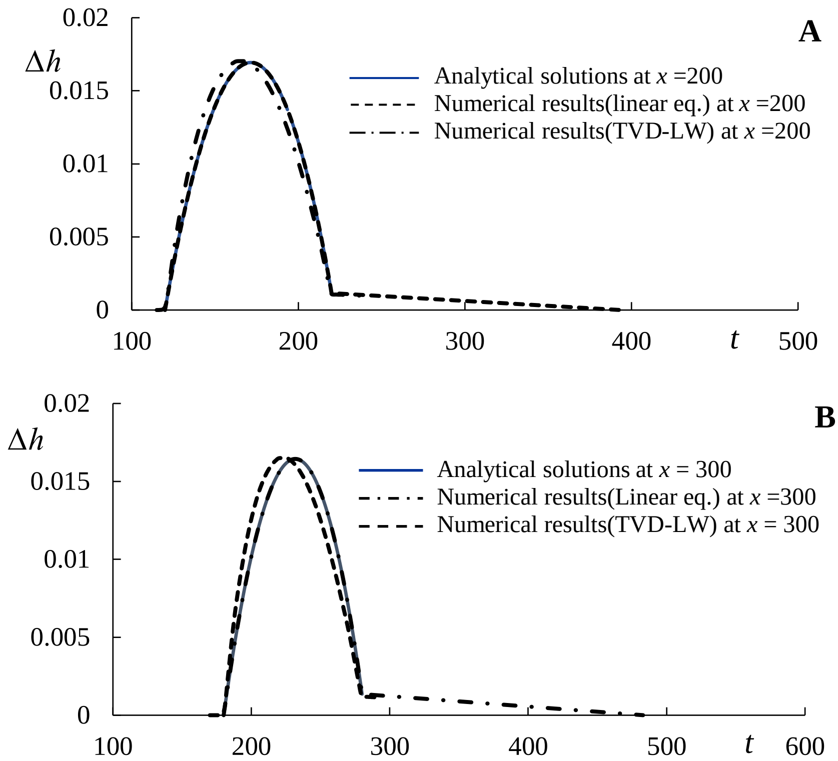


Figure.3.4.7. Comparison between the resultant hydrographs.

3.4.4. Characteristics of Analytical Solutions

In Fig.3.4.8, the third order derivatives with respect to time of resultant hydrograph shown in Fig.3.4.5 are expressed. Since the equations for upstream hydrograph have only up to the second order term of time, the values of third order derivative in the solution of zero order coefficient term become zero at any time. On the other hand, for solution of the sum from zero order to second order of coefficient, the value of third order derivatives has the change in significant value with time.

For the values of second order derivative with respect to time, the value of zero order coefficient solution is a constant value because of the upstream hydrograph condition in which time power series is up to second order like as a quadratic equation. On the contrary, for solution of the sum from zero order to second order coefficient terms, the absolute value of second order derivative becomes smaller than the zeroth order solution and it increases temporally.

In this way, solution of the sum from zero order to second order coefficient has a third order derivative with significant value in numerical calculation. This fact indicates that there is a possibility to identify the four unknowns at the upstream boundary condition using only one hydrograph at one site.

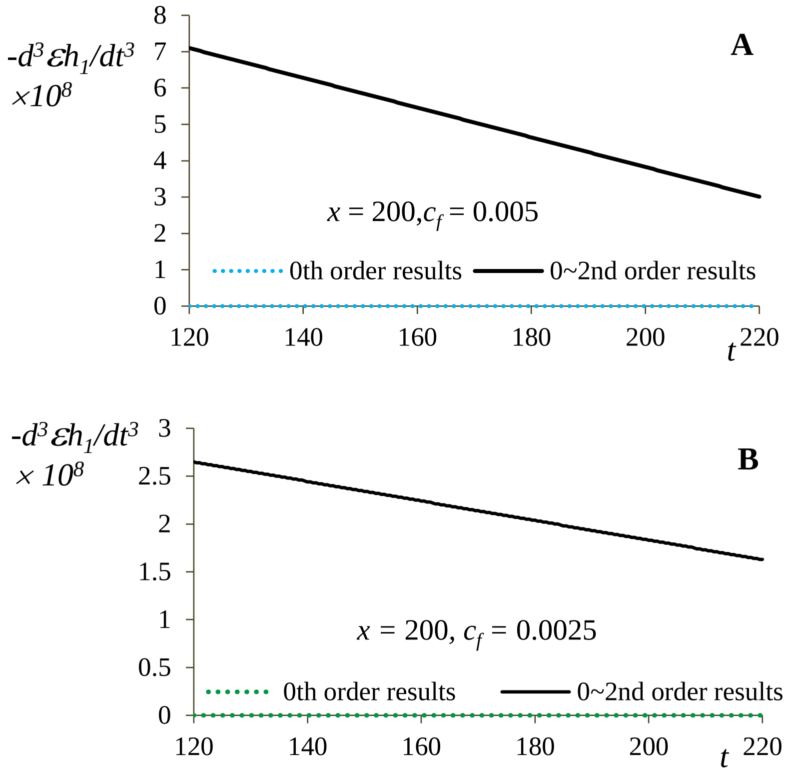
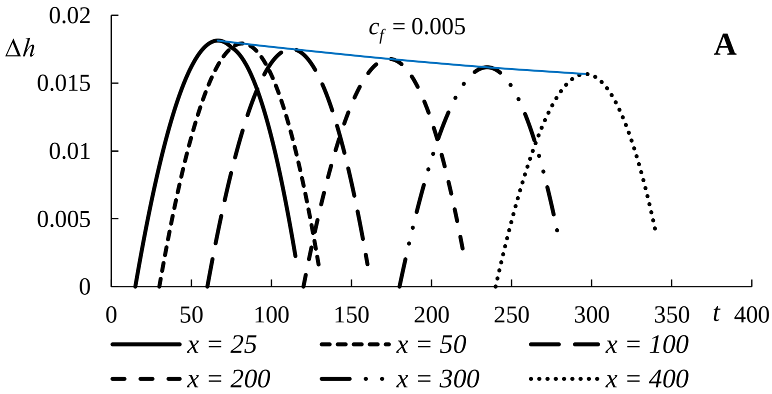


Figure.3.4.8. Results from the 3rd order derivatives of the solution.

From the hydrographs along downstream of the channel, it can be clearly seen that the dynamic wave will change into a kinematic wave where it moves far from the upstream region. The peak values of hydrographs become decline from upstream to downstream region and it can be clearly seen in Fig.3.4.9.



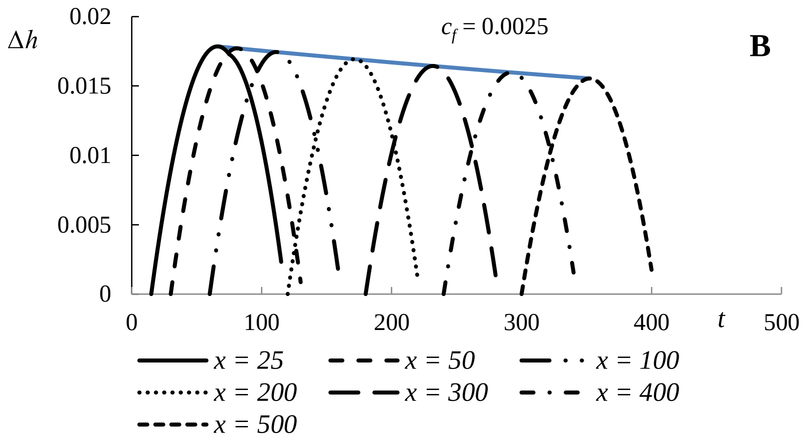


Figure.3.4.9. Attenuation of the hydrographs from the upstream.

The peak times occurring the peak water depth at different locations along the channel are depicted in Fig.3.4.10. According to these figures, the wave speed almost becomes constant.

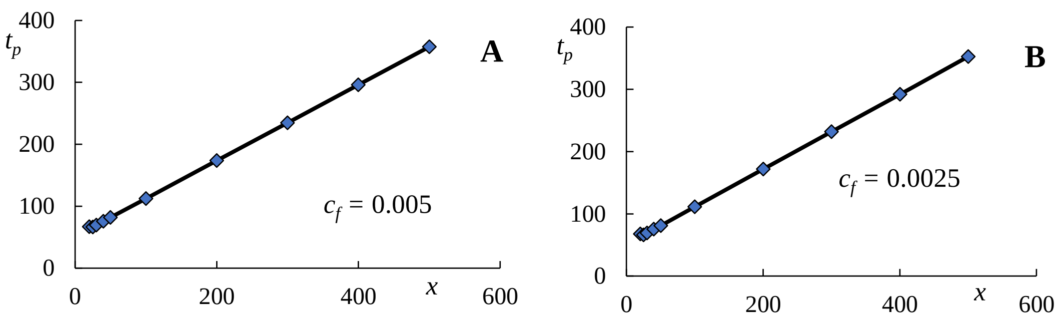


Figure.3.4.10. Relationship between the peak time and the channel reach.

3.5. Reproduction of Upstream Hydrographs using One-depth Hydrograph

3.5.1. Inversion Analysis to Produce Upstream Hydrographs

Since the approximate solutions obtained here for the region far from the upstream end include the fourth order power time, the temporal change of a hydrograph enables to calculate the derivatives on time up to the fourth order. By using the numerical values of the derivatives, parameters which are required to reproduce the two hydrographs at upstream end can be identified with the matrix analysis.

$$[P]=[A][Q] \tag{3.5.1}$$

$[P]$ is the 4×4 matrix which is composed of 4 sets of parameters (α_1, α_2, β_1 and β_2) given at the upstream end.

Once the parameters (α_1 , α_2 , β_1 and β_2) are given, the hydrographs and the derivatives on time at the downstream locations are calculated using analytical solutions.

$[Q]$ is the 4×4 matrix which is composed of the calculated results of (dh/dt , d^2h/dt^2 , d^3h/dt^3 and d^4h/dt^4) at the front of a hydrograph.

Since $[P]$ and $[Q]$ are known, the matrix $[A]$ can be determined uniquely using Eq. (3.5.2).

By using the numerical values of matrix *A* and the derivative values of the known hydrograph at downstream site, the parameter matrix *P* can be obtained to produce the hydrographs at the upstream boundary. In the backward calculation process, firstly, four sets of parameters are considered and downstream hydrographs are produced from the analytical solution. Then, values of the derivative with respect to time are calculated from the resultant hydrographs. The numerical values of matrix *A* can be obtained with the use of the sets of parameter values and the derivative values of resultant hydrographs by the matrix analysis.

$$P = \begin{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix}_{Set1} & \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix}_{Set2} & \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix}_{Set3} & \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix}_{Set4} \end{bmatrix}$$

$$Q = \begin{bmatrix} \begin{pmatrix} dh/dt \\ d^2h/dt^2 \\ d^3h/dt^3 \\ d^4h/dt^4 \end{pmatrix}_{Set1} & \begin{pmatrix} dh/dt \\ d^2h/dt^2 \\ d^3h/dt^3 \\ d^4h/dt^4 \end{pmatrix}_{Set2} & \begin{pmatrix} dh/dt \\ d^2h/dt^2 \\ d^3h/dt^3 \\ d^4h/dt^4 \end{pmatrix}_{Set3} & \begin{pmatrix} dh/dt \\ d^2h/dt^2 \\ d^3h/dt^3 \\ d^4h/dt^4 \end{pmatrix}_{Set4} \end{bmatrix}$$

$$P = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix}_{4\times 4} , \quad Q = \begin{bmatrix} dh/dt \\ d^2h/dt^2 \\ d^3h/dt^3 \\ d^4h/dt^4 \end{bmatrix}_{4\times 4} , \quad A = PQ^{-1} \qquad (3.5.2)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

The procedures for Inversion Analysis are explained below. For the location $x = 200$, the specified matrix is denoted as matrix A and for $x = 400$, that is denoted as matrix A_1.

Suppose that the matrix Q_{Set5} are obtained from one depth hydrograph at a downstream site. By using matrix A and matrix Q_{Set5}, the matrix P can be calculated as follow. The same procedure is applied to identify the matrix P_1.

$$Q_{Set5} = \begin{bmatrix} dh/dt \\ d^2h/dt^2 \\ d^3h/dt^3 \\ d^4h/dt^4 \end{bmatrix}_{Set5(x=200)} , \quad P_{Set5} = AQ_{Set5} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix}_{Set5} \tag{3.5.3}$$

$$Q_{1Set5} = \begin{bmatrix} dh/dt \\ d^2h/dt^2 \\ d^3h/dt^3 \\ d^4h/dt^4 \end{bmatrix}_{Set5(x=400)} , \quad P_{1Set5} = A_1Q_{1Set5} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix}_{Set5} \tag{3.5.4}$$

The reproduced upstream hydrographs obtained using matrix P and matrix P_1 are shown in Fig.3.5.1.

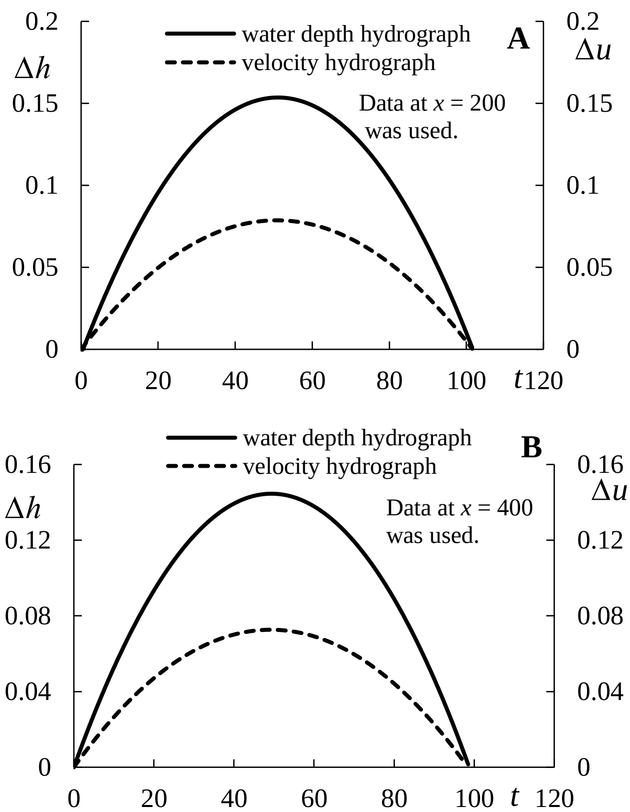


Figure.3.5.1. Upstream boundary hydrographs reproduced through Inversion Analysis.

The calculated depth and velocity hydrographs at the upstream end are compared with the upstream hydrographs using known parameters. These comparisons are shown in Fig.3.5.2. It can be pointed out that although the calculated results are slightly different from the known hydrographs, the Inversion Analysis proposed here can be used to reproduced the two boundary hydrographs using only one depth hydrograph at one site.

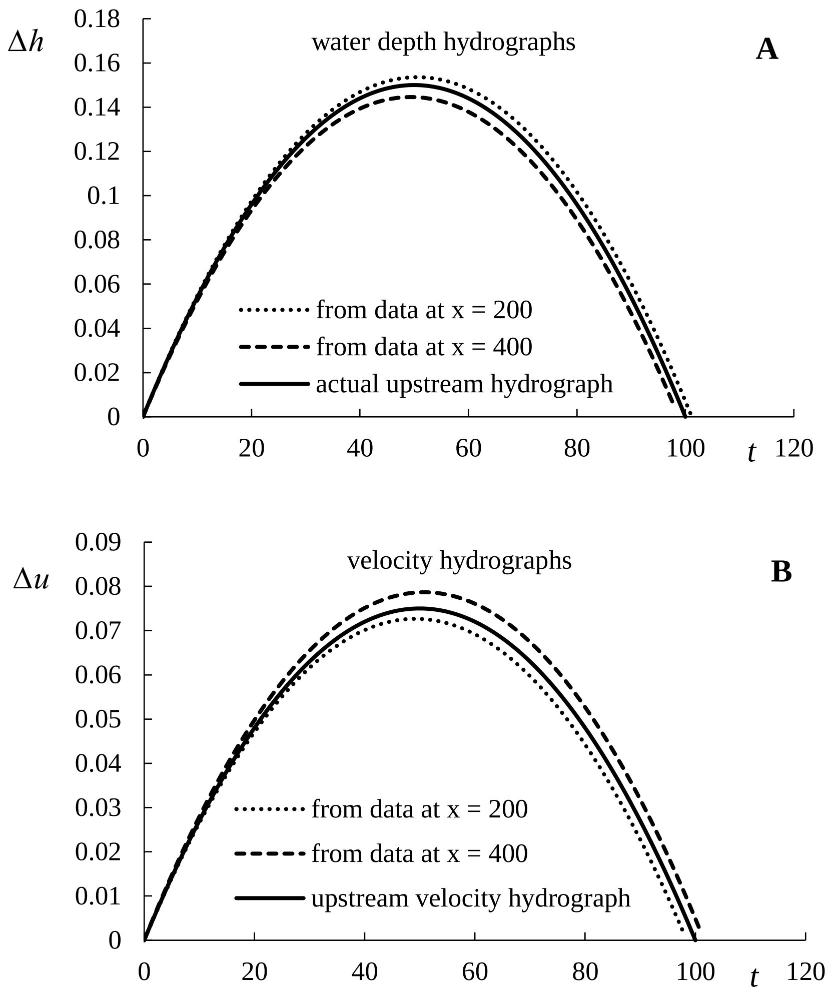


Figure.3.5.2. Comparison of the hydrographs at upstream boundary.

3.6. Summary

Some considerations on the reproduction of flood waves in the supercritical flow state by means of one depth hydrograph are made using the analytical solutions of the linearized shallow water equations. Based on MOC, the analytical solutions on the propagations of hydrographs under the two hydrographs of depth and velocity given at the upstream end are derived for the downstream region. The analytical solutions are verified through the comparisons to the numerical results.

Then, the computational method to conduct the inversion analysis, which was proposed in cooperation with the analytical solutions indicating the possibility of the inversion analysis.

[References]

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3. Hosoda, T., Shirai, H., Onda, S. and Shibayama, Y., "Reproduction of Flood Flow Using Only One-depth Hydrograph in a River Course"- Theoretical Background and Computational Method, Proc. 34th IAHR World Congress, Brisbane, pp.202-209,2011.

CHAPTER 4

SOME CONSIDERATIONS ON DAM-BREAK FLOW PROBLEMS WITH THE APPROXIMATE SOLUTIONS

4.1. Objectives and Previous Research

Dam-break flows are caused due to an instantaneous dam collapse and can lead to serious problems in the downstream area. Thus, dam-break flows have been a major concern in the hydraulic engineering field over a century.

Ritter found an analytical solution in 1892 on the flow structure of an ideal fluid released after an instantaneous dam failure based on the shallow water equations or *Saint-Venant* equations for a channel of prismatic and rectangular cross section without considering the bottom slope and friction. The equations and its basic features are described in 4.2.

It is well known based on previous experimental observations that several features such as the depth profile near the front of positive wave and front speed cannot be accounted by the *Ritter*'s solution, indicating the important effect of the bottom shear stresses to the flow structure.

The effect of flow resistance on the dam break flow was studied by *Dressler* (1952), *Whitham*(1955) and *Chanson*(2009). *Dressler* derived the approximate equations of the shallow flow equations with the friction coefficient as a perturbation parameter, and solved the equations of 1st order. But the depth and velocity profile near the water front were not analysed exactly.

Whitham and *Chanson* considered that near the water front of a dam break flow the bottom friction forces are almost balanced by pressure forces and the inertia effect is therefore ignored. They obtained the simplified solution near the tip portion.

However, in this chapter, without assuming the balance between the bottom friction and pressure forces, the method of analysis to derive the analytical solutions is systematically proposed.

4.2. Basic Equations to Derive the Approximate Solutions

Basic equations to derive the approximate solutions are the shallow water equations Eq. (4.2.1) and (4.2.2). The differential form of the momentum equation is applied and the bottom slope is set to be zero.

$$\frac{\partial h}{\partial t}+\frac{\partial hu}{\partial x}=0 \tag{4.2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial h}{\partial x}=-c_f\frac{u^2}{h} \tag{4.2.2}$$

where, x = the spatial coordinate,

t = the time,

h = the water depth,

u = the depth averaged velocity,

g = the gravity acceleration,

c_f = the friction coefficient.

When the bottom friction term which appears on the right side of Eq. (4.2.1) is neglected, the analytical solutions of dam break flows after an instantaneous dam collapse or gate removal can be derived based on MOC. The solutions are well known as *Ritter*'s solutions, which are given as Eq. (4.2.3) and (4.2.4).

$$h=\frac{1}{9g}\left(2\sqrt{gh_0}-\frac{x}{t}\right)^2 \tag{4.2.3}$$

$$u=\frac{2}{3}\sqrt{gh_0}+\frac{2}{3}\frac{x}{t} \tag{4.2.4}$$

Fig.4.2.1 shows the schematic illustration of Eq. (4.2.3) and (4.2.4), indicating that the positive wave front with zero depth moves in the downstream direction with the speed $2\sqrt{gh_0}$ and the negative wave front $-\sqrt{gh_0}$.

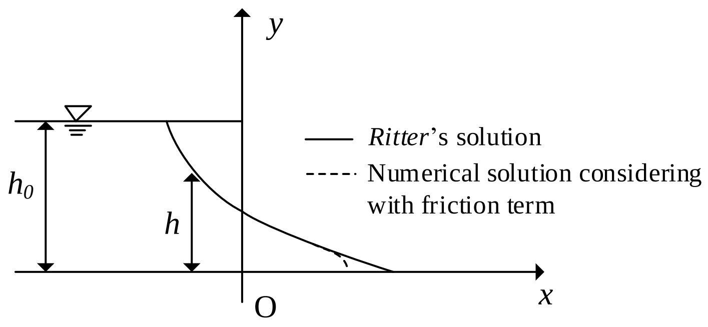

Figure.4.2.1. Schematic diagram of dam-break flow.

(O = initial partition position)

Eq. (4.2.3) and (4.2.4) indicate that the depth at the origin does not change temporarily keeping the constant value of $(4/9)h_0$. The Froude number at the origin also does not change from the initial value of $Fr_0 = 1$.

In this study, the basic equations (4.2.1) and (4.2.2) in (x,t) system are transformed into the equations in the moving coordinate system (ξ,τ) defined as Eq. (4.2.5).

$$\xi = x - ct\,,\ \tau = t\,,\ c \equiv -\sqrt{gh_0} \qquad (4.2.5)$$

$\xi = 0$ at $\tau = 0$ coincides with the origin in (x,t) system and $\xi =$ constant indicates the negative characteristic lines with the speed of $-\sqrt{gh_0}$ in (x,t) system. This means that the negative wave front of dam break flows remains at the origin in (ξ,τ) system as shown in Fig.4.2.2.

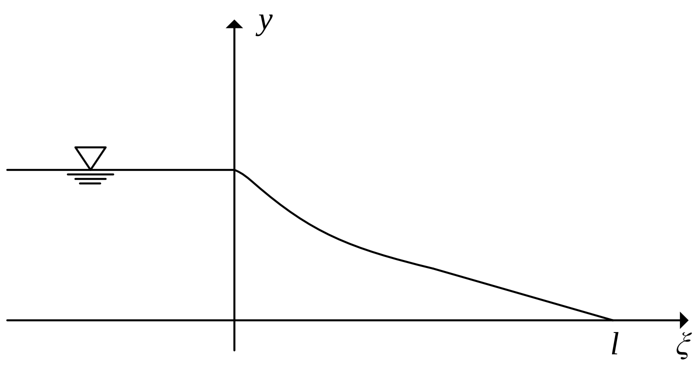

Figure.4.2.2. Dam-break flow in moving coordinate system.

Using Eq. (4.2.5) with the mathematical relations Eq. (4.2.6), the basic equations (4.2.1) and (4.2.2) are denoted as Eq. (4.2.7) and (4.2.8).

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}+\frac{\partial(-c)}{\partial \xi}\ ,\quad \frac{\partial}{\partial x}=\frac{\partial}{\partial \xi} \tag{4.2.6}$$

$$\frac{\partial h}{\partial \tau}+\frac{\partial h\left(u-c\right)}{\partial \xi}=0 \tag{4.2.7}$$

$$\frac{\partial u}{\partial \tau}+\left(u-c\right)\frac{\partial u}{\partial \xi}+g\frac{\partial h}{\partial \xi}=-c_f\frac{u^2}{h} \tag{4.2.8}$$

ξ is denoted as Eq. (4.2.9).

$$\xi \equiv x+\sqrt{gh_0}\,t \tag{4.2.9}$$

Introducing the non-dimensional spatial coordinate ξ' which related to the front position *l* in Eq. (4.2.10), the basic equations (4.2.7) and (4.2.8) are further transformed into the Eq. (4.2.11) and (4.2.12) in $\left(\xi',\tau'\right)$ system.

$$\xi'=\frac{\xi}{l}\ ,\ \tau'=\tau \tag{4.2.10}$$

$$\frac{\partial h}{\partial \tau'}-\frac{\xi'}{l}\frac{dl}{d\tau}\frac{\partial h}{\partial \xi'}+\frac{1}{l}\frac{\partial h\left(u-c\right)}{\partial \xi'}=0 \tag{4.2.11}$$

$$\frac{\partial u}{\partial \tau'}-\frac{\xi'}{l}\frac{dl}{d\tau}\frac{\partial u}{\partial \xi'}+\left(u-c\right)\frac{1}{l}\frac{\partial u}{\partial \xi'}+g\frac{1}{l}\frac{\partial h}{\partial \xi'}=-c_f\frac{u^2}{h} \tag{4.2.12}$$

Ritter's solutions of Eq. (4.2.3) and (4.2.4) can be expressed as Eq. (4.2.13) in $\left(\xi',\tau'\right)$ system.

$$h=h_0\left(1-\xi'\right)^2\ ,\quad u=2\sqrt{gh_0}\,\xi' \tag{4.2.13}$$

4.3. Derivation of the Approximate Solutions of First Order Term

In this study, the power series of ξ' is used as the mathematical expression of *h* and *u* as follows:

$$h=h_0\left(1+a_1\left(\tau'\right)\xi'+a_2\left(\tau'\right)\xi'^2+a_3\left(\tau'\right)\xi'^3+...\right) \tag{4.3.1a}$$

$$u=u_0\left(b_1\left(\tau'\right)\xi'+b_2\left(\tau'\right)\xi'^2+b_3\left(\tau'\right)\xi'^3+...\right) \tag{4.3.1b}$$

where, u_0 = an unknown scale of velocity. It will be shown later that u_0 disappears since $1/u_0$ appears in the coefficients $b_1, b_2, ...$.

$$\frac{\partial h}{\partial \tau'} = h_0\left(\frac{da_1}{d\tau'}\xi' + \frac{da_2}{d\tau'}\xi'^2 + \frac{da_3}{d\tau'}\xi'^3 + ...\right) \tag{4.3.2a}$$

$$\frac{\partial h}{\partial \xi'} = h_0\left(a_1(\tau') + 2a_2(\tau')\xi' + 3a_3(\tau')\xi'^2 + ...\right) \tag{4.3.2b}$$

Substituting Eq. (4.3.2a) and (4.3.2b) into Eq. (4.2.11) and Eq. (4.2.12), the relations for each ξ'^i can be obtained as follows:

[Continuity Equation]

$$\xi'^0 : -\frac{1}{l}h_0u_0b_1 - \frac{c}{l}h_0a_1 = 0 \tag{4.3.3a}$$

$$\xi'^1 : \frac{da_1}{d\tau'} - \frac{1}{l}\frac{dl}{d\tau}a_1 + 2\frac{1}{l}u_0\left(b_2 + a_1b_1\right) - 2\frac{c}{l}a_2 = 0 \tag{4.3.3b}$$

$$\xi'^2 : \frac{da_2}{d\tau'} - 2\frac{1}{l}\frac{dl}{d\tau}a_2 + 3\frac{1}{l}u_0\left(b_3 + a_1b_2 + a_2b_1\right) - 3\frac{c}{l}a_3 = 0 \tag{4.3.3c}$$

[Momentum Equation]

$$\xi'^0 : -\frac{c}{l}u_0b_1 + \frac{g}{l}h_0a_1 = 0 \tag{4.3.4a}$$

$$\xi'^1 : u_0\frac{db_1}{d\tau'} - \frac{1}{l}\frac{dl}{d\tau}u_0b_1 + \frac{1}{l}u_0^2b_1^2 - \frac{c}{l}u_0\left(2b_2 + a_1b_1\right) + \frac{g}{l}h_0\left(2a_2 + a_1^2\right) = 0 \tag{4.3.4b}$$

$$\xi'^2 : u_0\left(\frac{db_2}{d\tau'} + a_1\frac{db_1}{d\tau'}\right) - \frac{1}{l}\frac{dl}{d\tau}u_0\left(a_1b_1 + 2b_2\right) + \frac{1}{l}u_0^2\left(3b_1b_2 + a_1b_1^2\right)$$

$$-\frac{c}{l}u_0\left(3b_3 + 2a_1b_2 + a_2b_1\right) + \frac{g}{l}h_0\left(3a_3 + 3a_1a_2\right) + c_f\frac{u_0}{h_0}b_1^2 = 0 \tag{4.3.4c}$$

Using c_f as a small parameter of the perturbation expansion, the coefficients for each ξ'^i are approximated as Eq. (4.3.5a~c), (4.3.6a~c) and (4.3.7).

$$a_1(\tau') = -2 + c_fa_1^{(1)}(\tau') + c_f^2a_1^{(2)}(\tau') + ... \tag{4.3.5a}$$

$$a_2(\tau') = 1 + c_fa_2^{(1)}(\tau') + c_f^2a_2^{(2)}(\tau') + ... \tag{4.3.5b}$$

$$a_3(\tau') = c_fa_3^{(1)}(\tau') + c_f^2a_3^{(2)}(\tau') + ... \tag{4.3.5c}$$

$$b_1(\tau') = \frac{-2c}{u_0} + c_fb_1^{(1)}(\tau') + c_f^2b_1^{(2)}(\tau') + ... \tag{4.3.6a}$$

$$b_2(\tau')=c_f b_2^{(1)}(\tau')+c_f{}^2 b_2^{(2)}(\tau')+\ldots \tag{4.3.6b}$$

$$b_3(\tau')=c_f b_3^{(1)}(\tau')+c_f{}^2 b_3^{(2)}(\tau')+\ldots \tag{4.3.6c}$$

$$l(\tau')=-3c\tau'+c_f l^{(1)}(\tau')+c_f{}^2 l^{(2)}(\tau')+\ldots \tag{4.3.7}$$

The following conditions for a water front with zero depth are also combined with Eq. (4.3.3a~c) and (4.3.4a~c) to derive the approximate solutions.

$$1+a_1(\tau')+a_2(\tau')+a_3(\tau')=0 \tag{4.3.8}$$

$$\frac{dl}{d\tau'}=\sqrt{gh_0}+u_0\left(b_1(\tau')+b_2(\tau')+b_3(\tau')\right) \tag{4.3.9}$$

After some manipulations, setting the coefficients of the terms with the same power of ξ'^i for $c_f{}^i$ leads to the relations among $a_j^{(1)}$, $b_j^{(1)}$ and $l^{(1)}$ as described in Eq. (4.3.10a~c) and Eq. (4.3.11a~c).

[Continuity Equation]

$$\xi'^0, c_f{}^1 : u_0 b_1^{(1)}=ca_1^{(1)} \tag{4.3.10a}$$

$$\xi'^1, c_f{}^1 : -3c\tau'\frac{da_1^{(1)}}{d\tau'}-ca_1^{(1)}+2\frac{dl^{(1)}}{d\tau}+2u_0\left(b_2^{(1)}-2b_1^{(1)}\right)-2ca_2^{(1)}=0 \tag{4.3.10b}$$

$$\xi'^2, c_f{}^1 : -3c\tau'\frac{da_2^{(1)}}{d\tau'}-2\frac{dl^{(1)}}{d\tau}+3u_0\left(b_3^{(1)}-2b_2^{(1)}+b_1^{(1)}\right)-3ca_3^{(1)}=0 \tag{4.3.10c}$$

[Momentum Equation]

$$\xi'^0, c_f{}^1 : u_0 b_1^{(1)}=ca_1^{(1)} \tag{4.3.11a}$$

$$\xi'^1, c_f{}^1 : -3cu_0\tau'\frac{db_1^{(1)}}{d\tau'}+cu_0 b_1^{(1)}+2c\frac{dl^{(1)}}{d\tau}+2c^2 a_1^{(1)}-2cu_0 b_2^{(1)}$$

$$+gh_0\left(2a_2^{(1)}-4a_1^{(1)}\right)=0 \tag{4.3.11b}$$

$$\xi'^2, c_f{}^1 : -3cu_0\tau'\frac{db_2^{(1)}}{d\tau'}+6cu_0\tau'\frac{db_1^{(1)}}{d\tau'}-4c\frac{dl}{d\tau}^{(1)}+c^2 a_1^{(1)}-4c^2 a_2^{(1)}+3c^2 a_3^{(1)}$$

$$-cu_0\left(3b_3^{(1)}-4_1b_2^{(1)}+b_1^{(1)}\right)-12\frac{c^3}{h_0}\tau'=0 \tag{4.3.11c}$$

$a_j^{(1)}$, $b_j^{(1)}$ and $l^{(1)}$ in the above equations are represented by the following power series on τ'.

$$a_1^{(1)}=a_{1,1}^{(1)}\tau'+a_{1,2}^{(1)}\tau'^2+\ldots \tag{4.3.12a}$$

$$a_2^{(1)}=a_{2,1}^{(1)}\tau'+a_{2,2}^{(1)}\tau'^2+\ldots \tag{4.3.12b}$$

$$a_3^{(1)}=a_{3,1}^{(1)}\tau'+a_{3,2}^{(1)}\tau'^2+\ldots \tag{4.3.12c}$$

$$b_1^{(1)}=b_{1,1}^{(1)}\tau'+b_{1,2}^{(1)}\tau'^2+\ldots \tag{4.3.13a}$$

$$b_2^{(1)}=b_{2,1}^{(1)}\tau'+b_{2,2}^{(1)}\tau'^2+\ldots \tag{4.3.13b}$$

$$b_3^{(1)}=b_{3,1}^{(1)}\tau'+b_{3,2}^{(1)}\tau'^2+\ldots \tag{4.3.13c}$$

$$l^{(1)}=l_{,2}^{(1)}\tau'+l_{,3}^{(1)}\tau'^2+\ldots \tag{4.3.14}$$

Substituting Eq. (4.3.12a~c), Eq. (4.3.13a~c) and Eq. (4.3.14) into Eq. (4.3.10a~b), (4.3.11a~b) and using the constraints for the wave front Eq. (4.3.8) and (4.3.9) lead to the coefficient equations given by Eq. (4.3.15a~b) and (4.3.16a~b). By solving Eq. (4.3.15a~b), (4.3.16a~b), (4.3.17) and (4.3.18) simultaneously, the solutions of the coefficients can be calculated.

$$-4ca_{1,1}^{(1)}-ca_{2,1}^{(1)}+u_0b_{2,1}^{(1)}+2l_{,2}^{(1)}=0 \tag{4.3.15a}$$

$$-2ca_{1,1}^{(1)}+ca_{2,1}^{(1)}-u_0b_{2,1}^{(1)}+2l_{,2}^{(1)}=0 \tag{4.3.15b}$$

$$3ca_{1,1}^{(1)}-3ca_{2,1}^{(1)}-3ca_{3,1}^{(1)}-6u_0b_{2,1}^{(1)}+3u_0b_{3,1}^{(1)}-4l_{,2}^{(1)}=0 \tag{4.3.16a}$$

$$8ch_0a_{1,1}^{(1)}-4ch_0a_{2,1}^{(1)}+3ch_0a_{3,1}^{(1)}+h_0u_0b_{2,1}^{(1)}-3h_0u_0b_{3,1}^{(1)}-8h_0l_{,2}^{(1)}=12c^2 \tag{4.3.16b}$$

$$a_{1,1}^{(1)}+a_{2,1}^{(1)}+a_{3,1}^{(1)}=0 \tag{4.3.17}$$

$$2l_{,2}^{(1)}\tau'=u_0\left(b_{1,1}^{(1)}+b_{2,1}^{(1)}+b_{3,1}^{(1)}\right)\tau' \tag{4.3.18}$$

$$a_{1,1}{}^{(1)} = -\frac{3}{2}\frac{c}{h_0}, a_{2,1}{}^{(1)} = \frac{1}{2}\frac{c}{h_0}, a_{3,1}{}^{(1)} = \frac{c}{h_0} \tag{4.3.19}$$

$$b_{1,1}{}^{(1)} = -\frac{3}{2}\frac{c^2}{h_0 u_0}, b_{2,1}{}^{(1)} = -\frac{c^2}{h_0 u_0}, b_{3,1}{}^{(1)} = -2\frac{c^2}{h_0 u_0} \tag{4.3.20}$$

$$l_{,2}{}^{(1)} = -\frac{9}{4}\frac{c^2}{h_0} \tag{4.3.21}$$

4.4. Comparison between Numerical Results and Approximate Solutions of First Order

In order to consider the applicability and limitation of the approximate solutions, the derived approximate solutions were compared with the numerical results. The numerical analysis of the dam break flow in the semi-finite region was carried out by using the finite volume method. For higher accuracy, the discretization based on the TVD-MUSCL method was applied and the *Adams-Bashforth* method was used for time integration. The same calculation method was applied for the whole area.

The two different grid spacings of $\Delta x = 0.05$ (m) and $\Delta x = 0.025$ (m) with $h_0 = 0.1$ (m), $c_f = 0.01$ were considered and the results are shown in Fig.4.4.1. Since both calculated results are in good agreement each other, $\Delta x = 0.05$ (m) and $\Delta t = 10^{-4}$ (sec) were used for the subsequent calculations.

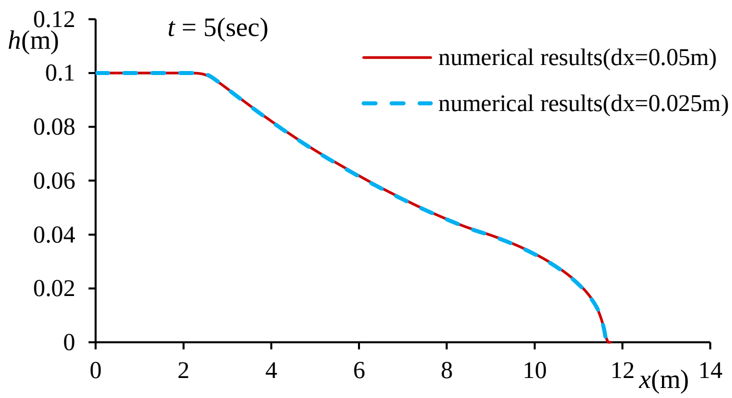

Figure.4.4.1. Numerical results with different grid sizes.

Fig.4.4.2 shows the comparison between the numerical results and theoretical results (Ritter's solution) when bottom shear stress term is neglected. The two results are in good agreement and it can be considered that the validity of the numerical analysis method used in this study to some extent.

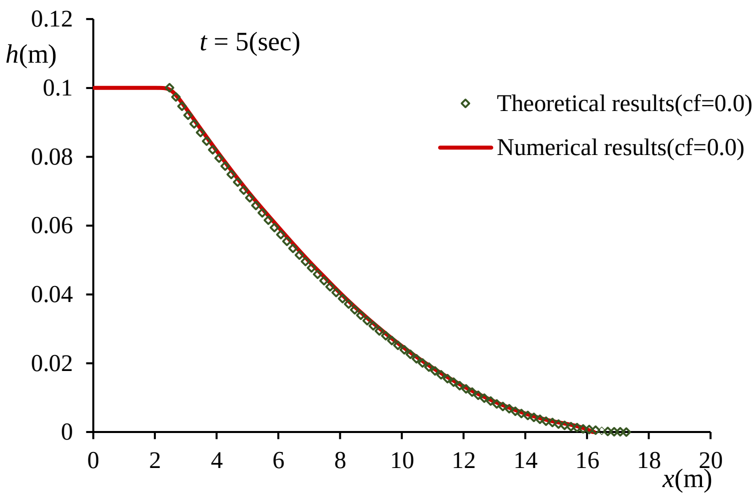

Figure.4.4.2. Comparison between numerical results and theoretical results.

(The origin is the left end of calculation area)

The relationship between the front position and the time is shown in Fig.4.4.3. The approximate solution for the front position is given by Eq. (4.4.1). Fig.4.4.3 shows that the moving speed of the front from the approximate solutions is much slower than that of the theoretical solution in which the bottom shear stress is ignored (Ritter's solution), and it shows a value close to the numerical results.

$$l(\tau') = 3\sqrt{gh_0}\tau' + c_f l^{(1)}(\tau') = 3\sqrt{gh_0}\tau' - c_f \frac{9}{4} g\tau'^2 \tag{4.4.1}$$

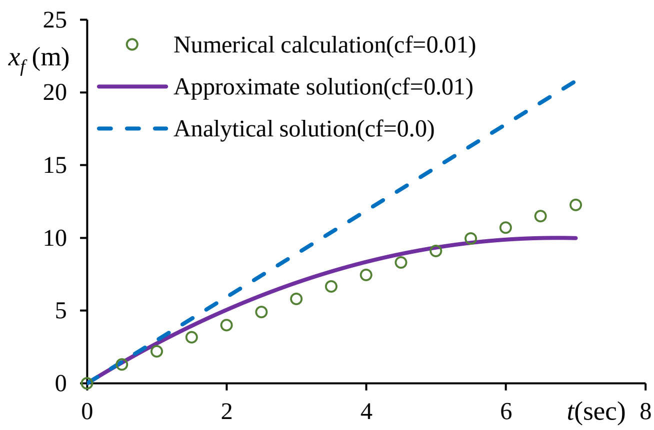

Figure.4.4.3. Relationship of front position with time.

However, because only the first power of c_f is considered in Eq. (4.4.1), it may cause the unrealistic behaviour such as the decrease of front position with the increase of time. This indicates the need to derive the approximate solutions that consider higher order terms of c_f with higher order terms of ξ as shown in Eq. (4.4.2).

$$l(\tau')=3\sqrt{gh_0}\tau'+c_f l_{,2}^{(1)}\tau'^2+c_f^{\,2}l_{,3}^{(2)}\tau'^3+c_f^{\,3}l_{,4}^{(3)}\tau'^4+\ldots \tag{4.4.2}$$

in which, $l_{,3}^{(2)}$ and $l_{,4}^{(3)}$ are the parameters of the power expansion series and constant.

The comparison of depth profiles calculated by using both methods are shown in Fig.4.4.4. The approximate solution of water depth is given by Eq. (4.4.3).

$$h=h_0\left\{1+\left(-2+c_f a_1^{(1)}(\tau')\right)\xi'+\left(1+c_f a_2^{(1)}(\tau')\right)\xi'^2+c_f a_3^{(1)}(\tau')\xi'^3\right\}$$

$$h=h_0\left\{1+\left(-2-c_f\frac{3}{2}\frac{c}{h_0}\tau'\right)\xi'+\left(1+c_f\frac{1}{2}\frac{c}{h_0}\tau'\right)\xi'^2+c_f\frac{c}{h_0}\tau'\xi'^3\right\} \tag{4.4.3}$$

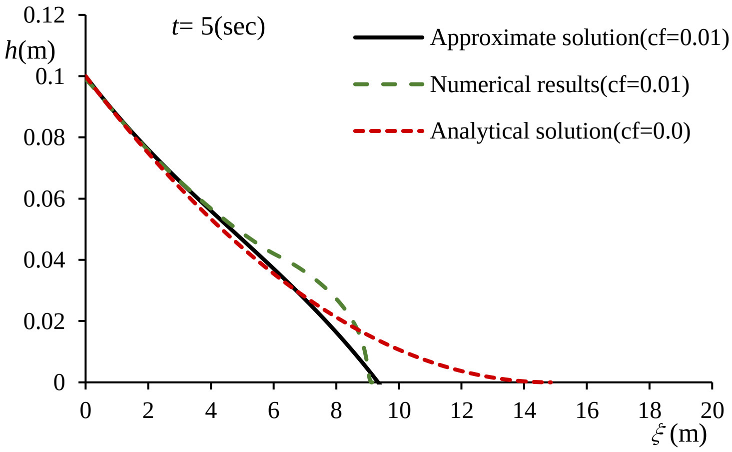

Figure.4.4.4. Comparison of the water depth profiles.

(Horizontal axis is the distance from the front of the negative wave)

Compared to the theoretical solutions with and without the bottom shear stress, the front positions and profiles of the approximate solutions are considerably close to the numerical results. But the large negative water depth gradient near the front found in the numerical analysis results is not sufficiently reproduced in the approximate solution. In order to examine this in more detail, Fig.4.4.5 shows the distributions of water depth considering only the first-order term, second-order term, and third order term with respect to ξ' in the approximate solution given as Eq. (4.4.3). It can be seen that the water depth distributions considering the first order term and the second order term of ξ' conform different ways with the numerical results near the front and the solution with third order term of ξ' can be considered as a solution to satisfy the condition of zero water depth at the front.

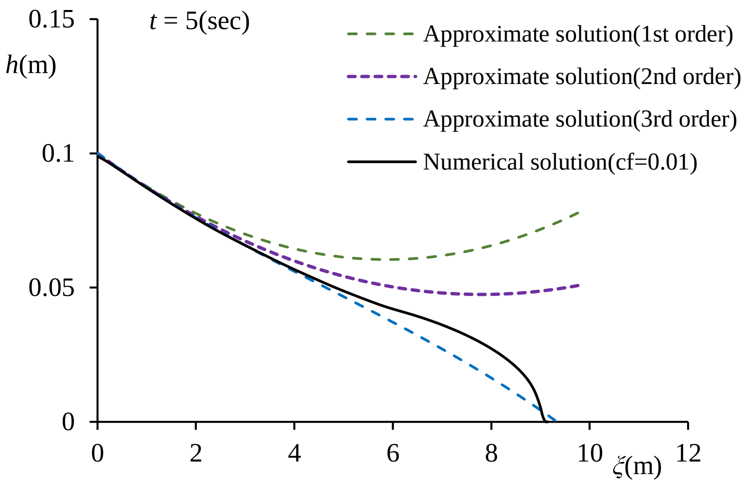

Figure.4.4.5. Characteristics of the approximate solutions.

(Horizontal axis is the distance from the front of the negative wave)

From Fig.4.4.5, it can be clearly seen that considering up to the third order term of ξ' is not sufficient to reproduce the large negative slope of the water depth at the tip of the front, which may be the reason why the approximate solutions are not compatible with the numerical results.

Fig.4.4.6 shows the distribution of $1-\xi'^n$. It shows that when the number n increases, the negative slope of $1-\xi'^n$ increases sharply at $\xi'=1$. It also points out that an approximate solution considering the higher order ξ' terms should be studied in the future for the improvement of water depth distribution near the front.

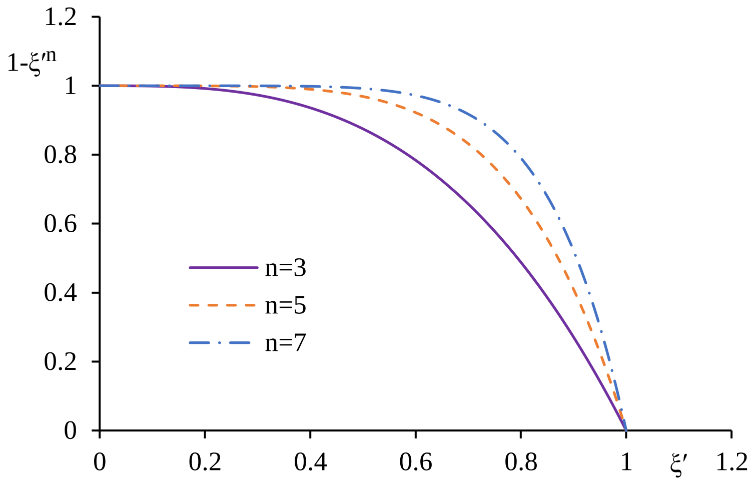

Figure.4.4.6. Analysis of tip behaviour.

The flow velocity distribution is also considered and the comparison of the results are shown in Fig.4.4.7. The approximate solution shown in Fig,4.4.7 is given by Eq. (4.4.4).

$$u = u_0\left\{\left(-\frac{2c}{u_0}+c_f b_1^{(1)}(\tau')\right)\xi' + c_f b_2^{(1)}(\tau')\xi'^2 + c_f b_3^{(1)}(\tau')\xi'^3\right\}$$

$$u = \left(-2c - c_f\frac{3}{2}\frac{c^2}{h_0}\tau'\right)\xi' - c_f\frac{c^2}{h_0}\tau'\xi'^2 - c_f 2\frac{c^2}{h_0}\tau'\xi'^3 \tag{4.4.4}$$

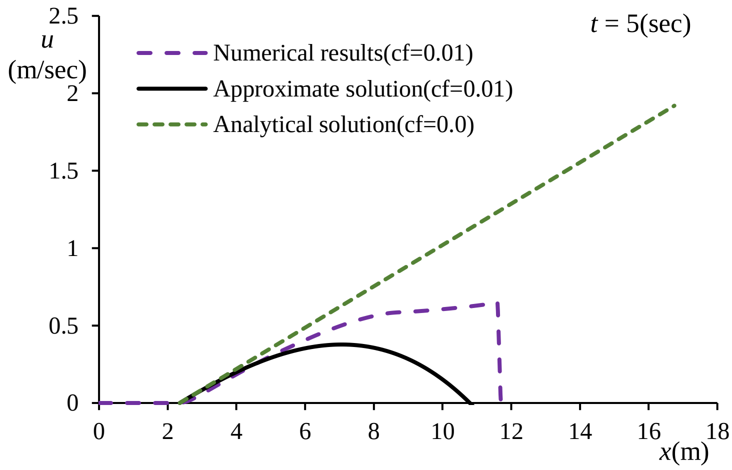

Figure.4.4.7. Comparison of flow velocity distribution.

Similar to the discussion of water depth profile, the result from approximate solution is close to the numerical results as compared with the analytical solution without the friction term. It should be noted that the approximate solution with higher order terms is needed to consider to get a better result.

4.5. Summary

This chapter is concerned with the derivation of the approximate solutions based on the shallow water equations including the bottom shear stress term for the dam break flow in the semi-infinite region. A method that can analyse the flows between positive and negative fronts without dividing the region as *Chanson* does is proposed and the approximate solutions also derived and compared to the numerical results. Although the approximate solutions of water depth profile derived in this research agreed with the numerical results to some extent compared to the Ritter's solution, it is pointed out that the large negative gradient of water depth cannot be reproduced at the front position.

So, it is concluded that it is worth trying to include the higher order terms with respect to c_f and ξ' to obtain the better profiles near the wave front.

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CHAPTER 5
CONCLUSIONS

In order to develop the fundamental theory of unsteady open channel flows, some basic flows were dealt with using the one-dimensional shallow water equations in this thesis. The new findings obtained in this thesis are summarized as follows:

5.1. Propagation of Flood Waves

The previous flood wave theories have been dealing with the waves with large wave length or large time period in base flows of low Froude numbers, making clear the limitation and drawbacks of the previous theories. In chapter 2, the analytical solutions on the hydrograph propagation were derived applying the wave front expansion method to shallow water equations. The linear analytical solutions with up to fifth order power terms were derived. To validate the theoretical solutions, the results were compared to the numerical results obtained using MOC.

It was shown that the analytical solutions can reproduce the hydrograph propagation with small amplitude, small time period and different Froude numbers in sub-critical flows considering the terms up to third order. The non-linear solutions with the terms up to second order were also derived and were compared to the numerical results pointing out the necessity to include higher order terms.

5.2. Reproduction of Flash Waves by One Depth Hydrograph at One Site

In chapter 3, some considerations on the reproduction of flood waves in the supercritical flow state by means of one depth hydrograph were made using the analytical solutions of the linearized shallow water equations. Based on MOC, the analytical solutions on the propagations of hydrographs under the two hydrographs of depth and velocity given at the upstream end are derived for the downstream region. The analytical solutions are verified through the comparisons to the numerical results.

Then, the computational method to conduct the inversion analysis was proposed in cooperation with the theoretical consideration of the analytical solutions indicating the possibility of the inversion analysis.

5.4. Some Consideration on Dam-break Flow Problems

This chapter is concerned with the derivation of the approximate solutions based on the shallow water equations including the bottom shear stress term for the dam break flow in the semi-infinite region. A method that can analyse the flows between positive and negative fronts without dividing the region as *Chanson* applied is proposed. The approximate solutions are also derived and compared to the numerical results. Although the approximate solutions of water depth profile derived in this research agreed with the numerical results to some extent compared to the Ritter's solution, it is pointed out that the large negative gradient of water depth cannot be reproduced at the front position.

So, it is concluded that it is worth trying to include the higher order terms with respect to c_f and ξ' to obtain the better profiles near the wave front.

5.5. Recommendation for Future Studies

Regarding the fundamental theory on unsteady open channel flows, some developments were represented in this thesis. But, the necessity of further investigations was also pointed out, taking into account of the higher order terms for all chapters. So, I am intending to advance the fundamental theory of unsteady open channel flows in the future.