

BLOW ANALYTIC MODULI OF ANALYTIC FUNCTIONS OF TWO VARIABLES

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Dedicated to the Memory of Professor Etsuo Yoshinaga

1. INTRODUCTION

Let $f_1, f_2 : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}, 0)$ be germs of real analytic functions. We say that f_1, f_2 are blow analytic equivalent if there exist maps $\Phi, \phi, \beta_1, \beta_2$ such that the diagram is commute:

$$\begin{array}{ccccc} (\mathcal{M}_1, \beta_1^{-1}(0)) & \xrightarrow{\beta_1} & (\mathbb{R}^n, \mathbf{0}) & \xrightarrow{f_1} & (\mathbb{R}, 0) \\ \Phi \downarrow & & \downarrow \phi & & \\ (\mathcal{M}_2, \beta_2^{-1}(0)) & \xrightarrow{\beta_2} & (\mathbb{R}^n, \mathbf{0}) & \xrightarrow{f_2} & (\mathbb{R}, 0) \end{array}$$

where ϕ is a homeomorphism, Φ is an analytic isomorphism and $\beta_i (i = 1, 2)$ are compositions of blowing ups with smooth centers. T.C.Kuo proved the following theorem in [1].

Theorem 1.1. *Let $F(x, p) : (\mathbb{R}^n \times P, \mathbf{0} \times P) \rightarrow (\mathbb{R}, 0)$ be real analytic and let P be a subanalytic set. Suppose that for $p \in P$ fixed, $F_p : (\mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, 0)$, $F_p(x) = F(x, p)$ has an isolated singular point. Then there exists a finite filtration of P by subanalytic sets $P^{(i)} (i = 0, \dots, l)$*

$$P = P^{(0)} \supset P^{(1)} \supset \dots \supset P^{(l)} = \phi$$

such that

- (1) $\dim P^{(i)} > \dim P^{(i+1)}$, $P^{(i)} - P^{(i+1)}$ is smooth,
- (2) for $p, p' \in P^{(i)} - P^{(i+1)}$, F_p and $F_{p'}$ are blow analytic equivalent.

Let $\mathcal{A}_n := \{f | f : (\mathbb{R}^n, \mathbf{0}) \rightarrow (\mathbb{R}, 0); \text{analytic}\}$. If $f \in \mathcal{A}_n$ has an isolated singularity, then f is finitely determined, that is, there exists an integer $r \in \mathbb{N}$ such that the analytic type of f is determined by the Taylor polynomial of f with degree r . Let $J^r(n, 1)$ be the set of r -jets of the element of $\mathcal{A}_n$ and $L^r(n, n)$ be the set of r -jets of isomorphisms of $(\mathbb{R}^n, \mathbf{0})$. Then the Lie group $L^r(n, n)$ acts on $J^r(n, 1)$. Since $\text{codim Orb}(j^r f) < +\infty$,

There exists an analytic map $F : (\mathbb{R}^n \times \mathbb{R}^{\mu-1}, \mathbf{0} \times \mathbf{0}) \rightarrow (\mathbb{R}, 0)$ such that F is transversal to $\text{Orb}(j^r(f))$. We call F the transversal family of f . It is well known that we have

$$F(x, p) = f(x) + \sum_{i=1}^{\mu-1} p_i \alpha_i(x),$$

where $\alpha_1, \dots, \alpha_{\mu-1}$ are a basis of $\mathfrak{M}^2/\mathfrak{M}(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$.

Let $\mathcal{S}_f := \{p \in \mathbb{R}^{\mu-1} | f \text{ is blow analytic equivalent to } F_p\}$ as a germ at the origin. By Kuo's theorem above mentioned, it follows that $\mathcal{S}_f$ consists of finitely union of smooth manifolds. We propose the following problems:

Problems.

- (1) Estimate the dimension of $\mathcal{S}_f$.
- (2) $\mathcal{S}_f$ is smooth?
- (3) Classify $\mathcal{A}_n$ by the dimension of $\mathcal{S}_f$.
- (4) Prove that the upper semi-continuity of the $\dim \mathcal{S}_f$.

2. COMPLEX CASE

Let $f : (\mathbb{C}, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$ be a germ of an analytic function with isolated singularity and let $F : (\mathbb{C}^n \times \mathbb{C}^{\mu-1}, \mathbf{0} \times \mathbf{0}) \rightarrow (\mathbb{C}, 0)$ be the trasversal family of f . Let

$$\mathcal{S}_f := \{p \in \mathbb{C}^{\mu-1} | (\mathbb{C}^n, f^{-1}(0)) \text{ is relatively topological equivalent to } (\mathbb{C}^n, F_p^{-1}(0))\}$$

Let $f : (\mathbb{C}^2, \mathbf{0}) \rightarrow (\mathbb{C}, 0)$ be a germ of an analytic function and let $f(x, y) = \sum a_{i,j} x^i y^j$ be the Taylor expansion of f with respect to a coordinate system (x, y) of $\mathbb{C}^2$. The Newton polygon $\Gamma_+(f; (x, y))$ with respect to the coordinate system (x, y) is the set $\bigcup_{i,j \neq 0} \{(i, j) + \mathbb{R}_+^2\}$. Newton boundary $\Gamma(f; (x, y))$ of f is the union of caomcompact faces of the boundary of the Newton polygon of f . For a compact face γ of $\Gamma(f; (x, y))$, we define f_γ by $f_\gamma(x, y) = \sum_{(i,j) \in \gamma} a_{i,j} x^i y^j$.

The author prove the following result in [2].

Theorem 2.1. *Let $f(x, y)$ be a germ of a quasihomogeneous function of two complex variables with isolated singularity. Then we have $\mathcal{S}_f$ is a linear space in $\mathbb{C}^{\mu-1}$ generated by a basis of $\mathbb{C}[x_1, \dots, x_n]/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ in $\Gamma_+(f; (x, y))$.*

M.Oka proved the following result in [3].

Theorem 2.2. *Let $F(x, t) : (\mathbb{C}^2 \times \mathbb{C}, \mathbf{0} \times 0) \rightarrow (\mathbb{C}, 0)$ be analytic and suppose that $f_t = F|_{\mathbb{C}^2 \times \{0\}}$ has an isolated singularity. If the Milnor number of f_t is constant independent*

of $\forall t$ and f_0 is convenient, then there exists a coordinate system $\phi_t(x, y) = (x(t), y(t))$ which is analytic in t and satisfies the following conditions:

- (1) $\phi_t(0) = 0$, $\phi_0(x, y) = (x, y)$
- (2) $\Gamma(f_t; \phi_t) = \Gamma(f_0; \phi_0)$

M.Oka and Kushnirenko proved the following result in [4],[5].

Theorem 2.3. Suppose that $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ has an isolated singularity and has a non-degenerate Newton boundary. Then the Milnor number of f is the number of a basis of $\mathbb{C}[x_1, \dots, x_n] / (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ in $\Gamma_+(f)$.

Lê Dũng Tráng and C.P.Ramanujan proved the following result in [6].

Theorem 2.4. Let $F : (\mathbb{C}^n \times \mathbb{R}, 0 \times 0) \rightarrow (\mathbb{C}, 0)$ be analytic and $F_t = F|_{\mathbb{C}^n \times \{t\}}$ has an isolated singularity for $\forall t \in \mathbb{R}$. If the Milnor numbers of F_t are independent of t , then the relative topological types of $(\mathbb{C}^n, F_t^{-1}(0))$ are independent of t .

From the above three results, it follows that

Theorem 2.5. Let f be a germ of complex analytic function with isolated singularity at the origin, $f(0) = 0$ and suppose that f has the non-degenerate Newton boundary. Then we have S_f = a linear space in the moduli space of the transversal family of f generated by a basis of $\mathbb{C}[x_1, \dots, x_n] / (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ in $\Gamma_+(f; (x, y))$.

We will draw an analogy in case of real analytic functions in what follows.

3. REAL CASE

In what follows, suppose that $f \in \mathcal{A}_n$ has an isolated singularity. Let $F : (\mathbb{R}^n \times I, 0 \times I) \rightarrow (\mathbb{R}, 0)$ be analytic and $F_0 = f$, where I is the open interval $(-1, 1)$.

Definition 3.1. We say that F admits a blow analytic trivialization along I if there exist a local homeomorphism ϕ , an analytic isomorphism Φ and successive blowing ups $\beta_i (i = 1, \dots, \gamma)$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 (\mathcal{M}_\gamma, \beta^{-1}(0 \times I)) & \xrightarrow{\beta_\gamma} \dots \xrightarrow{\beta_0} & (\mathbb{R}^n \times I, 0 \times I) & \xrightarrow{f} & (\mathbb{R}, 0) \\
 \uparrow \Phi & & \uparrow \phi & \searrow \text{proj} & \uparrow f \\
 & & & & I \\
 (\mathcal{M}_\gamma, \beta^{-1}(0 \times I)) & \xrightarrow{\beta_\gamma} \dots \xrightarrow{\beta_0} & (\mathbb{R}^n \times I, 0 \times I) & \xrightarrow{\text{proj } f} & (\mathbb{R}^n, 0),
 \end{array}$$

where $\beta_i : \mathcal{M}_i \rightarrow \mathcal{M}_{i-1}$ is the blowing up with a smooth center $S_{i-1} \subset \mathcal{M}_i$ ($\mathcal{M}_0 = \mathbb{R}^n \times I$) and the composition map $S_i \xrightarrow{\text{inclusion}} \mathcal{M}_i \xrightarrow{\beta_i} \dots \xrightarrow{\beta_1} \mathcal{M}_0 = \mathbb{R}^n \times I \rightarrow I$ is a submersion.

We have the following result.

Theorem 3.1. Suppose that $F : (\mathbb{R}^2 \times I, \mathbf{0} \times I) \rightarrow (\mathbb{R}, 0)$ admits a blow analytic trivialization along I and $F_0(x, y) = f(x, y)$ is convenient. Then there exists a coordinate system (x', y') of $\mathbb{R}^2$ (which is a small perturbation of the original coordinate (x, y)) and a real analytic family of local coordinates $\varphi_t(x', y') = (x(t), y(t))$ ($|t| \ll 1$) such that

- (1) $\varphi_t(0, 0) = (0, 0)$ and $\varphi_0(x', y') = (x', y')$
- (2) $\Gamma(F_t; \varphi_t) = \Gamma(f; (x', y'))$.

In Kuo's theorem, we can replace the condition (ii) to

- (1) For $P^{(i)} - P^{(i+1)} \ni \forall p, p'$ (close enough), F_p and $F_{p'}$ are jointed by a blow analytically trivial homotopy.

In fact, he have proved Theorem 1.1 under the above condition in [1]. From our result and Kuo's result, we have

Assertion 3.1. $\dim(\text{the topological component of } P^i - P^{(i+1)} \text{ which contains } f) \leq \text{the number of a basis of } \mathcal{A}_2 / (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \text{ in } \Gamma_+(f)$

The following result is deduced as the special case from the result in T.Fukui and E.Yoshinaga [6].

Theorem 3.2. Let $F : (\mathbb{R}^2 \times I, \mathbf{0} \times I) \rightarrow (\mathbb{R}, 0)$ be analytic. Suppose that the Newton boundary of F_t is independent of t and non-degenerate. Then F admits a blow analytic trivialization along I .

From this result and the above Assertion, we have

Assertion 3.2. Let $f : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}, 0)$ be a germ of analytic function with isolated singularity. If f has a non-degenerate Newton boundary, then $\dim(\text{the topological component of } P^{(i)} - P^{(i+1)} \text{ which contains } f) = \text{the number of a basis of } \mathcal{A}_2 / (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \text{ in } \Gamma_+(f)$.

In addition to the problems mentioned in the section 1, we propose the following problem:

Problem

- (5) Blow analytic constancy implies blow analytic triviality?

If this is true, then we have

Conjecture. Let $f : (\mathbb{R}^2, \mathbf{0}) \rightarrow (\mathbb{R}, 0)$ be a germ of analytic functions with isolated singularity. If f has a non-degenerate Newton boundary, then we have $\dim S_f = \text{the number of a basis of } \mathcal{A}_2 / (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \text{ in } \Gamma_+(f)$.

we expect the conjecture will be true in general.

4. OUTLINE OF A PROOF OF THE RESULT

Let $F : (\mathbb{R}^2 \times I, \mathbf{0} \times I)$ be analytic and $F_0(x, y) = f(x, y)$. Suppose that F admits a blow analytic trivialization along I and satisfies the commutative diagram mentioned in the previous section:

$$\begin{array}{ccccccc}
 (\mathcal{M}_\gamma, \beta^{-1}(o \times I)) & \xrightarrow{\beta_\gamma} \cdots \xrightarrow{\beta_0} & (\mathbb{R}^2 \times I, \mathbf{0} \times I) & \xrightarrow{f} & (\mathbb{R}, 0) \\
 \uparrow \Phi & & \uparrow \phi & \searrow \text{proj} & \uparrow f \\
 (\mathcal{M}_\gamma, \beta^{-1}(o \times I)) & \xrightarrow{\beta_\gamma} \cdots \xrightarrow{\beta_0} & (\mathbb{R}^2 \times I, \mathbf{0} \times I) & \xrightarrow{\text{proj } f} & (\mathbb{R}^2, 0),
 \end{array}$$

Note that β_0 is $\sigma \times id_I : \mathcal{N} \times I \rightarrow \mathbb{R}^2 \times I$, where σ is the blowing up of $\mathbb{R}^2$ with center the origin and $\mathcal{N} = \{([\xi, \eta], (x, y)) | \xi y - \eta x = 0\} \subset \mathbb{R}P^1 \times \mathbb{R}^2$. Let (x, y) be the coordinate system of $\mathbb{R}^2$ which is obtained by the coordinate transformation

$$\begin{cases} x = x \\ y = y - \pi(p)x \end{cases}$$

where $p \in \sigma^{-1}(0) \cong S^1$ is a point which is not contained in the centers of the blowing ups $\beta_1, \dots, \beta_\gamma$ and $\pi = \text{proj} : \mathbb{R}P^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}P^1$.

We set $(u_{0\pm}, v_{0\pm}) = (u'_{0\pm}, v'_{0\pm}) := (x, y)$, $o_{0\pm} = o'_{0\pm}$ = the origin of $\mathbb{R}^2$ and set $\mathcal{N}_{0\pm} = \mathbb{R}^2(x, y)$. Next we define inductively real analytic manifold $\mathcal{N}_{k\pm}$, real analytic maps $\sigma_{k\pm} : \mathcal{N}_{k\pm} \rightarrow \mathcal{N}_{(k-1)\pm}$ according to the sign $+$, $-$ respectively as follows. Let $\mathbb{R}^2(u_{k\pm}, v_{k\pm})$, $\mathbb{R}^2(u'_{k\pm}, v'_{k\pm})$ be copies of $\mathbb{R}^2$ and set $\mathcal{N}_{k\pm} = \mathbb{R}^2(u_{k\pm}, v_{k\pm}) \cup \mathbb{R}^2(u'_{k\pm}, v'_{k\pm})$. Let $o_{k\pm}$ (resp. $o'_{k\pm}$) be the origin of the patch $\mathbb{R}^2(u_{k\pm}, v_{k\pm})$ (resp. $\mathbb{R}^2(u'_{k\pm}, v'_{k\pm})$) and let $\sigma_{k+} : \mathcal{N}_{k+} \rightarrow \mathcal{N}_{(k-1)+}$ (resp. $\sigma_{k-} : \mathcal{N}_{k-} \rightarrow \mathcal{N}_{(k-1)-}$) be the blowing up of $\mathcal{N}_{(k-1)+}$ (resp. $\mathcal{N}_{(k-1)-}$) with center o_{k+} (resp. o'_{k-}) defined by

$$\begin{aligned}
 \sigma_{k+}(u_{k+}, v_{k+}) &= (u_{k+}, u_{k+}v_{k+}) = (u_{(k-1)+}, u_{(k-1)+}v_{(k-1)+}) \\
 &= (u'_{k+}v'_{k+}, v'_{k+}) = \sigma_{k+}(u'_{k+}, v'_{k+}) \\
 \sigma_{k-}(u_{k-}, v_{k-}) &= (u_{k-}, u_{k-}v_{k-}) = (u'_{(k-1)-}, u'_{(k-1)-}v'_{(k-1)-}) \\
 &= (u'_{k-}v'_{k-}, v'_{k-}) = \sigma_{k-}(u'_{k-}, v'_{k-}),
 \end{aligned}$$

and we set $\sigma_{0\pm} = id : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

At first we consider the analytic map $\sigma_{k+} : \mathcal{N}_{k+} \rightarrow \mathcal{N}_{(k-1)+}$. We set

$$\begin{aligned}
 m_{k+} &= \text{the order of } f \circ \sigma_{0+} \circ \cdots \circ \sigma_{k+}(u_{k+}, v_{k+}) \\
 \Delta'_{k+} &= \{(u_{k+}, v_{k+}) \in \Gamma(f \circ \sigma_{0+} \circ \cdots \circ \sigma_{k+}; (u_{k+}, v_{k+})) | u_{k+} + v_{k+} = m_{k+}\}
 \end{aligned}$$

Let Δ_{k+} be the face of $\Gamma(f; (x, y))$ corresponding to Δ'_{k+} by the map

$$\begin{pmatrix} u_{k+} \\ v_{k+} \end{pmatrix} \mapsto \begin{pmatrix} u_{k+} - kv_{k+} \\ v_{k+} \end{pmatrix}.$$

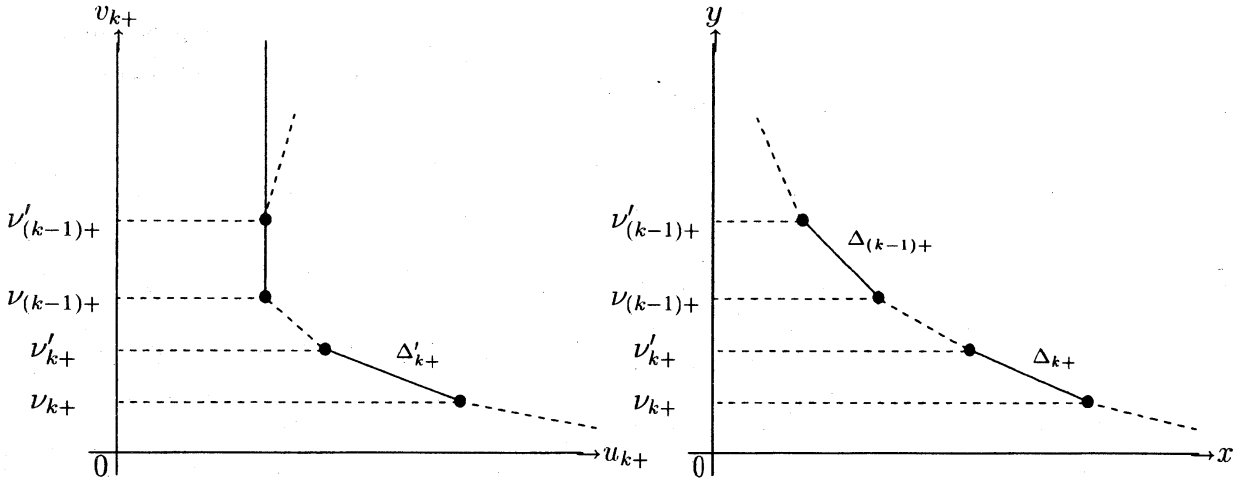
Next we set

$$\nu_{k+} = \min\{y \mid \text{there exists } x \text{ such that } (x, y) \in \Delta_{k+}\}$$

$$\nu'_{k+} = \max\{y \mid \text{there exists } x \text{ such that } (x, y) \in \Delta_{k+}\}$$

and there exists $\gamma \in \mathbb{N}$ such that

$$\nu_{0+} \geq \cdots \geq \nu_{(\gamma-1)+} = 0, \quad \nu'_{0+} \geq \cdots \geq \nu'_{(\gamma-1)+}$$



We have the following lemmata.

Lemma 4.1. Suppose that $F : (\mathbb{R}^2 \times I, \mathbf{0} \times I) \rightarrow (\mathbb{R}, 0)$ admits a blow analytic trivialization along I and suppose that

$$f_{\Delta_{0+}}(x, y) = x^\alpha y^\beta \left(\sum_{i=0}^{\gamma} a_i x^{\gamma-i} y^i \right), \quad a_0 a_\gamma \neq 0$$

as a germ at the origin. Then there exist germs $\varepsilon(t)$, $\delta(t)$ and $a_i(t)$ ($i = 1, \dots, \gamma$), at $t = 0$, of real analytic functions which satisfy the following conditions,

- (1) $F_{t, \Delta_t}(x, y) = (x - \delta(t)t)^\alpha (y - \varepsilon(t)x)^\beta \left(\sum_{i=0}^{\gamma} a_i(t) x^{\gamma-i} y^i \right)$
- (2) $\varepsilon(0) = \delta(0) = 0$ and $a_i(0) = a_i$ ($i = 1, \dots, \gamma$)
- (3) $\sum a_i(t) x^{\gamma-i} y^i$ does not divide by $(x - \delta(t)y)$ or $(y - \varepsilon(t)x)$ in $\mathbb{R}\{x, y\}$,

where Δ_t is the homogeneous face of $\Gamma(F_t; (x, y))$.

By Lemma 4.1 we can find germs of real analytic functions $\varepsilon_1(t)$, $\delta_1(t)$ and $a_i(t)$ ($i = 1, \dots, \gamma_1$) as in Lemma 4.1 so that

$$F_{t,\Delta_1}(x, y) = (x - \delta_1(t)y)^\alpha (y - \varepsilon_1(t)x)^\beta \left(\sum_{i=0}^{\gamma_1} a_i(t) x^{\gamma_1-i} y^i \right).$$

We set

$$\varphi_{1+}(x, y, t) = (x_1(t), y_1(t), t) = (x - \delta_1(t)y, y - \varepsilon_1(t)x, t).$$

Then $(x_1(t), y_1(t), t)$ is a real analytic family of coordinates of $\mathbb{R}^2$ and

$$\{(x_1, y_1) \in \Gamma(F_t; (x_1, y_1)) | x_1 + y_1 = \text{the order of } F_t \circ \varphi_{1+,t}^{-1}(x_1, y_1)\} = \Delta_{0+}$$

for $\forall t$ ($|t| < 1$), where $\varphi_{1+,t} = \varphi_{1+}|_{\mathbb{R}^2 \times \{t\}}$.

We have the following lemma by means of induction.

Lemma 4.2. *Suppose that $F \circ (\sigma_{(k-1)+} \times id_J) \circ \dots \circ (\sigma_{0+} \times id_J) : (\mathcal{N}_{(k-1)+}, o_{(k-1)+}) \rightarrow (\mathbb{R}, 0)$ admits a blow analytic trivialization along J , where $J = (-\varepsilon, \varepsilon)$ and suppose that $\Gamma(F_t; (x_{(k-1)+}, y_{(k-1)+})) \cap \{\nu_{(k-1)+} \leq y_{(k-1)+} \leq \nu'_{(k-1)+}\} = \Delta_{(k-1)+}$ for $\forall t \in J$ and $f_{\Delta_{(k-1)+}}$ has no power of x only. Then $F \circ (\sigma_{k+} \times id_J) \circ \dots \circ (\sigma_{0+} \times id_J) : (\mathcal{N}_{k+}, o_{k+}) \rightarrow (\mathbb{R}, 0)$ admits a blow analytic trivialization along J and there exist a positive number δ_{k+} and a bianalytic map*

$$\begin{aligned} \varphi_{k+} : \mathbb{R}^2(x, y) \times I_{\delta_{k+}} &\longrightarrow \mathbb{R}^2(x_{k+}, y_{k+}) \times I_{\delta_{k+}} \\ (x, y, t) &\longmapsto (x_{k+}(t), y_{k+}(t), t) \end{aligned}$$

such that

- (1) $\varphi_{k+}(x, y, 0) = (x, y, 0)$, $\varphi_{k+}(0, 0, t) = (0, 0, t)$
- (2) $\Gamma(F_t; (x_{k+}, y_{k+})) \cap \{\nu_{k+} \leq y_{k+} \leq \nu'_{k+}\} = \Delta_{k+}$
- (3) $\Gamma(F_t; (x_{k+}, y_{k+})) \cap \{\nu_{1+} \leq y_{k+}\} = \Gamma(F_t; (x_{1+}, y_{1+})) \cap \{\nu_{1+} \leq y_{1+}\}$,

where $I_{\delta_{k+}}$ is the open interval $(-\delta_{k+}, \delta_{k+})$.

Lemma 4.3. *If F admits a blow analytic trivialization along I and noncompact face of $\Gamma_+(F_t; (x, y))$ is independent of t , then we have*

$$\Gamma(F_t; (x, y)) = \Gamma(f; (x, y)) \quad \text{for } |t| < 1$$

We apply Lemma 4.2 for F and then for $k = 0, \dots, \gamma - 1$ and $|t| < 1$,

$$\Gamma(F_t; (x_{k+}, y_{k+})) \cap \{(x_{k+}, y_{k+}) | \nu_{k+} \leq y_{k+} \leq \nu'_{k+}\} = \Delta_{k+}$$

Next for the sign $-$, we proceed with the same argument as the sign $+$ and then for $l = 1, \dots, \xi - 1$ and $|t| < 1$,

$$\begin{aligned} \Gamma(F_t; (x_{l-}, y_{l-})) \cap \{(x_{l-}, y_{l-}) | \zeta_{l-} \leq x_{l-} \leq \zeta'_{l-}\} &= \Delta_{l-} \\ \Gamma(F_t; (x_{l-}, y_{l-})) \cap \{\zeta_{1-} \leq x_{l-}\} &= \Gamma(F_t; (x', y')) \cap \{\zeta_{1-} \leq x'\}, \end{aligned}$$

where $(x'(t), y'(t), t) = \varphi_{1-} \circ \varphi_{(\gamma-1)+} \circ \cdots \circ \varphi_{1+}(x, y, t)$ and

$$\zeta_{k-} = \min\{x \mid \text{there exists } y \text{ such that } (x, y) \in \Delta_{k-}\}$$

$$\zeta'_{k-} = \max\{x \mid \text{there exists } y \text{ such that } (x, y) \in \Delta_{k-}\}$$

$$\zeta_{0+} \geq \cdots \geq \zeta_{(\xi-1)+} = 0, \quad \zeta'_{0+} \geq \cdots \geq \zeta'_{(\xi-1)+}.$$

We set $\varphi(x, y, t) = (\varphi_t(x, y), t) = \varphi_{(\gamma-1)-} \circ \cdots \circ \varphi_{1-} \circ \varphi_{(\gamma-1)+} \circ \cdots \circ \varphi_{1+}(x, y, t)$ and then we have that the noncompact face of $\Gamma(F_t; \varphi_t)$ is independent of t for $|t| \ll 1$. Hence from Lemma 4.3, $\Gamma(F_t; \varphi_t)$ is independent of t for $|t| \ll 1$. This completes the proof.

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