

Scrambling and Complexity in AdS/CFT and Black Holes

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Abstract

The AdS/CFT correspondence and quantum entanglement (or quantum information). This is one of topics exciting many physicists in this decade. And black holes always play a role of inspiring lab in many cases. In this thesis, we will discuss about scrambling of quantum entanglement and complexity of states in AdS/CFT. In both topics, black holes are expected to be the fastest processors of quantum information. This thesis contains brief reviews of each topics and our own works. One work is an explicit calculation of fast scrambling time in two dimensional holographic CFTs and three dimensional BTZ black holes using the two-sided mutual information. This calculation is a first CFT counterpart of Shenker-Stanford's work in the gravity dual and an extension of their shock-wave approximation to finite time region in the gravity side. Another work is about so-called optimization procedure of Euclidean path-integrals in CFTs. The optimization procedure gives us a new way to extract dual geometries only from CFT data. And it also gives a new proposal for the definition of complexity of states in CFTs.

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Chapter 1

Introduction

”It from Qubit”— This is a famous slogan which describes well a recent trend fascinating many researchers in quantum information, condensed matter and high energy physics. In this decade, some measures and concepts developed in quantum information theory have been applied to understand some aspects of quantum field theory and gravity dual to some quantum field theory through the AdS/CFT correspondence [1] or holography (and ultimately quantum gravity). Such efforts have opened up new horizons and nowadays a big collaboration, the ”It from Qubit” Simons collaboration, takes the initiative.

A famous example initiating the philosophy of ”It from Qubit” is entanglement entropy and its holographic formula proposed by Ryu and Takayanagi [2]. Entanglement entropy (EE) is the von-Neuman entropy associated to a subsystem A .

$$S_A = -\text{tr} \rho_A \log \rho_A, \quad \rho_A = \text{tr}_{\bar{A}} \rho. \quad (1.0.1)$$

Here the Hilbert space is bipartite $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. This is a measure of entanglement between a subsystem A and its complement $\bar{A}$. Many applications of EE have been explored in quantum many-body systems and quantum field theories.

The holographic dual of EE is called holographic entanglement entropy (HEE) [2].

$$S_A = \min_{\substack{\gamma_A \\ \partial \gamma_A = \partial A \\ \gamma_A \sim A}} \frac{\text{Area}(\gamma_A)}{4G_N}, \quad (1.0.2)$$

where γ_A is a codimension-two space-like minimal surface anchored to $|A|$ on the AdS boundary. $\gamma_A \sim A$ is the homology condition on γ_A . It means that γ_A is homologous to A . The holographic relation is often called the Ryu-Takayanagi (RT) formula. And the minimal surface γ_A is called RT surface.

The bulk region surrounded by A and γ_A plus its domain of dependence is called entanglement wedge [10, 11, 12]. It is known as the gravity dual of the subsystem A (or the domain of dependence for A). And the causal wedge for A is nested in the entanglement wedge.

Some Properties of EE

A famous property of EE is the area law typically satisfied for ground states in local, gapless and relativistic field theory. Unlike thermal entropy which is extensive in system size (the volume law), The EEs for the states have a leading divergence proportional to the area of the subsystem.

$$S_A = a_{d-2} \cdot \frac{L^{d-2}}{\epsilon^{d-2}} + a_{d-4} \cdot \frac{L^{d-4}}{\epsilon^{d-4}} \cdots + \begin{cases} (-1)^{\frac{d-1}{2}} \cdot F & (d : \text{odd}), \\ (-1)^{\frac{d-2}{2}} \cdot A \cdot \log\left(\frac{L}{\epsilon}\right) & (d : \text{even}). \end{cases}$$

Generically, EEs in field theories have local divergences so we regularize them by a UV cutoff ϵ . The area law means that the main contribution comes from entanglement nearby the boundary ∂A of the subsystem. There are universal contributions not depending on choices of ϵ . For odd d , the constant terms F . For even d , the coefficients A of logarithmic terms. They have some use in several contexts as mentioned soon.

In two dimensional conformal field theories ($d = 2$), the EE for an interval in the vacuum is proportional to the central charge c .

$$S_A = \frac{c}{3} \log\left(\frac{L}{\epsilon}\right). \quad (1.0.3)$$

It seems to violate the area law 1.0.3 but it is proportional to the numbers of the edges of the interval. Using this property conversely, we can read off the central charges c . This property is useful to characterize criticality and degrees of freedom in the theories. And actually scalings of EEs in system size L cost for lattice simulation of density matrix

For $d = 3$, the EE for a sphere has the linear divergence in the radius L and the constant term.

$$S_A = a_1 \cdot \frac{L}{\epsilon} - \gamma_{top}. \quad (1.0.4)$$

The constant term γ_{top} is called the topological entanglement entropy [3, 4]. It can be used as an order parameter of quantum phase transition.

To discuss monotonicity of degrees of freedom in QFTs under renormalization group (RG) flows, the universal contributions of EE are also useful. Using the Lorentz invariance and

the strong subadditivity of EEs

$$S_{ABC} + S_B \leq S_{AB} + S_{BC}, \quad (1.0.5)$$

which holds for any states, we can prove the existence of monotonic functions under RG flows. This functions becomes central charges (for odd d) at UV and IR fixed points by construction and extrapolate "degrees of freedom" between the fixed points. In $d = 2$, we define the following function socalled the C -function.

$$C(L) = 3L \frac{dS_A}{dL}, \quad C(\epsilon) = c_{UV} < C(\infty) = c_{IR} \quad (1.0.6)$$

The proof of the monotonicity is called the c -theorem in $d = 2$ [5], F -theorem in $d = 3$ [6] and a -theorem in $d = 4$ [7]¹. The entropic proofs are simpler than other field theoretic proofs known in $d = 2$ [8] and in $d = 4$ [9]. For higher dimensional cases (exactly speaking including $d = 4$), there is no proof so far.

Note that, for generic states including thermal states, the EEs have volume law divergence like thermodynamical entropy.

$$S_A = a_{d-1} \cdot \left(\frac{L}{\epsilon} \right)^{d-1} + \dots \quad (1.0.7)$$

For thermal states, the difference between the EE for the subsystem and the complement is the thermal entropy in the large subsystem limit.

$$|S_A - S_{\bar{A}}| = S_{thermal} \quad (1.0.8)$$

For dynamical aspects of quantum systems, many interesting properties have been studied. For example, the first law of entanglement (in the dual gravity, the linearized Einstein equation), scrambling (and quantum chaos) and energy inequality (ANEC, QNEC).

Entanglement (Entropy) Probes Dual Spacetime

The RT formula tells us the connection between entanglement in the field theory side and geometry in the gravity side and entanglement (entropy) is a useful probe for dual spacetimes. It implies that we can interpret AdS/CFT as a geometrization of quantum states.²

¹In $d = 4$, additionally we need to impose the Markov property of vacuum.

²Another geometrization of quantum states is socalled the tensor network (TN) description of quantum states. The method has been developed in the condensed matter community to simulate quantum states efficiently in computational sense. The relation to AdS/CFT as a toy model of holography was pioneered by Swingle [13].

A famous example was pointed out by Maldacena [37] before the RT proposal. The connection between the eternal AdS black hole and the thermofield double (TFD) state in two copies of CFTs, CFT_L and CFT_R ,

$$|TFD\rangle_\beta = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\beta}{2} E_n} |n\rangle_L |n\rangle_R, \quad (1.0.9)$$

where β is a inverse temperature. The TFD state has a particular structure of entanglement such that the density matrix reduced to one CFT is the thermal density matrix and the TFD state has a time shift symmetry generated by $H_L - H_R$. Such properties imply the connectivity of space-like separated asymptotic boundaries in the two-sided black holes. From the viewpoint of holographic entanglement entropy, the connectivity is probed by minimal surfaces anchored on the two boundaries. The entanglement between two copies of CFTs is geometrized.

This aspect of AdS/CFT was emphasized van Raamsdonk [14, 15] and drastically pushed by Maldacena and Susskind as the EPR = ER conjecture [16].

Other Interesting Things from Quantum Information Theory

In quantum information theory, not only EE, many measures and concepts have been developed. Some of them have been applied or are applying to QFTs and AdS/CFT. For examples, Mutual information, Tripartite information, Relative entropy, Fidelity, Quantum information metric [17] and Complexity. As Concepts, Quantum error correction, Recovery map, Quantum channel (or Tensor network) Quantum operation, Quantum teleportation [18], Distillation of entanglement, Localizable entanglement and Entanglement of purification [19]. Anyway we have been applied or are trying to apply so many things. They elucidate new aspects of QFT and dual gravity through AdS/CFT from the view point of quantum information theory.

Interests of This Thesis

In this thesis, I will focus on the two following things among these interesting topics. "Scrambling" and "Complexity".

Scrambling is a delocalization of (localized) quantum information under time evolution. In scrambled systems, initial information cannot be recovered from any local measurements. It is important to understand thermalization of closed quantum systems and the black hole information problem. And recently the relation to quantum chaos especially butterfly effects has been actively studied.

Although scrambling saturates some types of quantum information, for instance, entanglement entropy, the quantum state still keeps evolving. In the black hole information context, the black hole interior still keeps growing. To track such evolution nontrivially, "Entanglement (Entropy) is Not Enough".

As a candidate of the measure for late time dynamics of black holes, holographic duals of complexity of a state were conjectured by Susskind and collaborators. Complexity of a state is the minimal numbers of simple operations to construct the state from a simple reference state. The complexity is conjectured to be dual to the maximal volume of the codimension-one space-like surface or the on-shell gravitational action on the Wheeler-De-Witt (WDW) patch. Due to many sample checks, the holographic conjectures are believed as a useful tools for dual gravitational spacetimes.

However, against the rapid development of the holographic conjectures of the complexity, the QFT definition is undeveloped yet. We think it is quite important and interesting to pursue the application of the new quantum information to QFTs and AdS/CFT.

Outline of The Thesis

The outline of this thesis is as follows ³.

In Chapter 2, we will review more about scrambling. Inspired by black hole information problems, the formulation of scrambling has been developed. Some important thought experiments will be discussed. From these discussions, it was conjectured that black holes are the fastest scramblers. Some recent developments about the fast scrambling in the context of AdS/CFT will be reviewed.

In Chapter 3, we introduce our work [20] about fast scrambling time detected by (two-sided) mutual information in the holographic setup and the CFT setup each other. The holographic setup is a generalization of Shenker and Stanford's work [21]. And the CFT calculation is a first explicit example for fast scrambling time in CFTs which is detected by mutual information. ⁴

In Chapter 4, we will review more about complexity. The maximum of complexity, its growth rate and the conjectures of the holographic complexity and the beginning efforts for the QFT definitions will be presented.

³For clear presentations, we will give you out-of-time-ordered (OTO) or out-of-historical-ordered presentations about many interesting works on scrambling and complexity (of course including our works). We hope this way would not be chaotic and keep on low complexity for you.

⁴In [22], which is bit earlier than our work, fast scrambling time in CFT was realized by using the two-point correlators on the perturbed background.

In Chapter 5, we introduce our works [23, 24] about optimization of Euclidean path integral and complexity in CFTs. The optimization procedure can be regarded as a direct method to extract holographic geometries only from CFT data. And this procedure might suggest one way to define the CFT analogue of complexity of states.

In Chapter 6, we will conclude the thesis.

Chapter 2

Scrambling

How fast are local degrees of freedom mixed up in a quantum state of a quantum system? This is an important question to understand dynamical properties of quantum systems (chaosity or integrability in the system) and also the black hole information problem.

2.1 Scrambling, Quantum Information and Black Holes

Consider time evolution of closed quantum systems. Because the time evolution is unitary, a system never forgets the initial state. But the information about the initial state might be mixed up as time evolves and eventually it becomes hard to distinguish different initial states from local measurements. The information is delocalized and it cannot be recovered from measurements in any (small) subsystems. Such delocalization phenomenon is called "scrambling". This is a strong form of thermalization applicable to closed quantum systems. The minimal time to forget the information about the initial state is called "scrambling time".

How do we formulate this phenomenon? Inspired by the black hole information problem, the formulation of scrambling has been developed. We need to see the delocalization of the initial localized degree of freedom after the time evolution.

2.1.1 Page Scrambling

One way is that any subsystems with the size less than half of the total system are (nearly) maximally entangled with the complements. The entanglement entropy for the subsystems are maximized. In other words, the information in the subsystems is completely shared with the complements and we cannot read the information from measurements localized in the

subsystems. The information is delocalized. This is called "Page scrambling" proposed by Don Page[25].

What Page showed is as follows. For any bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ ($\dim \mathcal{H}_A = |A| \leq |B|$), a randomly chosen pure state in $\mathcal{H}_{AB}$ is likely to be close to maximally entangled. It means the system is scrambled.

Let us discuss more about the Page's theorem. Choose a pure state $\rho(U)$ evolved by a random unitary matrix U from an initial state $|\psi_0\rangle$.

$$\rho(U) = U|\psi_0\rangle\langle\psi_0|U^\dagger. \quad (2.1.1)$$

In terms of the trace distance, we can show that the pure state is close to the maximally entangled state $\frac{I_A}{|A|}$ for $|B| \gg |A| \gg 1$ on average ¹.

$$\int dU \|\rho_A(U) - \frac{I_A}{|A|}\|_1 \leq \sqrt{\frac{|A|^2 - 1}{|A||B| + 1}} \simeq \sqrt{\frac{|A|}{|B|}}, \quad (2.1.2)$$

where $\|M\|_1 = \text{tr} \sqrt{MM^\dagger}$ is the L_1 norm. This inequality follows from the inequality for the L_1 norm and the L_2 norm ($\|M\|_2 = \sqrt{\text{tr} MM^\dagger}$).

$$\|M\|_2 \leq \|M\|_1 \leq \sqrt{d} \cdot \|M\|_2, \quad (2.1.3)$$

where d is the dimension of the Hilbert space we take the trace. In terms of the entanglement entropy, we can obtain a similar argument.

$$\int dU S_A = - \int dU \text{tr} \rho_A \log \rho_A = \log |A| - \frac{1}{2} \frac{|A|}{|B|} + \dots. \quad (2.1.4)$$

In the limit $|B| \gg |A| \gg 1$, the random average of the entanglement entropy takes the maximal value $\log |A|$ given by the maximally entangled state. The system is (Page-)scrambled by the random unitary matrix. The information about the initial state $|\psi_0\rangle$ cannot be recovered from the information in the small subsystem A .

For n -qubit system, the small subsystem A with a qubits has $|A| = 2^a$ ($a < \frac{n}{2}$, $|B| = 2^{n-a}$). In this case, the random averaged entanglement entropy is exponentially close to the number of EPR pairs in A .

$$\int dU S_A = a \log 2 - \frac{1}{2} \cdot 2^{-(n-2a)} + \dots. \quad (2.1.5)$$

Also the reduced density matrix is exponentially close to be maximally mixed.

$$\int dU \|\rho_A - \frac{I_A}{|A|}\|_1 \leq 2^{-(n-2a)}. \quad (2.1.6)$$

¹Here we take a Harr random unitary $U(N)$ matrix U such that $\int dU = 1$, $\int dU U_{ij} U_{kl}^\dagger = \frac{1}{N} \delta_{il} \delta_{jk}$, $\int dU U_{ij} U_{kl} U_{mn}^\dagger U_{op}^\dagger = \frac{1}{N^2-1} (\delta_{in} \delta_{kp} \delta_{jm} \delta_{lo} + \delta_{ip} \delta_{kn} \delta_{jo} \delta_{lm}) - \frac{1}{N(N^2-1)} (\delta_{in} \delta_{kp} \delta_{jo} \delta_{lm} + \delta_{ip} \delta_{kn} \delta_{jm} \delta_{lo})$.

The qubit system is scrambled.

Motivated by black hole information problem, let us identify B to a black hole and A to the Hawking radiation from it. As the black hole evaporates, the entropy S_A of the radiation will increase. Then when S_A reaches to half of the initial black hole entropy $S_0/2 \sim n/2$ which proportional to half of the horizon area, S_A will starts to decrease because $|A|$ become larger than $|B|$ and $S_A \simeq \log |B|$ for $|A| > |B|$. This time is called the Page time t_{Page} or the evaporation time. And the information $I_A = \log |A| - S_A$ starts to release out from the black hole. linearly grow in the number of qubits radiated.

$$I_A \simeq (a - b) \log 2 = \left(\frac{2a}{n} - 1 \right) \cdot n \log 2. \quad (2.1.7)$$

We can read off some information about A only from measurements in A .

2.1.2 The Hayden-Preskill Thought Experiment

Hayden and Preskill [26] proposed a famous thought experiment which is a simple arrangement of the black hole information problem in quantum informational fashion. We can extract some essence of scrambling from their discussion.

Consider information recovery problem from the Hawking radiation. First, Alice throws a quantum state (a Alice's secret) into a black hole. As time evolves, the state will be mixed up with the black hole and the Hawking radiation. Then, Bob tries to recover the information of the state by collecting qubits in the Hawking radiation before and after Alice threw the state. How much qubits does Bob need to collect to recover the Alice's secret?

The conclusion of their discussion is summarized as "black hole as a mirror". This means that only measuring the same number of qubits in the radiations as that Alice threw initially, Bob can recover the Alice's secret. The black hole behaves like a mirror which reflects the information Alice threw into.

Let us assume Bob knows the initial black hole state (say because Bob has been keeping track of the black hole since its formation) or, after the Page time, the early radiation emitted before Alice's throw. After the Page time, the "old" black hole is maximally entangled with the early radiation. They share the quantum information completely like EPR pairs. Then, Bob has access to the initial black hole B (or early radiation B') and the (late) radiation D after Alice's secret A was thrown into. If Alice encodes her secret into EPR pairs and threw one of each pair into the black hole, Bob can verify the information recovery using the qubits Alice additionally prepared (the reference state A'). The remaining black hole C is not accessible by Bob.

In this setup, we can convert the unitary evolution by U from AB to CD

$$U_{AB \rightarrow CD} = \sum_{i,j,k,l} U_{(ij)(kl)} |i\rangle_C |j\rangle_D \langle k|_A \langle l|_B, \quad (2.1.8)$$

into the isomorphic state in the system $A'B'CD$ ².

$$\begin{aligned} |U\rangle_{A'B'CD} &= (I_{A'B'} \otimes U_{AB \rightarrow CD}) \cdot \left(2^{-\frac{a}{2}} \sum_i |i\rangle_{A'} |i\rangle_A \right) \otimes \left(2^{-\frac{b}{2}} \sum_j |j\rangle_{B'} |j\rangle_B \right), \\ &= 2^{-\frac{a+b}{2}} \sum_{i,j,k,l} U_{(kl)(ij)} |i\rangle_{A'} |j\rangle_{B'} |k\rangle_C |l\rangle_D. \end{aligned} \quad (2.1.9)$$

Here each index runs as $i = 1, \dots, 2^a, j = 1, \dots, 2^b, k = 1, \dots, 2^c, l = 1, \dots, 2^d$. And, because the time evolution by U is unitary, the number of qubits are preserved, $a + b = c + d$. By construction, A' and B' are separated or have no entanglement each other. And each of them is maximally entangled with their compliments.

$$\rho_{A'B'}(|U\rangle) = \rho_{A'}(|U\rangle) \otimes \rho_{B'}(|U\rangle) = 2^{-a} \cdot I_{A'} \otimes 2^{-b} \cdot I_{B'}. \quad (2.1.10)$$

This means the mutual information $I_{A':B'}$ between A' and B' vanishes.

$$I_{A':B'} = S_{A'} + S_{B'} - S_{A'B'} = 0. \quad (2.1.11)$$

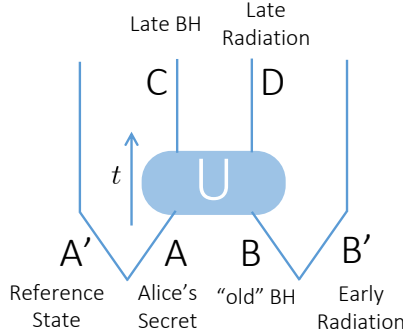


Figure 2.1: The setup for the Hayden-Preskill thought experiment [26]. When the total radiation $B'D$ becomes larger than half of the whole system $A'B'CD$, the Alice's secret shared with the reference state A' is transferred to $B'D$ only. Bob can recover the Alice's secret from the total radiation $B'D$.

²In quantum information theory, this relation is called the Choi-Jamiołkowski isomorphism or the state-channel duality.

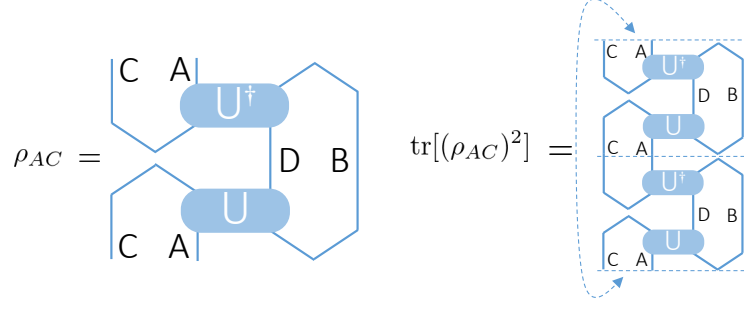


Figure 2.2: The quantum circuit diagrams for ρ_{AC} and $\text{tr}[(\rho_{AC})^2]$. To estimate the fast scrambling time, the latter is important. It is related to unitary two-design [26, 31, 30].

From the late radiation, Bob retrieves qubits to recover the Alice's secret. When Bob collects more qubits than the Alice's qubits ($d > a$), the qubits in the subsystem $B'D$ accessible by Bob becomes more than half of the system $A'B'CD$.

$$(b + d) - \frac{1}{2}(a + b + c + d) = d - a > 0. \quad (2.1.12)$$

Then, using the Page's theorem, we find that A' and $B'D$ are maximally entangled or equivalently A' and C are separated (no entanglement).

$$\int dU \|\rho_{A'C} - \rho_{A'} \otimes \rho_C\|_1 \geq \sqrt{\frac{(2^{2a} - 1)(2^{2c} - 1)}{2^{2a} \cdot 2^{2b} - 1}} \simeq 2^{c-b} = 2^{-(d-a)}. \quad (2.1.13)$$

Remind that, for any unitary matrices U , each subsystem A', B', C, D are maximally entangled with the complements by construction.

$$\rho_A(U) = 2^{-a} \cdot I_A, \quad \rho_C(U) = 2^{-c} \cdot I_C, \dots \quad (2.1.14)$$

The mutual information $I_{A':C} = S_A + S_C - S_{AC}$ between A' and C approaches to zero on average.

$$\int dU I_{A':C} = a + c - \int dU S_{A'C} \leq a + c + \int dU \log \text{tr} \rho_{A'C}^2 \simeq 0 + O(2^{-(d-a)}). \quad (2.1.15)$$

And it means that the average mutual information between A' and $B'D$ is maximized.

$$\int dU I_{A':B'D} = 2a - \int dU I_{A':C} \simeq 2a + O(2^{-(d-a)}). \quad (2.1.16)$$

Therefore, after emitting qubits more than a qubits ($d > a$), the late black hole C forgot the information of the qubits Alice's threw into. The information of Alice's secret was released

completely into the early and late radiations $B'D$. It was reflected back to the outside of the black hole like a mirror ³.

This result has another interpretation from the viewpoint of multi-partite entanglement [28]. The tripartite information $I_3(A' : B' : D) = I_{A':B'} + I_{A':D} - I_{A':B'D}$ is a multi-partite entanglement measure for three-parties. Using $I_{A':B'} = 0$ and assuming $b > d$ or $I(A' : D) = 0$ (small late radiation),

$$\int dU I_3(A' : B' : D) = \int dU (I_{A':D} - I_{A':B'D}) \simeq -2a < 0. \quad (2.1.17)$$

The information of the Alice's secret encoded in A' is delocalized (scrambled) as it transferred to $B'D$ through the random unitary channel U . If Bob does not know about either B' or D , he would lose some information of A' . The negativity of tri-partite information means the monogamy of mutual information. This property is not true for general states. Only some class of states satisfy it. For example, a GHZ state $(|0000\rangle + |1111\rangle)/\sqrt{2}$ gives a positive value $I_3 = \log 2 > 0$. Interestingly, the holographic mutual information also satisfies the monogamy property [29].

How fast is this scrambling process? Hayden and Preskill [26] pointed out that it can be estimated by using unitary two-design to implement the Harr-random unitary circuit U . Consider an ensemble $\mathcal{E} = (p_i, U_i)$ of unitary operators U_i acting on the Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes k}$ with probability p_i . The unitary k -design is defined as the ensemble which gives same averaged values for all quantum states $\rho \in \mathcal{H}$ as the Harr unitary ensemble with accuracy ϵ .

$$\sum_i p_i (U_i)^{\otimes k} \rho (U_i^\dagger)^{\otimes k} = \int_{Harr} dU (U)^{\otimes k} \rho (U^\dagger)^{\otimes k}. \quad (2.1.18)$$

The k -design diagnoses a level of the randomness of the ensemble ⁴. To construct the two design in n -qubit systems, we need the circuit depth which scales as $O(\log n \log \frac{1}{\epsilon})$ [31]. We can conclude that the scrambling time t_{sc}/β scales as logarithmic in the system size or the maximal entropy $S_{max} = \frac{n}{2} \log 2$. This logarithmic time is quite faster than the Page time $\sim n$ when black holes release their information.

³To recover the Alice's secret from the radiations, Bob needs to decode the information by acting a certain unitary operator V on $B'D$ like $\sum_i |i\rangle_{A'} |0\rangle'_B |i\rangle_D \rightarrow |0\rangle_{A'} \sum_i U_{B'} |i\rangle_{B'} |i\rangle_D$ and then acting $U_{B'}^\dagger$. The decoding is nothing but a quantum error-correcting code [26]. Very recently the efficient decoding was discussed in [27].

⁴For recent applications of the unitary k -design to quantum chaos and complexity, see [30]. In fact, the k -designs are some types of the higher point versions of the out-of-time-ordered correlators (OTOCs).

2.1.3 Fast Scrambling Conjecture

What is the fastest time scale for scrambling? What is the fastest scrambler in nature? Sekino and Susskind [32] conjectured that the fastest scrambling time scales as logarithmic in degrees of freedom and black holes are the fastest scramblers in nature.

They consider to perturb a Schwarzschild black hole by a charged particle freely falling into the black hole. And they estimate the time when the perturbation spreads over the whole black hole horizon (stretched by string scale $l_s = \sqrt{\alpha'}$). Near the horizon, the spacetime can be written by the Rindler spacetime.

$$ds^2 = -dt^2 + dz^2 + (dx^i)^2 = -\rho^2 d\omega^2 + d\rho^2 + (dx^i)^2, \quad (2.1.19)$$

$$\rho^2 = z^2 - t^2, \quad z = \rho \cosh \omega, \quad t = \rho \sinh \omega.$$

where $i = 1, \dots, d-1 (=8 \text{ for superstring theory})$. In the light-cone coordinate, $X^\pm = \rho e^{\pm\omega}$. The stretched horizon is a surface at $\rho = l_s$. We can consider the freely falling particle with a charge q located at $(t, z = 1, x^i = 0)$. In this setup, it turns out that the horizon behaves as a electrically conducting surface (membrane). The electric field has the ρ -component as

$$E_\rho = E_z = \frac{q(z-1)}{((z-1)^2 + x^2)^{d/2}} = \frac{q(l_s \cosh \omega - 1)}{((l_s \cosh \omega - 1)^2 + x^2)^{d/2}}. \quad (2.1.20)$$

For large ω , the surface charge density σ is

$$\sigma = \frac{E_\rho}{\rho} \Big|_{\rho=l_s} \sim \frac{q}{4\pi} \frac{(l_s e^\omega / 2)^{-(d-1)}}{(1 + 4y^2)^{d/2}}, \quad (2.1.21)$$

In the last line, we rescale x^i as $x^i = l_s e^\omega y^i$. At a Rindler time ω , the charge density spread to a distance $\Delta x = l_s e^\omega$. Then the time ω_* to spread over the whole black hole horizon with radius r_s is

$$\omega_* \sim \log \frac{r_s}{l_s}. \quad (2.1.22)$$

In terms of the asymptotic observer time t , the Rindler time with the inverse temperature β is $\omega = \frac{2\pi t}{\beta}$. Assuming $l_s \sim l_{pl}$, the time scales as logarithmic in the black hole entropy $S_{BH} \sim \left(\frac{r_s}{l_{pl}}\right)^{d-1}$.

$$t_{sc} \sim \beta \log S_{BH}. \quad (2.1.23)$$

Sekino and Susskind also discussed for a D0-brane black hole or $U(n)$ matrix quantum mechanics. Based on the analyses, a naive estimation (mentioned soon), a thought experiment related to quantum cloning and black hole complementarity [33], the Hayden-Preskill thought experiment [26], they made the conjectures about the fast scrambling.

2.1.4 A Naive Estimation of Fast Scrambling Time

Consider a two-qubit (local) interaction in a n -qubit system. How many steps do we need to relate or correlate one-qubit with all other qubits at least? Let us try a naive estimation.

Consider to put qubits on a d -dimensional lattice (lattice spacing ϵ) and allow only nearest neighbor interactions. Naively speaking, the effect of one-qubit spread out ballistically. In the m -th step, the volume of the ball is proportional to $(\epsilon m)^d$ and the number of sites contained in the ball is roughly $\sim m^d$. Then, the order of the number of steps needed to cover all sites is polynomial at least.

$$m \gtrsim n^{1/d}. \quad (2.1.24)$$

If we focus on diffusive processes, the perturbation spreads to a distance $\gtrsim \sqrt{t}$. Then, the lower bound of scrambling time is doubled $m \sim n^{2/d}$. If we consider the limit $d \rightarrow \infty$, it may scale as logarithmic $m \gtrsim \log n$. This limit seems to be a kind of infinite nearest neighbors limit.

The limit can be achieved by random two-qubit interactions⁵. And actually also it seems to be indirectly achievable by mixing of entanglement. As a step, we consider to relate two qubits due to the interaction. In the second step, we can relate each qubit with other one qubit. Now at most 4-qubits are related. We repeat such step. Then, in the m -th step, we can relate 2^m -qubits at most. Therefore, the lowest order of the number of the steps m is logarithmic in the system size n .

$$m \gtrsim \log n. \quad (2.1.25)$$

This is a kind of minimal time that an effect caused by one-qubit spreads over all qubits. If we consider a parallel processing which acts two-qubit operations to all qubits simultaneously within time β , the scrambling time t_{sc} is bounded by

$$t_{sc} \gtrsim \beta \log n. \quad (2.1.26)$$

The fast scramblers might saturate the bound.

2.2 Scrambling in Holography

The fast scrambling conjecture proposed by Sekino and Susskind was based on some thought experiments and naive estimations as mentioned. Can we make any concrete calculation

⁵In the paper [32], it was illustrated by discussing charge diffusions on horizons of Schwarzschild black hole and D0-brane black hole or $U(n)$ matrix quantum mechanics. And as a recent remarkable example is the SYK model [34, 35].

in field theoretic models of the fastest scramblers? Shenker and Stanford [21] discussed a concrete setup in CFTs with a holographic dual (black holes with a perturbation) and performed a holographic calculation using the shock wave approximation of the perturbation.

Moreover, they pointed out a connection between a chaotic property "butterfly effect" in strongly coupled quantum field theories and a physics in black holes. The chaotic behavior is probed by the out-of-time ordered correlators (OTOCs) $\langle W(t)V(0)W(t)V(0) \rangle_\beta$ or expectation values of square of commutators $\langle [W(t), V(0)]^2 \rangle_\beta$.

2.2.1 A Holographic Setup for Scrambling : Shock-wave Approximation

A simple setup for scrambling in the quantum field theory side is to consider a perturbation of thermal state by a local operator W in CFTs. The thermal state can be purified by the thermofield double (TFD) state in two copy of CFTs, CFT_L and CFT_R .

$$|TFD\rangle_\beta = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\beta}{2} E_n} |n\rangle_L |n\rangle_R. \quad (2.2.1)$$

This is invariant under the time evolution by $H_L - H_R$. To consider a nontrivial time evolution, we consider the time evolution by $H_L + H_R$ ⁶.

We perturb the TFD state by a local operator $W_L(-t_\omega) = e^{it_\omega H_L} W_L e^{-it_\omega H_L}$ at time $-t_\omega$ in CFT_L .

$$|\Psi\rangle = e^{-it_\omega H_L} W_L e^{it_\omega H_L} |TFD\rangle_\beta \quad (2.2.2)$$

Here we denote $W_L = W \otimes I$, $W_R = I \otimes W$ and omitted the spacial positions of the operators. On the TFD state, the operators in CFT_L and CFT_R are related by the imaginary time evolution as $W_R(t) = W_L^\dagger(t + i\frac{\beta}{2})$. The operators live in a cylinder with circumference β parametrized by a complex coordinate.

What is the holographic dual of the perturbed state $|\Psi\rangle$? The holographic dual of the TFD state in d -dimensional CFT was known to be a $(d+1)$ -dimensional eternal AdS-Schwarzschild black hole with maximal extension to include two AdS boundaries [37]. For simplicity, we focus on the three bulk dimension, a (two-sided) BTZ black hole. In the coordinate covering an exterior of the BTZ black hole,

$$ds^2 = -\frac{r^2 - r_s^2}{R_{AdS}^2} dt^2 + R_{AdS}^2 \frac{dr^2}{r^2 - r_s^2} + r^2 d\phi^2, \quad (2.2.3)$$

⁶This setup is a kind of global quench [36]. The thermalization will be caused by mixing up the entanglement between two CFTs.

where $r_s^2 = 8G_N M_{BH} R_{AdS}^2$ and $\beta = \frac{2\pi R_{AdS}^2}{r_s}$. To cover the maximally extended geometry, it is convenient to use the Kruskal coordinate.

$$ds^2 = \frac{-4R_{AdS}^2 dudv + r_s^2(1 - uv)^2 d\phi^2}{(1 + uv)^2}. \quad (2.2.4)$$

The horizons are at $uv = 0$, the two AdS boundaries are at $uv = -1$ and the two singularities are at $uv = 1$. We can read off the isometry of the boost on the (u, v) -plane $(u, v) \rightarrow (e^\lambda u, e^{-\lambda} v)$ or the translation along the Schwarzschild time t ⁷.

To perturb this black hole geometry, we put a few particles into the black hole from the left boundary. Due to the boost isometry, the particles get heavily blue-shifted near the horizon. Then, at the $t = 0$ slice, the proper energy E_p at $t = 0$ will be exponentially amplified in t_ω from the initial energy E_0 at t_ω .

$$E_p \sim \frac{E_0 R_{AdS}}{r_s} e^{\frac{r_s}{R_{AdS}^2} t_\omega} = \frac{E_0}{4M_{BH}} e^{\frac{2\pi}{\beta}(t_\omega - t_{sc})}. \quad (2.2.5)$$

Here the time scale t_{sc} is $\sim \beta \log S_{BH}$. For sufficiently large t_ω ($> t_{sc}$), we cannot ignore the back-reaction from the boosted particles. Even if the initial perturbation is so small, finally the black hole background will be deformed into a different geometry. Stanford and Shenker approximated the back-reacted geometry by the shock-wave geometry.

$$ds^2 = \frac{-4R_{AdS}^2 dU dV + 4R_{AdS}^2 \alpha \delta(U) dU^2 + r_s^2(1 - UV)^2 d\phi^2}{(1 + UV)^2}, \quad (2.2.6)$$

where $U = u$, $V = v + \alpha\theta(u)$ are discontinuous coordinates. At the null surface $u = 0$, the v -direction is shifted by a parameter.

$$\alpha = \frac{E_0}{4M_{BH}} e^{\frac{2\pi}{\beta} t_\omega}. \quad (2.2.7)$$

This shock-wave geometry can be constructed by gluing two BTZ solutions of mass M_{BH} and $M_{BH} + E_0$ along the null surface $u_\omega = e^{-\frac{2\pi}{\beta} t_\omega} \simeq 0$. This description becomes exact in the limit $\frac{E_0}{M_{BH}} \rightarrow 0$ and $t_\omega \rightarrow \infty$. At the gluing surface $u = U = 0$, this geometry is continuous but the curvature has a delta functional divergence. From the Einstein equation, we can find the stress tensor localized on the null surface.

$$T_{uu} = \alpha \delta(u). \quad (2.2.8)$$

We can interpret it as a shell of null particles causes the impulsive shift of the v -direction.

⁷In another exterior patch, the time translation is flipped $t \rightarrow -t$. This isometry matches the symmetry of the TFD state $e^{it(H_L - H_R)} |TFD\rangle_\beta = |TFD\rangle_\beta$

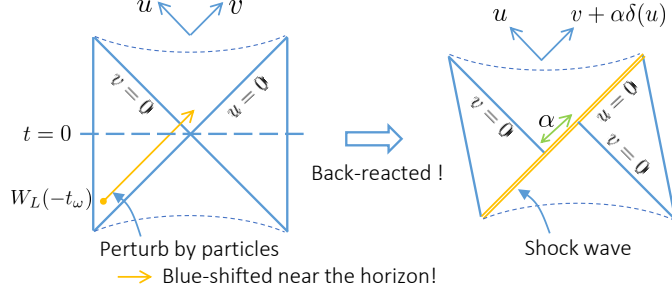


Figure 2.3: The shock-wave approximation [21]. The small perturbation at far past will be blue-shifted near the horizon and heavily deforms the black hole geometry as a shock-wave on the null surface $u = 0$.

2.2.2 Fast Scrambling Time from Two-sided Mutual information

Let us probe the perturbed state $|\Psi\rangle$ using the two-sided mutual information $I_{A:B} = S_A + S_B - S_{AB}$ between subsystems A in CFT_L and B in CFT_R . Another probe is the two-sided two-point correlator on the excited state.

$$\frac{\langle \Psi | V_L V_R | \Psi \rangle}{\langle \Psi | \Psi \rangle_\beta \langle V_L V_R \rangle_\beta} = \frac{\langle W_L^\dagger(-t_\omega) V_L V_R W_L(-t_\omega) \rangle_\beta}{\langle W_L^\dagger W_L \rangle_\beta \langle V_L V_R \rangle_\beta} = \frac{\langle V_R W_L^\dagger(-t_\omega) V_L W_L(-t_\omega) \rangle_\beta}{\langle W_L^\dagger W_L \rangle_\beta \langle V_L V_R \rangle_\beta}. \quad (2.2.9)$$

In the last step, we used commutativity between operators in the different CFTs. The numerator in the right hand side has a form nowadays called the out-of-time-ordered correlators (OTOCs). After Shenker and Stanford's work, the OTOCs became to be studied very actively to diagnose a chaotic property in quantum systems.

Actually the mutual information is the upper bound of the connected part of the two point correlators

$$I_{A:B} \geq \frac{(\langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle)^2}{2||\mathcal{O}_A||^2 ||\mathcal{O}_B||^2}. \quad (2.2.10)$$

for any operators $\mathcal{O}_A, \mathcal{O}_B$ in A and B [38]. Therefore, to see disruptions of correlations between two disjoint subsystems, it is sufficient to see vanishment (or decay) of the mutual information $I_{A:B} = 0$.

Holographically, the mutual information $I_{A:B} = S_A + S_B - S_{AB}$ is given by minimal lengths of geodesics connecting between the edge points of the subsystems A and B on the boundaries. The subsystems A and B are intervals $[0, \phi(< \pi)]$ at $t = 0$ on the left and right boundaries. For small intervals ($\sinh \frac{\pi \phi R_{AdS}}{\beta} < 1$) with any α , the dominant contribution always comes from geodesics disconnected the two AdS boundaries. Then, $I_{A:B} = 0$ for any α or t_ω . For large intervals ($\sinh \frac{\pi \phi R_{AdS}}{\beta} > 1$) with small α (small t_ω), S_{AB} can be dominated by the minimal lengths of geodesics connected the two AdS boundaries. But, as

α becomes larger with t_ω , the connected geodesics become longer and then the dominant contribution to S_{AB} will change to the disconnected geodesics contribution. Therefore the mutual information for large intervals behaves as follows.

$$I_{A:B} = \begin{cases} \frac{R_{AdS}}{G_N} \left[2 \log \left(\sinh \frac{\pi \phi R_{AdS}}{\beta} \right) - \log \left(1 + \frac{\alpha}{2} \right) \right] & (t_\omega < t_{sc}) \\ 0 & (t_\omega > t_{sc}) \end{cases}, \quad (2.2.11)$$

where t_{sc} is the time scale when $I_{A:B}$ vanishes. We can interpret t_{sc} as the scrambling time. For high temperature (small β), the scrambling time is

$$t_{sc} = \frac{\phi R_{AdS}}{2} + \frac{\beta}{2\pi} \log \frac{2S_{BH}}{\beta E_0} \quad (2.2.12)$$

Assuming that E_0 takes the smallest reasonable value $E_0 \sim 1/\beta$ and S_{BH} is large because of small G_N (or large N),

$$t_{sc} \simeq \frac{\beta}{2\pi} \log S_{BH} \quad (2.2.13)$$

This is the fast scrambling time conjectured by Sekino and Susskind [32]. As they conjectured, the physics of black holes plays an important role in the fastest scrambling process.

2.2.3 Out-of-Time-Ordered Correlators

As mentioned before, in the paper [21], Shenker and Stanford made an important suggestion about the relation between black hole physics and a chaotic phenomenon "butterfly effect" in quantum systems as the title of the paper says. The butterfly effect is a sensitive dependence on initial condition in chaotic systems. An initial small difference causes a huge difference after some time evolution. It can be measured by the trace distance between two perturbed states which are initially very close to, or equivalently the expectation values of square of commutators or the out-of-time ordered correlators (OTOCs).

First we perturb the TFD state by a local operator W . After time evolution by t , we try to probe the perturbed state $W(t)|TFD\rangle_\beta$ by a local operator V (at a different point from that we inserted W originally). Because under the time evolution, the size of the operator W grows as

$$W(t) = e^{-itH} W e^{itH} = W - it[H, W] - \frac{t^2}{2}[H, [H, W]] + \frac{it^3}{3!}[H, [H, [H, W]]] + \dots \quad (2.2.14)$$

This is a sum of products of local operators⁸. So even if W has no overlapping with V and $[W, V] = 0$ for $t = 0$, $W(t)$ can overlap with V and $[W(t), V] \neq 0$ for generic t . Then, we

⁸For example, in the Ising spin model $H = \sum_i (Z_i Z_{j+1} + g Z_i + h X_i)$ with the Pauli matrices X_i, Y_i, Z_i at i -th site, $[H, [H, [H, Z_j]]]$ is a linear combination of products of local operators $Y_j, I_{j-1} Y_j, Y_j I_{j+1}, X_{j-1} Y_j, Y_j X_{j-1}$ and $Z_{j-1} Y_j Z_{j+1}$. This commutator acts on three sites.

check the distinguishability when we change the ordering of operator insertions using the trace distance or the square of the commutator between $W(t)$ and V .

$$\begin{aligned}
& |W(t)V|TFD\rangle_\beta - VW(t)|TFD\rangle_\beta|^2 \\
&= \langle V^\dagger W^\dagger(t)W(t)V \rangle_\beta + \langle W^\dagger(t)V^\dagger VW(t) \rangle_\beta - \langle W^\dagger(t)V^\dagger W(t)V \rangle_\beta - \langle V^\dagger W^\dagger(t)VW(t) \rangle_\beta, \\
&= -\langle [W(t), V]^2 \rangle_\beta.
\end{aligned} \tag{2.2.15}$$

Here we take the expectation value on the TFD state in each line. In the second line, the first two terms in the right hand side are the norms of the states and they are time-ordered after cyclic permutations. The second two terms are out-of-time-ordered correlators (OTOCs). These correlators are inner products between two states with the different ordering of operator insertions. If these states become distinguishable, the OTOCs will decay from the values of the time-ordered correlators and the trace distance or the square of the commutator will grow.

Actually the growth can be exponentially in time in some strong coupled systems (fast scramblers) and the exponent λ_L can be a "Lyapunov exponent" as an analogy from classical chaos which diagnoses "butterfly effect"⁹. This early time behavior of the OTOCs or the square of the commutator can be used for diagnosing "quantum chaos" (not only scrambling).

To define the ordering of operators more precisely, we need to introduce a regulator for each operator. For example, for the first correlator, we can set

$$\langle V^\dagger W^\dagger(t)W(t)V \rangle_\beta = \langle W^\dagger(t)V^\dagger W(t)V \rangle_\beta \rightarrow \langle W^\dagger(t+i\epsilon_1)W(t+i\epsilon_2)V^\dagger(i\epsilon_3)V(i\epsilon_4) \rangle_\beta, \tag{2.2.16}$$

with $\epsilon_1 > \epsilon_3 > \epsilon_2 > \epsilon_4$. The regulators ϵ_j control the ordering of operators. We can set $\epsilon_1 > \epsilon_3 > \epsilon_4 > \epsilon_2$ for $\langle W^\dagger(t)V^\dagger VW(t) \rangle_\beta$, $\epsilon_1 > \epsilon_3 > \epsilon_2 > \epsilon_4$ for $\langle W^\dagger(t)V^\dagger W(t)V \rangle_\beta$ and $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$ for $\langle W^\dagger(t)VW(t)V^\dagger \rangle_\beta$.

Let us mention about some features of the (four-point) OTOCs in strong coupled thermal systems with inverse temperature β and a large number of degrees of freedom N (say the number of qubits). The time-ordered correlators will factorize for large t because the group of the operators the time-like separation.

$$\langle V^\dagger W^\dagger(t)W(t)V \rangle_\beta = \langle V^\dagger V \rangle_\beta \langle W^\dagger W \rangle_\beta + O(e^{-t/t_d}), \tag{2.2.17}$$

⁹Semiclassically, the square of commutator is the square of the Poisson bracket $i\hbar\{q(t), p\}_{P.B.} = i\hbar\frac{\partial q(t)}{\partial q(0)} \simeq [\hat{q}(t), \hat{p}]$ in a one particle quantum chaotic system [39]. The semiclassical quantity gives the time dependence of the change of the final position caused by small change of the initial position. Around the Ehrenfest time $t_E = \frac{1}{\lambda_L} \log \frac{1}{\hbar}$, it grows exponentially $\langle [\hat{q}(t), \hat{p}]^2 \rangle \simeq \hbar^2 e^{\lambda t} = e^{\lambda(t-t_E)}$.

where $t_d \sim \beta$ is a "dissipation" or "thermalization" time which is the same time when the connected part of the 2-point function $\langle VW(t) \rangle_\beta$ decays exponentially.

$$\langle VW(t) \rangle_\beta = \langle V \rangle_\beta \langle W \rangle_\beta + O(e^{-t/t_d}). \quad (2.2.18)$$

Around $t \sim t_d$, the time-ordered correlators will decay to a late time value controlled by $\langle V^\dagger V \rangle_\beta$, $\langle W^\dagger W \rangle_\beta$.

On the other hands, the OTOCs behave more complicated. For the four-point OTOCs, we have one more time scale, the scrambling time t_{sc} .

$$\frac{\langle V^\dagger W^\dagger(t) VW(t) \rangle_\beta}{\langle V^\dagger V \rangle_\beta \langle W^\dagger W \rangle_\beta} = \begin{cases} 1 + O(e^{-t/t_d}) & (t < t_d) \\ 1 - O(e^{\lambda_L(t-t_{sc})}) & (t_{sc} > t > t_d) \\ O(e^{-O(N)}) & (t > t_{sc}) \end{cases}. \quad (2.2.19)$$

The exponent λ_L for $t_{sc} > t > t_d$ is the Lyapunov exponent which characterizes the butterfly effect. For large- N strongly coupled systems, the fast scrambling time $t_{sc} \sim \beta \log N \sim \beta \log S$. and in $\lambda_L \sim \frac{2\pi}{\beta}$. The system has a large hierarchy between the times t_d and t_{sc} . In systems with large- N and the large hierarchy, a universal bound on the Lyapunov exponent Λ_L was proven in [40].

$$\lambda_L \leq \frac{2\pi}{\beta} + O\left(\frac{1}{N^2}\right). \quad (2.2.20)$$

This bound is called the chaos bound. It is saturated by states in holographic CFTs dual to black holes in Einstein gravity and also by a quantum mechanical model with random interaction, the SYK model [35, 34].

Chapter 3

Scrambling time from local perturbations of the eternal BTZ black hole

In this chapter, we introduce our work [20] about fast scrambling time in 2d holographic CFT and its gravity dual reading from mutual information. This work was a first explicit calculation in the CFT side and an extension of the shockwave approximation in the gravity side discussed by Shenker and Stanford [21].

[20] : JHEP **1508**, 011 (2015), arXiv:1503.08161 [hep-th]

3.1 Introduction and Summary

Consider a (1+1)d CFT in some state having non-trivial quantum correlations. If the system is perturbed at some instant of time t_ω and evolve unitarily afterwards, it is natural to ask whether there exists any time scale when its subsystems become uncorrelated. Since the mutual information $I_{A:B} = S_A + S_B - S_{A \cup B}$ between two such subsystems A and B provides an upper bound for the connected two-point functions of operators acting on these subsystems¹ [38]

$$I_{A:B} \geq \frac{(\langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle)^2}{2 \|\mathcal{O}_A\|^2 \|\mathcal{O}_B\|^2}, \quad (3.1.1)$$

¹This bound is proved for finite dimensional Hilbert spaces. We are not aware of an extension of this result to QFTs/CFTs, but we expect it to hold when regulating and normalising appropriately the relevant quantities in the continuum limit.

it is natural to study the vanishing of this quantity to answer this question. The study of the time dependence in this measure of entanglement can help us to understand the time scales controlling how quantum systems get thermalized, which is one of the most important problems in non-equilibrium physics.

A particular situation of the above scenario is when a perturbation acts on a thermal state. In holographic theories, thermal states are believed to have a gravity dual in terms of black holes [41, 37]. Black hole physics suggests that the speed at which the system forgets initial conditions, i.e. the perturbation, is the fastest among all physical diffusive processes. This gave rise to the notion of fast scramblers and the scrambling conjecture [26, 32]. The main goal of this chapter is to provide a first principle derivation for the time scale at which this phenomenon occurs for 2d CFTs in the large c limit in a concrete setup which allows both CFT and holographic computations.

Recently, Shenker and Stanford considered an excellent and tractable setup to study the fast scrambling phenomena in the context of an eternal black hole [21, 42, 43]. This involves a pair of non-interacting CFTs in an entangled state, the thermofield double state. Tracing any entire CFT Hilbert space, gives rise to a thermal density matrix in the remaining CFT. The perturbation is described by some boundary CFT operator and its gravity dual involved a shock-wave propagating in the black hole background. No matter how small the boundary perturbation is, the blue shift of energies when this perturbation reaches the horizon suggests the existence of a non-trivial backreaction.

In this work, we study such a setup for a perturbation localized in a point-like region, triggered by a primary operator in a given CFT. To obtain analytical results, we consider 2d large c CFTs and their gravity duals given by a perturbation of the BTZ black hole [44]. Recent developments in the calculation of 4-pt functions involving heavy and light operators in the large c limit of the dual 2d CFT [45, 46] (see also [47, 48]) allow us to analytically test these ideas.

Computations of time evolutions of entanglement entropy after local perturbations² by primary operators have been formulated in [50, 52] and have been applied to many examples for CFTs at zero temperature in [51, 54, 55, 56, 57, 58]. Entanglement entropy and mutual information at finite temperature CFTs has been analyzed for integral CFTs in [59]. On the

²Note that this setup looks similar to the local quenches in CFTs [49]. However, in the latter the local excitations are triggered by joining two semi-infinite lines and lead to local excitations in all sectors of a given CFT. Thus their behaviours differ from each other in integrable CFTs [50, 51, 52]. On the other hand, in large c CFTs, they behave similarly [53, 54, 55] in that both results for 2d CFTs show logarithmic time evolution of entanglement entropy.

other hand, the holographic calculations of time-evolutions of entanglement entropy after local perturbations have been analyzed in [53, 60, 61] at zero temperature and in [59] at finite temperature. In this chapter we will extend the discussion of local excitations to the thermofield double formalism of finite temperature CFTs in the large c limit.

Summary of results : Specifically, we perturb the thermofield double (TFD) state by a local primary operator ψ at time t_ω in the past and compute the mutual information between regions A and B belonging to opposite boundaries. We denote the two boundary times in the thermofield double by $t_L \equiv t_-$ and $t_R \equiv t_+$. When measuring the mutual information at $t_- = t_+ = 0$, we ask for the time scale t_ω^* when the mutual information vanishes

$$I_{A:B}(t_\omega^*) = 0. \quad (3.1.2)$$

Equivalently, we ask for the time scale t_ω^* at which correlations between A and B vanish. When both subsystems A and B are the intervals $(0 <) y \leq x \leq y + L$ and the perturbation is turned on at $x = 0$ and time $t_- = -t_\omega$, we obtain the following analytical result for $t_\omega^* \gg \beta$

$$t_\omega^* = y + \frac{L}{2} - \frac{\beta}{2\pi} \log \left(\frac{\beta \sin \pi \alpha_\psi}{\pi \epsilon} \right) + \frac{\beta}{\pi} \log \left(2 \sinh \frac{\pi L}{\beta} \right), \quad (3.1.3)$$

where β is the inverse temperature and $\alpha_\psi = \sqrt{1 - 24h_\psi/c}$ carries the information about the primary operator perturbation of conformal dimension h_ψ . The parameter ϵ represents a UV cut off for the local excitation, so that the excited state is localized around a region of size ϵ of the operator insertion. This makes the energy of the perturbation $E_\psi = \frac{\pi h_\psi}{\epsilon}$ finite. In the limit $h_\psi/c \ll 1$, which is the relevant one to match the butterfly effect discussed in [21, 42], this reduces to

$$t_\omega^* = f(L, \beta) + \frac{\beta}{2\pi} \log \left(\frac{\pi S_{density}}{4E_\psi} \right), \quad (3.1.4)$$

where $S_{density} = \frac{\pi c}{3\beta}$ is the entropy density of the original thermal system. The $\log S$ behavior in (3.1.4) is consistent with the fast scrambling conjecture [32, 21].

Given the bound (3.1.1), it should be possible to extract the same time scale from the condition of vanishing two sided 2-pt functions. We can explicitly check that this is true as observed in Shenker and Stanford [21].

In the second part of this work, we derive the same time scale from bulk holographic considerations and find a *perfect matching* between both calculations. Our holographic model is based on the description of the local boundary perturbation in terms of some free falling particle satisfying an initial condition guaranteeing such particle carries the right amount of

energy from the CFT stress tensor perspective. This is done by generalizing the model in [53, 59] to the two sided BTZ black hole, based on the back reaction description of point particles as quotients of AdS_3 . Applying the holographic entanglement entropy [2, 62] to evaluate the entanglement entropy and mutual information in our set-up, leads to the same scrambling time (3.1.3). At the same time, our setup and calculation may be useful for the interesting question regarding the dual CFT interpretation of a particle falling into a AdS black hole in future studies.

Our local boundary perturbation includes a regularization parameter ϵ describing its size. This parameter is holographically interpreted as the bulk position (distance from the boundary) from which the massive particle falls into the black hole. Our solution computes the back reacted geometry for any t_ω and approaches a localized shock-wave in the limit of large t_ω [63, 21].

This chapter is organized as follows: In section 2, we will analyze the time evolution of entanglement entropy in large c 2d CFTs at finite temperature, which agrees perfectly with a previous gravity dual computation. In section 3, we study local perturbations in finite temperature CFTs by employing the thermofield double formalism. We compute the mutual information from entanglement entropies. In section 4, we compute the scrambling time for the mutual information. In section 5, we introduce our holographic model. In section 6, we present our holographic computations of mutual information in a two sided AdS_3 black hole background with a local excitation.

Note : While finishing our main computations, the work of Roberts and Stanford [22] appeared. The latter has a detailed account of two point functions in the presence of localized excitations over thermal states and briefly mentions the behaviour of the mutual information in the same set-up. Thus, it has some overlap with our results. In this chapter, we literally evaluate the mutual information between the thermofield double in both 2d large c CFTs and their gravity duals independently and show their results perfectly agree. Our gravity solutions explicitly have the regularization parameter ϵ and our matching between gravity and CFT results holds while keeping this parameter small but non-zero. We would also like to mention that in the interesting paper [40] by Maldacena, Shenker and Stanford, the fast scrambling behavior of the correlations functions has been interpreted in terms of chaos.

3.2 Single sided entropy

To introduce our basic tools and fix the notation, we analyse the local perturbation to a thermal state in a single 2d CFT at finite temperature in this section.

Consider a thermal state ρ_β locally perturbed by a primary operator $\psi(0, -t_\omega)$ inserted at $x = 0$ at time $-t_\omega$. The time evolution of the resulting density matrix is given by

$$\rho(t) = \mathcal{N} e^{-iHt} \psi(0, -t_\omega) \rho_\beta \psi^\dagger(0, -t_\omega) e^{iHt}. \quad (3.2.1)$$

where H is the Hamiltonian of our system.

Denote by $\rho_A = \text{Tr}_{\bar{A}} \rho(t)$ the reduced density matrix on a finite interval A with endpoints³ $y, y + L$ satisfying $y, L > 0$. Its entanglement entropy S_A can be computed using the replica trick. We first compute the Renyi entropies

$$S_A^{(n)} \equiv \frac{1}{1-n} \log \text{Tr} \rho_A^n(t). \quad (3.2.2)$$

The entanglement entropy is obtained by taking the limit $S_A = \lim_{n \rightarrow 1} S_A^{(n)}$.

The trace of the reduced density matrix $\text{Tr} \rho_A^n(t)$ requires the calculation of the normalized 4-point function

$$\text{Tr} \rho_A^n(t) = \frac{\langle \Psi(x_1, \bar{x}_1) \sigma_n(x_2, \bar{x}_2) \tilde{\sigma}_n(x_3, \bar{x}_3) \Psi^\dagger(x_4, \bar{x}_4) \rangle}{(\langle \psi(x_1, \bar{x}_1) \psi^\dagger(x_4, \bar{x}_4) \rangle_{C_1})^n} \quad (3.2.3)$$

with the insertion points

$$\begin{aligned} x_1 = -i\epsilon, \quad x_2 = y - t_\omega - t_-, \quad x_3 = y + L - t_\omega - t_-, \quad x_4 = +i\epsilon \\ \bar{x}_1 = +i\epsilon, \quad \bar{x}_2 = y + t_\omega + t_-, \quad \bar{x}_3 = y + L + t_\omega + t_-, \quad \bar{x}_4 = -i\epsilon. \end{aligned} \quad (3.2.4)$$

The operator Ψ stands for the product of the operators ψ_i in each of the i -th copies of the theory⁴

$$\Psi = \psi_1 \cdot \psi_2 \cdots \psi_n \quad (3.2.5)$$

and has conformal dimension $h_\Psi = n h_\psi$, where h_ψ is the conformal dimension of the original perturbation ψ . Notice ϵ is a parameter smearing the local operator perturbation and all the time evolution is carried by the twist operators $\sigma_n, \tilde{\sigma}_n$ which are initially inserted at both ends of the interval when cyclically gluing the different cylinder copies that give rise to the manifold C_n . Finally, the conformal dimension $\Delta_\sigma = 2H_\sigma$ of the twist operators is

$$H_\sigma = \frac{c}{24} \left(n - \frac{1}{n} \right). \quad (3.2.6)$$

³Notice that the perturbation is originally inserted outside of the interval A .

⁴This correlator is formally computed in the cyclic orbifold CFT^n/Z_n .

We compute (3.2.3) analogously to [55] but with an additional composition of a map from the cylinder to the plane

$$w(x) = e^{\frac{2\pi}{\beta}x}, \quad (3.2.7)$$

to take care of the thermal nature of the original state, as well as the map

$$z(w) = \frac{(w_1 - w)w_{34}}{w_{13}(w - w_4)} \quad (3.2.8)$$

that brings the points $w_1 \rightarrow 0$, $w_2 \rightarrow z$, $w_3 \rightarrow 1$, $w_4 \rightarrow \infty$.

The transformation properties of primary operators determine the resulting trace to be⁵

$$\text{Tr} \rho_A^n(t) = \left| \frac{\beta}{\pi \varepsilon_{UV}} \sinh \left(\frac{\pi x_{23}}{\beta} \right) \right|^{-4H_\sigma} |1 - z|^{4H_\sigma} G(z, \bar{z}) \quad (3.2.9)$$

where we used the 2-pt function on the cylinder C_1

$$\langle \psi(x_1, \bar{x}_1) \psi(x_4, \bar{x}_4) \rangle_{C_1} = \left| \frac{\beta}{\pi} \sinh \left(\frac{\pi x_{14}}{\beta} \right) \right|^{-4h_\psi} \quad (3.2.10)$$

and introduced the canonical 4-point function

$$\begin{aligned} G(z, \bar{z}) &\equiv \lim_{z_4 \rightarrow \infty} |z_4|^{4h_\psi} \langle \psi(z_4, \bar{z}_4) \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \psi(0, 0) \rangle \\ &\equiv \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) | \psi \rangle \end{aligned} \quad (3.2.11)$$

defined in terms of the cross-ratio

$$z = \frac{w_{12}w_{34}}{w_{13}w_{24}}, \quad 1 - z = \frac{w_{14}w_{23}}{w_{13}w_{24}}. \quad (3.2.12)$$

We defined $w_{ij} = w_i - w_j$ in all the above formulas and the same conventions hold for x_{ij} and z_{ij} .

The Renyi entropies are computed by inserting (3.2.9) into (3.2.2)

$$S_A^{(n)} = \frac{c(n+1)}{6} \log \left(\frac{\beta}{\pi \varepsilon_{UV}} \sinh \frac{\pi L}{\beta} \right) + \frac{1}{n-1} \log(|1 - z|^{4H_\sigma} G(z, \bar{z})), \quad (3.2.13)$$

where ε_{UV} is the UV cut off of the CFT i.e. the lattice spacing. The first term is the standard Renyi entropy for an interval L in a 2d CFT at finite temperature $T = 1/\beta$; the second term captures the extra contribution due to the local operator insertion. In particular, the dependence on the conformal dimension of the local operator h_ψ is encoded in $G(z, \bar{z})$.

In general, the extra contribution to the Renyi entropies requires the knowledge of the full four-point function $G(z, \bar{z})$, i.e. the dynamical details of the particular 2d CFT under

⁵We already used the regularized twist operators so that ε_{UV} is the standard UV cut-off.

consideration. To make further progress, we consider the large c limit⁶. Notice that in the limit $n \rightarrow 1$, the twist operators $\sigma_n, \tilde{\sigma}_n$ become light

$$H_\sigma/c = \frac{1}{24} \left(n - \frac{1}{n} \right) \rightarrow 0 \quad \text{as } n \rightarrow 1 \quad (3.2.14)$$

If h_ψ/c remains fixed in the large c limit, the 4-pt function (3.2.11) becomes a 4-pt function involving two heavy and two light operators. This is precisely the set-up considered in [45, 46] to compute the dominant (vacuum) contribution to $G(z, \bar{z})$ ⁷. Using their results, one derives [55]

$$\log G(z, \bar{z}) \simeq -\frac{c(n-1)}{6} \log \left(\frac{z^{\frac{1}{2}(1-\alpha_\psi)} \bar{z}^{\frac{1}{2}(1-\bar{\alpha}_\psi)} (1-z^{\alpha_\psi}) (1-\bar{z}^{\bar{\alpha}_\psi})}{\alpha_\psi \bar{\alpha}_\psi} \right) + \mathcal{O}((n-1)^2) \quad (3.2.15)$$

where

$$\alpha_\psi = \sqrt{1 - \frac{24h_\psi}{c}}, \quad (3.2.16)$$

encodes all the dependence on the conformal dimension of the local operator h_ψ . Finally, we can compute the variation in the entanglement entropy due to the insertion of the local primary operator to be

$$\Delta S_A = \frac{c}{6} \log \left(\frac{z^{\frac{1}{2}(1-\alpha_\psi)} \bar{z}^{\frac{1}{2}(1-\bar{\alpha}_\psi)} (1-z^{\alpha_\psi}) (1-\bar{z}^{\bar{\alpha}_\psi})}{\alpha_\psi \bar{\alpha}_\psi (1-z)(1-\bar{z})} \right), \quad (3.2.17)$$

where we subtracted the entanglement entropy of the interval L at finite temperature $T = 1/\beta$

$$S_{\text{thermal}} = \frac{c}{3} \log \left(\frac{\beta}{\pi \varepsilon_{UV}} \sinh \frac{\pi L}{\beta} \right). \quad (3.2.18)$$

As explained in [59], to extract a non-trivial contribution to the entanglement entropy in the CFT at finite temperature we must take the smearing parameter ϵ small but finite. This still allows us to work with completely analytic formulas at order ϵ . Then, in the small ϵ/β limit⁸, the cross-ratios are

$$\begin{aligned} z &\simeq 1 + \frac{2\pi i \epsilon}{\beta} \frac{\sinh \frac{\pi L}{\beta}}{\sinh \frac{\pi(y+L-t_- - t_\omega)}{\beta} \sinh \frac{\pi(y-t_- - t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2) \\ \bar{z} &\simeq 1 - \frac{2\pi i \epsilon}{\beta} \frac{\sinh \frac{\pi L}{\beta}}{\sinh \frac{\pi(y+L+t_- + t_\omega)}{\beta} \sinh \frac{\pi(y+t_- + t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2) \end{aligned} \quad (3.2.19)$$

⁶We assume a class of CFTs allowing such a limit.

⁷We refer the readers to [64, 65] for the description of the regularisation of twist operators used to compute entanglement entropy.

⁸We are assuming that $y - t_- - t_\omega$ and $y + L - t_- - t_\omega$ are larger than the smearing parameter ϵ in units of β .

Due to the non-trivial monodromy properties of $G(z, \bar{z})$, we must carefully deal with the sign of the imaginary part of the cross-ratios [55, 22]⁹. Notice the imaginary part of $\bar{z}$ never changes sign, for $t_- + t_\omega \geq 0$. Thus, we conclude $\bar{z} \simeq 1$ for all such times. On the other hand, the imaginary part of z does flip sign whenever $t_- + t_\omega \in (y, y + L)$. Thus, we either have $(z, \bar{z}) \rightarrow (1, 1)$ for $t_- + t_\omega < y$ and $t_- + t_\omega > y + L$ or $(z, \bar{z}) \rightarrow (e^{2\pi i}, 1)$ for $y < t_- + t_\omega < y + L$. Using these phases in (3.2.17), we reach our first important result

$$\begin{aligned} \Delta S_A &= 0, \quad t_- + t_\omega < y \text{ and } t_- + t_\omega > y + L \\ \Delta S_A &= \frac{c}{6} \log \left[\frac{\beta}{\pi\epsilon} \frac{\sin \pi\alpha_\psi}{\alpha_\psi} \frac{\sinh\left(\frac{\pi(y+L-t_- - t_\omega)}{\beta}\right) \sinh\left(\frac{\pi(t_- + t_\omega - y)}{\beta}\right)}{\sinh\left(\frac{\pi L}{\beta}\right)} \right] \quad y < t_- + t_\omega < y + L. \end{aligned} \quad (3.2.20)$$

Thus, there is no variation in the entanglement entropy S_A either till the perturbation reaches region A ($t_- + t_\omega < y$) or when it leaves region A ($t_- + t_\omega > y + L$). While the perturbation can causally be in region A , the variation in entanglement reaches a maximum at $t_- + t_\omega = y + \frac{L}{2}$

$$(\Delta S_A)_{\max} = \frac{c}{6} \log \left[\frac{\beta}{2\pi\epsilon} \frac{\sin \pi\alpha_\psi}{\alpha_\psi} \tanh \frac{\pi L}{2\beta} \right]. \quad (3.2.21)$$

In the high temperature limit (or large interval L), the increase in entanglement due to the perturbation equals

$$\Delta S_A \simeq \frac{c}{6} \log \left[\frac{\beta}{2\pi\epsilon} \frac{\sin \pi\alpha_\psi}{\alpha_\psi} \right] \quad \beta \rightarrow 0, \quad \frac{\beta}{\epsilon} \gg 1. \quad (3.2.22)$$

In section 3.6.1 we will match this result with the gravity dual computation using the holographic entanglement entropy. Let us now proceed with the entanglement entropies that involve intervals in both CFTs.

3.3 Two-sided entropies

Entanglement entropy and mutual information in the thermofield double state involving intervals in both CFTs were discussed in detail in [66] and their time evolution in [36, 67]. Here we only briefly review this setup and extend by the insertion of a local operator to one of the CFTs.

Consider two non-interacting 2d CFTs, CFT_L and CFT_R with isomorphic Hilbert spaces $\mathcal{H}_L$ and $\mathcal{H}_R$. The thermofield double (TFD) state is a particular entangled state in the total

⁹We choose a reference phase to be consistent with causality and make entanglement entropies real and non-negative.

Hilbert space $\mathcal{H}_{tot} = \mathcal{H}_L \otimes \mathcal{H}_R$.

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\beta}{2}E_n} |n\rangle_L |n\rangle_R, \quad (3.3.1)$$

where $|n\rangle_{L,R} \in \mathcal{H}_{L,R}$ are the eigenstates in each Hilbert space and Z_β is a partition function in total Hilbert space (and also in each Hilbert space).

$$Z_\beta = \sum_n e^{-\beta E_n} \quad (3.3.2)$$

If we have the total Hamiltonian $H_{tot} = H_L + H_R$, this partition function is a partition function on a cylinder C_1 with circumference β

$$Z_\beta = \sum_{n,m} \langle n|_L \langle n|_R e^{-\frac{\beta}{2}(H_L+H_R)} |m\rangle_L |m\rangle_R = \text{Tr}_{tot} \left[e^{-\frac{\beta}{2}(H_L+H_R)} \right] \quad (3.3.3)$$

Tracing out the Hilbert space $\mathcal{H}_R$ from the pure state density matrix, we can get the thermal density matrix in CFT_L with temperature β

$$\rho_L = \text{Tr}_R |\Psi_\beta\rangle \langle \Psi_\beta| = \frac{1}{Z_\beta} \sum_n e^{-\beta E_n} |n\rangle_L \langle n|_L \quad (3.3.4)$$

A general time evolution of the TFD state is obtained by applying the evolution operator to both CFTs

$$|\Psi_\beta(t_-, t_+)\rangle = e^{-it_- H_L + it_+ H_R} |\Psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-i(t_+ - t_- - i\frac{\beta}{2})E_n} |n\rangle_L |n\rangle_R \quad (3.3.5)$$

One can immediately check that setting $t_- = t_+ = t$, that can be seen as evolving with Hamiltonian $H_L - H_R$, leaves the TFD state invariant (the symmetry of the TFD state). On the other hand, setting $t_- = -t_+ = t$ yields the time dependent state corresponding to evolution with $H_L + H_R$ as in [36]. These two configurations should be kept in mind since we leave general t_- and t_+ in our formulas so that our formalism can be used to extract the evolution after a local excitation with any of the two total Hamiltonians¹⁰.

In the TFD formalism, we can also relate one-sided and two-sided correlators by the analytical continuation of t . For example, consider the following one-sided correlator

$$\langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_L^\dagger(x_2, t) | \Psi_\beta \rangle = \sum_{n,m} e^{-\beta E_n + it(E_n - E_m)} \langle n|_L \mathcal{O}_L(x_1, 0) |m\rangle_L \langle m|_L \mathcal{O}_L^\dagger(x_2, 0) |n\rangle_L \quad (3.3.6)$$

¹⁰In fact our results also hold for evolution of the TFD state with H_L or H_R only and one can extract these formulas by setting $t_- = t$ and $t_+ = 0$ or $t_- = 0$ and $t_+ = t$ respectively.

On the other hand, we can deform the two-sided correlator as follows

$$\begin{aligned}
\langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_R(x_2, t) | \Psi_\beta \rangle &= \sum_{n, m} e^{-\frac{\beta}{2}(E_n + E_m) + it(E_n - E_m)} \langle n |_L \mathcal{O}_L(x_1, 0) | m \rangle_L \langle n |_R \mathcal{O}_R(x_2, 0) | m \rangle_R \\
&= \sum_{n, m} e^{-\beta E_n + i(t - i\frac{\beta}{2})(E_n - E_m)} \langle n |_L \mathcal{O}_L(x_1, 0) | m \rangle_L \langle m |_R \mathcal{O}_R^\dagger(x_2, 0) | n \rangle_R
\end{aligned} \tag{3.3.7}$$

Therefore, the one-sided and two-sided correlators can be related through the analytical continuation $t \rightarrow t + i\frac{\beta}{2}$

$$\langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_L(x_2, t) | \Psi_\beta \rangle = \langle \Psi_\beta | \mathcal{O}_L(x_1, 0) \mathcal{O}_R^\dagger(x_2, t - i\frac{\beta}{2}) | \Psi_\beta \rangle \tag{3.3.8}$$

If the operators $\mathcal{O}_L$ in CFT_L are located at $\tau = 0$, the operators $\mathcal{O}_R$ are located at $\tau = \frac{\beta}{2}$, or at the opposite side on the cylinder. We can express the correlators in the TFD state as the correlators on the cylinder C_1 (see Fig. 3.1).

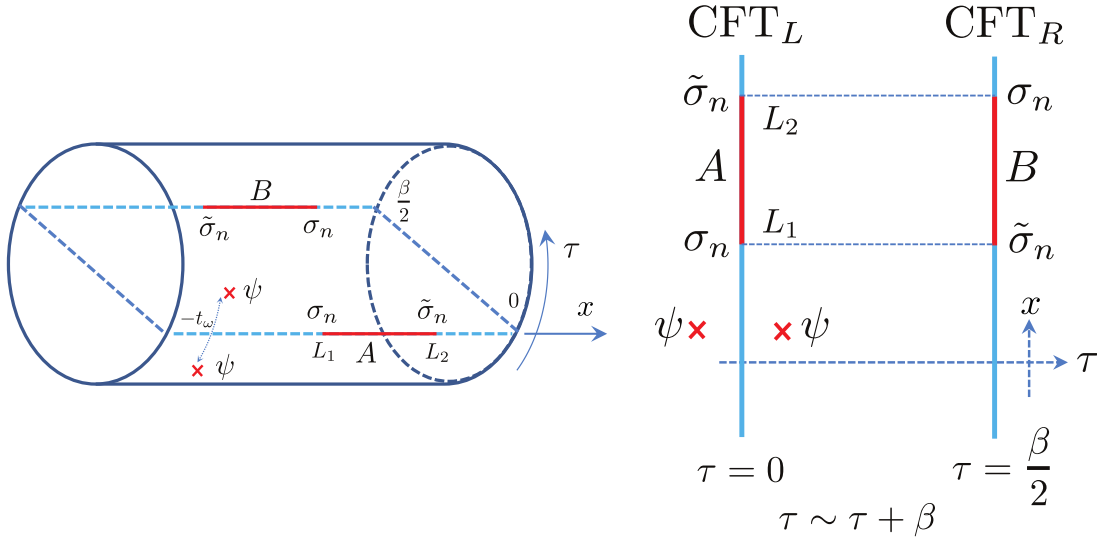


Figure 3.1: Our setup in the computation of the mutual information[20]. We have two intervals A and B of size $L_2 - L_1 = L$ in each CFT and a local operator inserted at time t_w in the past. The operators are separated by distance 2ϵ and in the CFT formulas we use $L_1 = y$ and $L_2 = y + L$.

Let us also mention a simple fact related to the symmetry of the TFD state. Namely, in the CFT, we compute the entanglement entropies as well as the mutual information in a state

$$|\tilde{\psi}\rangle = e^{-iH_L t_w} \mathcal{O}_L(x) e^{iH_L t_w} |\psi_\beta\rangle \tag{3.3.9}$$

with the TFD state $|\psi_\beta\rangle$. Since $H_L - H_R$ leaves the TFD invariant, the above state is equivalent to

$$|\tilde{\psi}\rangle = e^{-i(H_L - H_R)t_\omega} \mathcal{O}_L(x) |\psi_\beta\rangle. \quad (3.3.10)$$

Thus, the mutual information computed at $t_- = t_+ = 0$ in these two states has exactly the same functional dependence on t_ω . This can be confirmed from our explicit formulas for $I_{A:B}$ in this section.

Notice that in the previous single sided entanglement entropy calculations, we used translation invariance to write the time dependence of the operator insertions in (3.2.4) as a function of $t_- + t_\omega$. When computing two-sided observables, the same shift will be applied on the CFT time t_+ in the opposite boundary. This is consistent with the TFD path integral construction [37] and it also appears naturally in our holographic dual model.

3.3.1 Semi-infinite intervals

Before proceeding with finite entangling regions, consider A and B to be semi-infinite intervals $x \in [0, \infty)$. We want to clarify the difference between previous results for the second ($n = 2$) Renyi mutual information in this setup [59] and our current mutual information ($n = 1$) discussion. In the large central charge c limit, and after the insertion of a local operator, the second Renyi entanglement entropy of the union $S_{A \cup B}^{(2)}$ grows linearly with time. Equivalently, the change in the second Renyi mutual information for semi-infinite intervals decreases linearly with time [59]

$$\Delta I_{A \cup B}^{(2)} \simeq -\frac{8\pi h_\psi}{\beta} t. \quad (3.3.11)$$

This holds for late times in the regime where $1 \ll h_\psi \ll c$. Below, we want to compare this behavior with a large c computation of the mutual information ($n \rightarrow 1$) with twist operators and for heavy local operators $h_\psi \sim \mathcal{O}(c)$ (as in [22]).

To compute the entanglement entropy $S_{A \cup B}$ between two semi-infinite intervals A and B with starting point $L_1 = y > 0$ on each boundary CFT, we must calculate

$$\text{Tr} \rho_{A \cup B}^n(t) = \frac{\langle \Psi(x_1, \bar{x}_1) \sigma_n(x_2, \bar{x}_2) \tilde{\sigma}_n(x_6, \bar{x}_6) \Psi^\dagger(x_4, \bar{x}_4) \rangle}{(\langle \psi(x_1, \bar{x}_1) \psi^\dagger(x_4, \bar{x}_4) \rangle_{C_1})^n} \quad (3.3.12)$$

with the insertion points

$$\begin{aligned} x_1 = -i\epsilon, \quad x_2 = y - t_\omega - t_-, \quad x_6 = y + i\frac{\beta}{2} - t_+ - t_\omega, \quad x_4 = +i\epsilon \\ \bar{x}_1 = +i\epsilon, \quad \bar{x}_2 = y + t_\omega + t_-, \quad \bar{x}_6 = y - i\frac{\beta}{2} + t_+ + t_\omega, \quad \bar{x}_4 = -i\epsilon. \end{aligned} \quad (3.3.13)$$

Notice the edge of region B (the location x_6) was shifted by $i\frac{\beta}{2}$, in accordance with (3.3.8) and the dependence on t_+ is also through $t_+ + t_\omega$.

We follow the same strategy as before : after mapping the cylinder to a plane by $w = e^{\frac{2\pi}{\beta}x}$, it is the cross-ratio $z, \bar{z}$ on the plane that controls the relevant 4-pt function

$$z = \frac{w_{12}w_{64}}{w_{16}w_{24}} \simeq 1 + \frac{2\pi i\epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y - t_- - t_\omega)}{\beta} \cosh \frac{\pi(y - t_+ - t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2), \quad (3.3.14)$$

$$\bar{z} = \frac{\bar{w}_{12}\bar{w}_{64}}{\bar{w}_{16}\bar{w}_{24}} \simeq 1 - \frac{2\pi i\epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y + t_- + t_\omega)}{\beta} \cosh \frac{\pi(y + t_+ + t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2). \quad (3.3.15)$$

As before, the sign of the imaginary part of z flips for $t_\omega + t_- > y$ from positive to negative, whereas that of $\bar{z}$ is negative for all t_ω and $t_\pm$. To the first order in ϵ , we find $(z, \bar{z}) \rightarrow (1, 1)$ for $t_\omega + t_- < y$ and $(z, \bar{z}) \rightarrow (e^{2\pi i}, 1)$ for $t_\omega + t_- > y$. Using the same method as for the single sided case, we obtain the entanglement entropy $S_{A \cup B}$ for the semi-infinite intervals to be

$$S_{A \cup B} = \begin{cases} \frac{c}{3} \log \left[\frac{\beta}{\pi \epsilon_{UV}} \cosh \frac{\pi \Delta t}{\beta} \right] & (t_- + t_\omega < y) \\ \frac{c}{6} \log \left[\left(\frac{\beta}{\pi \epsilon_{UV}} \right)^2 \frac{\beta}{\pi \epsilon} \frac{\sin \pi \alpha_\psi}{\alpha_\psi} \cosh \frac{\pi \Delta t}{\beta} \sinh \frac{\pi(t_\omega + t_- - y)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y)}{\beta} \right] & (y < t_- + t_\omega) \end{cases} \quad (3.3.16)$$

where $\Delta t = t_- - t_+$. In particular, when $t_- = t_+ = 0$ and t_ω is very large ($y \ll t_\omega$), $S_{A \cup B}$ grows linearly with t_ω

$$S_{A \cup B} \sim \frac{\pi c}{3\beta} (t_\omega - y) + \frac{c}{6} \log \left[\left(\frac{\beta}{\pi \epsilon_{UV}} \right)^2 \frac{\beta}{4\pi \epsilon} \frac{\sin \pi \alpha_\psi}{\alpha_\psi} \right]. \quad (3.3.17)$$

We can find that the first term behaves like thermal entropy which is proportional to $1/\beta$. Equivalently the mutual information decreases linearly with t_ω but now with a coefficient proportional to the central charge c . As explained in [59], this behavior is interpreted as the destruction of the entanglement between CFT_L and CFT_R and the broken "entanglement bond" reconnects between the subsystem $A \cup B$ and its complement. The unit cost of this reconnection process is proportional to $\frac{\pi c}{3\beta}$ (which is the entropy density $S_{density}$).

3.3.2 Mutual information for finite intervals

In this section we compute the mutual information between finite regions A and B in opposite boundaries in the TFD state at large central charge. The setup is depicted on Fig. 3.1.

The mutual information is defined as

$$I_{A:B} = S_A + S_B - S_{A \cup B} \quad (3.3.18)$$

where $S_{A \cup B}$ stands for the entanglement entropy of the union of the two intervals. Each of the three entropies is computed using the replica trick in terms of the correlators of the local operators Ψ (3.2.5) and twist fields inserted at the endpoints of the entangling regions. Since we already computed S_A , we focus on S_B and $S_{A \cup B}$.

S_B

The calculation of the entanglement entropy S_B in the second CFT_R is analogous to the one for S_A . It involves the same type of normalized correlation function

$$\text{Tr} \rho_B^n(t) = \frac{\langle \Psi(x_1, \bar{x}_1) \sigma_n(x_5, \bar{x}_5) \tilde{\sigma}_n(x_6, \bar{x}_6) \Psi^\dagger(x_4, \bar{x}_4) \rangle}{(\langle \psi(x_1, \bar{x}_1) \psi^\dagger(x_4, \bar{x}_4) \rangle_{C_1})^n} \quad (3.3.19)$$

but with the insertion points for the twist operators conveniently shifted by $\pm i\frac{\beta}{2}$ as reviewed above

$$\begin{aligned} x_1 = -i\epsilon, \quad x_5 = y + L + i\frac{\beta}{2} - t_+ - t_w, \quad x_6 = y + i\frac{\beta}{2} - t_+ - t_w, \quad x_4 = +i\epsilon \\ \bar{x}_1 = +i\epsilon, \quad \bar{x}_5 = y + L - i\frac{\beta}{2} + t_+ + t_w, \quad \bar{x}_6 = y - i\frac{\beta}{2} + t_+ + t_w, \quad \bar{x}_4 = -i\epsilon. \end{aligned} \quad (3.3.20)$$

As stated earlier, $t_+ + t_w$ is the shifted time being used in the right CFT. To compute the 4-pt correlator, we compose the map (3.2.7) with

$$z(w) = \frac{(w_1 - w)(w_6 - w_4)}{(w_1 - w_6)(w - w_4)}. \quad (3.3.21)$$

The corresponding cross-ratios equal

$$\begin{aligned} z = z_5 \simeq 1 - \frac{2\pi i\epsilon}{\beta} \frac{\sinh \frac{\pi L}{\beta}}{\cosh \frac{\pi(y-t_+-t_w)}{\beta} \cosh \frac{\pi(y+L-t_+-t_w)}{\beta}} + \mathcal{O}(\epsilon^2), \\ \bar{z} = \bar{z}_5 \simeq 1 + \frac{2\pi i\epsilon}{\beta} \frac{\sinh \frac{\pi L}{\beta}}{\cosh \frac{\pi(y+t_++t_w)}{\beta} \cosh \frac{\pi(y+L+t_++t_w)}{\beta}} + \mathcal{O}(\epsilon^2). \end{aligned} \quad (3.3.22)$$

Notice that the signs of the imaginary parts are the same for all t_+ and t_w . Thus, in the small ϵ limit, $(z, \bar{z}) \rightarrow (1, 1)$ for all t_+ . This reflects the intuition that the local perturbation turned on on the left CFT has no effect, at lowest order in ϵ , in the quantum entanglement measured in the right CFT. Using the expansion in these cross-ratios, we derive

$$S_B = \frac{c}{3} \log \left(\frac{\beta}{\pi \varepsilon_{UV}} \sinh \frac{\pi L}{\beta} \right) \quad \forall t_+ \quad (3.3.23)$$

Thus, quantum entanglement in the region B remains thermal for all t_+ at lowest order in ϵ , i.e. $\Delta S_B = 0$.

$S_{A \cup B}$

The most interesting piece in the mutual information is $S_{A \cup B}$. Following [66], this requires the calculation of the 6-pt function

$$\text{Tr} \rho_{A \cup B}^n(t) = \frac{\langle \psi(x_1, \bar{x}_1) \sigma_n(x_2, \bar{x}_2) \tilde{\sigma}_n(x_3, \bar{x}_3) \sigma_n(x_5, \bar{x}_5) \tilde{\sigma}_n(x_6, \bar{x}_6) \psi^\dagger(x_4, \bar{x}_4) \rangle}{(\langle \psi(x_1, \bar{x}_1) \psi^\dagger(x_4, \bar{x}_4) \rangle_{C_1})^n} \quad (3.3.24)$$

where the different insertion points correspond to the different interval endpoints

$$\begin{aligned} x_1 &= -i\epsilon, & x_2 &= y - t_- - t_\omega, & x_3 &= y + L - t_- - t_\omega, & x_4 &= +i\epsilon \\ \bar{x}_1 &= +i\epsilon, & \bar{x}_2 &= y + t_- + t_\omega, & \bar{x}_3 &= y + L + t_- + t_\omega, & \bar{x}_4 &= -i\epsilon \\ x_5 &= y + L + i\frac{\beta}{2} - t_+ - t_\omega, & x_6 &= y + i\frac{\beta}{2} - t_+ - t_\omega, \\ \bar{x}_5 &= y + L - i\frac{\beta}{2} + t_+ + t_\omega, & \bar{x}_6 &= y - i\frac{\beta}{2} + t_+ + t_\omega. \end{aligned} \quad (3.3.25)$$

Following the same strategy as before, we compose the two maps

$$w(x) = e^{\frac{2\pi}{\beta}x} \quad \text{and} \quad z(w) = \frac{(w_1 - w)w_{34}}{w_{13}(w - w_4)}, \quad (3.3.26)$$

and use the transformation properties of primary operators, to write the trace (3.3.24) as

$$\text{Tr} \rho_{A \cup B}^n = \left| \frac{\beta}{\pi \varepsilon_{UV}} \sinh \left(\frac{\pi L}{\beta} \right) \right|^{-8H_\sigma} |1 - z|^{4H_\sigma} |z_{56}|^{4H_\sigma} \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle \quad (3.3.27)$$

where the cross-ratios $(z, \bar{z})$ are given in (3.2.19), and $z_i \equiv z(w_i)$.

In the following, we discuss two different CFT channels: S and T-channel, where we compute this 6-pt function on the plane in the large c limit (see a detailed discussion in [47]). The corrections to the particular channel choice are suppressed by $e^{-O(c)}$ factors. We will explicitly see how these channels match the two different bulk geodesics determining the holographic entanglement entropy in our holographic discussions. The upshot is that S and T-channel correspond to the disconnected and connected geodesics for the holographic calculation of $S_{A \cup B}$, respectively.

S-channel : Let us introduce a resolution of the identity

$$\langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle = \sum_{\alpha} \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) | \alpha \rangle \langle \alpha | \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle \quad (3.3.28)$$

where the sum runs over all possible intermediate states.

Consider the first 4-pt function $\langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) | \alpha \rangle$. As we have seen in (3.2.19), the relevant limit corresponding to $\epsilon \rightarrow 0$, is either $(z, \bar{z}) \rightarrow (1, 1)$ or $(z, \bar{z}) \rightarrow (e^{2\pi i}, 1)$. Thus, in either limit, the correlation function can be computed using the OPE of twist operators [68, 69]

$$\sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \sim \mathbb{I} + \mathcal{O}((z-1)^r) \quad r \in \mathbb{Z}^+ \quad (3.3.29)$$

Ignoring the terms proportional to $(z-1)$ and focusing in the dominant contribution due to the identity operator, we reach the important conclusion that the summation over the entire set of intermediate steps is restricted to $|\alpha\rangle = |\psi\rangle$ due to the orthogonality of 2-pt functions in any CFT. We stress that we could have reached the same conclusion in the limit of small $L_2 - L_1$, but this is *not* required in our set-up.

Thus, our 6-pt function can then be approximated by

$$\langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle \simeq \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) | \psi \rangle \langle \psi | \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle \quad (3.3.30)$$

The first 4-pt function equals $G(z, \bar{z})$ in (3.2.11), whereas the second 4-pt function factor will be proportional to the same function but evaluated at a different cross-ratio. To see this, consider the map taking $z_1 \rightarrow 0$ and $z_4 \rightarrow \infty$

$$\tilde{z}(x) = \frac{(z_1 - x)(z_6 - z_4)}{(z_1 - z_6)(x - z_4)}. \quad (3.3.31)$$

This allows us to write the desired correlator as

$$\langle \psi | \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle = |1 - \tilde{z}_5|^{4H_\sigma} |z_{56}|^{-4H_\sigma} \langle \psi | \sigma_n(\tilde{z}_5, \bar{\tilde{z}}_5) \tilde{\sigma}_n(1, 1) | \psi \rangle. \quad (3.3.32)$$

Thus, the leading contribution in this channel is

$$\text{Tr} \rho_{A \cup B}^n \simeq \left| \frac{\beta}{\pi \varepsilon_{UV}} \sinh \left(\frac{\pi L}{\beta} \right) \right|^{-8H_\sigma} |1 - z|^{4H_\sigma} |1 - \tilde{z}_5|^{4H_\sigma} G(z, \bar{z}) G(\tilde{z}_5, \bar{\tilde{z}}_5) + \dots \quad (3.3.33)$$

where the dots stand for the contributions coming from the subleading terms in the OPE of the twist operators (3.3.29). Interestingly, since the cross-ratio $\tilde{z}_5$ equals z_5 , the cross-ratio determining S_B , we reach the conclusion that

$$S_{A \cup B} = S_A + S_B, \quad \text{and} \quad I_{A:B} = 0. \quad (3.3.34)$$

This channel reproduces the bulk expectation coming from geodesics joining points in the same boundary, leading to a vanishing mutual information.

T-channel : We could also introduce the resolution of the identity as follows

$$\langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(1, 1) \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle = \sum_{\alpha} \langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(z_6, \bar{z}_6) | \alpha \rangle \langle \alpha | \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(1, 1) | \psi \rangle. \quad (3.3.35)$$

Notice the correlations involve twist operators inserted in different boundaries. Thus, we expect this channel to reproduce the bulk contribution from geodesics connecting both boundaries. Remember that in the small ϵ limit, we already argued that $z_5 \rightarrow 1$. Thus, we can use the same OPE argument as above to conclude that the dominant contribution comes from $|\alpha\rangle = |\psi\rangle$. By definition, this gives

$$\langle \psi | \sigma_n(z_5, \bar{z}_5) \tilde{\sigma}_n(1, 1) | \psi \rangle = G(z_5, \bar{z}_5) \quad (3.3.36)$$

The remaining correlation is again proportional to the same function, but evaluated at a different cross-ratio. This is proved by considering the map

$$\tilde{z}(x) = \frac{(z_1 - x)(z_6 - z_4)}{(z_1 - z_6)(x - z_4)}. \quad (3.3.37)$$

which allows us to derive

$$\langle \psi | \sigma_n(z, \bar{z}) \tilde{\sigma}_n(z_6, \bar{z}_6) | \psi \rangle = |1 - \tilde{z}_2|^{4H_\sigma} |z_{26}|^{-4H_\sigma} G(\tilde{z}_2, \bar{\tilde{z}}_2) \quad (3.3.38)$$

where $z_2 = z(w_2) = z$ as in (3.2.19) and $\tilde{z}_2 = \tilde{z}(z_2)$. Thus, after some manipulations we have

$$\text{Tr} \rho_{A \cup B}^n \simeq \left| \frac{\beta}{\pi \varepsilon_{UV}} \sinh \left(\frac{\pi L}{\beta} \right) \right|^{-8H_\sigma} \left| \frac{x}{1-x} \right|^{4H_\sigma} |1 - z_5|^{4H_\sigma} |1 - \tilde{z}_2|^{4H_\sigma} G(\tilde{z}_2, \bar{\tilde{z}}_2) G(z_5, \bar{z}_5) + \dots \quad (3.3.39)$$

where $(x, \bar{x})$ are the cross-ratios computed out of the insertion points of the four twist operators

$$x = \frac{z_{23} z_{56}}{z_{25} z_{36}} = \frac{w_{23} w_{56}}{w_{25} w_{36}} = \frac{2 \sinh^2 \frac{\pi L}{\beta}}{\cosh \frac{2\pi L}{\beta} + \cosh \frac{2\pi(t_- - t_+)}{\beta}} = \bar{x}, \quad (3.3.40)$$

what allows us to write the dominant contribution from the T-channel as

$$\text{Tr} \rho_{A \cup B}^n \simeq \left| \frac{\beta}{\pi \varepsilon_{UV}} \cosh \left(\frac{\pi \Delta t}{\beta} \right) \right|^{-8H_\sigma} |1 - \tilde{z}_2|^{4H_\sigma} G(\tilde{z}_2, \bar{\tilde{z}}_2) |1 - z_5|^{4H_\sigma} G(z_5, \bar{z}_5) + \dots, \quad (3.3.41)$$

where $\Delta t = t_- - t_+$ and the cross-ratios

$$\begin{aligned}
z_5 &= 1 - \frac{2\pi i \epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y+L-t_- - t_\omega)}{\beta} \cosh \frac{\pi(y+L-t_+ - t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2), \\
\bar{z}_5 &= 1 + \frac{2\pi i \epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y+L+t_- + t_\omega)}{\beta} \cosh \frac{\pi(y+L+t_+ + t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2), \\
\tilde{z}_2 &= 1 + \frac{2\pi i \epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y-t_- - t_\omega)}{\beta} \cosh \frac{\pi(y-t_+ - t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2), \\
\tilde{\bar{z}}_2 &= 1 - \frac{2\pi i \epsilon}{\beta} \frac{\cosh \frac{\pi(t_- - t_+)}{\beta}}{\sinh \frac{\pi(y+t_- + t_\omega)}{\beta} \cosh \frac{\pi(y+t_+ + t_\omega)}{\beta}} + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{3.3.42}$$

Now, using that at large central charge and for two heavy and two light operators we have the identity [45, 46]

$$|1 - z|^{4h} G(z, \bar{z}) \simeq \left(\frac{z^{\frac{1-\alpha}{2}} (1 - z^\alpha) \bar{z}^{\frac{1-\alpha}{2}} (1 - \bar{z}^\alpha)}{\alpha^2 (1 - z)(1 - \bar{z})} \right)^{-2h} \tag{3.3.43}$$

as well as (3.3.42) we can extract the behaviour of $S_{A \cup B}$ for any time regime. Let us analyse this carefully below assuming as before that $0 < y < y + L$.

It is clear that the monodromies of the correlator are determined depending on the relation of $t_- + t_\omega$ with y and $y + L$. From (3.3.42) the signs of the imaginary parts of $\bar{z}_5$ and $\tilde{\bar{z}}_2$ do not change with time and we have $\bar{z}_5 \simeq 1$ and $\tilde{\bar{z}}_2 \simeq 1$. On the other hand $\tilde{z}_2 \simeq e^{2\pi i}$ when $t_- + t_\omega > y$ and $z_5 \simeq e^{-2\pi i}$ when $t_- + t_\omega > y + L$. This gives us three possible contributions:

$$\begin{aligned}
S_{A \cup B} &\simeq \frac{2c}{3} \log \left| \frac{\beta}{\pi \varepsilon_{UV}} \cosh \left(\frac{\pi \Delta t}{\beta} \right) \right| \quad t_- + t_\omega < y \\
S_{A \cup B} &\simeq \frac{2c}{3} \log \left| \frac{\beta}{\pi \varepsilon_{UV}} \cosh \left(\frac{\pi \Delta t}{\beta} \right) \right| \\
&+ \frac{c}{6} \log \left(\frac{\beta}{\pi \epsilon} \frac{\sin \pi \alpha_\psi}{\alpha_\psi} \frac{\sinh \frac{\pi(t_- + t_\omega - y)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right) \quad y < t_- + t_\omega < y + L
\end{aligned} \tag{3.3.44}$$

and for $t_- + t_\omega > y + L$ we can rewrite our trace as

$$\begin{aligned}
\text{Tr} \rho_{A \cup B}^n &\simeq \left| \frac{\beta}{\pi \varepsilon_{UV}} \cosh \left(\frac{\pi \Delta t}{\beta} \right) \right|^{-4H_\sigma} \left(\frac{\beta}{\pi \epsilon} \frac{\sin \pi \alpha_\psi}{\alpha_\psi} \frac{\sinh \frac{\pi(t_- + t_\omega - y)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right)^{-2H_\sigma} \\
&\times \left| \frac{\beta}{\pi \varepsilon_{UV}} \cosh \left(\frac{\pi \Delta t}{\beta} \right) \right|^{-4H_\sigma} \left(\frac{\beta}{\pi \epsilon} \frac{\sin \pi \alpha_\psi}{\alpha_\psi} \frac{\sinh \frac{\pi(t_- + t_\omega - y - L)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y - L)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right)^{-2H_\sigma}
\end{aligned} \tag{3.3.45}$$

The entanglement entropy $S_{A \cup B}$ in this time regime can then be written as

$$S_{A \cup B} \simeq \frac{c}{6} \log \left(\frac{\sinh \frac{\pi(t_- + t_\omega - y)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y)}{\beta} \sinh \frac{\pi(t_- + t_\omega - y - L)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y - L)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta} \cosh \frac{\pi \Delta t}{\beta}} \right) + \frac{2c}{3} \log \left| \frac{\beta}{\pi \epsilon_{UV}} \cosh \left(\frac{\pi \Delta t}{\beta} \right) \right| + \frac{c}{3} \log \left(\frac{\beta \sin \pi \alpha_\psi}{\pi \epsilon \alpha_\psi} \right) \quad t_- + t_\omega > y + L \quad (3.3.46)$$

Notice that (3.3.45) resembles the contributions from two different pieces. In the holographic part, these will be interpreted as the contributions from two bulk geodesics connecting points in opposite boundaries.

3.3.3 The evolution of the Mutual information

The evolution of the mutual information after turning on the local excitation can now be computed in the large central charge limit. At early times $t_- + t_\omega < y$, the single sided entropies are thermal $S_A \simeq S_B = S_{\text{thermal}}$. Thus, the mutual information equals

$$I_{A:B}^0 \equiv \frac{2c}{3} \log \left(\frac{\sinh \frac{\pi L}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right). \quad (3.3.47)$$

This is clearly finite and depends on the Hamiltonian driving the evolution. If we use the bulk isometry $H_L - H_R$, then $t_- = t_+$ and $\Delta t = 0$, giving rise to a time independent mutual information, as it should. Notice that positivity of the mutual information in this case requires $\pi L/\beta \gtrsim 1$. Whereas for the $H_L + H_R$ Hamiltonian, we recover the mutual information time decrease discussed in [36].

These results can be understood using causality considerations : for $t_- + t_\omega < y$, the perturbation did not enter into region A and could not possibly disturb the original thermal entanglement. Once the excitation reaches region A ($y < t_- + t_\omega < y + L$), using (3.2.20), (3.3.23) and (3.3.44), the mutual information evolves as

$$I_{A:B} \simeq I_{A:B}^0 + \frac{c}{6} \log \left[\frac{\sinh \frac{\pi(y+L-t_- - t_\omega)}{\beta} \cosh \frac{\pi \Delta t}{\beta}}{\cosh \frac{\pi(t_+ + t_\omega - y)}{\beta} \sinh \frac{\pi L}{\beta}} \right] \quad (3.3.48)$$

Note how the dependence on the conformal dimension h_ψ of the perturbation cancels, between the contributions in S_A and $S_{A \cup B}$, in this regime. It would be interesting to understand the mechanism behind this large c behaviour of the mutual information.

In the last region $t_- + t_\omega > y + L > y$ the mutual information equals

$$I_{A:B} \simeq I_{A:B}^0 - \frac{c}{3} \log \left(\frac{\beta \sin \pi \alpha_\psi}{\pi \epsilon \alpha_\psi} \right) - \frac{c}{6} \log \left(\frac{\sinh \frac{\pi(t_- + t_\omega - y)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y)}{\beta} \sinh \frac{\pi(t_- + t_\omega - y - L)}{\beta} \cosh \frac{\pi(t_+ + t_\omega - y - L)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta} \cosh \frac{\pi \Delta t}{\beta}} \right) \quad (3.3.49)$$

It is important to stress that when extracting the answer for the mutual information for various times $t_{\mp}$ as well as t_{ω} , one has to maximise the mutual information between the S and the T channel answers so that it is always non-negative.

3.4 Scrambling time

Shenker and Stanford [21] defined the scrambling time t_{ω}^* as the time scale at which the perturbation has destroyed all the preexistent correlations. In our notation, their condition reduces to setting $t_- = t_+ = 0$ and to study the vanishing of the mutual information

$$I_{A:B}(t_{\omega}^*) = 0. \quad (3.4.1)$$

Evaluating our previous results (3.3.47), (3.3.48) and (3.3.49) for $t_- = t_+ = 0$, we obtain

$$I_{A:B} \simeq \frac{2c}{3} \log \sinh \frac{\pi L}{\beta}, \quad t_{\omega} < y \quad (3.4.2)$$

$$I_{A:B} \simeq \frac{c}{6} \log \left[\frac{\left(\sinh \frac{\pi L}{\beta} \right)^3 \sinh \frac{\pi(y+L-t_{\omega})}{\beta}}{\cosh \frac{\pi(t_{\omega}-y)}{\beta}} \right], \quad y < t_{\omega} < y + L \quad (3.4.3)$$

$$I_{A:B} \simeq \frac{2c}{3} \log \sinh \frac{\pi L}{\beta} - \frac{c}{3} \log \left(\frac{\beta \sin \pi \alpha_{\psi}}{\pi \epsilon} \right) - \frac{c}{6} \log \left(\frac{\sinh \frac{2\pi(t_{\omega}-y)}{\beta} \sinh \frac{2\pi(t_{\omega}-y-L)}{\beta}}{4} \right), \quad t_{\omega} > y + L \quad (3.4.4)$$

Notice the mutual information is a monotonically decreasing function of t_{ω} . Thus, starting with a positive mutual information, i.e. $\pi L/\beta \gtrsim 1$, there is a single root t_{ω}^* where (3.4.1) holds. After that, by switching channels, the mutual information remains zero.

The first question to answer is whether $t_{\omega}^* \in (y, y + L)$ or whether $t_{\omega}^* > y + L$. Clearly, the second condition can only hold if the mutual information is positive at the transition. This requirement gives rise to the constraint

$$t_{\omega}^* > y + L \Rightarrow I_{A:B}(y + L + \epsilon) > 0 \Rightarrow \frac{\beta}{\pi \epsilon} \left(\frac{\sin \pi \alpha_{\psi}}{\alpha_{\psi}} \right)^2 < \frac{\sinh^3 \frac{\pi L}{\beta}}{\cosh \frac{\pi L}{\beta}}, \quad (3.4.5)$$

where we already used $\epsilon \ll \beta$, as in our previous CFT analysis¹¹. Because of working in this region of parameter space ($\epsilon \ll \beta$), we conclude that only small perturbations ($\alpha_{\psi} \rightarrow 1$) allow scrambling time scales $t_{\omega}^* > y + L$. Since the function $\sin(\pi \alpha_{\psi})/\alpha_{\psi}$ is monotonically

¹¹If $\epsilon \ll \beta$ breaks down, then condition (3.4.5) is modified.

decreasing in α_ψ (or increasing in h_ψ), the smaller the perturbation is, the easier it is to fulfil condition (3.4.5) for generic values of L/β . Since this is the regime considered in [21], we will study the scrambling time under these circumstances¹².

For $t_\omega > y + L$ and $\Delta t = 0$, the mutual information (3.3.49) becomes

$$I_{A:B}(t_\omega) \simeq \frac{c}{6} \log \frac{\sinh^4 \frac{\pi L}{\beta} \left(\frac{\beta \sin \pi \alpha_\psi}{\pi \epsilon} \right)^{-2}}{\cosh \left(\frac{\pi(t_\omega - y)}{\beta} \right) \sinh \frac{\pi(t_\omega - y)}{\beta} \sinh \frac{\pi(t_\omega - y - L)}{\beta} \cosh \frac{\pi(t_\omega - y - L)}{\beta}} \quad (3.4.6)$$

This vanishes when

$$\left(\frac{2\pi\alpha_\psi\epsilon \sinh^2 \left(\frac{\pi L}{\beta} \right)}{\beta \sin(\pi\alpha_\psi)} \right)^2 = \sinh^2 \frac{2\pi(t_\omega^* - y)}{\beta} \cosh \frac{2\pi L}{\beta} \left(1 - \frac{\tanh \frac{2\pi L}{\beta}}{\tanh \frac{2\pi(t_\omega^* - y)}{\beta}} \right). \quad (3.4.7)$$

Notice that $t_\omega^* - y > L$ guarantees the positivity of the left hand side. Condition (3.4.7) gives rise to a quadratic equation in $\sinh^2 \frac{2\pi(t_\omega^* - y)}{\beta}$. It can be shown that there is a *unique* consistent root, in agreement with our previous arguments. In the limit $t_\omega^*/\beta \gg 1$, this root reduces the scrambling time t_ω^* to

$$t_\omega^* = y + \frac{L}{2} - \frac{\beta}{2\pi} \log \left(\frac{\beta \sin \pi \alpha_\psi}{\pi \epsilon} \right) + \frac{\beta}{\pi} \log \left(2 \sinh \frac{\pi L}{\beta} \right) \quad (3.4.8)$$

Due to the non-compactness of the 2d CFT, no recurrences were expected to be seen in our calculation. Working in the small h_ψ/c limit, as required by our analysis, then

$$\frac{\beta \sin \pi \alpha_\psi}{\pi \epsilon} \frac{1}{\alpha_\psi} \sim \frac{E_\psi}{S_{\text{density}}}, \quad \text{where} \quad S_{\text{density}} = \frac{\pi c}{3\beta}, \quad (3.4.9)$$

and $E_\psi = \frac{\pi h_\psi}{\epsilon}$ is the total energy of our local excitation given by integrating the energy density as in [54]. In this limit, the scrambling time reduces to

$$t_\omega^* = y + \frac{L}{2} + \frac{\beta}{2\pi} \log \left(\frac{\pi S_{\text{density}}}{4E_\psi} \right) + \frac{\beta}{\pi} \log \left(2 \sinh \frac{\pi L}{\beta} \right). \quad (3.4.10)$$

The $\log S$ dependence is indeed consistent with the original scrambling conjecture [32, 21].

3.5 Holographic description

In this section we compute the mutual information using the AdS_3 gravity dual of the 2d CFT set-up discussed in the previous section. The starting thermal state ρ_β in the quantum

¹²One can equally study the conditions under which $t_\omega^* \in (y, y + L)$. These generically require heavier perturbations. We do not understand this regime, which appears precisely when the mutual information (3.3.48) is α_ψ independent.

theory is described in the semiclassical approximation of the correspondence by a black hole in the gravity side, the BTZ black hole [44]. Observables involving a single Hilbert space are described by the metric outside of the event horizon, whereas observables depending on both Hilbert spaces require the Kruskal-like extension of the BTZ black hole, as described in [37].

To compute the time evolution of the holographic entanglement entropy, we approximate the local CFT perturbation with conformal dimension $\Delta (= 2h_\psi)$ by a bulk free falling massive point particle with mass $m = \Delta/R$ [53]. Next, we compute its back-reaction on the BTZ background in Kruskal coordinates using the coordinate transformation in [70]. Finally, we compute the entanglement entropy and the mutual information using the holographic prescription [2, 62].

In three dimensions, the back-reacted metric of a point particle at $r = 0$ in global coordinates is known

$$ds^2 = - (r^2 + R^2 - \mu) d\tau^2 + \frac{R^2 dr^2}{r^2 + R^2 - \mu} + r^2 d\varphi^2, \quad (3.5.1)$$

where the mass of the point particle is related to μ by $\mu = 8G_N R^2 m = \frac{24h_\psi}{c} R^2$ and R is the radius of AdS_3 . Depending on the mass of the particle, the background describes a conical singularity or a BTZ black hole.

Now, the holographic entanglement entropy is given by $\frac{R}{4G_N} = \frac{c}{6}$ times the length L_γ of geodesic γ which connects the boundaries of the subsystem A for which we define the entanglement entropy S_A . In the above metric, the entanglement entropy of the boundary region A with endpoints $(r_\infty^{(1)}, \tau_\infty^{(1)}, \varphi_\infty^{(1)})$ and $(r_\infty^{(2)}, \tau_\infty^{(2)}, \varphi_\infty^{(2)})$ is [53]

$$S_A = \frac{c}{6} \log \left[\frac{2r_\infty^{(1)} \cdot r_\infty^{(2)}}{R^2} \frac{\cos(|\Delta\tau_\infty|a) - \cos(|\Delta\varphi_\infty|a)}{a^2} \right], \quad (3.5.2)$$

where $a \equiv \sqrt{1 - \frac{\mu}{R^2}} = \alpha_\psi$ carries the information on the perturbation, as in the CFT discussion, $\Delta\tau_\infty = \tau_\infty^{(2)} - \tau_\infty^{(1)}$ and $\Delta\varphi_\infty = \varphi_\infty^{(2)} - \varphi_\infty^{(1)}$ satisfies $0 < |\Delta\varphi_\infty| < \pi$.

Mapping the static $r = 0$ geodesic to one starting at some distance ϵ from the boundary and falling into the horizon afterwards can approximate the local perturbation turned on in the boundary theory. This is precisely the approach followed in [59] to describe the time dependent evolution of entanglement entropy in the bulk for locally perturbed thermal states. To describe the evolution across the horizon, which is required to study two sided correlation functions, one must use Kruskal coordinates. This is one of the tasks we will undertake in this section.

3.5.1 Free falling particle in Kruskal coordinates

The geodesic of a free falling particle in the AdS-Schwarzschild patch of the BTZ black hole

$$ds^2 = \frac{R^2}{z^2} \left[- (1 - Mz^2) dt_-^2 + \frac{dz^2}{1 - Mz^2} + d\theta^2 \right], \quad \theta \sim \theta + 2\pi \quad (3.5.3)$$

was already computed in [59] to be

$$t_- = \tilde{\tau}, \quad \theta = 0, \quad 1 - Mz^2 = (1 - M\epsilon^2) \cosh^{-2} \left(\sqrt{M}(\tilde{\tau} + t_\omega) \right). \quad (3.5.4)$$

The only addition in the expression above is the shift $\tilde{\tau} \rightarrow \tilde{\tau} + t_\omega$ to account for the initial boundary condition $z(-t_\omega) = \epsilon$ guaranteeing the particle's energy

$$E = \frac{mR}{\epsilon} \sqrt{1 - M\epsilon^2}, \quad (3.5.5)$$

matches the energy of the CFT perturbation in the small ϵ limit¹³. We extend this result to the entire eternal black hole by working in Kruskal coordinates.

One way to achieve this goal is to map the global AdS₃ description (3.5.1) to Kruskal coordinates. A second one is to solve the geodesic equation directly in Kruskal coordinates. We check below that, as expected, both approaches agree.

Free falling particle in Kruskal coordinates : The Kruskal extension of the BTZ metric (3.5.3) is given by

$$ds^2 = R^2 \frac{-4dudv + (-1 + uv)^2 d\phi^2}{(1 + uv)^2} = R^2 \frac{-4dT^2 + 4dX^2 + (1 - T^2 + X^2)^2 d\phi^2}{(1 + T^2 - X^2)^2}, \quad (3.5.6)$$

where $u = T - X \in \mathbb{R}$, $v = T + X \in \mathbb{R}$ with their range satisfying $-1 < uv < 1$ and $\phi \sim \phi + 4\pi^2/\beta$. The conformal boundary, horizons and singularities are at $uv = -1$, $uv = 0$ and $uv = 1$, respectively, with the left and right Kruskal regions defined by

$$\begin{aligned} \text{Left: } R_- &= \{0 \leq u, -1 \leq uv \leq 0\} \\ \text{Right: } R_+ &= \{u \leq 0, -1 \leq uv \leq 0\} \end{aligned} \quad (3.5.7)$$

Both coordinate systems are related to the AdS-Schwarzschild patches via

$$\begin{aligned} u &= \pm \sqrt{\frac{z_H - z}{z_H + z}} e^{t_\mp/z_H} & v &= \mp \sqrt{\frac{z_H - z}{z_H + z}} e^{-t_\mp/z_H} \\ T &= \pm \sqrt{\frac{1 - \sqrt{M}z}{1 + \sqrt{M}z}} \sinh(\sqrt{M}t_\mp) & X &= \mp \sqrt{\frac{1 - \sqrt{M}z}{1 + \sqrt{M}z}} \cosh(\sqrt{M}t_\mp) \end{aligned} \quad (3.5.8)$$

¹³In the absence of matter fields and working in the large c limit, as we do, the satisfaction of this condition guarantees that our holographic model should capture the bulk description of the identity conformal block, which in 2d includes the stress tensor.

Using these maps (3.5.8), we can rewrite the geodesic (3.5.4) in the parametric form

$$\begin{aligned} X(\tilde{\tau}) &= -\frac{\sqrt{1-M\epsilon^2} \frac{\cosh(\sqrt{M}\tilde{\tau})}{\cosh(\sqrt{M}(\tilde{\tau}+t_\omega))}}{1 + \sqrt{1-(1-M\epsilon^2) \cosh^{-2}\left(\sqrt{M}(\tilde{\tau}+t_\omega)\right)}}, \\ T(\tilde{\tau}) &= \frac{\sqrt{1-M\epsilon^2} \frac{\sinh(\sqrt{M}\tilde{\tau})}{\cosh(\sqrt{M}(\tilde{\tau}+t_\omega))}}{1 + \sqrt{1-(1-M\epsilon^2) \cosh^{-2}\left(\sqrt{M}(\tilde{\tau}+t_\omega)\right)}}, \end{aligned} \quad (3.5.9)$$

The initial condition $(t_-, z) = (-t_\omega, \epsilon)$ is mapped to

$$(T_0, X_0) = \sqrt{\frac{1-\sqrt{M}\epsilon}{1+\sqrt{M}\epsilon}} \left(-\sinh\left(\sqrt{M}t_\omega\right), -\cosh\left(\sqrt{M}t_\omega\right) \right). \quad (3.5.10)$$

This allows us to determine $t_\omega = t_\omega(T_0, X_0)$. Similarly, $\tilde{\tau} = \tilde{\tau}(T, X)$ can be determined from (3.5.8). Altogether, we can solve for $T(X)$ as

$$T(X) = -\frac{\sinh\left(\sqrt{M}t_\omega\right)}{\sqrt{1-M\epsilon^2}} \pm \sqrt{\left(X + \frac{\cosh\left(\sqrt{M}t_\omega\right)}{\sqrt{1-M\epsilon^2}}\right)^2 - \frac{M\epsilon^2}{1-M\epsilon^2}} \quad (3.5.11)$$

Proceeding in a similar way, we can obtain the geodesic $v = v(u)$ that reduces to

$$v(u) = -\frac{a_1 u - 1}{u + a_2}, \quad (3.5.12)$$

with

$$a_1 = \frac{1 - u_0 v_0}{2u_0} = \frac{e^{\sqrt{M}t_\omega}}{\sqrt{1-M\epsilon^2}}, \quad a_2 = \frac{1 - u_0 v_0}{2v_0} = -\frac{e^{-\sqrt{M}t_\omega}}{\sqrt{1-M\epsilon^2}}. \quad (3.5.13)$$

Checking equations of motion : Consider the relativistic action for a particle of mass m moving in the background metric (3.5.6) at constant ϕ . Working in the gauge where the parameter along the curve equals u , this effective action reduces to

$$S = -2mR \int \frac{\sqrt{v'} du}{1 + uv}. \quad (3.5.14)$$

Its equation of motion

$$v''(uv + 1) - 2v'(uv' - v) = 0, \quad (3.5.15)$$

has general solution

$$v(u) = \frac{C_2 + (C_1 + C_2^2)u}{1 + C_2 u}, \quad v'(u) = \frac{C_1}{(1 + C_2 u)^2}. \quad (3.5.16)$$

This agrees with the geodesic solution (3.5.12) if the integration constants are matched as

$$C_1 = -\frac{v_0(u_0 v_0 + 1)^2}{u_0(u_0 v_0 - 1)^2} = M\epsilon^2 e^{2\sqrt{M}t_\omega}, \quad C_2 = \frac{2v_0}{1 - u_0 v_0} = -\sqrt{1 - M\epsilon^2} e^{\sqrt{M}t_\omega}. \quad (3.5.17)$$

In T, X coordinates these extended geodesics have two branches which meet at the point

$$X_m = \frac{\sqrt{M}\epsilon - \cosh(\sqrt{M}t_\omega)}{\sqrt{1 - M\epsilon^2}}, \quad T_m = -\frac{\sinh(\sqrt{M}t_\omega)}{\sqrt{1 - M\epsilon^2}}. \quad (3.5.18)$$

They cross the future and past horizons at

$$X_{h^\pm} = -\frac{1}{2}\sqrt{1 - M\epsilon^2} e^{\mp\sqrt{M}t_\omega} \quad (3.5.19)$$

and hit the past and future singularities at

$$X_{s^\pm} = \pm \sinh(\sqrt{M}t_\omega) \quad (3.5.20)$$

All these features can be seen in the sample geodesic plotted on the $T - X$ Kruskal diagram (Fig.3.2).

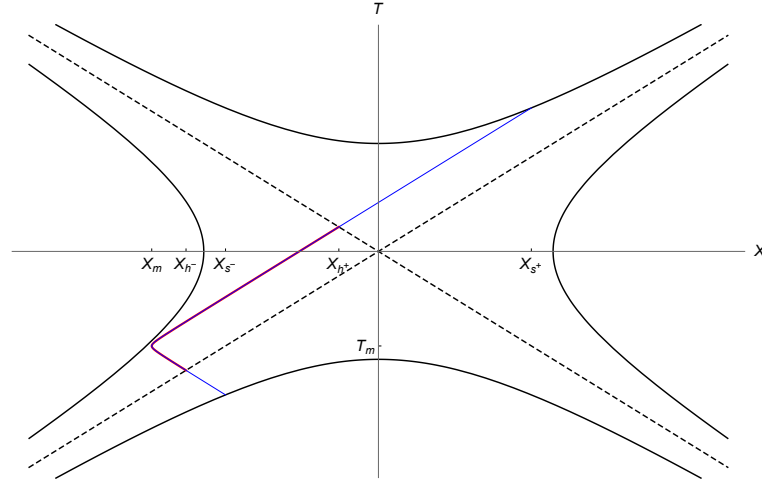


Figure 3.2: Plot shows our time-like geodesic on Kruskal diagram[20]. The red part is given by (3.5.9) and the full geodesic (blue) by (3.5.11). Plot for $M = 10$, $\epsilon = 0.01$ and $t_\omega = 0.25$

3.5.2 Back-reacted metric

The back reaction of the free falling particle in Kruskal coordinates can be obtained by following [70] and rewriting the metric (3.5.1) in Kruskal coordinates, but taking into the account the specific initial conditions discussed above. To solve this problem when $t_\omega = 0$,

in [59], we considered a boost in the plane $X_1 - X_3$ with rapidity $\lambda_2 = \lambda_2(M, \epsilon)$. In order to introduce the further parameter t_ω , we will consider a preliminary boost in $X_0 - X_3$, since this corresponds to the natural boost action on the light-like coordinates $u - v$ which captures the blue-shift of energy near the horizon stressed in [21]. In practice we then apply two particular boosts into the embedding coordinates of AdS_3 such that the identification between global and Kruskal coordinates becomes

$$\begin{aligned}
\sqrt{R^2 + r^2} \sin \tau &= \cosh \lambda_1 X_0 + \sinh \lambda_1 X_3 \\
&= R \frac{e^{\lambda_1 u} + e^{-\lambda_1 v}}{1 + uv} \\
\sqrt{R^2 + r^2} \cos \tau &= \cosh \lambda_2 X_1 - \sinh \lambda_2 (\sinh \lambda_1 X_0 + \cosh \lambda_1 X_3) \\
&= \frac{R \cosh \lambda_2 (1 - uv)}{1 + uv} \left(\cosh \phi - \tanh \lambda_2 \frac{e^{\lambda_1 u} - e^{-\lambda_1 v}}{1 - uv} \right) \\
r \sin \varphi &= X_2 = R \frac{1 - uv}{1 + uv} \sinh \phi \\
r \cos \varphi &= -\sinh \lambda_2 X_1 + \cosh \lambda_2 (\sinh \lambda_1 X_0 + \cosh \lambda_1 X_3) \\
&= \frac{R \cosh \lambda_2 (1 - uv)}{1 + uv} \left(\frac{e^{\lambda_1 u} - e^{-\lambda_1 v}}{1 - uv} - \tanh \lambda_2 \cosh \phi \right)
\end{aligned} \tag{3.5.21}$$

Solving for $r = r(u, v, \phi)$

$$r = \left| \frac{R(1 - uv) \cosh \lambda_2}{1 + uv} \right| \sqrt{\frac{\sinh^2 \phi}{\cosh^2 \lambda_2} + \left(\frac{e^{\lambda_1 u} - e^{-\lambda_1 v}}{1 - uv} - \tanh \lambda_2 \cosh \phi \right)^2} \tag{3.5.22}$$

we can determine λ_1 and λ_2 requiring that the location of the static particle in global AdS_3 , $r = 0$, gets mapped into the free falling geodesic $v(u)$ in (3.5.12). This fixes both boost parameters to be

$$\lambda_1 = \sqrt{M} t_\omega, \quad \tanh \lambda_2 = \sqrt{1 - M\epsilon^2}. \tag{3.5.23}$$

These boost parameters determine the explicit map between global AdS_3 and a free falling particle in Kruskal coordinates :

$$r = \frac{R}{\sqrt{M}\epsilon} \left| \frac{1 - uv}{1 + uv} \right| \sqrt{M\epsilon^2 \sinh^2 \phi + \left(\frac{e^{\sqrt{M} t_\omega u} - e^{-\sqrt{M} t_\omega v}}{1 - uv} - \sqrt{1 - M\epsilon^2} \cosh \phi \right)^2} \tag{3.5.24}$$

and also

$$\tan \tau = \sqrt{M}\epsilon \frac{\frac{e^{\sqrt{M} t_\omega u} + e^{-\sqrt{M} t_\omega v}}{1 - uv}}{\cosh \phi - \sqrt{1 - M\epsilon^2} \frac{e^{\sqrt{M} t_\omega u} - e^{-\sqrt{M} t_\omega v}}{1 - uv}}, \tag{3.5.25}$$

$$\tan \varphi = \sqrt{M}\epsilon \frac{\sinh \phi}{\frac{e^{\sqrt{M} t_\omega u} - e^{-\sqrt{M} t_\omega v}}{1 - uv} - \sqrt{1 - M\epsilon^2} \cosh \phi}. \tag{3.5.26}$$

Using this map, we can compute the exact back-reacted metric corresponding to a free falling particle in the eternal black hole satisfying the initial condition (u_0, v_0) .

Our analysis is valid for any value of t_ω . This allows us to compare with some approaches in the literature where the back reaction of the local perturbation in the CFT was approximated by a shock-wave, i.e. a BTZ spacetime in the presence of some non-trivial stress tensor localized at the horizon. Since our approach in 3d was based on computing the explicit backreaction of some point particle moving in some geodesic into the BTZ geometry, we can study the limit $t_\omega/\beta \gg 1$ in our geodesic analysis in subsection 5.1. In particular, figure 3.2 illustrates how our particle geodesic approaches a null geodesic on the horizon for such large t_ω . Thus, our back-reacted metric in this particular limit should indeed correspond to a shock-wave propagating in the BTZ background as originally described in [21]. The advantage of the shock-wave description is that it also applies in higher dimensions, whereas our finite t_ω results show the agreement between CFT and bulk computations also hold beyond this regime.

3.6 Bulk mutual information

The mutual information $I_{A:B}$ between regions A and B in the left and right boundaries, respectively

$$I_{A:B} = S_A + S_B - S_{A \cup B}, \quad (3.6.1)$$

can now be computed by applying (3.5.2) to the three different bulk geodesics providing the relevant minimal surface computing entanglement entropy in the bulk. All we need to know are the locations of the endpoints in the limits of small ϵ and z_∞ that we will insert into (3.5.2).

To keep the gravity formulas compact, the endpoints of the intervals will be denoted by L_i , $i \in \{1, 2\}$, where $0 < L_1 < L_2$. To compare with the CFT formulas one can substitute $L_1 = y$ and $L_2 = y + L$.

3.6.1 Geodesic in the left boundary

The two endpoints of the entanglement region A in the left boundary are $(t_-, z_1, \theta_1) = (t_-, z_\infty, L_1)$ and $(t_-, z_2, \theta_1) = (t_-, z_\infty, L_2)$. It is convenient to compute their image in global AdS_3 using the asymptotic maps to the right and left regions. Proceeding this way, their

radial coordinates in global AdS₃ satisfy

$$r^{(1)}r^{(2)} \simeq \left(\frac{R}{M\epsilon z_\infty}\right)^2 D_1 D_2 \quad (3.6.2)$$

where

$$D_i = |\cosh \sqrt{M}L_i - \cosh \sqrt{M}(t_- + t_w)| \quad i = 1, 2 \quad (3.6.3)$$

whereas the other coordinates are

$$\tan \tau^{(i)} \simeq \sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}(t_- + t_w) \right)}{\cosh \left(\sqrt{M}L_i \right) - \cosh \left(\sqrt{M}(t_- + t_w) \right)} \quad (3.6.4)$$

$$\tan \varphi^{(i)} \simeq \sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}L_i \right)}{\cosh \left(\sqrt{M}(t_- + t_w) \right) - \cosh \left(\sqrt{M}L_i \right)} \quad (3.6.5)$$

with $i = 1, 2$.

The length of the geodesics depends on the value of the time argument $t_- + t_w$. When $t_- + t_w < L_1 < L_2$, then the boundary points equal

$$\tau^{(i)} \simeq \sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}(t_- + t_w) \right)}{D_i}, \quad \varphi^{(i)} \simeq \pi - \sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}L_i \right)}{D_i}. \quad (3.6.6)$$

These determine the coordinate intervals to be

$$\begin{aligned} |\Delta\tau| &\simeq \frac{\sqrt{M}\epsilon}{D_1 D_2} |D_2 - D_1| \sinh \sqrt{M}(t_- + t_w), \\ |\Delta\varphi| &\simeq \frac{\sqrt{M}\epsilon}{D_1 D_2} \left| D_1 \sinh \sqrt{M}L_2 - D_2 \sinh \sqrt{M}L_1 \right|. \end{aligned} \quad (3.6.7)$$

Due to the identity

$$D_1 D_2 (|\Delta\varphi|^2 - |\Delta\tau|^2) = 4M\epsilon^2 \sinh^2 \frac{\pi\Delta L}{\beta}, \quad (3.6.8)$$

the geodesic length is

$$\begin{aligned} L_\gamma &\simeq \log \left[\frac{2r^{(1)}r^{(2)}}{R^2} \frac{\cos(a|\Delta\tau|) - \cos(a|\Delta\varphi|)}{a^2} \right] \simeq \log \left[\frac{r^{(1)}r^{(2)}}{R^2} (|\Delta\varphi|^2 - |\Delta\tau|^2) \right] \\ &\simeq 2 \log \left(\frac{\beta}{\pi z_\infty} \sinh \frac{\pi L}{\beta} \right). \end{aligned} \quad (3.6.9)$$

This reproduces the thermal entanglement entropy (3.2.18) computed in the CFT in the same time interval once both UV cut-offs are identified $\epsilon_{UV} = z_\infty$.

Similarly, when $t_- + t_w > L_2 > L_1$, the boundary points equal

$$\tau^{(i)} \simeq \pi - \sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}(t_- + t_w) \right)}{D_i}, \quad \varphi^i \simeq \sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}L_i \right)}{D_i}. \quad (3.6.10)$$

These are different from (3.6.6), but give rise to the same intervals (3.6.7). Thus, the length of the bulk geodesic joining them equals (3.6.9). This matches our CFT again.

Finally, when $L_2 > t_- + t_\omega > L_1$, the boundary points are

$$\begin{aligned}\tau^{(1)} &\simeq \pi - \sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_- + t_\omega)}{D_1}, & \varphi^{(1)} &\simeq \sqrt{M}\epsilon \frac{\sinh \sqrt{M}L_1}{D_1}, \\ \tau^{(2)} &\simeq \sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_- + t_\omega)}{D_2}, & \varphi^{(2)} &\simeq \pi - \sqrt{M}\epsilon \frac{\sinh \sqrt{M}L_2}{D_2}.\end{aligned}\tag{3.6.11}$$

From them we can easily get the absolute values of the intervals

$$\begin{aligned}|\Delta\tau| &\simeq \pi - \frac{\sqrt{M}\epsilon}{D_1 D_2} (D_1 + D_2) \sinh \sqrt{M}(t_- + t_\omega), \\ |\Delta\varphi| &\simeq \pi - \frac{\sqrt{M}\epsilon}{D_1 D_2} \left(D_1 \sinh \sqrt{M}L_2 + D_2 \sinh \sqrt{M}L_1 \right).\end{aligned}\tag{3.6.12}$$

Notice that in the small ϵ limit we are working on, $|\Delta\tau|$ and $|\Delta\varphi|$ are close to each other

$$\delta = |\Delta\varphi| - |\Delta\tau| = \frac{\sqrt{M}\epsilon}{D_1 D_2} \left[(D_1 + D_2) \sinh \sqrt{M}(t_- + t_\omega) - D_1 \sinh \sqrt{M}L_2 - D_2 \sinh \sqrt{M}L_1 \right].\tag{3.6.13}$$

This allows us to write the length of the bulk geodesic between these two boundary points as

$$\begin{aligned}L_\gamma &\simeq \log \left[\frac{2r^{(1)}r^{(2)}}{R^2} \frac{\cos(a|\Delta\tau|) - \cos(a|\Delta\varphi|)}{a^2} \right] \simeq \log \left[\frac{2r^{(1)}r^{(2)}}{R^2} \frac{\sin \pi a}{a} \delta \right] \\ &\simeq \log \left[\left(\frac{\beta}{\pi z_\infty} \sinh \frac{\pi \Delta L}{\beta} \right)^2 \frac{\beta}{\pi \epsilon} \frac{\sin \pi a}{a} \frac{\sinh \frac{\pi(t_\omega + t_- - L_1)}{\beta} \sinh \frac{\pi(L_2 - t_\omega - t_-)}{\beta}}{\sinh \frac{\pi \Delta L}{\beta}} \right].\end{aligned}\tag{3.6.14}$$

where $\Delta L = L_2 - L_1$. This also perfectly matches our CFT result (3.2.20) after employing the Ryu-Takayanagi formula.

3.6.2 Geodesic in the right boundary

The two endpoints of the entanglement region B in the right boundary are $(t_+, z_1, \theta_1) = (t_+, z_\infty, L_1)$ and $(t_+, z_2, \theta_1) = (t_+, z_\infty, L_2)$. Their radial coordinates satisfy

$$r^{(1)}r^{(2)} \simeq \left(\frac{R}{M\epsilon z_\infty} \right)^2 D_1 D_2\tag{3.6.15}$$

where

$$D_i = |\cosh \sqrt{M}L_i + \cosh \sqrt{M}(t_+ + t_\omega)| \quad i = 1, 2\tag{3.6.16}$$

whereas the other coordinates are

$$\tan \tau^{(i)} \simeq -\sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}(t_+ + t_\omega) \right)}{D_i}, \quad \tan \varphi^{(i)} \simeq -\sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}L_i \right)}{D_i} \quad (3.6.17)$$

In this case, no matter what the value of t_+ is, the boundary points are identified as

$$\tau^{(i)} \simeq -\sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}(t_+ + t_\omega) \right)}{D_i}, \quad \varphi^{(i)} \simeq \pi - \sqrt{M}\epsilon \frac{\sinh \left(\sqrt{M}L_i \right)}{D_i}. \quad (3.6.18)$$

These give rise to the intervals

$$\begin{aligned} |\Delta\tau| &\simeq \frac{\sqrt{M}\epsilon}{D_1 D_2} |D_1 - D_2| \sinh \sqrt{M}(t_+ + t_\omega), \\ |\Delta\varphi| &\simeq \frac{\sqrt{M}\epsilon}{D_1 D_2} \left| D_1 \sinh \sqrt{M}L_2 - D_2 \sinh \sqrt{M}L_1 \right|. \end{aligned} \quad (3.6.19)$$

Using the identity

$$D_1 D_2 (|\Delta\varphi|^2 - |\Delta\tau|^2) = 4M\epsilon^2 \sinh^2 \frac{\pi\Delta L}{\beta}, \quad (3.6.20)$$

the geodesic length equals

$$\begin{aligned} L_\gamma &\simeq \log \left[\frac{2r^{(1)}r^{(2)}}{R^2} \frac{\cos(a|\Delta\tau|) - \cos(a|\Delta\varphi|)}{a^2} \right] \simeq \log \left[\frac{r^{(1)}r^{(2)}}{R^2} (|\Delta\varphi|^2 - |\Delta\tau|^2) \right] \\ &\simeq 2 \log \left(\frac{\beta}{\pi z_\infty} \sinh \frac{\pi\Delta L}{\beta} \right). \end{aligned} \quad (3.6.21)$$

This reproduces the well-known thermal answer obtained in the CFT [64]

$$S_B \simeq \frac{c}{3} \log \left(\frac{\beta}{\pi z_\infty} \sinh \frac{\pi\Delta L}{\beta} \right) = S_{\text{thermal}}, \quad (3.6.22)$$

which also agrees with the CFT expression for S_B in (3.3.23).

3.6.3 Geodesics across the horizon and Mutual Information

We want to compute the geodesic length between two opposite boundary points located at the same space like location but with different time labels $t_\mp$. We will describe the calculation once and apply it to the two cases of interest afterwards. The product of the radial coordinates equals

$$r^{(1)}r^{(2)} \simeq \left(\frac{R}{M\epsilon z_\infty} \right)^2 D_1 D_2 \quad (3.6.23)$$

where

$$D_1 = |\cosh \sqrt{M}L_i - \cosh \sqrt{M}(t_- + t_\omega)| \quad (3.6.24)$$

$$D_2 = |\cosh \sqrt{M}L_i + \cosh \sqrt{M}(t_+ + t_\omega)| \quad (3.6.25)$$

where L_i labels the space like location in both boundaries, i.e. either L_1 or L_2 . The other coordinates for the left boundary point are

$$\tan \tau^{(1)} \simeq \sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_- + t_\omega)}{\cosh \sqrt{M}L_i - \cosh \sqrt{M}(t_- + t_\omega)} \quad (3.6.26)$$

$$\tan \varphi^{(1)} \simeq \sqrt{M}\epsilon \frac{\sinh \sqrt{M}L_i}{\cosh \sqrt{M}(t_- + t_\omega) - \cosh \sqrt{M}L_i}. \quad (3.6.27)$$

At early times, $L_i > t_\omega$, these are given by

$$\tau^{(1)} \simeq \sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_- + t_\omega)}{\cosh \sqrt{M}L_i - \cosh \sqrt{M}(t_- + t_\omega)} \quad (3.6.28)$$

$$\varphi^{(1)} \simeq \pi - \sqrt{M}\epsilon \frac{\sinh \sqrt{M}L_i}{\cosh \sqrt{M}L_i - \cosh \sqrt{M}(t_- + t_\omega)} \quad (3.6.29)$$

whereas at late times,

$$\tau^{(1)} \simeq \pi - \sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_- + t_\omega)}{\cosh \sqrt{M}(t_- + t_\omega) - \cosh \sqrt{M}L_1} \quad (3.6.30)$$

$$\varphi^{(1)} \simeq \sqrt{M}\epsilon \frac{\sinh \sqrt{M}L_1}{\cosh \sqrt{M}(t_- + t_\omega) - \cosh \sqrt{M}L_1} \quad (3.6.31)$$

The remaining coordinates for the right boundary point are

$$\tan \tau^{(2)} \simeq -\sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_+ + t_\omega)}{\cosh \sqrt{M}L_i + \cosh \sqrt{M}(t_+ + t_\omega)} \quad (3.6.32)$$

$$\tan \varphi^{(2)} \simeq -\sqrt{M}\epsilon \frac{\sinh \sqrt{M}L_i}{\cosh \sqrt{M}(t_+ + t_\omega) + \cosh \sqrt{M}L_i} \quad (3.6.33)$$

In this case, they are always given by

$$\tau^{(2)} \simeq -\sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_+ + t_\omega)}{\cosh \sqrt{M}L_i + \cosh \sqrt{M}(t_+ + t_\omega)} = -\sqrt{M}\epsilon \frac{\sinh \sqrt{M}(t_+ + t_\omega)}{D_2} \quad (3.6.34)$$

$$\varphi^{(2)} \simeq \pi - \sqrt{M}\epsilon \frac{\sinh \sqrt{M}L_i}{D_2} \quad (3.6.35)$$

Let us compute the length of the geodesic in the early time regime $L_i > t_\omega$. In this case, the interval differences are

$$\begin{aligned} |\Delta\tau| &= |\tau^{(1)} - \tau^{(2)}| \simeq \frac{\sqrt{M}\epsilon}{D_1 D_2} \left| D_2 \sinh \sqrt{M}(t_- + t_\omega) + D_1 \sinh \sqrt{M}(t_+ + t_\omega) \right|, \\ |\Delta\varphi| &= |\varphi^{(1)} - \varphi^{(2)}| \simeq \frac{\sqrt{M}\epsilon}{D_1 D_2} \left| D_2 \sinh \sqrt{M}L_i - D_1 \sinh \sqrt{M}L_i \right| \end{aligned} \quad (3.6.36)$$

Plugging this into the geodesic length (3.5.2), we obtain

$$\begin{aligned} L_\gamma &\simeq \log \left[\frac{2r^{(1)}r^{(2)}}{R^2} \frac{\cos(a|\Delta\tau|) - \cos(a|\Delta\varphi|)}{a^2} \right] \simeq \log \left[\frac{r^{(1)}r^{(2)}}{R^2} (|\Delta\varphi|^2 - |\Delta\tau|^2) \right] \\ &\simeq 2 \log \left[\frac{\beta}{\pi z_\infty} \cosh \frac{\pi \Delta t}{\beta} \right] \end{aligned} \quad (3.6.37)$$

In the late time regime, the interval differences equal

$$|\Delta\tau| \simeq \pi - \frac{\sqrt{M}\epsilon}{D_1 D_2} \left(D_2 \sinh \sqrt{M}(t_- + t_\omega) - D_1 \sinh \sqrt{M}(t_+ + t_\omega) \right) \quad (3.6.38)$$

$$|\Delta\varphi| \simeq \pi - \frac{\sqrt{M}\epsilon}{D_1 D_2} \left(D_1 \sinh \sqrt{M}L_i + D_2 \sinh \sqrt{M}L_i \right). \quad (3.6.39)$$

Since they are very close, we have

$$|\Delta\tau| \simeq |\Delta\varphi| - \delta \quad (3.6.40)$$

where

$$\delta \simeq \frac{\sqrt{M}\epsilon}{D_1 D_2} \left[D_2 (\sinh \sqrt{M}(t_- + t_\omega) - \sinh \sqrt{M}L_i) - D_1 (\sinh \sqrt{M}(t_+ + t_\omega) + \sinh \sqrt{M}L_i) \right] \quad (3.6.41)$$

This allows to write the geodesic length as

$$\begin{aligned} L_\gamma &\simeq \log \left[\frac{2r^{(1)}r^{(2)}}{R^2} \frac{\cos(a|\Delta\tau|) - \cos(a|\Delta\varphi|)}{a^2} \right] \simeq \log \left[\frac{2r^{(1)}r^{(2)}}{R^2} \frac{\sin \pi a}{a} \delta \right] \\ &\simeq \log \left[\frac{\beta^{2\frac{1}{2}} \left(1 + \cosh \frac{2\pi \Delta t}{\beta} \right)}{\pi^2 z_\infty^2} \frac{2}{\sqrt{M}\epsilon} \frac{\sin \pi a}{a} \frac{\sinh \frac{\pi(t_- + t_\omega - L_i)}{\beta} \cosh \frac{\pi(L_i - t_+ - t_\omega)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right] \end{aligned} \quad (3.6.42)$$

where $\Delta t = t_- - t_+$. These geodesics can now be used to compute the entanglement entropy of the union $S_{A \cup B}$.

In particular we will be interested in large $t_\omega > L_2 > L_1$ when $S_A = S_B = S_{\text{thermal}}$. In this case, there is a competition between the two geodesics connecting points in opposite boundaries and the geodesics connecting points in the same boundary giving rise to $2S_B$. The length of the new geodesics is

$$L_\gamma^1 \simeq \log \left[\left(\frac{\beta \cosh \frac{\pi \Delta t}{\beta}}{\pi z_\infty} \right)^2 \frac{\beta}{\pi \epsilon} \frac{\sin \pi a}{a} \frac{\sinh \frac{\pi(t_- + t_\omega - L_1)}{\beta} \cosh \frac{\pi(L_1 - t_+ - t_\omega)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right] \quad (3.6.43)$$

$$L_\gamma^2 \simeq \log \left[\left(\frac{\beta \cosh \frac{\pi \Delta t}{\beta}}{\pi z_\infty} \right)^2 \frac{\beta}{\pi \epsilon} \frac{\sin \pi a}{a} \frac{\sinh \frac{\pi(t_- + t_\omega - L_2)}{\beta} \cosh \frac{\pi(L_2 - t_+ - t_\omega)}{\beta}}{\cosh \frac{\pi \Delta t}{\beta}} \right] \quad (3.6.44)$$

where again $\Delta t = t_- - t_+$.

Summarizing, the holographic entanglement entropy of the union of two intervals on the left and the right boundary is given by

$$S_{A \cup B} \simeq \frac{c}{6} (L_\gamma^1 + L_\gamma^2), \quad (3.6.45)$$

which matches with the CFT result (3.3.46).

Finally from the above holographic results, we obtain the holographic mutual information $I_{A:B} = S_A + S_B - S_{A \cup B}$ and this again reproduces the CFT result (3.3.49) perfectly. As a consequence the scrambling time derived in the CFT (3.4.8) also holds as a result in gravity replacing $\alpha_\psi \rightarrow a$, as stressed below (3.5.2).

Chapter 4

Complexity of States

Quantum states have many information not only (bi-partite) entanglement. Complexity of states characterizes more fine-grained information than entanglement entropy. In recent context, it has been featured as a diagnostic tool for late time dynamics of scrambled quantum system and black holes in dual gravity. In this chapter, we review some properties about complexity of states in such systems, the holographic complexity conjectures and attempts for the QFT definition of the complexity.

4.1 Complexity of states

(Computational) Complexity is a concept coming from computer science. It is the difficulty of carrying out a task. The ingredients to define the complexity are a system, a space of states, a concept of a simple state, a concept of simple operations and a task. Typically, a task is represented as a sequence of simple operations on a simple state to obtain a state. The complexity is defined as the minimal number of simple operations required to carry out the task. Computer scientists discuss about the scaling of the complexity in the system size. If it scales as polynomial, they say computationally efficient (logarithmic). If it scales as exponential, performing the task is computationally quite hard.

We will review some properties about the complexity which is useful for latter discussions.

4.1.1 Some Properties of Complexity of states

Complexity of a state is defined as a minimal number of simple operations to construct a state from a simple reference state.

As a simple example, let us consider a n -coin system. If we assign 0 to the head and 1 to

the tale, the states are represented by sequences of binary digits (classical bits) with length n . For example, $0110 \cdots 01$. We can represent 2^n states in this system. The simplest state is $000 \cdots 00$ or $111 \cdots 11$. The most complicated states are states with half of coins flipped from the reference state. So the maximal complexity C_{max} is $\frac{n}{2}$. And the maximal (Shannon) entropy S_{max} is $\frac{n}{2} \log 2$ given by the uniform probability distribution ($p_j = 2^{-n}$). They are same order in the system size n .

Next, let us consider its quantum version (n -qubit system). We can superpose states. The states can be represented by unit vectors in the 2^n dimensional Hilbert space.

$$|\Psi\rangle = \sum_{i_1, \dots, i_n=0,1} C_{i_1 \dots i_n} |i_1, \dots, i_n\rangle = \sum_{j=1}^{2^n} \alpha_j |j\rangle. \quad (4.1.1)$$

Here we rename the states like $|1\rangle = |000 \cdots 00\rangle$ and normalize the states as $\sum_j \alpha_j = 1$. Instead of n -binary digits, the states are characterized by 2^n complex numbers. It makes huge difference between the classical and quantum cases. In the quantum case, we consider unitary two-qubit operations not only one-qubit operations as simple operations. This is because one-qubit operations does not affect entanglement. If we start from a simple reference state, for example $|000 \cdots 00\rangle$, we cannot make superposed states by using one-qubit operations. We need one- and two-qubit operations. They are called quantum gates in quantum information theory. Acting sequences of the quantum gates, called quantum circuits, on a reference state we can construct all quantum states in the qubit system. The complexity of the state is defined as the minimal number of gates or the minimal circuit depth .

The maximal (von-Neumann) entropy S_{max} is $\frac{n}{2} \log 2$ as similar as that in the classical case (linear or polynomial scaling in n). This is realized by a maximal entangled state $\alpha_j = 1$. On the other hand, it turns out that the maximal complexity C_{max} scales exponential $\sim 2^n \sim e^n$ in the system size n . This exponential scaling means that simulating such highly complex states is computationally hard. Unlike the classical case, the scaling of S_{max} and C_{max} are quite different.

The thermalization time t_{them} is defined as a time when the entropy is maximized. Similarly we can define the complexity time t_{comp} when the complexity is maximized. Classically these times are not so different. Their scalings are bounded by polynomial in n , $t_{them} \sim t_{comp} < n^\#$. In contrast, in the quantum case, the states can be much complicated and the complexity can keep growing after the thermalization time $t_{them} < n^p$. The complexity time t_{comp} scales as exponential $\sim e^n$.

In fact, there is much longer time scale, the Poincare recurrence time t_{rec} when the state goes back to the initial state. In the classical case, the largest scaling of t_{rec} is exponential

$\sim e^n$ because there are 2^n possible states¹. In the quantum case, naively more horrible, the scaling may be doubly exponential $\sim e^{e^n}$. It seems out of reach of our study. If we can reach, that's really impressive.

	n -bits	n -qubits
S_{max}	$\frac{n}{2} \log 2$	$\frac{n}{2} \log 2$
C_{max}	$\frac{n}{2} \sim S_{max}$	$e^n \sim e^{S_{max}}$
t_{therm}	$< n^\#$	$< n^\#$
t_{comp}		e^n

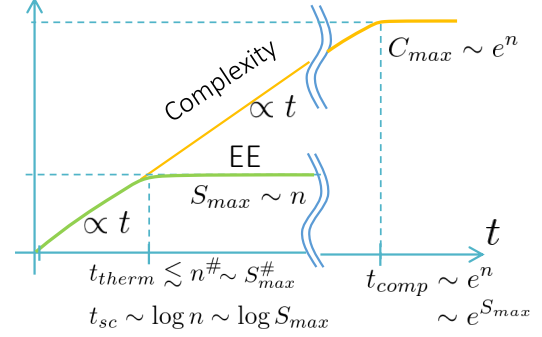


Table 4.1: n -bits system v.s. n -qubit system. Figure 4.1: Time scales in (strongly) chaotic quantum systems.

A rough estimation of the exponential scaling for generic states

Let us try to approximate a unitary transformation on n qubits by using g types of unitary two-qubit gates². Suppose each gate works on at most f qubits. In a quantum circuit with m gates starting from a reference state $|000 \cdots 00\rangle$, at most $({}_nC_f)^{gm} = O(n^{fgm})$ different states may be realizable.

On the other hand, we can estimate the number of all possible states with tolerance ϵ by counting the number of patches required to cover the state space. The state space is a $2^{n+1} - 1$ dimensional unit sphere and the patches are $2^{n+1} - 2$ dimensional sphere with radius ϵ . The number of patches needed to cover the state space is lower bounded by

$$\frac{S_{2^{n+1}-1}(1)}{V_{2^{n+1}-2}(\epsilon)} = \frac{\sqrt{\pi}(2^{n+1}-1)\Gamma(2^n - \frac{1}{2})}{\Gamma(2^n)} \cdot \epsilon^{-2^{n+1}+1} \geq \sqrt{\frac{\pi}{2}} \epsilon^{-2^{n+1}+1}. \quad (4.1.2)$$

Here we used $\Gamma(2^n) \geq \Gamma(2^n)/2^n$. Therefore, we can estimate the scaling of the number of gates.

$$m = \frac{2^n \log(\frac{1}{\epsilon})}{\log n}. \quad (4.1.3)$$

The Llyod bound

Is there any speed limits to process computational tasks? The question can be access by

¹Of course, in integrable systems, t_{rec} is much shorter but now we consider (a kind of) chaotic systems. In this case, states will return after going around all possible states ultimately.

²This estimation is from a famous quantum information textbook by Nielsen and Chuang [71].

studying the bounds on the growth rate of the complexity $\frac{dC}{dt}$ in time (step). Here we discuss the bounds controlled by quantum physics.

Consider a time evolution of a state in a two state system.

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{Et}{\hbar}}|E\rangle), \quad |\langle\psi(t)|\psi(0)\rangle| = \cos \frac{tE}{2\hbar} \quad (4.1.4)$$

The minimal time t_{ortho} to reach an orthogonal state is

$$t_{ortho} = \frac{\pi\hbar}{2E}. \quad (4.1.5)$$

For the orthogonality time t_{ortho} , two bounds have been proven. One is the Aharonov-Anandan-Bohm bound [72, 73]

$$t_{ortho} \geq \frac{\pi\hbar}{2\Delta E}, \quad (4.1.6)$$

where ΔE is the standard deviation of the energy. Another one is the Margolus-Levitin bound [74]

$$t_{ortho} \geq \frac{\pi\hbar}{2\langle E \rangle}, \quad (4.1.7)$$

where $\langle E \rangle$ is the expectation value of the energy exceeded from the ground state energy. These bounds are saturated by the two state system 4.1.4. Their relation differs by states we consider.

Referring to these bounds, Llyod made a conjecture of a bound on the rate of computation. A crucial assumption is that all gates can be implemented quantum mechanically but the input and output at any steps are classical states because his interest was classical computers. All classical states are orthogonal and have no superpositions with each other. To process an initial state $|0\rangle$ to an final state $|\Psi\rangle$, we suppose to act some quantum gates implemented by a time-evolution operator $e^{i\Delta t H}$. At each step to operate gates takes some time Δt . The minimal time to perform the task is controlled by the energy $E = \langle H \rangle$ as the above bounds imply.

$$\Delta t \geq \frac{\pi\hbar}{2E}. \quad (4.1.8)$$

The number of operations per unit time step is the computation speed for the task or the time rate of change of the complexity $C(|\Psi(t)\rangle)$ to construct the state $|\Psi\rangle$. Thus, it is upper bounded by

$$\frac{dC(|\Psi(t)\rangle)}{dt} \leq \frac{2E_\Psi}{\pi\hbar}, \quad (4.1.9)$$

up to the numerical coefficient depending on details of quantum gates. The computational speed is bounded from above by the instantaneous energy in unit time step.

This is the bound of the grow rate of the complexity discussed in the context of AdS/CFT initiated by Brown-Roberts-Susskind-Swingle-Zhao[75, 76]. Of course, for quantum computers, the input and output states can be quantum or superposed. Then, we don't need to wait until the orthogonal time. In general situations, there is no reason why we believe the Lloyd bound holds. But the Lloyd bound can be an useful tool to diagnose some features of quantum systems ³.

4.2 Holographic Dual of Complexity?

4.2.1 Entanglement (Entropy) Is Not Enough

As suggested by the Ryu-Takayanagi formula, entanglement in quantum field theories is closely related to spacetime in the dual gravitational theories. Also for their dynamical aspects we have checked many correspondences. Entanglement entropy seems quite useful to probe dual gravitational spacetime and the dynamics. However, it is not true that EE can probe any dual spacetimes. For example, as we can learn from Hartman-Maldacena [36] and Shenker-Stanford [21], thermalization or scrambling prevents us from investigating the interior of black holes by using holographic entanglement entropy. After scrambled or thermalized, the EE will be saturated and the holographic spacetime and the dynamics becomes almost invisible from the entropic point of view.

But, of course, this is not the end of the story about dynamics in (strongly coupled or chaotic) quantum systems and the dual gravity. The quantum state still keeps evolving. Some information or measures should know about the late time dynamics. From this viewpoint, Susskind said "Entanglement (Entropy) Is Not Enough" [78].

How do we know about the dynamics after scrambling time? An interesting suggestion comes from the dynamics of the eternal AdS-Schwarzschild black hole [78]. In the black hole, space-like surfaces connecting the two AdS boundaries can probe the interior of the black hole which is called the wormhole or the Einstein-Rosen (ER) bridge of the black hole. An important observation is that the wormhole keeps growing after the thermalization time, whereas the EE is saturated and the codimension-two minimal surfaces contributing to the holographic EE become disconnected. The size of the wormhole may give some hints about the late time dynamics of the black hole.

How do we measure the size of the wormhole? Let us try to measure the spatial volume of the wormhole (codimension-one surface). To fix the target uniquely as possible, we take the

³For some discussions about the Lloyd bounds and (holographic) complexity, see [77] for example.

maximal volume ending on the boundary time slices at (t_L, t_R) . Actually this slice is a "nice slice" which avoids the singularities. For large temperature or late time ($|t_L + t_R| \gg \beta$), the maximal volume grows linearly⁴

$$V(t_L, t_R) = \frac{8\pi G_N R_{AdS}}{D-2} M_{BH} |t_L + t_R| + O(1), \quad (4.2.1)$$

up to some time independent terms. This means the growth rate of the volume of the wormhole is time independent.

$$\frac{dV}{dt} \sim M_{BH} \sim TS_{BH}. \quad (4.2.2)$$

where $t = t_L + t_R$. It is proportional to the mass of the black hole or the (thermo-dynamical) energy (= the temperature $\times$ the entropy). With proper normalization, it seems to be the saturating value of the Lloyd bound of the growth rate of the complexity.

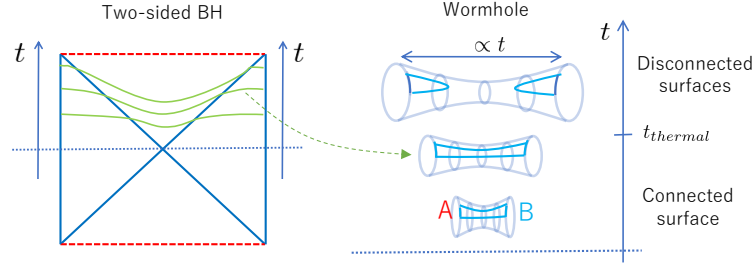


Figure 4.2: The EE saturates at the thermalization time but the wormhole keeps growing.

4.2.2 Complexity = Volume Conjecture

Motivated by the wormhole growth at late time, the "Complexity = Volume" (CV) conjecture was proposed [79, 80, 78].

$$C_V(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \frac{Vol(\mathcal{B})}{G_N R_{AdS}}, \quad (4.2.3)$$

where $\mathcal{B}$ is the maximal volume codimension-one surface anchored on the boundary time slice Σ where the state in the boundary theory is realized. Starting from two-sided and one-sided AdS black holes and (multiple) shock-wave geometries, the CV conjecture has been studied for many spacetimes.

We would like to mention another quantum informational measure which is conjectured to be dual to the maximal volume of the codimension-one surface. That is the quantum

⁴For low temperature or early time ($|t_L + t_R| \ll \beta$), the growth is quadratic.

information metric (or the Bures metric) between two quantum states close each other [17, 81]. It is just a second order deviation of an inner product of two states generalized for mixed states. For one parameter family of quantum states $|\Psi(\lambda)\rangle$, the information metric $G_{\lambda,\lambda}$ is defined as the second order expansion coefficient of the inner product with the perturbed states $|\Psi(\lambda + \delta\lambda)\rangle$.

$$|\langle\Psi(\lambda)|\Psi(\lambda + \delta\lambda)\rangle| = 1 - G_{\lambda,\lambda} \cdot (\delta\lambda)^2 + O((\delta\lambda)^3). \quad (4.2.4)$$

Unlike the complexity, in the quantum field theory side, this quantity and the computational method are well-defined.

4.2.3 Complexity = Action Conjecture

In the CV conjecture, we choose a length scale by hand to make the volume dimensionless. And it is not clear why the maximal volume surface plays an important role. Instead of the volume, we can take another choice, due to dimensional analysis,

$$C \sim \frac{V}{G_N R_{AdS}} = \frac{V R_{AdS}}{G_N R_{AdS}^2} \sim \frac{\Lambda}{G_N} \int \sqrt{g} dV. \quad (4.2.5)$$

This is an on-shell Einstein-Hilbert action with a cosmological constant Λ . The Ricci scalar is $R \sim \Lambda$ from the on-shell condition.

In [75, 76], to improve such unsatisfactory points, the authors proposed another conjecture for holographic dual of complexity, the "Complexity = Action" (CA) conjecture. In the CA conjecture, the complexity C_A is dual to a gravitational action I_{WDW} on a codimension-zero patch, so-called "the Wheeler-DeWitt (WDW) patch" M_{WDW} .

$$C_A(\Sigma) = \frac{I_{WDW}}{\pi\hbar}. \quad (4.2.6)$$

The WDW patch M_{WDW} is the domain of dependence of the Cauchy surface ending on the boundary time slice Σ where the state is realized. The patch is a collection of possible foliations of the codimension-one surface anchored on the boundary time slice Σ . The CA conjecture has no choice of the surface and the dimensionful coefficient appeared in the CV conjecture.

However, to define the action properly, we need to regularize the divergence coming from the null boundaries of the WDW patch and the joints of the null boundaries ⁵. The WDW

⁵In addition to these contributions, we can care about the contribution from the AdS boundaries. We have some choices.

action with the boundary terms is

$$I_{WDW} = \frac{1}{16\pi G_N} \int_{WDW} d^{d+1}x \sqrt{|-g|} (R - 2\Lambda) + \frac{1}{8\pi G_N} \int d^d x \sqrt{|h|} K \quad (4.2.7)$$

$$- \frac{1}{8\pi G_N} \int d^{d-1}\theta \sqrt{\gamma} \kappa + \frac{1}{8\pi G_N} \int d^{d-1}x \sqrt{\sigma} \eta + \frac{1}{8\pi G_N} \int d^{d-1}x \sqrt{\sigma} a,$$

plus the matter terms if we need. The two terms in the first line might be familiar, the Einstein-Hilbert term with cosmological constant and the Gibbons-Hawking boundary term. In the second line, we have the one surface term for null boundaries and the two space-like joint terms for the space-like joint of space-like and time-like boundaries and the space-like joint involving one or two null boundaries. The Affine parameters for the null surfaces might give us an ambiguity. For more detail treatments, see [82, 83, 84, 85, 86] for example.

When we estimate the growth rate of the WDW action $\frac{dI_{WDW}}{dt}$, we will compare contributions from two WDW patches anchored on the boundary Cauchy surface at t and $t + \delta t$. The difference between these contributions give us the growth rate. Actually, in the difference, many boundary and joint terms can be canceled each other.

As a simple example, consider a $(D = d + 1)$ -dimensional Reissner-Nordstrom-AdS black hole (two-sided) which the metric is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2, \quad (4.2.8)$$

$$f(r) = 1 - \frac{8\pi}{(D-2)\Omega_{D-2}} \frac{2G_N M_{BH}}{r^{D-3}} + \frac{8\pi}{(D-2)\Omega_{D-2}} \frac{G_N Q^2}{r^{2(D-3)}} + \frac{r^2}{R_{AdS}^2}. \quad (4.2.9)$$

The horizon radius $r_{\pm}$ ($r_+ \geq r_-$) satisfies $f(r_{\pm}) = 0$. For (3+1)-dimensional small charged black holes ($r_+ \ll R_{AdS}$), we have $r_+ + r_- = 2G_N M_{BH}$, $r_+ r_- = G_N Q^2$ and the chemical potential $\mu = \frac{Q}{r_+}$. And the small black hole is extremal at $M_{BH} = \frac{Q}{\sqrt{G_N}}$.

For the static and neutral AdS black hole ($Q = 0$), the contributions from the exterior regions are canceled each other and that of the region behind the past horizon decays to zero for $t_L \gg \beta$. Then, only the region behind the future horizon contributes to the growth rate of the WDW action the total growth rate of the WDW action.

$$\begin{aligned} \frac{dI_{WDW}}{dt_L} + \frac{dI_{WDW}^{bdy}}{dt_L} &= \frac{1}{16\pi G_N} \int_{WDW} \sqrt{|g|} (R - 2\Lambda) + \frac{1}{8\pi G_N} \int_{\partial WDW} \sqrt{|h|} K \quad (4.2.10) \\ &= -\frac{\Omega_{D-2}}{8\pi G_N} \frac{r_h^2}{R_{AdS}^2} + \left[-\frac{D-1}{D-2} M_{BH} + \frac{\Omega_{D-2} r^{D-3}}{8\pi G_N} \left((D-2) + (D-1) \frac{r^2}{R_{AdS}^2} \right) \right]_{r=0}^{r=r_h} \\ &= 2M_{BH}. \end{aligned}$$

In the last line, we used the relation at the horizon $r = r_h = r_+$ ($f(r_h) = 0$). Therefore, the holographic complexity has the simple and time-independent growth rate at late time.

$$\frac{dC_A}{dt_L} = \frac{2M_{BH}}{\pi\hbar}. \quad (4.2.11)$$

For static and neutral AdS black holes at late time $t_L \gg \beta$, the Lloyd bound saturates.

For the charged AdS black hole ($Q \neq 0$), as same as the previous case, the contribution comes only from the future horizon

$$\frac{dI_{WDW}}{dt_L} = \frac{r_+ - r_-}{G_N} \left(1 + \frac{r_+^2 + r_+ r_- + r_-^2}{R_{AdS}^2} \right) = \frac{Q^2}{r_-} + \frac{Q^2}{r_+}. \quad (4.2.12)$$

In the line line, we used $f(r_{\pm}) = 0$. In the limit $Q \rightarrow 0$, the growth rate reduces to the uncharged case. For small charged black holes ($r_+ \gg R_{AdS}$),

$$\left. \frac{dI_{WDW}}{dt_L} \right|_{r_+ \gg R_{AdS}} = \frac{r_+ - r_-}{G_N} = 2\sqrt{M_{BH}^2 - \frac{Q^2}{G_N}}. \quad (4.2.13)$$

Near the extremal value $M_{BH} = \frac{Q}{\sqrt{G_N}}$, the growth becomes slower. Because for a given chemical potential $\mu = \frac{Q}{r_+}$ the empty AdS $M = Q = 0$ is the ground state semiclassically, we also obtain the following form of the growth rate of the complexity.

$$\left. \frac{dC_A}{dt_L} \right|_{r_+ \gg R_{AdS}} = \frac{2}{\pi\hbar} [(M_{BH} - \mu Q) - (M_{BH} - \mu Q)_{g.s.}]. \quad (4.2.14)$$

where $g.s.$ means that we take the value for the ground state. We can rewrite it using the thermodynamical relation $d(M - \mu Q) = TdS$.

$$\frac{dC_A}{dt_L} = \frac{2}{\pi\hbar} \int_{g.s.}^S TdS. \quad (4.2.15)$$

From this result, the authors [75, 76] proposed the Llyod bound for charged cases ⁶.

$$\frac{dC_A}{dt_L} \leq \frac{2}{\pi\hbar} [(M_{BH} - \mu Q) - (M_{BH} - \mu Q)_{g.s.}]. \quad (4.2.16)$$

In this sense, the black holes seem to be the fastest computer in physics.

However, for large ($r_+ \gg R_{AdS}$) and intermediate ($r_+ \sim R_{AdS}$) charged black holes, the situation becomes more complicated. For a fixed $\sqrt{G_N}\mu$, the extremality holds for the following curve in the (M_{BH}, Q) -plane.

$$\begin{aligned} G_N^2 M_{BH}^2|_{extremal} &= \frac{R_{AdS}^2}{27} (G_N \mu^2 - 1)(2G_N \mu^2 + 1)^2, \\ G_N Q^2|_{extremal} &= \frac{R_{AdS}^2}{3} G_N \mu^2 (G_N \mu^2 - 1). \end{aligned} \quad (4.2.17)$$

⁶For rotating cases, they just replaced μ to the angular velocity Ω and Q to the angular momentum J . They checked that the saturation of such bound for the rotating BTZ black hole ($D = 2 + 1$).

For $\sqrt{G_N}\mu \leq 1$, there is no positive extremal mass and then the empty AdS $M_{BH} = Q = 0$ is the ground state ($(M_{BH} - \mu Q)_{g.s.} = 0$). For $\sqrt{G_N}\mu > 1$, the extremal black holes with positive mass and charge can exist. In this case, we suppose that the extremal black holes are the ground states for large chemical potential $\sqrt{G_N}\mu > 1$.

For large ($r_+ \gg R_{AdS}$) charged black holes with large chemical potential $\sqrt{G_N}\mu > 1$,

$$f(r) \simeq \frac{r^2}{R_{AdS}^2} - \frac{2G_N M_{BH}}{r} + \frac{G_N Q^2}{r}, \quad \mu = \frac{Q}{r_+}, \quad (4.2.18)$$

and the extremal mass is $M_Q = \frac{2}{3} \left(\frac{G_N}{3}\right)^{1/4} \sqrt{\frac{Q^3}{R_{AdS}}}$. If the ground state is supposed to be the extremal black holes, in the near extremal region,

$$(M_{BH} - \mu Q) - (M_{BH} - \mu Q)_{g.s.} = 2(M_{BH} - M_Q) + O((M_{BH} - M_Q)^2). \quad (4.2.19)$$

On the other hand, the growth rate of action is much bigger than the

$$\frac{dI_{WDW}}{dt_L} = \sqrt{6M_Q(M_{BH} - M_Q)} - O(M_{BH} - M_Q), \quad (4.2.20)$$

in the near extremal region. Therefore, for large charged black holes, the charged version of the Lloyd bound is violated. Actually, for $\sqrt{G_N}\mu \leq 1$, we also find the violation ⁷.

The reason why the violation occurs for large charged black holes discussed in the same paper [75, 76]. The authors suspected the naive assumption for the ground state. We assumed that the extremal black holes are the ground state. But, in some UV-completed theories, the charged black holes can grow hair. For example in the theories with light charged particles with a large chemical potential, it has hair of the charge particles extending to the outside of the horizon. Then, $(M_{BH} - \mu Q)$ can be negative. In such case, we cannot trust the above naive calculation ⁸.

Oppositely speaking, the apparent violation of the Lloyd bound can be used to diagnose the development of black hole hair. Along this direction, the authors suggested the relation to the weak-gravity conjecture[87].

4.3 QFT definition of Complexity?

As we reviewed a bit, the holographic conjectures for complexity of states (the CV and CA conjectures) seem to be knocking on the door for new research directions connecting further

⁷The same form of the violation occurs in the CV conjecture too.

⁸Sufficient supersymmetries protect the black holes from the formation of hair. If theories we consider are consistently truncated from the UV-complete theories with sufficient supersymmetry, the above naive calculation might not violate the Lloyd bound.

quantum information to AdS/CFT and black holes. However, the field theoretical definition of the complexity is almost undeveloped yet. We briefly review a few proposals for the complexity in QFTs or CFTs [88, 89] (see also [90, 91]).

In addition to them, another candidate of the complexity in CFTs was proposed by us [23, 24]. Our proposal will be presented in the next chapter.

4.3.1 The Finsler Geometry Approach

As explained, complexity of a state $|\psi\rangle$ is defined as the minimal number of simple unitary operations to construct the state $|\psi\rangle$ from a simple reference state $|\psi_0\rangle$. In the starting point, we have a given gate set or set of simple unitary operators and $|\psi_0\rangle$. Then, we try to look for an optimal circuit which is a sequence of the unitary operators U acting on $|\psi_0\rangle$ to realize $|\psi\rangle$ with a tolerance ϵ measured by a distance measure.

$$||\psi\rangle - U|\psi_0\rangle|^2 < \epsilon^2. \quad (4.3.1)$$

Inspired by Nielsen's works [92, 93, 94] for a geometrical approach to find the minimal size quantum circuit implemented by a specific n -qubit unitary operation U , let us consider the unitary operator generated by a time-dependent Hamiltonian represented by a linear combination of Hermitian operators (in n -qubit systems, generalized Pauli operators)

$$U(s) = \mathcal{P} \exp \left[\int_0^s d\tilde{s} \sum_I Y^I(\tilde{s}) M_I \right], \quad (4.3.2)$$

with the boundary conditions $U(s=0) = I$, $U(s=1) = U$. $\mathcal{P}$ is a time ordering (left to right in the circuit), and Y^I is the control function forming a $(4^n - 1)$ -dimensional vector space. In the vector space, $Y^I(s) = \text{tr}[\partial_s U(s) U^{-1}(s) M_I]$ is a tangent vector or a velocity to a trajectory of the unitary $U(s)$. Now, the problem to find the optimal circuit implemented by U can be replaced by the variational problem to minimize a cost for the trajectories of the unitaries $U(s)$ in the $(4^n - 1)$ -dimensional space for n -qubit systems.

$$\mathcal{D}(U(s)) = \int_0^1 ds F(U(s), \partial_s U(s)). \quad (4.3.3)$$

About the cost function F , we can focus on functions with desired properties:

- Smoothness : $F \in C^\infty$,
- Positivity : $F(U, v) \geq 0$ with equality if and only if $v = 0$,

- Positive homogeneity : $F(U, \lambda v) = \lambda F(U, v)$ for any positive real λ ,
- Triangle inequality : $F(U, v + v') \leq F(U, v) + F(U, v')$ for all tangent vectors v and v' .

Then, $\mathcal{D}$ defines a length functional for a Finsler manifold with a Finsler metric F (a kind of generalization of Riemannian manifold for various notions of norm). If we take the following norm, the Finsler geometry gives us a standard Riemannian geometry.

$$F_2(U, Y) = \sqrt{\sum_I (Y^I(s))^2}. \quad (4.3.4)$$

In this formalism, the optimal choice of circuit is given by the geodesic in the Finsler geometry.

The authors in [88] tried to apply the Nielsen's geometrical method to the complexity in quantum field theories. For a simple analysis, they worked in the d dimensional massive free scalar theories regularized by putting on a lattice with lattice spacing δ . The system is the infinite numbers of coupled harmonic oscillators. The essence of their idea can be seen in a simpler system, a system of two coupled harmonic oscillators.

$$H = \frac{1}{2} [p_1^2 + p_2^2 + \omega^2(x_1^2 + x_2^2) + \Omega^2(x_1 - x_2)^2]. \quad (4.3.5)$$

where $x_{1,2}$, $p_{1,2}$ are spatial positions and momenta of each oscillator, ω is the mass and Ω is the inter-mass coupling. If we rewrite it by the normal positions and modes, it reduces to two decoupled simple harmonic oscillators.

$$H = \frac{1}{2} (\tilde{p}_+^2 + \tilde{\omega}_+^2 \tilde{x}_+^2 + \tilde{p}_-^2 + \tilde{\omega}_-^2 \tilde{x}_-^2). \quad (4.3.6)$$

where $\tilde{x}_\pm = \frac{1}{\sqrt{2}}(x_1 \pm x_2)$, $\tilde{\omega}_+^2 = \omega^2$, $\tilde{\omega}_-^2 = \omega^2 + 2\Omega^2$. Then, the ground state wave function is factorized in each mode.

$$\psi_0(\tilde{x}_+, \tilde{x}_-) = \psi_0(\tilde{x}_+) \psi_0(\tilde{x}_-) = \frac{(\tilde{\omega}_+ \tilde{\omega}_-)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}(\tilde{\omega}_+^2 \tilde{x}_+^2 + \tilde{\omega}_-^2 \tilde{x}_-^2)}. \quad (4.3.7)$$

In terms of the physical coordinate,

$$\psi_0(x_1, x_2) = \frac{(\omega_1 \omega_2 - \beta^2)^{1/4}}{\sqrt{\pi}} e^{-\frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) - \beta x_1 x_2}, \quad (4.3.8)$$

where $\omega_1 = \omega_2 = \frac{1}{2}(\tilde{\omega}_+ + \tilde{\omega}_-)$, $\beta = \frac{1}{2}(\tilde{\omega}_+ - \tilde{\omega}_-) < 0$.

To identify a simple reference state, we can choose the product state in the physical coordinate,

$$\psi_R(x_1, x_2) = \sqrt{\frac{\omega_0}{\pi}} e^{-\frac{\omega_0}{2}(x_1^2 + x_2^2)}. \quad (4.3.9)$$

where ω_0 is a free parameter characterizing the reference state.

Now we are in a stage to check how many operations we need to construct the target state $\psi_0 = U\psi_R$ from the reference state ψ_R . Our natural choice of the set of elementary unitary operators $\{M_I\}$ are the following five kinds of operators.

$$\begin{aligned} H &= e^{i\epsilon x_0 p_0}, J_a = e^{i\epsilon x_0 p_a}, K_a = e^{i\epsilon x_a p_0} \\ Q_{ab} &= e^{i\epsilon x_a p_b} (a \neq b), Q_{aa} = e^{\epsilon/2} e^{i\epsilon x_a p_a}, \end{aligned} \quad (4.3.10)$$

where $a, b = 1, 2$ and x_0 and p_0 are constant c-number and $\epsilon \ll 1$ is an infinitesimal parameter we introduced to realize continuous sequence of operations. These operations generate a phase change, translations of positions and momenta and entangling for two oscillators ($x_1 \rightarrow \epsilon x_2$) and scaling of x_1 .

As a example of a circuit to construct the state, we can find

$$\psi_0 = Q_{22}^{\alpha_3} Q_{21}^{\alpha_2} Q_{11}^{\alpha_1} \psi_R, \quad (4.3.11)$$

with

$$\alpha_1 = \frac{1}{2\epsilon} \log \left(\frac{\omega_1}{\omega_0} \right), \quad \alpha_2 = \frac{1}{\epsilon} \sqrt{\frac{\omega_0}{\omega_1}} \frac{\beta}{\sqrt{\omega_1 \omega_1 - \beta}}, \quad \alpha_3 = \frac{1}{2\epsilon} \log \left(\frac{\omega_1 \omega_2 - \beta}{\omega_0 \omega_1} \right). \quad (4.3.12)$$

Then, the depth of the circuit is

$$\mathcal{D}(U) = |\alpha_1| + |\alpha_2| + |\alpha_3| = \frac{1}{\epsilon} \left[\frac{1}{2} \log \left(\frac{\omega_1 \omega_2 - \beta}{\omega_0^2} \right) + \sqrt{\frac{\omega_0}{\omega_1}} \frac{|\beta|}{\sqrt{\omega_1 \omega_1 - \beta}} \right]. \quad (4.3.13)$$

But we have no reason why this depth is minimal. Actually, another circuit can give the minimal value.

In the paper [88], from the variational analysis of the geodesic in the Finsler geometry, the minimal depth of the circuit is given. It gives the complexity of the state ψ_0 measured from ψ_R .

$$C = \mathcal{D}_{min} = \frac{1}{2} \sqrt{\sum_{k=\pm} \log^2 \left(\frac{\omega_k}{\omega_0} \right)} = \frac{2^{1/2}}{2} \log \left(\frac{m}{\omega_0} \right). \quad (4.3.14)$$

For N coupled harmonic oscillators, it turns out that

$$C = \mathcal{D}_{min} = \frac{1}{2} \sqrt{\sum_{k=0}^{N-1} \log^2 \left(\frac{\omega_k}{\omega_0} \right)} = \frac{N^{1/2}}{2} \log \left(\frac{m}{\omega_0} \right). \quad (4.3.15)$$

It scales as the square root of the volume $N^{1/2} \sim (V/\delta^{d-1})^{1/2}$ while the holographic results scale as V/δ^{d-1} . This difference comes from the choice of the cost function F . If we take

$F_1 = \sum_I |Y^I|$ not $F_2 = \sum_I (Y^I)^2$, we can get the same scaling in the volume. And the complexities have logarithmic terms controlled by ω_0 which we introduced as a free parameter of the reference state. By tuning ω_0 , we can remove the logarithmic terms as same as the holographic cases.

4.3.2 The Fubini-Study Metric Approach

In another approach for the complexity in QFTs in [89], the authors proposed to use the quantum information metric or the Fubini-Study metric. In this approach, we consider unitary operators U arising from iterating generators $G(s)$ in a set of Hermitian operators $\mathcal{G}$.

$$U(s) = \mathcal{P}e^{-i \int_{s_i}^{\tilde{s}} d\tilde{s} G(\tilde{s})}, \quad (4.3.16)$$

where $\tilde{s} \in [s_i, s_f]$ parametrizes the evolution along a path. We try to find the minimal path to achieve the state $|\psi_0\rangle \simeq U|\psi_R\rangle$ from the reference state $|\psi_R\rangle$. The line element by the Fubini-Study metric

$$ds_{FS}(\tilde{s}) = ds \sqrt{|\partial_{\tilde{s}}|\psi(\tilde{s})\rangle|^2 - |\langle\psi(\tilde{s})|\partial_{\tilde{s}}|\psi(\tilde{s})\rangle|^2} = ds \sqrt{\langle G^2(\tilde{s}) \rangle_{\psi(\tilde{s})} - \langle G(\tilde{s}) \rangle_{\psi(\tilde{s})}^2}, \quad (4.3.17)$$

gives us the circuit depth of a path going via intermediate states $|\psi(\tilde{s})\rangle$.

$$l(|\psi(\tilde{s})\rangle) = \int_{s_i}^{s_f} ds_{FS}(\tilde{s}). \quad (4.3.18)$$

They proposed the complexity can be defined as the minimal length measured by the Fubini-Study metric.

$$C(|\psi\rangle) = \min_{G(\tilde{s})} [l(|\psi(\tilde{s})\rangle)]. \quad (4.3.19)$$

This approach gives similar results as the previous Finsler geometry approach up to difference of the regularization schemes (except for an overall factor).

Chapter 5

Optimization of Path-Integrals and Complexity in CFTs

In this chapter, we introduce our work [23, 24] about so-called "Optimization of Euclidean Path-Integrals" in CFTs. The optimization procedure gives us a direct method to extract dual gravitational information (for example, metric, entanglement wedge, holographic entanglement entropy) only from CFT data. To formulate the optimization procedure itself, we do not assume the AdS/CFT correspondence. We hope that this work gives some hints to understand the basic mechanism of AdS/CFT or holography. And, in this chapter, we also propose the definition of complexity of states using the optimization procedure.

[23] : Phys. Rev. Lett. **119**, no. 7, 071602 (2017), arXiv:1703.00456 [hep-th]

[24] : JHEP **1711**, 097 (2017), arXiv:1706.07056 [hep-th]

5.1 Introduction

The AdS/CFT correspondence [1] has been the most powerful tool to understand quantum nature of gravity. Nevertheless, we still do not understand its basic mechanism nor how spacetime in gravity emerges from conformal field theories (CFTs). Recently, possible candidates which might explain the basic mechanism of the AdS/CFT correspondence have begun to be actively investigated. Among them, a very attractive candidate is the idea of emergent spacetimes from tensor networks, as first conjectured by Swingle [13], for the description of CFT states in terms of MERA (multi-scale entanglement renormalization ansatz) [95, 96]¹.

¹For recent developments we would like to ask readers to refer to e.g. [97, 98, 99, 100, 101, 102, 103, 104, 105, 106].

One strong evidence for this correspondence between holography and tensor networks, apart from the symmetry considerations, is the fact that the holographic entanglement entropy formula [2, 62] can naturally be explained in this approach by counting the number of entangling links in the networks.

However, up to now, most arguments in these directions have been limited to studies of discretized lattice models so that we can apply the idea of tensor networks directly. Therefore, they at most serve as toy models of AdS/CFT as they do not describe the genuine CFTs which are dual to the AdS gravity (though they provide us with deep insights of holographic principle such as quantum error corrections [102, 107]). Clearly, it is then very important to develop a continuous analogue of tensor networks related to AdS/CFT. There already exists a formulation called cMERA (continuous MERA) [97], whose connection to AdS/CFT has been explored in [99, 100, 101, 108, 104, 106]. Nevertheless, explicit formulations of cMERA are so far only available for free field theories [97] (see [99, 100, 109, 110] for various studies) which is the opposite regime from the strongly interacting CFTs which possess gravity duals, the so-called holographic CFTs. A formal construction of cMERA for general CFTs can be found in [108, 106].

The main aim of this work is to introduce and explore a new approach which realizes a continuous limit of tensor networks and allows for field theoretic computations. Essentially, we reformulate the conjectured relation between tensor networks and AdS/CFT from the viewpoint of Euclidean path-integrals. Indeed, the method called tensor network renormalization (TNR) [111, 112] shows that an Euclidean path-integral computation of a ground state wave function can be regarded as a tensor network description of MERA. In this argument, one first discretizes the path-integral into a lattice version and rewrites it as a tensor network. Then, an optimization by contracting tensors and removing unnecessary lattice sites, finally yields the MERA network. The ‘optimization’ here refers to some efficient numerical algorithm.

In our approach we will reformulate this idea, but in such a way that we remain working with the Euclidean path-integral. More precisely, we perform the optimization by changing the structure (or geometry) of lattice regularization. The first attempt in this direction was made in [106] by introducing a position dependent UV cut off. In this work, we present a systematic formulation of optimization by introducing a metric on which we perform the path-integral. The scaling down of this metric corresponds to the optimization assuming that there is a lattice site on a unit area cell.

To evaluate the amount of optimization we made, we consider a functional I_Ψ of the

metric for each quantum state $|\Psi\rangle$. This functional, which might appropriately be called “Path-integral Complexity”, describes the size of our path-integration and corresponds to the computational complexity in the equivalent tensor network description². In 2D CFTs, we can identify this functional I_Ψ with the Liouville action. The optimization procedure is then completed by minimizing this complexity functional I_Ψ , and we argue that the minimum value of I_Ψ is a candidate for complexity of a quantum state in CFTs. Below, we will perform a systematic analysis of our complexity functional for various states in 2D CFTs, lower dimensional example of $\text{NAdS}_2/\text{CFT}_1$ (SYK) as well as in higher dimensions, where we will find an interesting connection to the gravity action proposal [75, 76].

Our new path-integral approach has a number of advantages. Firstly, we can directly deal with any CFTs, including holographic ones, as opposed to tensor network approaches which rely on lattice models of quantum spins. Secondly, in the tensor network description there is a subtle issue that the MERA network can also be interpreted as a de Sitter space [98, 103], while the refined tensor networks given in [102, 105] are argued to describe Euclidean hyperbolic spaces. In our Euclidean approach we can avoid this issue and explicitly verify that the emergent space coincides with a hyperbolic space, i.e. the time slice of AdS.

This chapter is organized as follows: In section 5.2, we present our formulation of an optimization of Euclidean path-integrals in CFTs and relate to the analysis of computational complexity and tensor network renormalization. We will also start with an explicit example for a vacuum of a 2D CFT. In section 5.3, we will investigate the optimization procedure in 2D CFTs for more general states such as finite temperature states and primary states. In section 5.4, we apply our optimization procedure to reduced density matrices. We show that the holographic entanglement entropy and entanglement wedge naturally arise from this computation. In section 5.5, we explicitly evaluate the Liouville action for the optimized solutions and point out that, due to the conformal anomaly, we need to consider a difference of Liouville action, which corresponds to a relative complexity. In section 5.6, we apply our optimization to one dimensional nearly conformal quantum mechanics like SYK models. In section 5.7, propose and provide various support for generalization of our optimization

²The relevance of computational complexity in holography was recently pointed out and holographic complexity was conjectured to be the volume of maximal time slice in gravity duals [80, 79] (for recent progresses see e.g. [113, 114, 115, 85, 116]) and the gravity action in Wheeler-De Witt patch in [75, 76] (for recent progresses see e.g. [117, 82, 118, 83, 84, 119, 120, 121, 122]). We would also like to mention that for CFTs, the behavior of the complexity is very similar to the quantum information metric under marginal deformations as pointed out in [17] (refer to [113, 81, 116] for recent developments), where the metric is argued to be well approximated by the volume of maximal time slice in AdS.

to higher dimensional CFTs. We also compare our results with existing literature of holographic complexity. In section 5.8, we discuss the time evolution of thermo-field dynamics in 2D CFTs as an example of time-dependent states. Finally, in section 5.9 we summarize our findings and conclude.

5.2 Formulation of the Path-Integral Optimization

Here we introduce our idea of optimization of Euclidean path-integrals[23, 24]. We consider a discretized version of Euclidean path-integral which produces a quantum wave functional in QFTs, having in mind a numerical computation of path-integrals. The UV cut off (lattice constant) is written as ϵ throughout this chapter. The optimization here means the most efficient procedure to perform the path-integral in its discretized form³. In other words, it is the most efficient algorithm to numerically perform the path-integrals which leads to the correct wave functional.

5.2.1 General Formulation

We can express the ground state wave functional in a d dimensional QFT on R^d in terms of a Euclidean path-integral as follows:

$$\Psi_0[\tilde{\varphi}(x)] = \int \left(\prod_x \prod_{\epsilon \leq z < \infty} D\varphi(z, x) \right) e^{-S_{QFT}(\varphi)} \times \prod_x \delta(\varphi(\epsilon, x) - \tilde{\varphi}(x)). \quad (5.2.1)$$

Here we write the coordinate of R^d as (z, x) , where $-z(\equiv \tau)$ is the Euclidean time and x is the $d - 1$ dimensional space coordinate of R^{d-1} . We set $z = \epsilon$ at the final time when the path-integral is completed for our convenience. However, we can shift this value as we like without changing our results as is clear from the time translational invariance. Now we perform our discretization of path-integral in terms of the lattice constant ϵ . We start with the square lattice discretization as depicted in the left picture of Fig.5.1. To optimize the path-integral we can omit any unnecessary lattice sites from our evaluation. Since only the low energy mode $k \ll 1/\tau$ survives after the path-integral for the period τ , we can estimate that we can combine $O(\tau/\epsilon)$ lattice sites into one site without losing so much accuracy. It is then clear that the optimization via this coarse-graining procedure leads to the middle picture in Fig.5.1, which coincides with the hyperbolic plane.

³Please distinguish our optimization from other totally different procedures such as the optimization of parameter of tensors in tensor networks. Instead, as we will see later in this section, our optimization changes the tensor network structures as in tensor network renormalization [111, 112].

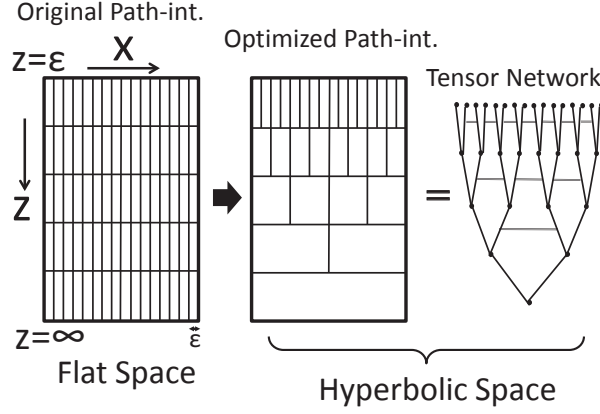


Figure 5.1: Computation of a ground state wave function from Euclidean path-integral (left) and its optimization (middle), which is described by a hyperbolic geometry [24]. The right figure schematically shows its tensor network expression.

One useful way to systematically quantify such coarse-graining procedures is to introduce a metric on the d dimensional space (z, x) (on which the path integration is performed) such that we arrange one lattice site for a unit area. In this rule, we can write the original flat space metric before the optimization as follows:

$$ds^2 = \frac{1}{\epsilon^2} \cdot \left(dz^2 + \sum_{i=1}^{d-1} dx^i dx^i \right). \quad (5.2.2)$$

Consider now the optimization procedure in this metric formulation. The basic rule is to require that the optimized wave functional Ψ_{opt} is proportional to the correct ground state wave function (i.e. the one (5.2.1) for the metric (5.2.2)) even after the optimization i.e. $\Psi_{\text{opt}}[\varphi(x)] \propto \Psi_0[\varphi(x)]$. The optimization can then be realized by modifying the background metric for the path-integration

$$ds^2 = g_{zz}(z, x) dz^2 + g_{ij}(z, x) dx^i dx^j + 2g_{zj}(z, x) dz dx^j, \quad (5.2.3)$$

$$g_{zz}(z = \epsilon, x) = \epsilon^{-2}, \quad g_{ij}(z = \epsilon, x) = \delta_{ij} \cdot \epsilon^{-2}, \quad g_{iz}(z = \epsilon, x) = 0,$$

where the last constraints argue that the UV regularization agrees with the original one (5.2.2) at the end of the path-integration (as we need to reproduce the correct wave functional after the optimization).

In conformal field theories, because there are no coupling RG flows, we should be able to complete the optimization only changing the background metric as in (5.2.3). However, in non-conformal field theories, actually we need to modify external fields J (such as mass parameter or other couplings of various interactions) in a position dependent way $J(z, x)$. The same is true for CFT states in the presence of external fields.

To finalize the optimization procedure, we should provide a sufficient condition for the metric to be "maximally" optimized. Thus, we assume that for each quantum state $|\Psi\rangle$, there exists a functional $I_\Psi[g_{ab}(z, x)]$ whose minimization with respect to the metric g_{ab} gives such maximal optimization⁴. In this way, once we know the functional I_Ψ , we can finalize our optimization procedure. As we will see shortly, in 2D CFTs we can explicitly identify this functional $I_\Psi[g_{ab}(z, x)]$.

5.2.2 Connection to Computational Complexity

At an intuitive level, the optimization corresponds to minimizing the number of path-integral operations in the discretized description. As we will explain in subsection 5.2.4, we can map this discretized Euclidean path-integration into a tensor network computation. Tensor networks are a graphical description of wave functionals in quantum many-body systems in terms of networks of quantum entanglement (see e.g.[123, 124]). The optimization of tensor network was introduced in [111, 112], called tensor network renormalization. We are now considering a path-integral counterpart of the same optimization here. In the tensor network description, the optimization corresponds to minimizing the number of tensors. We can naturally identify this minimized number as a computational complexity of the quantum state we are looking at.

Let us briefly review the relevant facts about the computational complexity of a quantum state (for example, see [125, 126, 127, 128]). In quantum information theory, a quantum state made of qubits can be constructed by a sequence of simple unitary operations acting on a simple reference state. The sequence is called a quantum circuit and the unitary operations are called quantum gates. As a simple choice, we use 2-qubit gates for simple unitary operations and a direct product state for a simple state which has no real space entanglement (Fig.5.2). The quantum circuit (gate) complexity of a quantum state is then defined as a minimal number of the quantum gates needed to create the state starting from a reference state. Because the quantum circuit is a model of quantum computation, here we refer to the complexity as the computational complexity⁵.

Based on the above considerations as well as the evidence provided in the following section, we are naturally lead to a conjecture that a computational complexity C_Ψ of a state

⁴In non-conformal field theories or in the presence of external fields in CFTs, this functional depends on gauge fields for global currents and scalar fields etc. as $I[g_{ab}(z, x), A_a(z, x), J(z, x), \dots]$.

⁵The relevance of computational complexity in AdS/CFT was recently pointed out and holographic computations of complexity have been proposed in [80, 79, 75, 76]

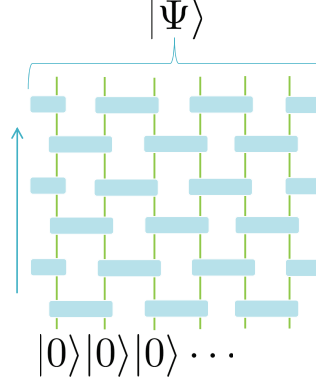


Figure 5.2: A quantum circuit representation for a quantum state in a qubit system [24]. A quantum state $|\Psi\rangle$ can be constructed by simple local (2-qubit) unitary operations from a simple reference state, for example, a product state $|0\rangle|0\rangle|0\rangle\dots$.

$|\Psi\rangle$ is obtained from the functional introduced before by a minimization:

$$C_\Psi = \text{Min}_{g_{ab}(z,x)} [I_\Psi[g_{ab}(z,x)]] . \quad (5.2.4)$$

In other words, the functional $I_\Psi[g_{ab}(z,x)]$ for any $g_{ab}(z,x)$ estimates the amount of complexity for that network corresponding to the (partially optimized) path-integral on the space with the specified metric. Understanding of the properties of this complexity functional I_Ψ , which might appropriately be called “Path-integral Complexity”, is the central aim of this work. As we will soon see, this functional will be closely connected to the mechanism of emergent space in the AdS/CFT.

5.2.3 Optimization of Vacuum States in 2D CFTs

Let us first see how the optimization procedure works for vacuum states in 2D CFTs. We will study more general states later in later sections.

In 2D CFTs, we can always make the general metric into the diagonal form via a coordinate transformation. Thus the optimization is performed in the following ansatz:

$$\begin{aligned} ds^2 &= e^{2\phi(z,x)}(dz^2 + dx^2), \\ e^{2\phi(z=\epsilon,x)} &= 1/\epsilon^2, \end{aligned} \quad (5.2.5)$$

where the second condition specifies the boundary condition so that the discretization is fine-grained when we read off the wave function after the full path-integration. Obviously this is a special example of the ansatz (5.2.3). Thus the metric is characterized by the Weyl scaling function $\phi(z,x)$.

Remarkably, in 2D CFTs, we know how the wave function changes under such a local Weyl transformation. Keeping the universal UV cut off ϵ , the measure of the path-integrations of quantum fields in the CFT changes under the Weyl rescaling [129]:

$$[D\varphi]_{g_{ab}=e^{2\phi}\delta_{ab}} = e^{S_L[\phi]-S_L[0]} \cdot [D\varphi]_{g_{ab}=\delta_{ab}}, \quad (5.2.6)$$

where $S_L[\phi]$ is the Liouville action⁶ [130] (see also [129, 131])

$$S_L[\phi] = \frac{c}{24\pi} \int_{-\infty}^{\infty} dx \int_{\epsilon}^{\infty} dz [(\partial_x \phi)^2 + (\partial_z \phi)^2 + \mu e^{2\phi}]. \quad (5.2.7)$$

The constant c is the central charge of the 2D CFT we consider. The kinetic term in S_L represents the conformal anomaly and the potential term arises the UV regularization which manifestly breaks the Weyl invariance. In our treatment, we simply set $\mu = 1$ below by suitable shift of ϕ .

Therefore, the wave functional $\Psi_{g_{ab}=e^{2\phi}\delta_{ab}}(\tilde{\varphi}(x))$ obtained from the Euclidean path-integral for the metric (5.2.5) is proportional to the one $\Psi_{g_{ab}=\delta_{ab}}(\tilde{\varphi}(x))$ for the flat metric (5.2.2) thanks to the conformal invariance. The proportionality coefficient is given by the Liouville action as follows⁷

$$\Psi_{g_{ab}=e^{2\phi}\delta_{ab}}(\tilde{\varphi}(x)) = e^{S_L[\phi]-S_L[0]} \cdot \Psi_{g_{ab}=\delta_{ab}}(\tilde{\varphi}(x)). \quad (5.2.8)$$

Let us turn to the optimization procedure [23, 24]. we argue that the optimization is equivalent to minimizing the normalization factor $e^{S_L[\phi]}$ of the wave functional, or equally the complexity functional I_{Ψ_0} for the vacuum state $|\Psi_0\rangle$ in 2D CFTs, can be identified as follows⁸

$$I_{\Psi_0}[\phi(z, x)] = S_L[\phi(z, x)]. \quad (5.2.9)$$

The intuitive reason is that this factor is expected to be proportional to the number of repetition of the same operation (i.e. the path-integral in one site). In 2D CFTs, we believe this is only one quantity which we can come up with to measure the size of path-integration. Indeed it is proportional to the central charge, which characterizes the degrees of freedom.

⁶Here we take the reference metric is flat $ds^2 = dz^2 + dx^2$. Later in section (5.5), we will present the Liouville action for a more general reference metric.

⁷ Here we compare the optimized metric $g_{ab} = e^{2\phi}\delta_{ab}$ with $g_{ab} = \delta_{ab}$. To be exact we need to take the latter to be the original one (5.2.2) i.e. $g_{ab} = \epsilon^{-2}\delta_{ab}$. However the different is just a constant factor multiplication and does not affect our arguments. So we simply ignore this.

⁸In two dimensional CFTs, as we will explain in section 5.5, due to the conformal anomaly we actually need to define a relative complexity by the difference of the Liouville action between two different metrics. However this does not change our argument in this section.

Thus the optimization can be completed by requiring the equation of motion of Liouville action S_L and this reads

$$4\partial_w\partial_{\bar{w}}\phi = e^{2\phi}, \quad (5.2.10)$$

where we introduced $w = z + ix$ and $\bar{w} = z - ix$.

With the boundary condition $e^{2\phi(z=\epsilon, x)} = \epsilon^{-2}$, we can easily find the suitable solution to (5.2.10):

$$e^{2\phi} = \frac{4}{(w + \bar{w})^2} = z^{-2}, \quad (5.2.11)$$

which leads to the hyperbolic plane metric

$$ds^2 = \frac{dz^2 + dx^2}{z^2}. \quad (5.2.12)$$

This justifies the heuristic argument to derive a hyperbolic plane H_2 in Fig.5.1.

Indeed, this hyperbolic metric is the minimum of S_L with the boundary condition. To see this, we rewrite

$$S_L = \frac{c}{24\pi} \int dx dz [(\partial_x \phi)^2 + (\partial_z \phi + e^\phi)^2] - \frac{c}{12\pi} \int dx [e^\phi]_{z=\epsilon}^{z=\infty} \geq \frac{cL}{12\pi\epsilon}, \quad (5.2.13)$$

where $L \equiv \int dx$ is the length of space direction and we assume the IR behavior $e^{2\phi(z=\infty, x)} = 0$. The final inequality in (5.2.13) is saturated if and only if

$$\partial_x \phi = \partial_z \phi + e^\phi = 0, \quad (5.2.14)$$

and this leads to the solution (5.2.11).

In this way, we observe that the time slice of AdS_3 dual to the 2D CFT vacuum emerges after the optimization. We will see more evidences throughout this chapter that geometries obtained from our optimization coincides with the time slice in AdS/CFT . This is consistent with the idea of tensor network description of AdS/CFT and can be regarded as its continuous version. We would like to emphasize that the above argument only depends on the central charge c of the 2D CFT we consider. Therefore this should be applied to both free and interacting CFTs including holographic ones.

It is also interesting to note that the optimized value of S_L , i.e. our complexity C_{Ψ_0} , scales linearly with respect to the momentum cut off ϵ^{-1} and the central charge c as

$$C_{\Psi_0} = \text{Min}_\phi [S_L[\phi]] = \frac{cL}{12\pi\epsilon}, \quad (5.2.15)$$

and this qualitatively agrees with the behavior of the computational complexity [80, 79] of a CFT ground state and the quantum information metric [17] for the same state, both of

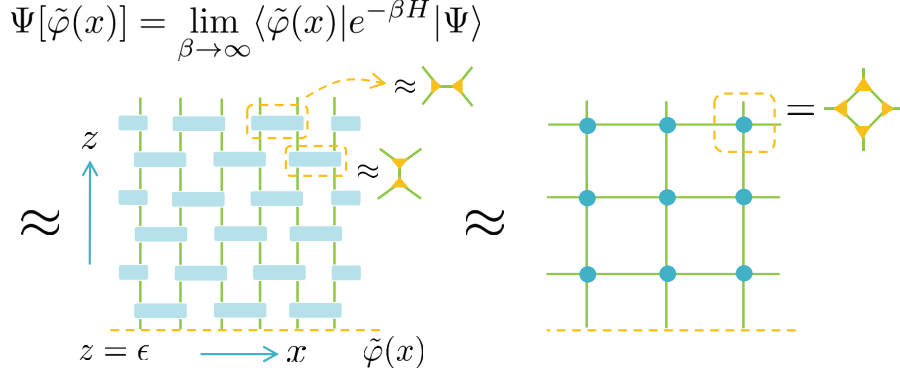


Figure 5.3: The Euclidean path integral for the ground state wave functional $\Psi[\tilde{\varphi}(x)]$ can be approximately described by a tensor network on a square lattice [24].

which are given by the volume of time slice of AdS. In this relation, our minimization of S_L nicely corresponds to the optimization of the quantum circuits which is needed to define the complexity.

5.2.4 Tensor Network Renormalization and Optimization

Our identification of the Liouville action with a complexity i.e.(5.2.9) is partly motivated by an interesting connection between the tensor network renormalization (TNR) [111, 112] and our optimization procedure of Euclidean path-integral. This is because the number of tensors in TNR is an estimation of complexity and the Liouville action has a desired property in this sense, e.g. it is obvious that the Liouville potential term $\int e^{2\phi}$ (i.e. the volume) measures the number of unitary tensors in TNR. Soon later this argument was sharpened in the quite recent paper [132] where the number of isometries is argued to explain the kinetic term $\int (\partial\phi)^2$ in Liouville theory.

An Euclidean path-integral on a semi-infinite plane (or cylinder) with a boundary condition on the edge gives us a ground state wave functional in a quantum system. The path-integral can be approximately described by a tensor network which is a collection of tensors contracted with each other. Using the Suzuki-Trotter decomposition [133, 134] and the singular value decomposition of the tensors, we can rewrite the Euclidean path-integral into a tensor network on a square lattice (Fig.5.3). Tensor network renormalization (TNR) is a procedure to reorganize the tensors to ones on a coarser lattice by inserting projectors (isometries) and unitaries (disentangers) with removing short-range entanglement.⁹ This is

⁹Note that by adding a dummy or ancilla state $|0\rangle$ we can equivalently regard an isometry as a unitary.

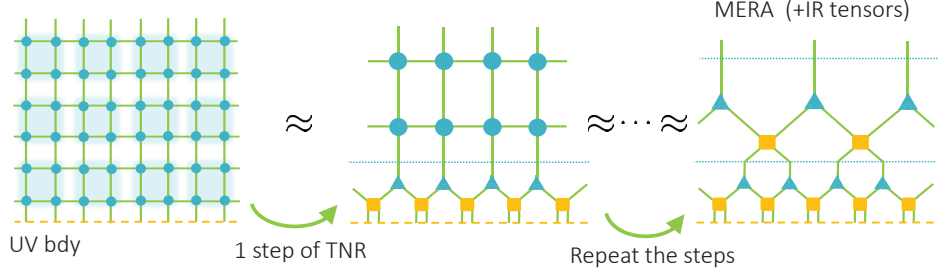


Figure 5.4: The tensor network renormalization (TNR) gradually makes the coarse-grained tensor network with removing short-range entanglement [24]. From the UV boundary, isometries (coarse-graining) and unitaries (disentangers) accumulate and the MERA network grows with the TNR steps.

a step of TNR (Fig.5.4). Repeating this procedure, we can generate a RG flow properly and end up with a tensor network at the IR fixed point. For the ground state wave functional in a CFT, it ends up with a MERA (Multi-scale Entanglement Renormalization Ansatz) network made of isometries and disentangers. The MERA network clearly contains smaller numbers of the tensors than ones in the tensor network on the original square lattice before the coarse-graing. In this sense, this MERA network is an optimal tensor network to approximately describe the Euclidean path-integral.

Our optimization procedure is motivated by TNR. In our procedure (Fig.5.1), the tensor network on the square lattice corresponds to the Euclidean path-integral on flat space with a UV cutoff ϵ . Changing the tensor network with inserting isometries and entanglers corresponds to deforming the back-ground metric for the path-integral. And the MERA network, which is the tensor network after the TNR procedure, approximately corresponds to the optimized path-integral.

Actually, it is not difficult to estimate the amount of complexity for each tensor network during the TNR optimization procedure, by identifying the complexity with the number of tensors, both isometries (coarse-graining) and unitaries (disentangers). For simplicity, consider an Euclidean path-integral for the ground state wave function in a 2d CFT, which is performed on the upper half plane ($\epsilon < z < \infty, -\infty < x < +\infty$). First we consider the original square lattice. Since we suppose that each tensor have unit area, the uniform metric is given by $e^{2\phi(z)} = \epsilon^{-2}$ as in (5.2.2). Therefore, the total number of tensors, which are only unitaries, is estimated from the total volume:

$$\int_{-\infty}^{\infty} dx \int_{\epsilon}^{\infty} dz \frac{1}{\epsilon^2} = \int_{-\infty}^{\infty} dx \int_{\epsilon}^{\infty} dz e^{2\phi}. \quad (5.2.16)$$

Then, performing the TNR procedure, the number of the tensors or the square lattice sites is reduced by the factor $(1/2)^2$ per step. On the other hand, the isometries and disentanglers accumulate from the UV boundary [111, 112]. Refer to Fig.5.4.

At the k -th step of TNR, the total area changes into

$$\int_{-\infty}^{\infty} dx \int_{2^k \epsilon}^{\infty} dz \frac{1}{(2^k \epsilon)^2} + \sum_{s=1}^k \int_{-\infty}^{\infty} dx \int_{2^{s-1} \epsilon}^{2^s \epsilon} dz \left(\frac{1}{(2^{s-1} \epsilon) \cdot (2^s \epsilon)} + \frac{1}{(2^s \epsilon)^2} \right). \quad (5.2.17)$$

The first term is the contribution from the tensors on the coarser lattice. The second term is the contribution from the MERA network. For the s -th layer of the MERA network, we have $dx dz / ((2^{s-1} \epsilon) \cdot (2^s \epsilon))$ isometries and $dx dz / (2^s \epsilon)^2$ per unit cell. This contribution is depicted in Fig.5.5.

This network corresponds to the metric

$$e^{2\phi} = \begin{cases} (2^k \epsilon)^{-2} & (z \geq 2^k \epsilon). \\ z^{-2} & (z < 2^k \epsilon). \end{cases} \quad (5.2.18)$$

Obviously, the first and third term in (5.2.17) are approximated by the Liouville potential integral $\int e^{2\phi}$ [23]. The second term arises because of the non-zero gradient of ϕ and is estimated by the kinetic term $\int (\partial\phi)^2$ [132].

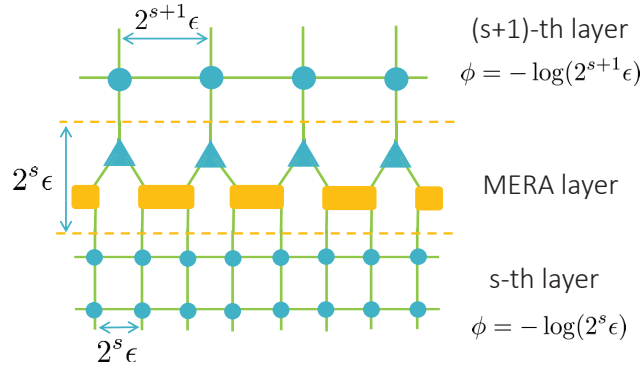


Figure 5.5: The tensor network produced when we have a shift of ϕ at a specific layer [24]. This also represents the one step (s -th) contribution in the process of tensor network renormalization, which finally reaches the MERA network. This corresponds to s -th terms $\int_{2^{s-1}\epsilon}^{2^s\epsilon} dz(\dots)$ in (5.2.17).

5.3 Optimizing Various States in 2D CFTs

Here we would like to explore optimizations in 2D CFTs for more general quantum states. First it is useful to remember that the general solutions to the Liouville equation (5.2.10) is well-known (see e.g.[135, 129]):

$$e^{2\phi} = \frac{4A'(w)B'(\bar{w})}{(1 - A(w)B(\bar{w}))^2}. \quad (5.3.1)$$

Note that functions $A(w)$ and $B(\bar{w})$ describe the degrees of freedom of conformal mappings. For example, if we choose

$$A(w) = w, \quad B(\bar{w}) = -1/\bar{w}, \quad (5.3.2)$$

then we reproduce the solution for vacuums states (5.2.11).

5.3.1 Finite Temperature States

Consider a 2D CFT state at a finite temperature $T = 1/\beta$. In the thermofield double description [37], the wave functional is expressed by an Euclidean path-integral on a strip defined by $-\frac{\beta}{4}(\equiv z_1) < z < \frac{\beta}{4}(\equiv z_2)$ in the Euclidean time direction, more explicitly

$$\begin{aligned} \Psi[\tilde{\varphi}_1(x), \tilde{\varphi}_2(x)] &= \int \left(\prod_x \prod_{-\frac{\beta}{4} < z < \frac{\beta}{4}} D\varphi(z, x) \right) e^{-S_{CFT}(\varphi)} \\ &\times \prod_{-\infty < x < \infty} \delta(\varphi(z_1, x) - \tilde{\varphi}_1(x)) \delta(\varphi(z_2, x) - \tilde{\varphi}_2(x)). \end{aligned} \quad (5.3.3)$$

where $\tilde{\varphi}_1(x)$ and $\tilde{\varphi}_2(x)$ are the boundary values for the fields of the CFT (i.e. $\tilde{\varphi}(x)$) at $z = \mp \frac{\beta}{4}$ respectively.

Minimizing the Liouville action S_L leads to the solution in (5.3.2) given by:

$$A(w) = e^{\frac{2\pi i w}{\beta}}, \quad B(\bar{w}) = -e^{\frac{2\pi i \bar{w}}{\beta}}. \quad (5.3.4)$$

This leads to

$$e^{2\phi} = \frac{16\pi^2}{\beta^2} \frac{e^{\frac{2\pi i}{\beta}(w+\bar{w})}}{\left(1 + e^{\frac{2\pi i}{\beta}(w+\bar{w})}\right)^2} = \frac{4\pi^2}{\beta^2} \sec^2\left(\frac{2\pi z}{\beta}\right). \quad (5.3.5)$$

If we perform the following coordinate transformation

$$\tan\left(\frac{\pi z}{\beta}\right) = \tanh\left(\frac{\rho}{2}\right), \quad (5.3.6)$$

then we obtain the metric

$$ds^2 = d\rho^2 + \frac{4\pi^2}{\beta^2} \cosh^2 \rho \, dx^2, \quad (5.3.7)$$

which coincides with the time slice of eternal BTZ black hole (i.e. the Einstein-Rosen bridge) [37].

5.3.2 CFT on a Cylinder and Primary States

Now consider 2D CFTs on a cylinder (with the circumference 2π), where the wave functional is defined on a circle $|w| = 1$ at a fixed Euclidean time. After the optimization procedure, we obtain the geometry $A(w) = w$ and $B(\bar{w}) = \bar{w}$ given by

$$e^{2\phi(w, \bar{w})} = \frac{4}{(1 - |w|^2)^2}, \quad (5.3.8)$$

which is precisely the Poincare disk and is the solution to (5.2.10).

Then we consider an excited state given by a primary state $|\alpha\rangle$. This is created by acting a primary operator $O_\alpha(w, \bar{w})$ with the conformal dimension $h_\alpha = \bar{h}_\alpha$ at the center $w = \bar{w} = 0$. Its behavior under the Weyl re-scaling is expressed as

$$O(w, \bar{w}) \propto e^{-2h_\alpha \phi}. \quad (5.3.9)$$

Thus the dependence of the wave function on ϕ looks like

$$\Psi_{g_{ab}=e^{2\phi}\delta_{ab}}(\tilde{\varphi}) \simeq e^{S_L} \cdot e^{-2h_\alpha \phi(0)} \cdot \Psi_{g_{ab}=\delta_{ab}}(\tilde{\varphi}). \quad (5.3.10)$$

This shows that the complexity function should be taken to be

$$I_\alpha[\phi(w, \bar{w})] = S_L[\phi(w, \bar{w})] - 2h_\alpha \phi(0). \quad (5.3.11)$$

The equation of motion of I_α leads to

$$4\partial_w \partial_{\bar{w}} \phi - e^{2\phi} + 2\pi(1 - a)\delta^2(w) = 0, \quad (5.3.12)$$

where we set

$$a = 1 - \frac{12h_\alpha}{c}. \quad (5.3.13)$$

The solution can be found as

$$A(w) = w^a, \quad B(\bar{w}) = \bar{w}^a, \quad (5.3.14)$$

which leads to the expression:

$$e^{2\phi} = \frac{4a^2}{|w|^{2(1-a)}(1 - |w|^{2a})^2}. \quad (5.3.15)$$

Since the angle of w coordinate is 2π periodic, this geometry has the deficit angle $2\pi(1 - a)$.

Now we compare this geometry with the time slice of the gravity dual predicted from $\text{AdS}_3/\text{CFT}_2$. It is given by the conical deficit angle geometry (5.3.15) with the identification

$$a = \sqrt{1 - \frac{24h_\alpha}{c}}. \quad (5.3.16)$$

Thus, the geometry from our optimization (5.3.13) agrees with the gravity dual (5.3.16) up to the first order correction when $h_\alpha \ll c$, i.e. the case where the back-reaction due to the point particle is very small.

It is intriguing to note that if we consider the quantum Liouville theory rather than the classical one, we find the perfect matching. In the quantum Liouville theory, we introduce a parameter γ such that $c = 1 + 3Q^2$ and $Q \equiv \frac{2}{\gamma} + \gamma$. The chiral conformal dimension of the primary operator $e^{\frac{2\beta}{\gamma}\phi}$ is given by $\frac{\beta(Q-\beta)}{2}$. If the central charge is very large so that the 2D CFT has a classical gravity dual, we find

$$a \simeq 1 - \beta\gamma \simeq \sqrt{1 - \frac{24h_\alpha}{c}}, \quad (5.3.17)$$

which indeed agrees with the gravity dual (5.3.16) even when h_α/c is finite.

This agreement may suggest that the actual optimized wave functional is given by a ‘quantum’ optimization defined as follows:

$$\Psi_{\text{opt}}[\tilde{\varphi}] = \left[\int D\phi(x, z) e^{-S_L[\phi]} (\Psi_{g_{ab}=\delta_{ab}}[\tilde{\varphi}])^{-1} \right]^{-1}. \quad (5.3.18)$$

If we take the semi-classical approximation when c is large, we reproduce our classical optimization. It is an important future problem to understand the exact form of the proposal at the quantum level.

5.4 Reduced Density Matrices and EE

Consider an optimization of path-integral representation of reduced density matrix ρ_A in a two dimensional CFT defined on a plane $\mathbb{R}^2$. We simply choose the subsystem A to be an interval $-l \leq x \leq l$ at $z(= -\tau) = \epsilon$. ρ_A is defined from the CFT vacuum by tracing out the complement of A (the upper left picture in Fig.5.6).

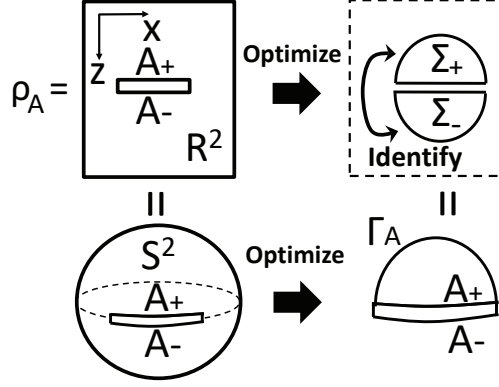


Figure 5.6: The optimization of path-integral for a reduced density matrix [24]. The upper left picture is the definition of ρ_A in terms of the path-integral in flat space. This is conformally mapped into a sphere with a open cut depicted in the lower left picture. The upper right one is the one after the optimization and is equivalent to a geometry which is obtained by pasting two identical entanglement wedges along the geodesic (=the half circle) as shown in the lower right picture.

5.4.1 Optimizing Reduced Density Matrices

The optimization procedure is performed by changing the background metric as in (5.2.5), where the boundary condition of ϕ is imposed around the upper and lower edge of the slit A . Refer to Fig.5.6 for a sketch of this procedure. The plane R^2 is conformally mapped into a sphere S^2 . Therefore the optimization is done by shrinking the sphere with an open cut down to a much smaller one so that the Liouville action is minimized.

To make the analysis clear, let us divide the final manifold into two halves by cutting along the horizontal line $z = 0$, denoted by Σ_+ and Σ_- . The boundary of $\Sigma_{\pm}$ consist of two parts:

$$\partial\Sigma_{\pm} = A_{\pm} \cup \Gamma_A, \quad (5.4.1)$$

where Γ_A in both $\partial\Sigma_+$ and $\partial\Sigma_-$ are identified so that the topology of the final optimized manifold $\Sigma_+ \cup \Sigma_-$ is a disk with the boundary $A_+ \cup A_-$. On the boundary $A_+ \cup A_-$ we have $e^{2\phi} = 1/\epsilon^2$.

The optimization of each of $\Sigma_{\pm}$ is done by minimizing the Liouville action with boundary contributions. The boundary action in the Liouville theory [146] reads

$$S_{Lb} = \frac{c}{12\pi} \int_{\partial\Sigma_{\pm}} ds [K_0 \phi + \mu_B e^{\phi}], \quad (5.4.2)$$

where K_0 is the (trace of) extrinsic curvature of the boundary $\partial\Sigma_{\pm}$ in the flat space. If we

describe the boundary by $x = f(z)$, then the extrinsic curvature in the flat metric $ds^2 = dz^2 + dx^2$, is given by $K_0 = -\frac{f''}{(1+(f')^2)^{3/2}}$. On the other hand, the final term is the boundary Liouville potential. Since Σ_+ and Σ_- are pasted along the boundary smoothly, we set $\mu_B = 0$ for our ρ_A optimization.¹⁰

Now, to satisfy the equation of motion at the boundary Γ_A , we impose the Neumann boundary condition¹¹ of ϕ . This condition (when $\mu_B = 0$) is explicitly written as

$$(n^x \partial_x + n^z \partial_z) \phi + K_0 = 0. \quad (5.4.3)$$

where $n^{x,z}$ is the unit vector normal to the boundary in the flat space. Actually this is simply expressed as $K = 0$, where K is the extrinsic curvature in the curved metric (5.2.5). This fact can be shown as follows. Consider a boundary $x = f(z)$ in the two dimensional space defined by the metric $ds^2 = e^{2\phi(z,x)}(dz^2 + dx^2)$. The out-going normal unit vector N^a is given by

$$N^z = e^{-\phi(z,x)} n^z = \frac{-f'(z) e^{-\phi(z,x)}}{\sqrt{1 + f'(z)^2}}, \quad N^x = e^{-\phi(z,x)} n^x = \frac{e^{-\phi(z,x)}}{\sqrt{1 + f'(z)^2}}, \quad (5.4.4)$$

where n^a is the normal unit vector in the flat space $ds^2 = dz^2 + dx^2$. The extrinsic curvature (=its trace part) at the boundary is defined by $K = h^{ab} \nabla_a N_b$, where all components are projected to the boundary whose induced metric is written as h_{ab} . Explicitly we can calculate K as follows:

$$K = \frac{e^{-\phi(z,x)}}{\sqrt{1 + f'(z)^2}} \left[\partial_x \phi - f' \partial_z \phi - \frac{f''}{1 + (f')^2} \right] = e^{-\phi(z,x)} [n^a \partial_a \phi + K_0]. \quad (5.4.5)$$

In this way, the Neumann boundary condition requires that the curve Γ_A is geodesic. By taking the bulk solution given by the hyperbolic space $\phi = -\log z + \text{const.}$, the geodesic Γ_A is given by the half circle $z^2 + x^2 = l^2$. Thus, this geometry obtained from the optimization of ρ_A , coincides with (two copies of) the entanglement wedge [2, 62, 10, 11, 12].

Note that if we act a local operator inside the entanglement wedge in the original flat space, then this excitation survives after the optimization procedure. However, if we act the operator outside, then the excitation is washed out under the optimization procedure and does not reflect the reduced density matrix ρ_A as long as we neglect its back-reaction.

¹⁰Non-zero μ_B leads to a jump of the extrinsic curvature which will be used later.

¹¹On the cuts $A_\pm$ we imposed the Dirichlet boundary condition. The reason why we imposed the Neumann one on Γ_A is simply because the manifold is smoothly connected to the other side on Γ_A .

5.4.2 Entanglement Entropy

Next we evaluate the entanglement entropy by the replica method. Consider an optimization of the matrix product ρ_A^n . We assume an analytical continuation of n with $|n - 1| \ll 1$. The standard replica method leads to a conical deficit angle $2\pi(1 - n) \equiv 2\delta$ at the two end points of the interval A . Thus, after the optimization, we get a geometry with the corner angle $\pi/2 + \pi(n - 1)$ instead of $\pi/2$ (the lower right picture in Fig. 5.7). This modification of the boundary Γ_A is equivalent to shifting the extrinsic curvature from $K = 0$ to $K = \pi(n - 1)$. Indeed, if we consider the boundary given by $x^2 + (z - z_0)^2 = l^2$, we get $K = z_0/l$. When z_0 is infinitesimally small, we get $x \simeq l + (z_0/l) \cdot z + O(z^2)$ near the boundary point $(z, x) = (0, l)$. Therefore the corner angle is shifted to be $\pi/2 - \delta$ with $\delta \simeq -z_0/l$ (for the definition of δ , refer also to lower pictures in Fig.5.7). Therefore we find the relation $K \simeq -\delta$. If we set

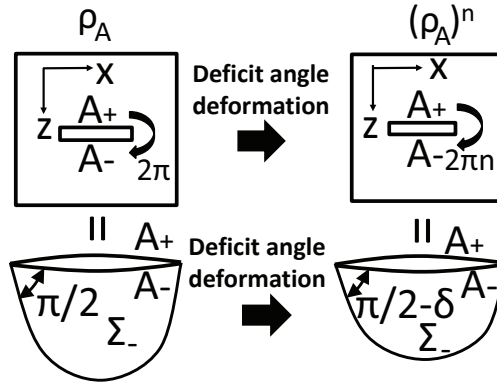


Figure 5.7: The computation of entanglement entropy using the optimized reduced density matrix [24]. Following the replica method shown in the upper pictures, we consider the evaluation of ρ_A^n . We assume the analytical continuation such that n is very close to 1 such that $\delta \equiv \pi(1 - n) \ll 1$. Thus this describes an infinitesimally small (negative) deficit angle deformation. After the optimization, we obtain the conical geometry in the lower right picture with $\delta = \pi(1 - n)$.

the boundary Liouville term in (5.4.2) non-zero $\mu_B \neq 0$, the boundary condition is modified from (5.4.3) i.e. $K = 0$ into $K + \mu_B = 0$. Thus the desired angle shift (or negative deficit angle) is realized by setting $\mu_B = \pi(1 - n)$. In the presence of infinitesimally small μ_B we can evaluate the Liouville action by a probe approximation neglecting all back reactions. By

taking a derivative with respect to n , we obtain the entanglement entropy¹² S_A :

$$S_A = -\partial_n \left[2 \times \frac{c\mu_B}{12\pi} \int_{\Gamma_A} ds e^\phi \right]_{n=1} = \frac{c}{6} \int_{\partial\Sigma_+} ds e^\phi = \frac{c}{3} \log \frac{l}{\epsilon}, \quad (5.4.6)$$

reproducing the well-known result [147]. The lower left expression (5.4.6) $\frac{c}{6} \int_{\partial\Sigma_+} e^\phi$ precisely agrees with the holographic entanglement entropy formula [2, 62] as Γ_A has to be the geodesic due to the boundary condition.

5.4.3 Subregion Complexity

Finally we would like to evaluate the value of Liouville action $S_L[\phi]$ in the reduced subregion. It is natural to argue that this provides a definition of complexity for the reduced density matrix ρ_A . For various earlier proposals for holographic subregion complexity refer to [113, 85].

As in the previous section we take A to be the interval $-l \leq x \leq l$. By computing the action for two copies of the half disk $x^2 + z^2 \leq l^2$ with the solution $\phi = -\log z$, we find

$$\begin{aligned} S_L &= \frac{c}{12\pi} \int dx dz [(\partial_z \phi)^2 + e^{2\phi}] + \frac{c}{6\pi} \int ds K_0 \phi \\ &= \frac{c}{6\pi} \int_\epsilon^l dz \frac{2\sqrt{l^2 - z^2}}{z^2} + \frac{c}{6\pi} \int_{-\pi/2}^{\pi/2} [-\log(l \cos \theta)] \\ &= \frac{c}{6\pi} \left[\frac{2l}{\epsilon} - \pi - \pi \log \left(\frac{l}{2} \right) \right]. \end{aligned} \quad (5.4.7)$$

It will be interesting to compute and explore it further for more general states and we leave it as an open future problem.

5.5 Evaluation of S_L in 2D CFTs

Here we first explain the properties of Liouville action S_L in general setups with boundaries. We will find that it depends on the reference metric and it does not seem to be possible to define its absolute value, which is due to the conformal anomaly in 2D CFTs. Rather we are lead to introduce an functional defined by a difference of Liouville action denoted by $I_L[g_2, g_1]$, where g_1 is the reference metric and g_2 is the final metric after the optimization. $I_L[g_2, g_1]$ is expected to measure of the complexity between the two path-integrals in g_1 and g_2 . Having them in mind, we proceed to explicit evaluations of $I_L[g_1, g_2]$ in various cases.

¹²The abuse of notation for the entanglement entropy and the Liouville action should be clear from the context.

5.5.1 General properties of the Liouville action

We start with a two dimensional space $\mathcal{M}$ described by the metric $ds^2 = g_{ab}dx^a dx^b$ ($a, b = 1, 2$), which is called the reference metric. We now perform the Weyl transformation and define the rescaled metric:

$$ds^2 = e^{2\phi} g_{ab} dx^a dx^b. \quad (5.5.1)$$

The Liouville action corresponding to this Weyl rescaling is given by

$$S_L[\phi, g_{ab}] = \frac{c}{24\pi} \int_{\mathcal{M}} d^2x \sqrt{g} \left[g^{ab} \partial_a \phi \partial_b \phi + e^{2\phi} + R_g \phi \right] + \frac{c}{12\pi} \int_{\partial\mathcal{M}} ds \sqrt{h} K_g \phi, \quad (5.5.2)$$

where

$$\begin{aligned} \mathcal{M} &= \text{The 2-dim manifold with co-ordinates } \{x, y\} \\ \partial\mathcal{M} &= \text{The boundary of } \mathcal{M} \text{ with the coordinate } s \\ f(x, y) &= \text{The equation for the boundary } \partial\mathcal{M} \\ R_g &= \text{Ricci scalar for the metric } g_{ab} \\ n_a &= \pm \frac{\partial_a f}{\sqrt{g^{bc} \partial_b f \partial_c f}} = \text{The unit normal to the boundary } \partial\mathcal{M} \\ h_{ab} &= g_{ab} - n_a n_b = \text{Induced metric on } \partial\mathcal{M}, \text{ such that } h^{ab} n_b = 0, \\ K_g &= g^{ab} \nabla_a n_b = \text{Trace of the extrinsic curvature of } \partial\mathcal{M} \end{aligned}$$

Now let us consider the following transformation parameterized by the function $A(x, y)$

$$\begin{aligned} \phi(x, y) &\rightarrow \tilde{\phi}(x, y) = \phi(x, y) - A(x, y) \\ g_{ab}(x, y) &\rightarrow \tilde{g}_{ab}(x, y) = e^{2A(x, y)} g_{ab}(x, y) \end{aligned} \quad (5.5.3)$$

such that the final metric in (5.5.1) is invariant.

Let us note the following relations¹³

$$\begin{aligned} \sqrt{g} &= e^{2A} \sqrt{\tilde{g}}, \quad \tilde{n}_a = e^A n_a, \quad \sqrt{h} = e^A \sqrt{\tilde{h}}, \\ \tilde{R}_{\tilde{g}} &= e^{-2A} [R_g - 2\nabla^2 A], \quad \tilde{K}_{\tilde{g}} = e^{-A} [K_g + n^a \partial_a A] \end{aligned} \quad (5.5.6)$$

¹³For deriving the last relation in (5.5.6) we used

$$\tilde{n}_a = \pm \frac{\partial_a f}{\sqrt{\tilde{g}^{bc} \partial_b f \partial_c f}} = \pm e^{-A} \frac{\partial_a f}{\sqrt{g^{bc} \partial_b f \partial_c f}} = e^{-A} n_a, \quad (5.5.4)$$

and also

$$\tilde{K}_{\tilde{g}} = \tilde{g}^{ab} \tilde{\nabla}_a \tilde{n}_b = \frac{1}{\sqrt{\tilde{g}}} \partial_a (\sqrt{\tilde{g}} \tilde{n}^a) = \frac{e^{2A}}{\sqrt{g}} \partial_a (e^A \sqrt{g} n^a) = e^{-A} [K_g + n^a \partial_a A] \quad (5.5.5)$$

Therefore the transformed Liouville action becomes

$$S_L[\tilde{\phi}, \tilde{g}_{ab}] = \frac{c}{24\pi} \int_{\mathcal{M}} d^2x \sqrt{\tilde{g}} \left[\tilde{g}^{ab} \partial_a \tilde{\phi} \partial_b \tilde{\phi} + e^{2\tilde{\phi}} + \tilde{R}_{\tilde{g}} \tilde{\phi} \right] + \frac{c}{12\pi} \int_{\partial\mathcal{M}} ds \sqrt{\tilde{h}} \tilde{K}_{\tilde{g}} \tilde{\phi} \quad (5.5.7)$$

Using the relations in (5.5.6) it can be checked that

$$\begin{aligned} & \int_{\mathcal{M}} d^2x \sqrt{\tilde{g}} \left[\tilde{g}^{ab} \partial_a \tilde{\phi} \partial_b \tilde{\phi} + e^{2\tilde{\phi}} + \tilde{R}_{\tilde{g}} \tilde{\phi} \right] \\ &= \int_{\mathcal{M}} d^2x \sqrt{g} \left[g^{ab} \partial_a \phi \partial_b \phi + e^{2\phi} + R_g \phi \right] - \int_{\mathcal{M}} d^2x \sqrt{g} g^{ab} \partial_a A \partial_b A \\ & \quad - \int_{\mathcal{M}} d^2x \sqrt{g} R_g A - 2 \int_{\mathcal{M}} d^2x \partial_a [\sqrt{g} g^{ab} (\phi - A) \partial_b A] \end{aligned} \quad (5.5.8)$$

and note that the last term on the RHS above is a total derivative, and contribute to the boundary term. Also, it can be further checked that

$$\begin{aligned} 2 \int_{\partial\mathcal{M}} ds \sqrt{\tilde{h}} \tilde{K}_{\tilde{g}} \tilde{\phi} &= 2 \int_{\partial\mathcal{M}} ds \sqrt{h} K_g \phi - 2 \int_{\partial\mathcal{M}} ds \sqrt{h} K_g A \\ & \quad + 2 \int_{\partial\mathcal{M}} ds \sqrt{h} (\phi - A) n^a \partial_a A \end{aligned} \quad (5.5.9)$$

The last term on the RHS of (5.5.9) and the last term, i.e. the boundary term, on the RHS of (5.5.8) will cancel each other. Therefore, we can combine (5.5.8) and (5.5.9) to obtain

$$\begin{aligned} S_L[\tilde{\phi}, \tilde{g}_{ab}] &= S_L[\phi, g_{ab}] - \frac{c}{24\pi} \int_{\mathcal{M}} d^2x \sqrt{g} \left[g^{ab} \partial_a A \partial_b A + R_g A \right] \\ & \quad - \frac{c}{12\pi} \int_{\partial\mathcal{M}} ds \sqrt{h} K_g A. \end{aligned} \quad (5.5.10)$$

Note that the extra terms involving A , the second third term, on the RHS of (5.5.10) looks similar to Liouville action for the field A except for the missing potential e^{2A} term. This motivates us to add an extra term : $-\int_{\mathcal{M}} d^2x \sqrt{g}$ in the Liouville action S_L in (5.5.2), such that we now define an “improved Liouville action” $I_L[g_1, g_2]$ (g_1 is the final metric and g_2 is the reference metric) as follows

$$\begin{aligned} I_L[e^{2\phi} g, g] &= \frac{c}{24\pi} \int_{\mathcal{M}} d^2x \sqrt{g} \left[g^{ab} \partial_a \phi \partial_b \phi + (e^{2\phi} - 1) + R_g \phi \right] \\ & \quad + \frac{c}{12\pi} \int_{\partial\mathcal{M}} ds \sqrt{h} K_g \phi. \end{aligned} \quad (5.5.11)$$

For this action, we can find the following relation:

$$I_L[e^{2\tilde{\phi}} \tilde{g}, \tilde{g}] = I_L[e^{2\phi} g, g] - I_L[\tilde{g}, g]. \quad (5.5.12)$$

This expression is naturally interpreted that the action $I_L[e^{2\tilde{\phi}}\tilde{g}, \tilde{g}]$ measures the difference between the final metric $e^{2\tilde{\phi}}\tilde{g} = e^{2\phi}g$ and the reference metric $\tilde{g} = e^{2A}g$. In other words, this relation shows the chain rule:

$$I_L[g_1, g_2] + I_L[g_2, g_3] = I_L[g_1, g_3], \quad (5.5.13)$$

which includes the identity $I_L[g_2, g_1] = -I_L[g_1, g_2]$. The different between S_L and I_L does not depend on the Liouville field ϕ and thus the equations of motion from the variations of ϕ in both actions are the same. It is useful to note that if g_1 is the flat metric and the transformation from g_1 and g_2 is holomorphic, then we have $I_L[g_2, g_1] = 0$, in other words the second plus third term in (5.5.10) is vanishing.

In summary, $S_L[\phi, g]$ does not provide us with an absolute quantity which measures the complexity of the optimized state because it depends not only on the final metric $e^{2\phi}g$ but also on the reference metric g . Rather, we find it is convenient to look at the relative quantity $I_L[g_2, g_1]$ which is expected to measure the difference of complexity between the path-integral in g_2 and g_1 .

Before we move onto explicit evaluations of $I_L[g_2, g_1]$, we would like to mention that another potential source of ambiguity. We need to be careful with the fact that a constant shift of ϕ can change the action S_L when the Euler number is non-zero due to the Gauss-Bonnet term $\int R_g \phi + 2 \int K_g \phi$ in S_L . This is removed by placing the background charge as in the standard computation of correlation in Liouville CFTs [151] and we will follow this prescription.

5.5.2 Vacuum States

Let us start with vacuum states in 2D CFTs on a circle with the circumference 2π . In $\text{AdS}_3/\text{CFT}_2$, they are dual to the global AdS_3 . As we explained in section 5.3.2, we obtained the Poincare disk metric (5.3.8) after the optimization. This metric can be written in the following two ways:

$$ds^2 = e^{2\phi}(dr^2 + r^2 d\theta^2), \quad e^{2\phi} = \frac{4}{(1-r^2)^2}, \quad (5.5.14)$$

$$ds^2 = e^{2\tilde{\phi}}(dz^2 + d\theta^2), \quad e^{2\tilde{\phi}} = \frac{1}{\sinh^2 z}, \quad (5.5.15)$$

where θ has a periodicity 2π . We introduce the cut off $z = \epsilon$ and $r = r_0$ such that $\frac{2r_0}{1-r_0^2} = \frac{1}{\epsilon}$ or equally $r_0 \simeq 1 - \epsilon + \frac{\epsilon^2}{2} + \dots$. We express the flat metric for the polar coordinate (r, θ) and the Cartesian one (z, θ) by $g_{(r, \theta)}$ and $g_{(z, \theta)}$, respectively. Also the Poincare disk metric

$e^{2\phi}g_{(r,\theta)} = e^{2\tilde{\phi}}g_{(z,\theta)}$ is represented by g_{AdS} .

The Liouville action for (5.5.14) is evaluated as

$$\begin{aligned}
S_L[\phi, g_{(r,\theta)}] &= \frac{c}{24\pi} \int_0^{r_0} r dr \int_0^{2\pi} d\theta [(\partial_r \phi)^2 + e^{2\phi}] + \frac{c}{12\pi} \int ds K_0 \phi \\
&= \frac{c}{12} \int_0^{r_0} dr \left(\frac{4(r^3 + r)}{(1 - r^2)^2} \right) + \frac{c}{6} (\phi(r = r_0) - \phi(r = 0)) \\
&= \frac{c}{12} \left(\frac{2}{\epsilon} + 2 \log \epsilon - 2 + 2 \log 2 \right) + \frac{c}{6} (-\log \epsilon - \log 2) \\
&= \frac{c}{6} \left(\frac{1}{\epsilon} - 1 \right).
\end{aligned} \tag{5.5.16}$$

In the above, the contribution $\propto -\phi(r = 0) = -\log 2$ comes from the background charge, while another one $\propto \phi(r = r_0) = -\log \epsilon$ is the standard boundary contribution. Finally, as before, K_0 corresponds to the trace of the extrinsic curvature of the boundary evaluated in the flat metric.

Similarly we can evaluate the Liouville action for (5.5.15)

$$S_L[\tilde{\phi}, g_{(z,\theta)}] = \frac{c}{12} \int_{\epsilon}^{z_{\infty}} \left[\frac{\cosh^2 z + 1}{\sinh^2 z} \right] = \frac{c}{6} \left(\frac{1}{\epsilon} - 1 + \frac{z_{\infty}}{2} \right), \tag{5.5.17}$$

where $z_{\infty} (\rightarrow \infty)$ is the IR cut off in the z integral. Indeed this expression differs from (5.5.16).

We can also calculate the Liouville action for the Weyl scaling $g_{(r,\theta)} = e^{-2z} g_{(z,\theta)}$

$$S_L[-z, g_{(z,\theta)}] = \frac{c}{12} \cdot \int_{\epsilon}^{z_{\infty}} (1 + e^{-2z}) = \frac{c}{24\pi} (\pi + 2\pi z_{\infty}). \tag{5.5.18}$$

In terms of the improved Liouville action, we can summarize our results as follows:

$$I_L[g_{AdS}, g_{(r,\theta)}] = \frac{c}{6\epsilon} - \frac{5c}{24}, \tag{5.5.19}$$

$$I_L[g_{AdS}, g_{(z,\theta)}] = \frac{c}{6\epsilon} - \frac{c}{6}, \tag{5.5.20}$$

$$I_L[g_{(r,\theta)}, g_{(z,\theta)}] = \frac{c}{24}, \tag{5.5.21}$$

which indeed satisfy (5.5.13).

5.5.3 Primary States

As we have seen in section (5.3.2), for the primary states, the optimized metric is given by the conical geometry (denoted by g_C):

$$ds^2 = e^{2\phi}(dr^2 + r^2 d\theta^2), \quad e^{2\phi} = \frac{4a^2}{r^{2(1-a)}(1 - r^{2a})^2}, \tag{5.5.22}$$

$$ds^2 = e^{2\tilde{\phi}}(dz^2 + d\theta^2), \quad e^{2\tilde{\phi}} = \frac{a^2}{\sinh^2(az)}, \tag{5.5.23}$$

where we define $r = e^{-z}$ with $-z_\infty < z < -\epsilon$, $\delta(\equiv e^{-z_\infty}) < r < r_0$ and $0 \leq \theta < 2\pi$. The UV cut off $r = r_0$ is specified as

$$\frac{2ar_0^a}{1 - r_0^{2a}} = \frac{1}{\epsilon}, \quad (5.5.24)$$

which is solved as $r_0^a = -a\epsilon + \sqrt{1 + (a\epsilon)^2} \simeq 1 - a\epsilon + (a\epsilon)^2/2 + \dots$.

The Liouville action for (5.5.23) is evaluated as follows:

$$S_L[\tilde{\phi}, g_{(z,\theta)}] = \frac{ca^2}{12} \int_\epsilon^{z_\infty} dz \left[\frac{\cosh^2(az) + 1}{\sinh^2(az)} \right] = \frac{c}{24\pi} \left(\frac{4\pi}{\epsilon} - 4\pi a + 2\pi a^2 z_\infty \right). \quad (5.5.25)$$

On the other hand, the Liouville action for (5.5.22) becomes

$$\begin{aligned} S_L[\phi, g_{(r,\theta)}] &= \frac{c}{12} \int_\delta^{r_0} r dr [(\partial_r \phi)^2 + e^{2\phi}] + 4\pi(\phi(r_0) - \phi(\delta)) \\ &= \frac{c}{12} \int_\epsilon^{z_\infty} dz \left[(\partial_z \tilde{\phi} + 1)^2 + e^{2\tilde{\phi}} \right] + \frac{c}{6} \log \frac{\delta}{r_0} + \frac{c}{6} (\tilde{\phi}(\epsilon) - \tilde{\phi}(z_\infty)) \\ &= S_L[\tilde{\phi}, g_{(z,\theta)}] - \frac{c}{12} z_\infty. \end{aligned} \quad (5.5.26)$$

In summary, we obtain the relative actions:

$$\begin{aligned} I_L[g_C, g_{(r,\theta)}] &= \frac{c}{6\epsilon} - \frac{ca}{6} - \frac{c}{24} + \frac{c}{12}(a^2 - 1)z_\infty, \\ I_L[g_C, g_{(z,\theta)}] &= \frac{c}{6\epsilon} - \frac{ca}{6} + \frac{c}{12}(a^2 - 1)z_\infty, \\ I_L[g_{(r,\theta)}, g_{(z,\theta)}] &= \frac{c}{24}, \end{aligned} \quad (5.5.27)$$

which indeed satisfy (5.5.13). When $a = 1$, they are reduced to (5.5.19), (5.5.20), (5.5.21). For $a \neq 1$, the actions for g_C include IR divergences, which may be canceled by further adding a background charge at the conical singularity.

5.5.4 Finite Temperature State

Finally we turn to the thermofield double (TFD) states dual to BTZ black holes. As discussed in section 5.3.1, after the optimization we obtained the metric of Einstein-Rosen bridge (denoted by g_{ER}):

$$ds^2 = e^{2\phi}(dz^2 + d\theta^2), \quad e^{2\phi} = \frac{\left(\frac{2\pi}{\beta}\right)^2}{\cos^2\left(\frac{2\pi z}{\beta}\right)}, \quad (5.5.28)$$

$$ds^2 = e^{2\tilde{\phi}}(dr^2 + r^2 d\theta^2), \quad \tilde{\phi} = \phi - \log r, \quad (5.5.29)$$

where $r = e^{-z}$ with $-\beta/4 < z < \beta/4$ and $0 \leq \theta < 2\pi$.

For the metric (5.5.28), the action is evaluated as follows:

$$\begin{aligned}
S_L[\phi, g_{(z,\theta)}] &= \frac{c}{12} \int_{-\beta/4+\epsilon}^{\beta/4-\epsilon} dz \left[\frac{4\pi^2}{\beta^2} (\tan^2(2\pi z/\beta) + \cos^{-2}(2\pi z/\beta)) \right] \\
&= \frac{c}{6} \left(\frac{2\pi}{\beta} \right)^2 \cdot \int_0^{\beta/4-\epsilon} dz \left[\frac{1 + \sin^2(2\pi z/\beta)}{\cos^2(2\pi z/\beta)} \right] \\
&= \frac{c}{3\epsilon} - \frac{\pi^2 c}{6\beta}.
\end{aligned} \tag{5.5.30}$$

On the other hand for the metric (5.5.29), the action is evaluated as follows:

$$\begin{aligned}
S_L[\tilde{\phi}, g_{(r,\theta)}] &= \frac{c}{12} \int_{r_1}^{r_2} r dr \left[(\partial_r \tilde{\phi})^2 + e^{2\tilde{\phi}} \right] + \frac{c}{6} (\tilde{\phi}(r=r_2) - \tilde{\phi}(r=r_1)) \\
&= \frac{c}{12} \int_{-\beta/4+\epsilon}^{\beta/4-\epsilon} dz ((\partial_z \phi - 1)^2 + e^{2\phi}) + \frac{c}{6} (\tilde{\phi}(r=r_2) - \tilde{\phi}(r=r_1)) \\
&= \frac{c}{3\epsilon} - \frac{\pi^2 c}{6\beta} - \frac{c}{24} \beta,
\end{aligned} \tag{5.5.31}$$

where $r_1 = e^{-\frac{\beta}{4}+\epsilon}$ and $r_2 = e^{\frac{\beta}{4}-\epsilon}$.

The relative action is computed as follows

$$I_L[g_{ER}, g_{(z,\theta)}] = \frac{c}{3\epsilon} - \frac{\pi^2 c}{6\beta} - \frac{c}{24} \beta, \tag{5.5.32}$$

$$I_L[g_{ER}, g_{(r,\theta)}] = \frac{c}{3\epsilon} - \frac{\pi^2 c}{6\beta} - \frac{c}{24} \beta - \frac{c}{24} (e^{\beta/2} - e^{-\beta/2}), \tag{5.5.33}$$

$$I_L[g_{(r,\theta)}, g_{(z,\theta)}] = \frac{c}{24} (e^{\beta/2} - e^{-\beta/2}), \tag{5.5.34}$$

which again satisfy (5.5.13).

5.6 Application to NAdS₂/CFT₁

As recently discovered, to make sense of AdS₂/CFT₁, we need a conformal symmetry breaking term [35, 152, 153, 154], under the reparameterization of $\tilde{\tau} = \tilde{\tau}(\tau)$, often called NAdS₂/CFT₁ duality. The effective action is written as a Schwarzian derivative term $Sch[\tilde{\tau}, \tau]$ as explicitly realized in the Sachdev-Ye-Kitaev (SYK) model [155, 34]:

$$Sch[\tilde{\tau}, \tau] = -\frac{3}{2} \left(\frac{\partial_\tau^2 \tilde{\tau}}{\partial_\tau \tilde{\tau}} \right)^2 + \frac{\partial_\tau^3 \tilde{\tau}}{\partial_\tau \tilde{\tau}}. \tag{5.6.1}$$

For the one dimensional metric $ds^2 = e^{2\phi} d\tau^2$, we can identify $\frac{d\tilde{\tau}}{d\tau} = e^\phi$. Thus the conformal symmetry breaking term (5.6.1) looks like $N \int d\tau (\partial_\tau \phi)^2$, where N is a constant proportional

to degrees of freedom. Therefore we find (we shifted ϕ appropriately)

$$\begin{aligned}\Psi_{g_{\tau\tau}=e^{2\phi}}(\tilde{\varphi}(x)) &= e^{S_1[\phi]-S_1[0]} \cdot \Psi_{g_{\tau\tau}=1}(\tilde{\varphi}(x)), \\ S_1[\phi] &= N \int d\tau [(\partial_\tau \phi)^2 + 2e^\phi].\end{aligned}\tag{5.6.2}$$

By minimizing the action, this again leads to

$$ds^2 = e^{2\phi} d\tau^2 = \frac{d\tau^2}{\tau^2}.\tag{5.6.3}$$

This is consistent with the time slice of AdS_2 space-time. Note that if there is no conformal symmetry breaking effect, we cannot stabilize the optimization procedure. Also notice that in standard tensor network descriptions, it is very difficult to describe one dimensional quantum mechanics as we normally coarse-grain space directions to build an extra dimension in the network. In our path-integral approach the extra dimension arises naturally even in quantum mechanics.

5.7 Applications to Higher Dimensional CFTs

Higher dimensional generalizations of our optimization procedure do not seem to be straightforward as the generic metric cannot be expressed only by the Weyl scaling as in (5.2.5). Nevertheless, it is useful to see what optimization can lead to correct time slices of gravity duals by taking into account only the Weyl scaling degrees of freedom as a first step toward this direction. As we will see below, at least for pure AdS_{d+1} we can obtain expected results even from this limited range of optimization.

5.7.1 Our Formulation

For this we need a complexity functional $I[\phi]$ for the metric of the form:

$$ds^2 = e^{2\phi(x)} g_{ab} dx^a dx^b,\tag{5.7.1}$$

with x regarded as d dimensional vector (which includes “ z ” coordinate). We propose that for a vacuum state in a d dimensional CFT, the optimization can be done by minimizing the following functional $I_d^{bulk}[\phi, g]$ (N is a normalization factor proportional to the degrees of freedom):

$$I_d^{bulk}[\phi, g] = N \int_\Sigma dx^d \sqrt{g} \left[e^{d\phi} + e^{(d-2)\phi} (g^{ab} \partial_a \phi \partial_b \phi) + \frac{e^{(d-2)\phi}}{(d-1)(d-2)} R_g \right],\tag{5.7.2}$$

where R_g is the Ricci scalar for the reference metric g . Reader can regard this as a generalization of Liouville action to general dimensions, which is quadratic in derivative of ϕ field¹⁴. The computation of entanglement entropy shown later allows us to identify the normalization factor N in (5.7.2) for holographic CFTs:

$$N = \frac{(d-1)R^{d-1}}{16\pi G_N}, \quad (5.7.3)$$

where R is the AdS radius. In particular for $d = 2$ and $d = 4$ we find

$$\begin{aligned} N_{d=2} &= \frac{c}{24\pi}, \\ N_{d=4} &= \frac{3}{2\pi^2} a_4, \end{aligned} \quad (5.7.4)$$

in terms of the central charge c in 2D CFTs and a_4 in 4D CFTs.

Indeed, the minimization of $I_d^{bdy}[\phi, g]$ leads to the hyperbolic space H_d which is the time slice of pure AdS_{d+1} as we will see later. For the optimization of reduced density matrix we need to introduce the boundary term as in section 5.4. We argue it is given by

$$I_d^{bdy}[\phi, g] = 2N \int_{\partial\Sigma} dx^{d-1} \sqrt{\gamma} \left[\frac{K_g}{d-1} \frac{e^{(d-2)\phi}}{d-2} + \mu_B \frac{e^{(d-1)\phi}}{d-1} \right], \quad (5.7.5)$$

where γ_{ij} is the induced metric on the boundary $\partial\Sigma$. This again leads to the boundary condition $K + (d-1)\mu_B = 0$, where $K = e^{-\phi}((d-1)\partial_n\phi + K_0)$. We defined our optimization by minimizing the total functional $I_d^{tot}[\phi, g]$

$$I_d^{tot}[\phi, g] = I_d^{bulk}[\phi, g] + I_d^{bdy}[\phi, g]. \quad (5.7.6)$$

It is important to consider the limit $d \rightarrow 2$ in $I_d^{tot}[\phi, g]$ and explore the possibility of recovering the standard Liouville action for $d = 2$ dimensions. As it is obvious from the expression of $I_d^{bulk}[\phi, g]$ in (5.7.2), a naive limit of $d \rightarrow 2$ is singular as the third term on the RHS of (5.7.2) gives a contribution proportional to $1/(d-2)$. However, as it was mentioned in section 5.5.1 (see the discussion in the paragraphs following (5.5.11)), for an absolute measure of complexity in 2-dimensions the relative or the improved Liouville action $I_L(g_1, g_2)$, defined in (5.5.11), is more suitable compared to S_L , defined in (5.5.2). The advantage was mentioned to be the fact that $I_L(g_1, g_2)$ being a relative measure of complexity does not depend on the reference metric. It is interesting to note that the subtlety of taking

¹⁴The possibility of having terms in the complexity functional with higher than quadratic derivatives of ϕ is discussed in section 8.8 and in appendix C of [24]. In fact they will be important for reproducing correct anomalies for even dimensional CFTs in higher dimensions.

the $d \rightarrow 2$ limit works out perfectly if we subtract away the contribution of the reference metric while taking the above mentioned limit. Therefore we notice the following identity

$$\lim_{d \rightarrow 2} \left[I_d^{tot}[\phi, g] - I_d^{tot}[\phi = 0, g] \right] = I_L[e^{2\phi}g, g], \quad (5.7.7)$$

where g_{ab} is considered as the reference 2-dimensional metric and therefore following the discussion after (5.5.11), $I_L[e^{2\phi}g, g]$ computes the relative complexity of the generic metric $e^{2\phi}g_{ab}$ compared to that reference metric.

Notice that actually we can combine the functional $I_d + I_d^{bdy}$ into the Einstein-Hilbert action plus a cosmological constant on the final metric (5.7.1) which we write $\tilde{g} = e^{2\phi}g$ and $\tilde{\gamma} = e^{2\phi}\gamma$:

$$\begin{aligned} I_d^{tot} = & N \int_{\Sigma} dx^d \sqrt{\tilde{g}} \left[1 + \frac{R_{\tilde{g}}}{(d-1)(d-2)} \right] \\ & + 2N \int_{\partial\Sigma} dx^{d-1} \sqrt{\tilde{\gamma}} \left[\frac{K_g}{(d-1)(d-2)} + \frac{\mu_B}{d-1} \right]. \end{aligned} \quad (5.7.8)$$

From this manifestly covariant expression which only depends on $\tilde{g}$, as opposed to the 2D Liouville action, the invariance of the action by the change of reference metric is manifest:

$$I_d^{tot}[\phi - A, e^{2A}g] = I_d^{tot}[\phi, g]. \quad (5.7.9)$$

5.7.2 AdS_{d+1} from Optimization

Here we would like to confirm that the optimization leads to expected AdS geometries for vacuums states. This is almost obvious from the expression (5.7.8). However notice that we take only the Weyl mode ϕ dynamical.

Consider a CFT_d defined on R^d or $R \times S^{d-1}$. In these two cases the metrics are taken in the following form:

$$R^d : \quad ds^2 = e^{2\phi(z)} \left(dz^2 + \sum_{i=1}^{d-1} dx_i^2 \right), \quad (5.7.10)$$

$$R \times S^{d-1} : \quad ds^2 = e^{2\phi(r)} (dr^2 + r^2 d\Omega_{d-1}^2). \quad (5.7.11)$$

The values of the functional I_d in these cases read

$$R^d : \quad I_d^{bulk} = N \int dx^d [e^{d\phi} + e^{(d-2)\phi} (\partial_z \phi)^2], \quad (5.7.12)$$

$$R \times S^{d-1} : \quad I_d^{bulk} = N \int d\Omega_{d-1} \int dr \cdot r^{d-1} [e^{d\phi} + e^{(d-2)\phi} (\partial_r \phi)^2], \quad (5.7.13)$$

and their equation of motions are given by

$$\mathbf{R}^d : de^{2\phi} - (d-2)e^{(d-2)\phi}(\partial_z\phi)^2 - 2\partial_z^2\phi = 0, \quad (5.7.14)$$

$$\mathbf{R} \times \mathbf{S}^{d-1} : de^{2\phi} - (d-2)e^{(d-2)\phi}(\partial_r\phi)^2 - 2\partial_r^2\phi - \frac{2(d-1)}{r}\partial_r\phi = 0. \quad (5.7.15)$$

We can confirm that both of them have the hyperbolic space solutions:

$$\mathbf{R}^d : e^{2\phi(z)} = \frac{1}{z^2}, \quad (5.7.16)$$

$$\mathbf{R} \times \mathbf{S}^{d-1} : e^{2\phi(r)} = \frac{4}{(1-r^2)^2}. \quad (5.7.17)$$

They coincide with the time slice of AdS_{d+1} as expected.

5.7.3 Excitations in Global AdS_{d+1}

Now let us consider excitations in a d dimensional CFT on $\mathbf{R} \times \mathbf{S}^{d-1}$. We focus on the case $d = 3, 4$ and assume that the excitations are homogeneous and static. In $\text{AdS}_{d+1}/\text{CFT}_d$, such a state is dual to spherically symmetric solution given by the AdS_4 Schwarzschild black hole solution

$$ds^2 = -h(\rho)dt^2 + \frac{d\rho^2}{h(\rho)} + \rho^2 d\Omega_{d-1}^2. \quad (5.7.18)$$

with $h(\rho) = \rho^2 + 1 - M\rho^{-(d-2)}$. Here we are interested in the leading correction when M is very small. We focus on the time slice $t = 0$ and rewrite

$$ds^2 = \frac{d\rho^2}{h(\rho)} + \rho^2 d\Omega_{d-1}^2 = e^{2\phi}(dr^2 + r^2 d\Omega_{d-1}^2). \quad (5.7.19)$$

We can find an explicit relation between ρ and r and the function ϕ as follows (up to the linear order of M)

$$\begin{aligned} r &\simeq \frac{\rho}{1 + \sqrt{1 + \rho^2}} \cdot (1 + Mf(\rho)), \\ e^{\phi(r)} &\simeq (1 + \sqrt{1 + \rho^2}) \cdot (1 - Mf(\rho)), \end{aligned} \quad (5.7.20)$$

where the function $f(r)$ depends on the dimension d

$$\begin{aligned} d = 3 : \quad f(\rho) &= 1 - \frac{2\rho^2 + 1}{2\rho\sqrt{\rho^2 + 1}}, \\ d = 4 : \quad f(\rho) &= -\frac{1}{4} \left[\frac{3\rho^2 + 1}{\rho^2\sqrt{\rho^2 + 1}} + 3 \log \left[\frac{\rho}{\sqrt{\rho^2 + 1} + 1} \right] \right]. \end{aligned} \quad (5.7.21)$$

Finally we obtain the function $\phi(r)$ in (5.7.19) in the form

$$e^{\phi(r)} \simeq \frac{2}{(1-r^2)} \left(1 + M \cdot \eta(r) \right), \quad (5.7.22)$$

where the function $\eta(r)$ is explicitly given for each d :

$$d = 3 : \quad \eta(r) = \frac{(1-r)^3}{4r(1+r)}, \quad (5.7.23)$$

$$d = 4 : \quad \eta(r) = \frac{r^6 + 9r^4 - 9r^2 - 12(r^4 + r^2) \log(r) - 1}{16r^2(r^2 - 1)}. \quad (5.7.24)$$

On the other hand, the optimization of our functional (5.7.2) given by the differential equation (5.7.15) leads to the perturbative solution of the form:

$$e^\phi = \frac{2}{(1-r^2)} \left(1 + \tilde{M} \cdot h(r) \right) + O(\tilde{M}^2), \quad (5.7.25)$$

where we treat $\tilde{M}$ as an infinitesimally small parameter. We can analytically determine the function $h(r)$ and confirm that $h(r)$ is equal to $\eta(r)$ in (5.7.23) and (5.7.24) in each dimension up to a constant factor.

In this way we find that the first order back-reaction to the time slice metric in AdS gravity is correctly reproduced by our optimization procedure.

5.7.4 Holographic Entanglement Entropy

In this subsection we will show that the total action $I_d^{bulk} + I_d^{bdy}$ can reproduce the correct holographic entanglement entropy (HEE) [2] when the subsystem A is a round ball. We will also focus on $d = 3, 4$ case in the $\text{AdS}_{d+1}/\text{CFT}_d$.

We will closely follow the method that was explicitly used in the case of 2D CFT in section 5.4. We start with the holographic construction of density matrix and argue that this accurately reproduce the entanglement wedge as expected and following that we will compute the entanglement entropy holographically.

The metric of the manifold on which the path-integral for the density matrix is being computed, will be denoted by

$$ds^2 = e^{2\phi} \left(dz^2 + \sum_{i=1}^{d-1} dx_i^2 \right) = e^{2\phi} (dz^2 + dr^2 + r^2 d\Omega_{d-2}^2), \quad (5.7.26)$$

where, sometimes, we will also use the notation $ds^2 = e^{2\phi} g_{ab} dx^a dx^b$, with the understanding that the reference metric g_{ab} is the flat metric $g_{ab} dx^a dx^b = dz^2 + \sum_{i=1}^{d-1} dx_i^2 = dz^2 + dr^2 +$

$r^2 d\Omega_{d-2}^2$. Also, $d\Omega_{d-2}^2$ is the metric for $(d-2)$ -dimensional unit sphere. Therefore, we will have

$$\begin{aligned} d=3 &\Rightarrow d\Omega_1^2 = d\theta^2 \\ d=4 &\Rightarrow d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi_1^2 \end{aligned} \quad (5.7.27)$$

The round ball subsystem is defined by $A_D = \{x_i | r \leq \ell\}$, $r = \sqrt{\sum_{i=1}^{d-1} x_i^2}$, where ℓ is the radius of the circular disk.

Following the same steps, as depicted in section 5.4, we should proceed with the optimization that will lead us to identifying the boundaries $\Gamma_A^{(d)}$ of the bulk regions $\Sigma_{\pm}^{(d)}$. The boundary condition for ϕ should be imposed at the two edges of the slit composed by the boundary of the round ball A_D , i.e. at $r = \ell$ near $z = 0$. These arguments validate that we should also consider the boundary part of the action I_d^{bdy}

$$I_d^{bdy} = 2N \int_{\partial\Sigma} d^{d-1}x \sqrt{\gamma} \left[\frac{K_g}{d-1} \frac{e^{(d-2)\phi}}{d-2} + \mu_B \frac{e^{(d-1)\phi}}{d-1} \right] \quad (5.7.28)$$

Let us describe the boundary Γ_A as $r = f(z)$ with the normal vectors (normalized with respect to the full metric $e^{2\phi}g_{ab}$) given by

$$n^z = -\frac{e^{-\phi}f'(z)}{\sqrt{1+f'(z)^2}}, \quad n^r = \frac{e^{-\phi}}{\sqrt{1+f'(z)^2}}, \quad n^{\Omega_{d-2}} = 0. \quad (5.7.29)$$

The extrinsic curvature of the boundary is

$$K = \gamma^{ab} \nabla_a n_b = e^{-\phi} (K_0 + (d-1)n^a \partial_a \phi). \quad (5.7.30)$$

where K_0 is the extrinsic curvature for the same boundary but in the reference metric g_{ab} (which is flat metric in our case)

$$K_0 = -\frac{f''}{(1+(f')^2)^{3/2}}. \quad (5.7.31)$$

The boundary condition for the field on the edge of the slit (the round ball A_D in our case) is Dirichlet, however we need to impose Neumann boundary condition on the surface Γ_A and it leads us to the condition

$$K + (d-1)\mu_B = 0. \quad (5.7.32)$$

For the determination of the density matrix, since the two boundaries Γ_A in Σ_+ and that in Σ_- are pasted smoothly, we should consider $\mu_B = 0$. The optimization determines the bulk metric to be the hyperbolic one $\phi = -\log z$.

In order to fix the shape of Γ_A we should solve $K = 0$, which is precisely the condition of minimal surfaces. Thus we find that Γ_A is given by the half-sphere $z^2 + r^2 = \ell^2$. Accordingly, the holographic dual for the density matrix corresponds to the region $z^2 + r^2 \leq \ell^2$ and it agrees with the entanglement wedge.

Let us now consider the entanglement entropy and for that we need to consider ρ_A^n and finally analytically continue considering $|n - 1| \ll 1$, leading us to a conical geometry with deficit angle $2\pi(1 - n)$ along the entangling surface $r = \ell$. It is then natural to expect that the extrinsic curvature for the boundaries Γ_A will now become different from vanishing, i.e. $K \neq 0$. This can be estimated from considering that the boundaries Γ_A now changes from $z^2 + r^2 = \ell^2$ to $(z - z_0)^2 + r^2 = \ell^2$, with which we can now evaluate the extrinsic curvature

$$K = -(d - 1) \frac{z_0}{l}. \quad (5.7.33)$$

Also with infinitesimal z_0 we obtain $r \sim l + \frac{z_0}{l}z + \mathcal{O}(z^2)$ near the boundary point $\{z, r, \Omega^{(d-2)}\} = \{0, \ell, \Omega^{(d-2)}\}$. The corner angle at the intersection of $\partial\Sigma_{\pm}^{(d-1)}$ and the entangling surface $r = \ell$, also becomes $\pi/2 + z_0/l$. For the n-sheeted conical geometry we can interpret this corner angle as $z_0/l = \pi(1 - n)$ and thus obtain the relation

$$K = (d - 1)(n - 1). \quad (5.7.34)$$

We need to satisfy the Neumann boundary condition $K + (d - 1)\mu_B = 0$ at $\partial\Sigma_{\pm}^{(d-1)}$, and we implement this by setting

$$\mu_B = \pi(1 - n). \quad (5.7.35)$$

Note that this condition is true for any dimension d .

The entanglement entropy in 3-d CFT : Now one can explicitly check that the holographic entanglement entropy can be evaluated for $d = 3$ by considering $\mu_B = \pi(1 - n)$ in the boundary action in (5.7.28) as follows

$$\begin{aligned} S_A^{(d=3)} &= -\partial_n \left[2N\mu_B \int_{\partial\Sigma_+^{(2)}} \frac{e^{2\phi}}{2} + 2N\mu_B \int_{\partial\Sigma_-^{(2)}} \frac{e^{2\phi}}{2} \right] \\ &= 4N\pi^2 \left[\frac{\ell}{\epsilon} - 1 \right] \end{aligned} \quad (5.7.36)$$

where ϵ is the UV cut-off as the range of integration in r has been taken to be $\epsilon \leq r \leq \ell$.

The entanglement entropy in 4-d CFT : Similarly for $d = 4$ one obtains

$$\begin{aligned} S_A^{(d=4)} &= -\partial_n \left[2N\mu_B \int_{\partial\Sigma_+^{(3)}} \frac{e^{3\phi}}{3} + 2N\mu_B \int_{\partial\Sigma_-^{(3)}} \frac{e^{3\phi}}{3} \right] \\ &= \frac{8N\pi^2}{3} \left[\frac{\ell^2}{\epsilon^2} - \log\left(\frac{\ell}{\epsilon}\right) + \left(\frac{1}{2} + \log 2\right) \right] \end{aligned} \quad (5.7.37)$$

Finally, It can be checked that the expressions in both (5.7.36) and (5.7.37) do indeed reproduce the correct behavior for holographic entanglement entropy in higher dimensions, compare with [2], by choosing the normalization N as in (5.7.3) and (5.7.4).

Moreover, we confirm that for the spherical choice of the region the general formula for entanglement entropy reads

$$S_A^d = \frac{4\pi N}{d-1} \int_{\Gamma_A} e^{(d-1)\phi}. \quad (5.7.38)$$

5.7.5 Evaluation of Complexity Functional

As we argued for 2D CFTs (see discussions following (5.2.13)), the Liouville action, S_L when computed on-shell for the solutions gives us a measure of holographic computational complexity. Here we would like to examine an analogous quantity for the higher dimensional CFTs. Namely, we evaluate the complexity functional $I_d^{bulk} + I_d^{bdy}$ for optimized solutions corresponding to the global AdS_{d+1} . We focus on the $d = 3, 4$ case again.

3D CFT (d=3) : Consider the metric obtained by setting $d = 3$ in the solution of (5.7.17). The boundary condition for $r = r_0$ is chosen as in 2D case:

$$\frac{4r_0^2}{(1-r_0^2)^2} = \frac{1}{\epsilon^2} \Rightarrow r_0 = 1 - \epsilon + \frac{\epsilon^2}{2} + \mathcal{O}(\epsilon^3) \quad (5.7.39)$$

Then the bulk I_d^{bulk} and boundary I_d^{bdy} are evaluated as follows:

$$\begin{aligned} I_3^{bulk} &= 4\pi N \left[\frac{1}{\epsilon^2} - \frac{2}{\epsilon} + \frac{1}{2} + \log\left(\frac{2}{\epsilon}\right) \right], \\ I_3^{bdy} &= 8\pi N \left[\frac{1}{\epsilon} + \frac{\mu_B}{2\epsilon^2} \right]. \end{aligned} \quad (5.7.40)$$

Finally, adding the two contributions we obtain

$$I_3^{tot} = 4\pi N \left[\frac{1}{\epsilon^2} + \frac{1}{2} + \log\left(\frac{2}{\epsilon}\right) \right] + \frac{4\pi N \mu_B}{\epsilon^2}. \quad (5.7.41)$$

4D CFT (d=4) : In the same way, when $d = 4$, the solution (5.7.17) leads to

$$\begin{aligned} I_4^{bulk} &= 2\pi^2 N \left[\frac{2}{3\epsilon^3} - \frac{1}{\epsilon^2} + \frac{1}{\epsilon} - \frac{5}{12} \right], \\ I_4^{bdy} &= 4\pi^2 N \left[\frac{1}{2\epsilon^2} + \mu_B \left(\frac{1}{3\epsilon^3} + \frac{1}{8} \right) \right]. \end{aligned} \quad (5.7.42)$$

Totally we obtain

$$I_4^{tot} = 2\pi^2 N \left[\frac{2}{3\epsilon^3} + \frac{1}{\epsilon} - \frac{5}{12} \right] + 4\pi^2 N \mu_B \left(\frac{1}{3\epsilon^3} + \frac{1}{8} \right). \quad (5.7.43)$$

For the sake of comparing our results with the existing literature, which we do in the next sub-section, we have to set $\mu_B = 0$ and the normalization factor N in the above formulas should be taken as defined in (5.7.3) for holographic CFTs. Therefore, by simply setting $R_{AdS} = 1$, the complexity C_{Ψ_0} of the vacuum state $|\Psi_0\rangle$ computed from our complexity functional I_d^{tot} , is given as follows

$$\text{3D CFT: } C_{\Psi_0}^3 = \frac{1}{2G_N} \left[\frac{1}{\epsilon^2} + \frac{1}{2} + \log \left(\frac{2}{\epsilon} \right) \right] \quad (5.7.44)$$

$$\text{4D CFT: } C_{\Psi_0}^4 = \frac{\pi}{8G_N} \left[\frac{2}{\epsilon^3} + \frac{3}{\epsilon} - \frac{5}{4} \right]. \quad (5.7.45)$$

It is also helpful to remember that for 2D CFTs, according to (5.5.20), we find the complexity of the vacuum (if we choose $g_{(z,\theta)}$ as the reference metric)

$$\text{2D CFT: } C_{\Psi_0}^2 = I_L[g_{AdS}, g_{(z,\theta)}] = \frac{1}{4G_N} \left(\frac{1}{\epsilon} - 1 \right). \quad (5.7.46)$$

Notice that generally the complexity behaves like $C_{\Psi}^d \sim \epsilon^{-(d-1)}$ in any CFT_{d+1} and this is interpreted as the volume law divergence.

5.7.6 Comparison with Earlier Conjectures

It is worthwhile to compare our complexity functional I_d^{tot} evaluated specifically for dimensions $d = 3, 4$ in the previous subsection, with those in earlier conjectures.

(1) “Complexity = Volume” Conjecture

Recently there has been exciting developments in understanding computational complexity in quantum systems holographically, i.e. some geometric calculation in the gravity side has been proposed to be measuring the computational complexity. The computational complexity of any boundary state at any given time, i.e. on some spacelike slice, say Σ , of the

boundary, was first proposed in [80, 79] to be identified with the volume of a maximal volume space-like slice, say M_Σ , in the bulk where the bulk space like surface ends on the given boundary slice. We will refer to it as the CV-conjecture (complexity = volume),

$$C_V(\Sigma) = \frac{\mathcal{V}(M_\Sigma)}{G_N R_{AdS}}, \quad \text{such that } \partial M_\Sigma = \Sigma \quad (5.7.47)$$

where $\mathcal{V}(M_\Sigma)$ denotes the volume of the maximal time slice M_Σ , and R_{AdS} is some associated length scale in the bulk, conveniently taken to be the AdS radius in AdS/CFT. For TFD states in holographic CFTs, the gravity dual computation of C_V shows the linear growth [80, 79].

It may also be useful to mention that in the paper [17], it was found that the gravity dual of the information metric $G(\Sigma)$ in CFT_d , which is equivalent to an integral of two point functions of a marginal primary operator, is well approximated by $G(\Sigma) \simeq n_d \cdot \frac{\mathcal{V}(M_\Sigma)}{R_{AdS}^d}$ (n_d is a numerical constant). Indeed, the information metric in CFTs also have the linear growth under the time evolution of a TFD state.

(2) “Complexity = Action” Conjecture

Later, in [75, 76], it was also conjectured that the complexity is given by the action of a Wheeler-de Witt patch in the bulk bounded by the given space-like surface. One motivation for this conjecture was to remove the unpleasant feature about the CV-conjecture that it depends on the choice of a length scale R_{AdS} . We will refer to it as the CA-conjecture (Complexity = Action)

$$C_A(\Sigma) = \frac{I_{WDW}}{\pi} \quad (5.7.48)$$

where I_{WDW} is given by the Einstein-Hilbert action integrated only over the Wheeler-DeWitt (WDW) patch M_{WDW} , which extends from the boundary time slice Σ where we measure the complexity. The WDW patch is defined to be the bulk space-time region in the bulk which is union of all the space-like surfaces anchored at the boundary at a given time of the CFT. It is easy to visualize this for the eternal black-hole Penrose diagram (see figure 1 in [75]). In that case, once we pick two given times at the two boundary CFTs, say t_L and t_R respectively, the WDW patch is the bulk space-time region bounded by the null surfaces and such that it is union of all possible space-like surfaces anchored at the times t_L and t_R in the boundary.

Schematically, I_{WDW} can be written as

$$I_{WDW} = \frac{1}{16\pi G_N} \int_{M_{WDW}} d^{d+1}x \sqrt{-g} (R - 2\Lambda) + I_{WDW}^{bdy}, \quad (5.7.49)$$

where I_{WDW}^{bdy} contains the important boundary contributions coming from the null boundaries of the WDW patch M_{WDW} , also including the joint contributions coming from the intersections of the null boundaries [82].

In a related direction a quantity called complexity of formation was defined and studied in [83]. This quantity computes, following the CA conjecture, the difference of the action (only the bulk part) between the BTZ and two times that of the AdS (vacuum). The similar results were reproduced in [120] with a proposal of renormalized holographic complexity.

It is important to mention that, through our proposal we can only learn about the fixed time behavior of the computational complexity for the dual state in the CFT, whereas, the original motivation of proposing a holographic measure of complexity was to study its growth with time [75, 76, 82]. In order to compare the evaluation of complexity with our proposal with the same evaluated using other proposals, we need to therefore look into their constant time evaluations. In [85, 84] the authors investigated the constant time behavior of the holographic complexity. More specifically they studied the divergence structure, considering both the CV and CA-conjectures. Also a possible prescription to remove an ambiguity due to different parametrization of the null boundary surfaces in the WDW patch was found in [82]. This prescription was used to evaluate the holographic complexity in [84].

Comparisons with Our Results

We are finally ready to compare the evaluation of holographic complexity with our proposal against the same computed with the existing proposals in the literature .

First if we follow the “Complexity = Volume” conjecture (5.7.47), the complexity has the structure $C_V \sim c_v^{(1)} \cdot \epsilon^{-(d-1)} + c_v^{(3)} \cdot \epsilon^{-(d-3)} + \dots$. This behavior agrees with our results of complexity C_{Ψ_0} presented in (5.7.46), (5.7.44) and (5.7.45), though the relative coefficients do not coincide in general.

Next we turn to the “Complexity = Action” conjecture (5.7.48). The analysis in [85] evaluates it to be divergent, in fact a logarithmically enhanced divergence of the form $\log \epsilon^{-1} \cdot \epsilon^{(d-1)}$ for the CA-conjecture as opposed to the $1/\epsilon^{d-1}$ divergence for the CV-conjecture. On the other side, the [84] proposal, which introduces an additional boundary contribution, produces a surprising result for the $d = 2$ case i.e. bulk AdS_3 : for both Poincare and global AdS_3 , the leading divergence vanishes, leading to a constant holographic complexity. In higher dimensions $d = 3, 4$, the holographic complexity has a leading divergence of the form $1/\epsilon^{d-1}$ for both Poincare and global AdS_{d+1} . Therefore the divergence structure in [84] for $d > 2$ is the same as ours, whereas, they differ in the numerical coefficients in general.

Nevertheless, in the next subsection, we will point out an interesting relation between our complexity functional I_d^{tot} and the gravity action I_{WDW} in the WDW patch.

Since there is no precise definition of computational complexity in quantum field theories known at present, we cannot decide which of these prescriptions is true by consulting with rigorous results in field theory. However, notice that our proposal of computational complexity C_{Ψ_0} , defined in (5.2.4), is based on not any holography but a purely field theoretic argument as is clear in two dimensional CFT case, where it is related to the normalization of wave functional.

5.7.7 Relation to “Complexity = Action” Proposal

We have discussed in the previous subsection that, in [75, 76], it has been conjectured that the holographic complexity is measured by the bulk action being integrated over the WDW patch defined above including suitable boundary terms. Here we would like to compare this quantity with our complexity functional. For simplicity, we set $R_{AdS} = 1$ and thus $\Lambda = -\frac{d(d-1)}{2}$ below.

Consider the following class of $d + 1$ dimensional space-time:

$$ds^2 = -dt^2 + \cos^2 t \cdot e^{2\phi(x)} h_{ij} dx^i dx^j, \quad (5.7.50)$$

where t takes the range $-\pi/2 \leq t \leq \pi/2$ and $i, j = 1, 2, \dots, d$. The pure AdS_{d+1} which is a solution to the Einstein equation from I_{WDW} , is obtained when the metric $e^{2\phi(x)} h_{ij} dx^i dx^j$ coincides with a hyperbolic space H_d . For example, when $d = 2$, the Einstein equation just leads to $(\partial_1^2 + \partial_2^2)\phi = e^{2\phi}$ i.e. the Liouville equation. Note that in this pure AdS_{d+1} solution, the coordinate covered by (5.7.50) indeed represents the WDW patch.¹⁵ Motivated by this we identify this space (5.7.50) with M_{WDW} . However, note that for generic choices of ϕ and h_{ij} (5.7.50) does not represent the WDW patch in the original definition in [75, 76]. They coincide only on-shell.

Now we would like to evaluate the gravity action (5.7.49) within the WDW patch, integrating out the time t coordinate. Here we can ignore the contribution from the boundary as the Gibbons-Hawking term of this boundary turns out to be vanishing. We finally find that the final action is proportional to our complexity functional $I_d^{tot}[\phi, g]$ (5.7.2) with the normalization (5.7.3) up to surface terms at the AdS boundary $z = 0$ due to partial

¹⁵In Euclidean signature obtained by $t \rightarrow i\tau$, this leads to the hyperbolic slice of H_{d+1} given by $ds^2 = R_{AdS}^2 (d\tau^2 + \cosh^2 \tau \cdot e^{2\phi} dy d\bar{y})$.

integrations:

$$I_d^{WDW} = (d-2) \cdot n_d \cdot I_d^{tot}[\phi, g] + (\text{IR Surface Term}), \quad (5.7.51)$$

where the numerical constant n_d is defined by

$$n_d = \int_{-\pi/2}^{\pi/2} dt (\cos t)^{d-2} = \frac{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \quad (5.7.52)$$

In the above computation, by introducing the Gibbons-Hawking term for the d dimensional boundary time like surface given by $z = \epsilon$, the surface terms on this surface which are produced by the partial integrations of bulk action are all cancelled with the Gibbons-Hawking term. Therefore in the surface terms in (5.7.51) is localized at the IR boundary, which is at $z = \infty$ and gives the vanishing contribution for the Poincare AdS coordinate.

For example, when $d = 3$ with $h_{ij} = \delta_{ij}$ (setting $x_3 = z$), so that it fits with the Poincare AdS₄, we find

$$I_3^{WDW} = \frac{1}{16\pi G_N} \int_{\epsilon}^{\infty} dz \int d^2x \int_{-\pi/2}^{\pi/2} dt \left[6e^{3\phi} (\cos^3 t - \cos t \cos 2t) - 2e^{\phi} \cos t ((\partial_i \phi)^2 + 2\partial_i \partial_i \phi) \right], \quad (5.7.53)$$

which reproduces (5.7.51) after we integrate t and perform a partial integration with the boundary term at $z = \epsilon$ cancelled by the Gibbons-Hawking term at $z = \epsilon$.

When $d = 2$ we find

$$I_2^{WDW} = \frac{1}{8G_N} \int dz dx [-(\partial_1^2 + \partial_2^2)\phi], \quad (5.7.54)$$

which indeed leads to vanishing action up to partial integrations, where again the boundary term at $z = \epsilon$ is cancelled by the Gibbons-Hawking term at $z = \epsilon$. Therefore we simply find $I_2^{WDW} = -\frac{1}{8G_N} \int dx (\partial_z \phi)_{z=\infty}$, where note that $z = \infty$ is the IR boundary. Since we have $\phi = -\log z$ and $\phi = -\log \sinh z$ for Poincare and global AdS₃, we get

$$\begin{aligned} I_2^{WDW} &= 0 \quad \text{for CFT}_2 \text{ vacuum on } \mathbb{R}^1 \text{ dual to the Poincare AdS}_3, \\ I_2^{WDW} &= \frac{\pi}{4G_N} \quad \text{for CFT}_2 \text{ vacuum on } \mathbb{S}^1 \text{ dual to the Poincare AdS}_3. \end{aligned}$$

Interestingly, this agrees with the evaluation of holographic complexity with the prescription in [84].

The above relation shows that there is no difference with respect to the equation of motion for ϕ between the ‘‘Complexity = Action’’ approach and our proposal. However in

the $d = 2$ limit they differ significantly due to the $(d - 2)$ factor in (5.7.51). In our proposal, the complexity functional for 2D CFTs is obtained as $\lim_{d \rightarrow 2} (I_d^{tot}[\phi, g] - I_d^{tot}[0, g]) = \lim_{d \rightarrow 2} [(I_d^{WDW} - I_d^{WDW}|_{\phi=0})/(d - 2)]$, which coincides with the Liouville action $I_L[\phi, g]$. On the other hand, there are no bulk contributions in I_2^{WDW} as clear from (5.7.54). This is the essential reason why the former have the UV divergence $O(\epsilon^{-1})$, while the latter does not.

5.8 Discussions: Time Evolution of TFD States and Phase Transitions

So far we have studied stationary quantum states in CFTs. For further understandings of the dynamics of CFTs, we would like to turn to time dependent states in this section, focusing on 2D CFTs for simplicity. In particular we consider a simple but non-trivial class of time-dependent states, given by the time evolution of thermo field double states (TFD states) in 2D CFTs:

$$|TFD(t)\rangle = \frac{1}{\sqrt{Z_\beta(t)}} \sum_n e^{-\frac{\beta}{4}(H_1+H_2)} e^{-it(H_1+H_2)} |n\rangle_1 |n\rangle_2, \quad (5.8.1)$$

where the total Hilbert space is doubled $H_{tot} = H_1 \otimes H_2$ (H_1 is the original CFT Hilbert space and H_2 is its identical copy). Its density matrix¹⁶ is given by $\rho(t) = |TFD(t)\rangle\langle TFD(t)|$ and if we trace out H_2 , then the reduced density matrix ρ_1 is time-independent, given by the standard canonical distribution $\rho_1 \propto e^{-\beta H_1}$. However the TFD state $|TFD(t)\rangle$ shows very nontrivial time evolution and is closely related to quantum quenches as pointed out in [36].

5.8.1 Optimizing TFD State

Motivated by this, let us study the path-integral expression of $|TFD(t)\rangle$. First we can create the initial TFD state $|TFD(0)\rangle$ by the Euclidean path-integral for the range of Euclidean time τ :

$$-\frac{\beta}{4} \leq \tau \leq \frac{\beta}{4}. \quad (5.8.2)$$

After this path-integration, we can perform the Lorentzian path-integral by it on each CFT. This integration contour is depicted as the left picture in Fig.5.8. However, as we will see later, there is an equivalent but more useful contour given by the right picture in Fig.5.8.

¹⁶However note that $\rho(t)$ can not be obtained from the analytic continuation $\tau = it$ of Euclidean TFD density matrix $\rho(\tau) = |TFD(\tau)\rangle\langle TFD(\tau)|$ defined by the Euclidean path-integral for the Euclidean time region $-\beta/4 - \tau \leq \tau \leq \beta/4 + \tau$. Instead it is obtained from $\rho'(\tau) = |TFD(\tau)\rangle\langle TFD(-\tau)|$.

This is because we can exchange the Euclidean time evolution $e^{-\beta(H_1+H_2)/4}$ with the real time one $e^{-it(H_1+H_2)}$.

Now we consider an optimization of this path-integral. For the Euclidean part we can apply the same argument as before and minimize the Liouville action. Next we need to consider an optimization of the real time evolution. However, we would like to argue that this Lorentzian path-integral cannot be optimized. A heuristic reason for this is that if the final state even after a long time evolution, is still sensitive to the initial state as opposed to the Euclidean path-integral. On the other hand, if we perform an Euclidean time evolution for a period $\Delta\tau$, then the final state is insensitive to the high momentum mode $k \gg 1/\Delta\tau$ of the initial state. Once we assume this argument, we can understand the reason why we place the Lorentzian time evolution in the middle sandwiched by the Euclidean evolution as this obviously reduces the value of S_L . It is an intriguing future problem to verify these intuitive arguments using the tensor network framework.

Assuming that the above prescription of optimization is correct at least semi-quantitatively, we can find the following solution (remember we set $z = -\tau$):

$$e^{2\phi} = \begin{cases} \frac{4\pi^2}{\beta^2} \cos^{-2} \left(\frac{2\pi \text{Re}[z]}{\beta} \right), & (-\frac{\beta}{4} - it < z < -it, \quad it < z < it + \frac{\beta}{4}) \\ \frac{4\pi^2}{\beta^2}, & (-it < z < it). \end{cases} \quad (5.8.3)$$

This is depicted in Fig.5.9.

It is also intriguing to estimate the complexity. For the Euclidean part, we proposed that it is given by the Liouville action as we explained before. For Lorentzian part, there is no obvious candidate. However since we assumed that ϕ is constant during the real time evolution, we can make a natural identification: the Liouville potential term gives the

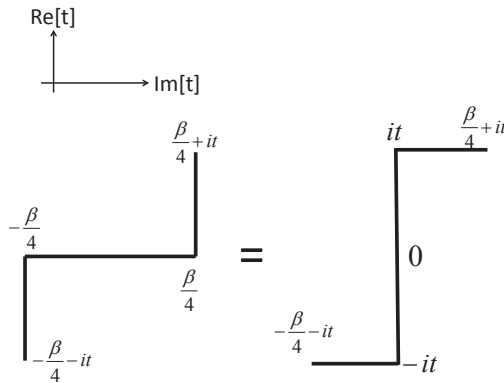


Figure 5.8: The choices of path-integral contour for the TFD states [24]. We employ the time coordinate in the first CFT H_1 . The left and right choices are equivalent.

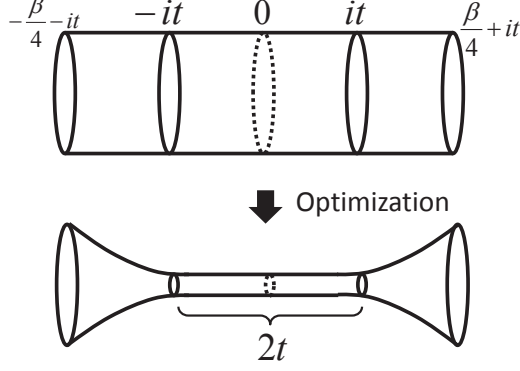


Figure 5.9: The optimization of path-integral for the TFD states [24]. The Euclidean path-integral is optimized by minimizing the Liouville action. We assumed that the Lorentzian one cannot be optimized.

complexity. This is clear from the fact that the complexity should be proportional to the number of tensors.

Thinking this way, we find

$$S_L(t) = S_L(0) + \left(\frac{2\pi}{\beta}\right)^2 \cdot \frac{c}{6} \cdot t. \quad (5.8.4)$$

This linear t growth is consistent with the basic idea in [79]. Since the energy in our 2D CFT at finite temperature $T = 1/\beta$ is given by

$$E_{CFT} = \frac{\pi^2}{3} c T^2, \quad (5.8.5)$$

we find

$$\frac{dS_L(t)}{dt} = 2E_{CFT}. \quad (5.8.6)$$

Interestingly this growth is equal to a half of the gravity action I_{WDW} on the WDW patch for holographic complexity found in [75, 76], where the holographic complexity C_A is conjectured to be $C_A = \frac{I_{WDW}}{\pi}$ (5.7.48). Note that we are shifting both the time in the first and second CFT at the time time. This relation $\frac{dI_{WDW}}{dt} = 2\frac{dS_L}{dt}$ may be natural because the partition function of CFTs $Z \sim e^A$ is the square of that of wave functional in CFTs $|\Psi|^2 \sim e^{2S_L}$.

It is also intriguing to consider a pure state which looks thermal when we coarse-grain its total system. One typical such example in CFTs is obtained by regularizing a boundary state $|B\rangle$

$$|\Psi_B\rangle = \mathcal{N}_B e^{-\beta H/4} |B\rangle, \quad (5.8.7)$$

where $\mathcal{N}_B$ is the normalization such that $\langle \psi_B | \psi_B \rangle = 1$. This can also be regarded as an approximation of global quenches [157, 158]. This quantum state is dual to a single-sided black hole [36] shows the linear growth of holographic entanglement entropy which matches with the 2D CFT result in [157]. This state after our path-integral optimization is clearly given by a half of TFD (5.5.28) for $0 < z < \beta/4 - \epsilon$. The boundary at $z = 0$ corresponds to that of the boundary state $|B\rangle$ which matches with the AdS/BCFT formulation [159, 160]. Thus the growth of the complexity functional is simply given by a half of the TFD case (5.8.6).

5.8.2 Comparison with Eternal BTZ black hole

The time evolution of TFD state provides an important class of time-dependent states and here we would like to discuss possible connections between our optimization procedure and its gravity dual given by the eternal BTZ black hole. In this section we set $\beta = 2\pi$ for simplicity.

First let us try to assume that the dual geometry for this time-dependent quantum states has a property that each time slice is given by a space-like geometry which is a solution to Liouville equation. Any solution to the Liouville equation is always a hyperbolic space with a constant curvature. Such a hyperbolic space at each time t is obtained by taking a union of all geodesic which connects two points at the time t with the same space coordinate in the two different boundaries, given explicitly by

$$ds^2 = e^{2\phi(z)}(dz^2 + dx^2),$$

$$e^{2\phi(z)} = \frac{\left(\frac{1}{\cosh t}\right)^2}{\sin^2\left(\frac{z}{\cosh t}\right)}. \quad (5.8.8)$$

By the transformation

$$\cosh \rho = \frac{\frac{1}{\cosh t}}{\sin\left(\frac{z}{\cosh t}\right)}, \quad (5.8.9)$$

the metric is rewritten as

$$ds^2 = \cosh^2 \rho \left(dx^2 + \frac{\sinh^2 \rho}{\sinh^2 \rho \cosh^2 \rho + \tanh^2 t \cosh^2 \rho} d\rho^2 \right). \quad (5.8.10)$$

Indeed the whole BTZ metric

$$ds^2 = -\sinh^2 \rho dt^2 + d\rho^2 + \cosh^2 \rho dx^2, \quad (5.8.11)$$

can be rewritten into the metric

$$ds^2 = \frac{1}{\sin\left(\frac{z}{\cosh \tau}\right)^2 \cdot \cosh^2 \tau} \left[-d\tau^2 + dx^2 + (z \tanh \tau (d\tau) - dz)^2 \right], \quad (5.8.12)$$

via the coordinate transformation

$$\cosh \rho = \frac{1}{\cosh \tau \cdot \sin\left(\frac{z}{\cosh \tau}\right)}, \quad \tanh t = \frac{\tanh \tau}{\cos\left(\frac{z}{\cosh \tau}\right)}. \quad (5.8.13)$$

However if we evaluate its action (as in the computation of (5.5.31)) we find (we recovered β dependence)

$$S_L = \frac{8\pi}{\epsilon} - \frac{4\pi^3}{\beta \cosh t}. \quad (5.8.14)$$

Thus there is no linear t growth. In this way, this surface does not seem to have an expected property which supports the linearly growing complexity argued in many papers [80, 79, 75, 76, 82].

Now we would like to turn to another candidate: maximal time slice, whose volume was conjectured to be one candidates of holographic complexity [80, 79]. Note that this maximal time slice is not a solution to Liouville equation as opposed to the previous hyperbolic space (5.8.8), which is constructed from geodesics.

The BTZ metric behind the horizon can be obtained by the analytic continuation $\rho = i\kappa$, $\tilde{t} = t + \frac{\pi i}{2}$. Maximal volume surface with boundary time t is determined by the equation

$$s^2(t) = \frac{\cosh^2 \rho \sinh^4 \rho}{\dot{\rho}^2 - \sinh^2 \rho} = \frac{\cos^2 \kappa \sin^4 \kappa}{-\dot{\kappa}^2 + \sin^2 \kappa}. \quad (5.8.15)$$

$s(t)^2$ increases monotonically as t (≥ 0) increases, with boundary value $s(0) = 0$ and $s(\infty) = 1/2$. The induced metric on the maximal volume time-slice is

$$ds^2 = \cosh^2 \rho \left[\frac{\sinh^2 \rho}{s(t)^2 + \sinh^2 \rho \cosh^2 \rho} d\rho^2 + dx^2 \right]. \quad (5.8.16)$$

The curvature of the maximal volume time slice is not constant, therefore the time slice is not hyperbolic. Then, we find that the volume term increases linearly in time. Finally we obtain

$$\frac{c}{24\pi} \frac{d(\text{Vol}(t))}{dt} \approx \frac{c}{12}, \quad (5.8.17)$$

at late time (here we used the same normalization as our proposal for the Liouville action). This behavior is in contrast to the previous hyperbolic time slice, where the action approaches monotonically to some constant value.

In summary, the above arguments imply that for a generic time dependent background, the assumption that a preferred time slice in a gravity dual is described by Liouville equation, is not compatible with the requirement that the Liouville action gives a measure of complexity. Thus an extension of our proposal in this chapter to time-dependent backgrounds looks highly non-trivial and deserves future careful studies.

5.8.3 Comment on Phase Transition

It is also intriguing to discuss how we can understand the confinement/deconfinement phase transition in our approach. For this, we focus on the initial state $|TFD(t = 0)\rangle$. Since our approach is based on pure states we need to consider the wave functional of TFD state (at temperature T) and see how the corresponding tensor network changes as a function of T . It is obvious that at high temperature, the connected network which looks like macroscopic wormhole is realized and this should be described by the optimized path-integral on the Einstein-Rosen bridge (5.3.5). As we make the temperature lower, the neck of bridge gets squeezed and eventually disconnected in a macroscopic sense. Here we mean by the macroscopic the quantum entanglement of order $O(c) = O(1/G_N)$. Refer to Fig.5.10.

Since the TFD state has non-zero (but sub-leading order $O(1)$) entanglement entropy between the two identical CFTs even at low temperature, there should be a microscopic bridge or wormhole (following ER=EPR conjecture [16]) which connects the two sides in the tensor network description. In this low temperature, the bridge is due to the singlet sector of the gauge theory and is in its confined phase. In large c holographic CFTs, there should be a phase transition of the macroscopic form of the tensor network at the value $\beta = 1/T = 2\pi$ predicted by AdS_3/CFT_2 . Naturally, we expect that the favored phase of a given quantum state Ψ is the one which has smaller complexity C_Ψ .

However, in the current form of our arguments based on the path-integral optimization, it is not straightforward to compare the value of the complexity (i.e. Liouville action) for

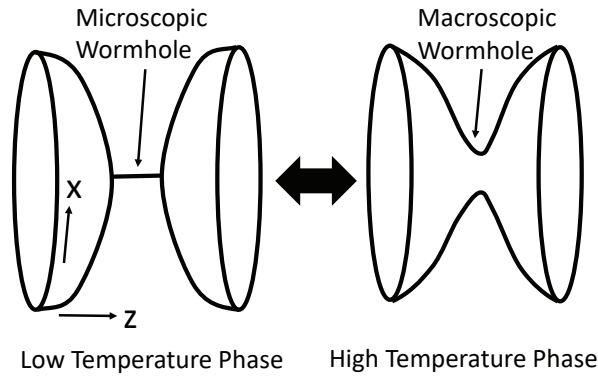


Figure 5.10: The expected description of confinement/deconfinement phase transition in our optimization of path-integral for the TFD states [24]. At low temperature the two CFTs are connected through a microscopic bridge with entanglement entropy $O(1)$ in the tensor network. At high temperature the bridge gets macroscopic and has entanglement entropy $O(c)$.

the confinement/deconfinement phase transition. This is because we can only define the difference of complexity which depends on the reference metric. In this phase transition, the topology of the reference space changes and it is difficult to know how to compare them precisely.

Nevertheless, it might be useful to try to roughly estimate the complexity. For this we assume that the complexity for the deconfined phase (denoted by C_{dec}) is estimated by the bridge solution (5.5.30) and that for the confined phase (denoted by C_{con}) is by the twice of the vacuum result (5.5.16), which leads to

$$C_{dec} \approx \frac{c}{3\epsilon} - \frac{\pi^2 c}{6\beta}, \quad C_{con} \approx \frac{c}{3\epsilon} - \frac{c}{3}. \quad (5.8.18)$$

Qualitatively, this has an expected behavior that $C_{dec} < C_{con}$ for $\beta \ll 2\pi$ and vice versa, though the phase transition temperature reads $\beta = \pi^2/2$, which is slightly different from the gravity result 2π .

Another interesting interpretation of the phase transition can be found from a property in the Liouville CFT. It is known that the (chiral) conformal dimension h of any local operators in Liouville theory has an upper bound (so called Seiberg bound [135]):

$$h \leq \frac{c-1}{24}, \quad (5.8.19)$$

which implies the non-normalizability of the corresponding state. The operator which violates the bound should be regarded as a (normalizable) quantum state. In the large c limit, this bound (5.8.19) agrees with the condition that the conical deficit angle parameter a given by (5.3.16), takes a real value, for which the metric is that for confined phase. When it is violated, a becomes imaginary and the metric changes into that for the deconfined phase (Einstein-Rosen bridge). This behavior seems to fit very nicely with the gravity dual prediction and to proceed this further is an important future problem.

5.9 Conclusions

In this work, we proposed an optimization procedure of Euclidean path-integrals for quantum states in CFTs. The optimization is described by a change of the background metric on the space where the path integral is performed. The optimization is completed by minimizing the complexity functional I_Ψ for a given state $|\Psi\rangle$, which is argued to be given by the Liouville action for 2D CFTs. The Liouville field ϕ corresponds to the Weyl scaling of the background metric. Since this complexity is defined from Euclidean path-integrals, we propose to call this “Path-Integral Complexity”.

Through calculations in various examples in 2d CFTs, we observed that optimized metrics for static quantum states coincide with those of time slices of their gravity duals. Thus we argued that our path-integral optimization offers a continuous version of the tensor network interpretation of $\text{AdS}_3/\text{CFT}_2$ correspondence. Moreover, we also find a simple formula to calculate the energy density for each quantum state.

At the same time, we provide a field theory framework for evaluating the computational complexity of any quantum states in CFTs. Note however, that in 2D CFTs, due to the conformal anomaly, the complexity functional (i.e. Liouville action) depends on the reference metric. Therefore, we proposed to use the difference of the action, which is expected to give a relative difference of complexity between the optimized network and the initial un-optimized one. We evaluated this quantity in several examples.

In order to calculate the entanglement entropy, we studied an optimization of reduced density matrices. After the optimization we find that the geometry is given by two copies of entanglement wedge and this nicely fits into the gravity dual. The entanglement entropy is finally reduced to the length of the boundary of the entanglement wedge and precisely reproduces the holographic entanglement entropy.

Even though in most parts of this chapter our analysis is devoted to static quantum states, we also discussed how our optimization of path-integrals can be applied to time-dependent backgrounds in 2D CFTs. Especially, we considered the time evolution of thermo-field states which describe finite temperature states as a basic example. Our heuristic arguments show that an wormhole throat region linearly grows under the time evolution, which is consistent with holographic predictions. Moreover, we discussed how to interpret the confinement/deconfinement phase transition in terms of tensor networks and our path-integral approach, whose details will be an interesting future problem. However, a precise connection between Liouville action and time-dependent states in 2D CFTs is still not clear and this was left as an important future problem.

In the latter half of this chapter, we investigated the application of our optimization method to CFTs in other dimensions than two. In one dimension, we find that 1D version of Liouville action naturally arises from the conformal symmetry breaking effect in $\text{NAdS}_2/\text{CFT}_1$ and this explains the emergence of extra dimension as in the $\text{AdS}_3/\text{CFT}_2$ case.

In higher dimensions, we expect that the optimization procedure gets very complicated as we need to change not only the scaling mode but also other components of the metric as opposed to the 2D case. We focused on the Weyl scaling mode and proposed a complexity

functional which looks like a higher dimensional version of Liouville action. However, notice the crucial difference from the 2D case that the higher dimensional action does not depend on the reference metric. We confirmed that this reproduces the correct time slice metric for the vacuum states and correct holographic entanglement entropy when the subsystem is a round sphere. We pointed out an interesting direct connection to earlier proposal of holographic complexity [75, 76], which may suggest we should optimize with respect to all components of the metric. We also analyzed the spherically symmetric excited states and found that the optimized metric agrees with the AdS Schwarzschild one up to the first order contribution of the mass parameter. We observed that for CFTs in any dimensions (including 2D), in order to take into account higher order back-reactions, we need to treat the Liouville mode ϕ in a quantum way. It is also possible to include higher derivative corrections without losing the above properties as discussed in appendix C of [24]. One advantage of this is that we can realize the higher dimensional conformal anomaly. However there is also a disadvantage that the action is no longer positive definite and cannot be minimized but extremized. These issues on higher dimensional CFTs should deserve further studies.

Last but not least, our approach based on the optimization of path-integrals is a modest but important step towards understanding of the basic mechanism of the AdS/CFT correspondence. For the future, apart from the questions we already mentioned above, there are many new directions for investigations like e.g. computation of correlation functions, generalizations to non-conformal field theories and understanding a precise connection to AdS/CFT including $1/c$ expansions etc.

Chapter 6

Conclusion

In this thesis, we discussed scrambling of quantum information and complexity of quantum states in AdS/CFT and black holes.

In the Chapter 2, we reviewed about scrambling of quantum information. Motivated by the black hole information problem, we considered some thought experiments (played by "Alice" and "Bob"). In the experiments, the information recovery from the Hawking radiation motivates us to formulate the scrambling phenomenon and estimate time to scramble up subsystems. The Page's formulation tells us that scrambling is a dynamical process to maximize entanglement entropies for small subsystems. The information of the black hole leaks out with the radiation. Typically, when black holes radiate half of the degrees of freedom is radiated (Page time), the information can be retrieval from the radiation only. This thought experiment was refined by Hayden and Preskill so that entanglement between the black hole and the early radiation assist the retrieval of the information initially threw into the black hole. It gives a much faster time than the Page's time. The time is logarithmic in the total system size. From some more thought experiments and naive estimation of maximal speed of information diffusion (and quantum mechanical examples), Sekino-Susskind argued that this fast scrambling time is actually the fastest in nature and black holes are the fastest scramblers. Recent development came from a holographic calculation in shocked black holes by Shenker-Stanford. Remarkably, not only realizing the fast scrambling time in the context of AdS/CFT, they pointed out the relation to a quantum chaotic behavior or the butterfly effect. It leads to a recent attractive quantity, the out-of-time-ordered correlators (OTOCs) diagnosing quantum chaos.

In the Chapter 3, we introduced our work on a explicit calculation of fast scrambling time in AdS/CFT. In two-dimensional holographic CFTs, we considered a heavy local operator

excitation on the TFD state dual to the two-sided BTZ black hole. The localized excitation spreads over the system with disrupting the entanglement in the TFD state. Then, after some time evolution, this disruption causes the vanishment of the two-sided mutual information. We estimated the vanishing time of the two-sided mutual information. From the calculation, we obtained the fast scrambling time as Shenker-Stanford showed in the gravity side. This work was a first explicit calculation of the fast scrambling time in CFTs by using the two-sided mutual information. Also we extended Shenker-Stanford analysis in the gravity side to finite time beyond the shock-wave approximation.

In Chapter 4, we reviewed about complexity of quantum states. The complexity is a quantum information beyond entanglement entropy in some sense. It can probe the dynamics of quantum states after thermalization or scrambling which the entropy saturates. Recently this quantity was connected to black hole late time dynamics or the wormhole growth by Susskind's keen insight. The holographic duals conjectured by Susskind and many collaborators have different codimension to holographic entanglement entropy. They are expected to probe broader regions of dual spacetime beyond the scope of the entropy. In contrast to the holographic studies, the definition of the complexity in quantum field theories is not explored yet. Only a few proposals have been in elemental trials. We are looking forward opening new horizons also in the field theory side.

In Chapter 5, we introduced our work on a new method, so-called the optimization procedure of Euclidean path-integrals, to extract dual geometrical information only from some data in CFTs. This method reproduces the metric of AdS from the ground state wave function in CFT, the entanglement wedge from the reduced density matrix and the holographic entanglement entropy from the entanglement entropy. The validity was tested only for simple states in CFTs. We would like to develop this method further, for example time-dependent cases and non-CFT cases. In addition to the well-developed quantities, we proposed a new definition of complexity of states in CFTs by using our method. The complexity in CFTs corresponds to the optimal action in our procedure. Especially in two dimension, it is the on-shell Liouville action and a naive estimation from the tensor network description exists. We can check the divergence structures as same as the holographic conjectures, but we can find the different coefficients in each term. Some connections to the Complexity = Action conjecture are discussed but we need more clear understandings. We hope to reveal some aspects of the optimization procedure and more connections to AdS/CFT in near future.

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