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# A Hardy type inequality and application to the stability of degenerate stationary waves

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## 1 Introduction

This note is a survey of our joint paper [2] on the stability problem of degenerate stationary waves for viscous conservation laws in the half space $x>0$:

$$\begin{aligned} &u_t+f(u)_x=u_{xx},\\ &u(0,t)=-1, \quad u(x,0)=u_0(x). \end{aligned} \tag{1.1}$$

Here $u_0(x)\to 0$ as $x\to\infty$, and $f(u)$ is a smooth function satisfying

$$f(u)=\frac{1}{q}(-u)^{q+1}(1+g(u)), \quad f''(u)>0 \ \text{ for } -1\le u<0, \tag{1.2}$$

where $q$ is a positive integer (degeneracy exponent) and $g(u)=O(|u|)$ for $u\to 0$. Notice that $1+g(u)>0$ for $-1\le u\le 0$. It is known that the corresponding stationary problem

$$\begin{aligned} &\phi_x=f(\phi),\\ &\phi(0)=-1, \quad \phi(x)\to 0 \ \text{ as } \ x\to\infty, \end{aligned} \tag{1.3}$$

admits a unique solution $\phi(x)$ (called *degenerate stationary wave*) which verifies $\phi(x) \sim -(1+x)^{-1/q}$. In particular, we have $\phi(x) = -(1+x)^{-1/q}$ when $g(u) \equiv 0$.

To discuss the stability of the degenerate stationary wave $\phi(x)$, it is convenient to introduce the perturbation $v$ by $u(x,t) = \phi(x) + v(x,t)$ and rewrite the problem (1.1) as

$$\begin{aligned} &v_t + (f(\phi+v) - f(\phi))_x = v_{xx}, \\ &v(0,t) = 0, \quad v(x,0) = v_0(x), \end{aligned} \tag{1.4}$$

where $v_0(x) = u_0(x) - \phi(x)$, and $v_0(x) \to 0$ as $x \to \infty$. The stability of degenerate stationary waves has been studied recently in [14, 2]. The paper [14] proved the following stability result: If the initial perturbation $v_0(x)$ is in the weighted space $L^2_\alpha$, then the perturbation $v(x,t)$ decays in $L^2$ at the rate $t^{-\alpha/4}$ as $t \to \infty$, provided that $\alpha < \alpha_*(q)$, where

$$\alpha_*(q) := (q + 1 + \sqrt{3q^2 + 4q + 1})/q.$$

The decay rate $t^{-\alpha/4}$ obtained in [14] would be optimal but the restriction $\alpha < \alpha_*(q)$ was not very sharp. This restriction has been relaxed to $\alpha < \alpha_c(q) := 3 + 2/q$ in our joint paper [2] by employing the space-time weighted energy method in [14] and by applying a Hardy type inequality with the best possible constant. Notice that $\alpha_*(q) < \alpha_c(q)$. This new stability result will be reviewd in this note.

It is interesting to note that a similar restriction on the weight is imposed also for the stability of degenerate shock profiles (see [9]). We remark that our stability result for degenerate stationary waves is completely different from those for non-degenerate case. In fact, for non-degenerate stationary waves, we have the better decay rate $t^{-\alpha/2}$ for the perturbation without any restriction on $\alpha$. See [4, 5, 13, 15] for the details. See also [6, 8, 10] for the related stability results for stationary waves.

To check the validity of our restriction $\alpha < \alpha_c(q) := 3 + 2/q$, it is important to discuss the dissipativity of the following linearized operator associated with (1.4):

$$Lv = v_{xx} - (f'(\phi)v)_x. \tag{1.5}$$

In a simpler situation including the case $g(u) \equiv 0$ in (1.2), we showed in [2] that the operator $L$ is uniformly dissipative in $L^2_\alpha$ for $\alpha < \alpha_c(q)$ but can not be dissipative for $\alpha > \alpha_c(q)$. This suggests that the exponent $\alpha_c(q)$ is the critical exponent of the stability problem of degenerate stationary waves. This result on the characterization of the dissipativity of $L$ is an improvement on the previous one in [14] and has been established again by using a Hardy type inequality with the best possible constant. This result will be also reviewd in this note.

**Notations.** For $1 \leq p \leq \infty$ and a nonnegative integer $s$, $L^p$ and $W^{s,p}$ denote the usual Lebesgue space on $\mathbb{R}_+ = (0, \infty)$ and the corresponding Sobolev space, respectively. When $p = 2$, we write $H^s = W^{s,2}$. We introduce weighted spaces. Let $w = w(x) > 0$ be a weight function defined on $[0, \infty)$ such that $w \in C^0[0, \infty)$. Then, for $1 \leq p < \infty$, we denote by $L^p(w)$ the weighted $L^p$ space on $\mathbb{R}_+$ equipped with the norm

$$\|u\|_{L^p(w)} := \Big( \int_0^\infty |u(x)|^p w(x)\, dx \Big)^{1/p}. \tag{1.6}$$

The corresponding weighted Sobolev space $W^{s,p}(w)$ is defined by $W^{s,p}(w) = \{u \in L^p(w);\ \partial_x^k u \in L^p(w) \text{ for } k \leq s\}$ with the norm $\|\cdot\|_{W^{s,p}(w)}$. Also, we denote by $W_0^{1,p}(w)$ the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(w)} := \|\partial_x u\|_{L^p(w)} = \Big( \int_0^\infty |\partial_x u(x)|^p w(x)\, dx \Big)^{1/p}. \tag{1.7}$$

When $p = 2$, we write $H^s(w) = W^{s,2}(w)$ and $H_0^1(w) = W_0^{1,2}(w)$. In the special case where $w = (1+x)^\alpha$ with $\alpha \in \mathbb{R}$, these weighted spaces are abbreviated as $L_\alpha^p$, $W_\alpha^{s,p}$, $W_{\alpha,0}^{1,p}$, $H_\alpha^s$ and $H_{\alpha,0}^1$, respectively.

## 2 Hardy type inequality

Our Hardy type inequality used in [2] is a simple modification of the original Hardy's inequality introduced in [1, 7] (see also [12]).

**Proposition 2.1.** *Let $\psi \in C^1[0, \infty)$ and assume either*
 (1) $\psi > 0$, $\psi_x > 0$ *and* $\psi(x) \to \infty$ *for* $x \to \infty$; *or*
 (2) $\psi < 0$, $\psi_x > 0$ *and* $\psi(x) \to 0$ *for* $x \to \infty$.
*Then we have*

$$\int_0^\infty v^2 \psi_x\, dx \leq 4 \int_0^\infty v_x^2\, \psi^2/\psi_x\, dx \tag{2.1}$$

*for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in H_0^1(w)$ with $w = \psi^2/\psi_x$. Here 4 is the best possible constant, and there is no function $v \in H_0^1(w)$, $v \neq 0$, which attains the equality in (2.1).*

*Proof.* The proof is quite simple. Let $v \in C_0^\infty(\mathbb{R}_+)$. A simple calculation gives

$$\begin{aligned}(v^2\psi)_x &= v^2\psi_x + 2vv_x\psi \\ &= \frac{1}{2}v^2\psi_x + \frac{1}{2}(v + 2v_x\psi/\psi_x)^2\psi_x - 2v_x^2\,\psi^2/\psi_x.\end{aligned} \tag{2.2}$$

Integrating (2.2) in $x$, we obtain

$$\int_0^\infty v^2\psi_x\,dx + \int_0^\infty (v + 2v_x\psi/\psi_x)^2\,dx = 4\int_0^\infty v_x^2\,\psi^2/\psi_x\,dx, \tag{2.3}$$

which gives the desired inequality (2.1). It follows from (2.3) that the equality in (2.1) holds if and only if $v + 2v_x\psi/\psi_x \equiv 0$. But we find that such a $v$ in $H_0^1(w)$ must be $v \equiv 0$.

We show the best possibility of the constant 4 in (2.1). We consider the case (1). Let us fix $a > 0$. Let $\epsilon > 0$ be a small parameter and put

$$v^\epsilon(x) = \begin{cases} 0, & 0 \le x < a, \\ (x-a)\psi(x)^{-1/2-\epsilon}, & a < x < a+1, \\ \psi(x)^{-1/2-\epsilon}, & a+1 < x. \end{cases} \tag{2.4}$$

Then we have after straigtforward computations that

$$\begin{aligned} \frac{\int_0^\infty (v_x^\epsilon)^2\psi^2/\psi_x\,dx}{\int_0^\infty (v^\epsilon)^2\psi_x\,dx} &= \frac{O(1) + (1/2+\epsilon)^2\frac{1}{2\epsilon}\psi(a+1)^{-2\epsilon}}{O(1) + \frac{1}{2\epsilon}\psi(a+1)^{-2\epsilon}} \\ &= \frac{O(\epsilon) + (1/2+\epsilon)^2\psi(a+1)^{-2\epsilon}}{O(\epsilon) + \psi(a+1)^{-2\epsilon}} \longrightarrow \frac{1}{4} \end{aligned}$$

for $\epsilon \to 0$. This shows that 4 in (2.1) is the best possible constant. The case (2) can be treated similarly if we take a test function $v^\epsilon(x)$ as

$$v^\epsilon(x) = \begin{cases} 0, & 0 \le x < a, \\ (x-a)(-\psi(x))^{-1/2-\epsilon}, & a < x < a+1, \\ (-\psi(x))^{-1/2-\epsilon}, & a+1 < x, \end{cases}$$

but we omit the details. This completes the proof of Proposition 2.1. □

The $L^p$ version of Proposition 2.1 is given as follows.

**Proposition 2.2.** *Let $\psi$ be the same as in Proposition 2.1. Let $1 < p < \infty$. Then we have*

$$\int_0^\infty |v|^p\psi_x\,dx \le p^p\int_0^\infty |v_x|^p|\psi|^p/\psi_x^{p-1}\,dx \tag{2.5}$$

*for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in W_0^{1,p}(w)$ with $w = |\psi|^p/\psi_x^{p-1}$. Here $p^p$ is the best possible constant, and there is no function $v \in W_0^{1,p}(w)$, $v \neq 0$, which attains the equality in (2.5).*

*Proof.* We only prove the inequality (2.5) and omit the other discussions. Let $1 < p < \infty$ and $v \in C_0^\infty(\mathbb{R}_+)$. A simple calculation gives

$$\begin{aligned}(|v|^p\psi)_x &= |v|^p\psi_x + p|v|^{p-2}vv_x\psi \\ &= \frac{1}{p}(|v|^p\psi_x - p^p|v_x|^p|\psi|^p/\psi_x^{p-1}) + R,\end{aligned} \tag{2.6}$$

where

$$R = \left(1 - \frac{1}{p}\right)|v|^p\psi_x + \frac{1}{p}p^p|v_x|^p|\psi|^p/\psi_x^{p-1} + p|v|^{p-2}vv_x\psi.$$

Integrating (2.6) in $x$, we obtain

$$\int_0^\infty |v|^p\psi_x\,dx + p\int_0^\infty R\,dx = p^p\int_0^\infty |v_x|^p|\psi|^p/\psi_x^{p-1}\,dx. \tag{2.7}$$

By applying the Young inequality $AB \le (1 - 1/p)A^{p/(p-1)} + (1/p)B^p$ for $A = |v|^{p-1}\psi_x^{(p-1)/p}$ and $B = p|v_x||\psi|/\psi_x^{(p-1)/p}$, we find that $R \ge 0$, which together with (2.7) gives the desired inequality (2.5). □

The following variant of Proposition 2.1 is useful in our application.

**Proposition 2.3.** *Let $\phi \in C^1[0,\infty)$, $\phi < 0$, $\phi_x > 0$, and $\phi(x) \to 0$ for $x \to \infty$. Let $\sigma \in \mathbb{R}$ with $\sigma \neq 0$, and define the weight functions $w$ and $w_1$ by*

$$w = (-\phi)^{-\sigma+1}/\phi_x, \quad w_1 = (-\phi)^{-\sigma-1}\phi_x. \tag{2.8}$$

*Then we have*

$$\int_0^\infty v^2 w_1\,dx \le \frac{4}{\sigma^2}\int_0^\infty v_x^2\,w\,dx \tag{2.9}$$

*for $v \in H_0^1(w)$. Here $4/\sigma^2$ is the best possible constant, and there is no function $v \in H_0^1(w)$, $v \neq 0$, which attains the equality in (2.9).*

*Proof.* We put $\psi = (-\phi)^{-\sigma}$ for $\sigma > 0$ and $\psi = -(-\phi)^{-\sigma}$ for $\sigma < 0$, and apply Proposition 2.1. This gives the desired conclusion. □

As a simple corollary of Proposition 2.3, we have:

**Corollary 2.4.** *Let $\alpha \in \mathbb{R}$ with $\alpha \neq 1$. Then we have*

$$\|v\|_{L^2_{\alpha-2}} \le \frac{2}{|\alpha - 1|}\|v_x\|_{L^2_\alpha} \tag{2.10}$$

*for $v \in H^1_{\alpha,0}$. Here the constant $2/|\alpha - 1|$ is the best possible, and there is no function $v \in H^1_{\alpha,0}$, $v \neq 0$, which attains the equality in (2.10).*

*Proof.* Let $\phi = -(1 + x)^{-1/q}$ with $q > 0$. We apply Proposition 2.3 for this $\phi$ and $\sigma = (\alpha - 1)q$. This gives the proof. □

# 3 Dissipativity of the linearized operator

Following [2], we discuss the dissipativity of the operator $L$ defined by (1.5) in the weighted space $L^2(w)$, where $w$ is given by (2.8) with $\phi$ being the the degenerate stationary wave. Note that our degenerate stationary wave $\phi$ is a smooth solution of (1.3) and verifies

$$-1 \leq \phi(x) < 0, \quad \phi_x(x) > 0, \quad \phi(x) \to 0 \text{ for } x \to \infty, \tag{3.1}$$

$$c(1+x)^{-1/q} \leq -\phi(x) \leq C(1+x)^{-1/q}. \tag{3.2}$$

Now, letting $w > 0$ be a smooth weight function depending only on $x$, we calculate the inner product $\langle Lv, v\rangle_{L^2(w)}$ for $v \in C_0^\infty(\mathbb{R}_+)$, where

$$\langle u, v\rangle_{L^2(w)} := \int_0^\infty uvw\, dx. \tag{3.3}$$

We multiply (1.5) by $v$. Then a simple computation gives

$$(Lv)v = \left(vv_x - \frac{1}{2}f'(\phi)v^2\right)_x - v_x^2 - \frac{1}{2}f''(\phi)\phi_x v^2.$$

Multiplying this equality by $w$, we obtain

$$\begin{aligned}(Lv)vw &= \left\{\left(vv_x - \frac{1}{2}f'(\phi)v^2\right)w - \frac{1}{2}v^2 w_x\right\}_x \\ &\quad - v_x^2 w + \frac{1}{2}v^2(w_{xx} + w_x f'(\phi) - wf''(\phi)\phi_x).\end{aligned} \tag{3.4}$$

Now we choose the weight function $w$ and the corresponding $w_1$ in terms of our degenerate stationary wave $\phi$ by (2.8), where $\sigma \in \mathbb{R}$. Then we have $w = (-\phi)^{-\sigma+1}/f(\phi)$ and $w_1 = (-\phi)^{-\sigma-1}f(\phi)$ by $\phi_x = f(\phi)$. After straightforward computations, we find that

$$w_{xx} + w_x f'(\phi) - wf''(\phi)\phi_x = 2(c_1(\sigma) - r(\phi))w_1, \tag{3.5}$$

where

$$\begin{aligned} c_1(\sigma) &:= \sigma(\sigma-1)/2 - q(q+1), \\ r(u) &:= (-u)^2 f''(u)/f(u) - q(q+1). \end{aligned} \tag{3.6}$$

Substituting (3.5) into (3.4) and integrating with respect to $x$, we get the following conclusion.

**Claim 3.1.** *Let $\phi$ be the degenerate stationary wave and define the weight functions $w$ and $w_1$ by (2.8) with $\sigma \in \mathbb{R}$. Then the operator $L$ defined in (1.5) verifies*

$$\langle Lv, v\rangle_{L^2(w)} = -\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)} - \int_0^\infty v^2 r(\phi) w_1 \, dx \qquad (3.7)$$

*for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in H_0^1(w)$, where $c_1(\sigma)$ and $r(\phi)$ are given in (3.6).*

The term $r(\phi)$ in (3.7) can be regarded as a small perturbation. In fact, a straightforward computation gives

$$r(u) = (-u)\{(-u)g''(u) - 2(q+1)g'(u)\}/(1+g(u)), \qquad (3.8)$$

which shows that $r(u) = O(|u|)$ for $u \to 0$. In particular, we have $r(u) \equiv 0$ if $g(u) \equiv 0$. With these preparations, we have the following result on the characterization of the dissipativity of $L$.

**Theorem 3.2.** *Assume (1.2). Let $\phi$ be the degenerate stationary wave and $L$ be the operator defined in (1.5). Let $w$ and $w_1$ be the weight functions in (2.8) with the parameter $\sigma \in \mathbb{R}$. Then we have:*
*(1) Let $-2q < \sigma < 2(q+1)$. Then, under the additional assumption that $r(u) \geq 0$ for $-1 \leq u \leq 0$, the operator $L$ is uniformly dissipative in $L^2(w)$. Namely, there is a positive constant $\delta$ such that*

$$\langle Lv, v\rangle_{L^2(w)} \leq -\delta(\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)}) \quad \text{for } v \in H_0^1(w). \qquad (3.9)$$

*(2) Let $\sigma > 2(q+1)$ or $\sigma < -2q$. Then the operator $L$ can not be dissipative in $L^2(w)$. Namely, we have $\langle Lv, v\rangle_{L^2(w)} > 0$ for some $v \in H_0^1(w)$ with $v \neq 0$.*

*Proof.* The proof is based on the equality (3.7) in Claim 3.1 and the Hardy type inequality (2.9) in Proposition 2.3.

Let $-2q < \sigma < 2(q+1)$. This is equivalent to $c_1(\sigma) < \sigma^2/4$. Therefore we can choose $\delta > 0$ so small that $\delta(1+\sigma^2/4) \leq \sigma^2/4 - c_1(\sigma)$. Since $r(\phi) \geq 0$ by the additional assumption on $r(u)$ and since $(\sigma^2/4)\|v\|^2_{L^2(w_1)} \leq \|v_x\|^2_{L^2(w)}$ by the Hardy type inequality (2.9), we have from (3.7) that

$$\begin{aligned}
\langle Lv, v\rangle_{L^2(w)} &\leq -\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)} \\
&= -\delta\|v_x\|^2_{L^2(w)} - (1-\delta)\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)} \\
&\leq -\delta\|v_x\|^2_{L^2(w)} - \{(1-\delta)\sigma^2/4 - c_1(\sigma)\}\|v\|^2_{L^2(w_1)} \\
&\leq -\delta(\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)})
\end{aligned} \qquad (3.10)$$

for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in H_0^1(w)$, where we have used the fact that $(1-\delta)\sigma^2/4 - c_1(\sigma) \geq \delta$. This completes the proof of the uniform dissipative case (1).

Next we consider the case where $\sigma > 2(q+1)$; the case $\sigma < -2q$ can be treated similarly and we omit the argument in this latter case. When $\sigma > 2(q+1)$, we have $c_1(\sigma) > \sigma^2/4$. Then we choose $\delta > 0$ so small that $c_1(\sigma) \geq \sigma^2/4 + 3\delta$. Since $r(u) = O(|u|)$ for $u \to 0$ and $\phi(x) \to 0$ for $x \to \infty$, we take $a = a(\delta) > 0$ so large that $|r(\phi)| \leq \delta$ for $x \geq a$. For this choice of $a$ and for $\epsilon > 0$, we take a test function $v^\epsilon$ as in (2.4):

$$v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x-a)(-\phi(x))^{\sigma(1/2+\epsilon)}, & a < x < a+1, \\ (-\phi(x))^{\sigma(1/2+\epsilon)}, & a+1 < x. \end{cases} \tag{3.11}$$

Then we have

$$\left| \int_0^\infty (v^\epsilon)^2 r(\phi) w_1 \, dx \right| \leq \delta \int_a^\infty (v^\epsilon)^2 w_1 \, dx = \delta \|v^\epsilon\|^2_{L^2(w_1)},$$

so that we have from (3.7) that

$$\langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)} \geq -\|v_x^\epsilon\|^2_{L^2(w)} + (c_1(\sigma) - \delta)\|v^\epsilon\|^2_{L^2(w_1)}. \tag{3.12}$$

Also, by straightforward computations, we find that

$$\begin{aligned} \frac{\|v_x^\epsilon\|^2_{L^2(w)}}{\|v^\epsilon\|^2_{L^2(w_1)}} &= \frac{O(1) + \sigma^2(1/2+\epsilon)^2 \frac{1}{2\sigma\epsilon}(-\phi(a+1))^{2\sigma\epsilon}}{O(1) + \frac{1}{2\sigma\epsilon}(-\phi(a+1))^{2\sigma\epsilon}} \\ &= \frac{O(\epsilon) + \sigma^2(1/2+\epsilon)^2(-\phi(a+1))^{2\sigma\epsilon}}{O(\epsilon) + (-\phi(a+1))^{2\sigma\epsilon}} \longrightarrow \frac{\sigma^2}{4} \end{aligned}$$

for $\epsilon \to 0$. Thus we have $\|v_x^\epsilon\|^2_{L^2(w)}/\|v^\epsilon\|^2_{L^2(w_1)} \leq \sigma^2/4 + \delta$ for a suitably small $\epsilon = \epsilon(\delta) > 0$. Consequently, we have from (3.12) that

$$\begin{aligned} \frac{\langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)}}{\|v^\epsilon\|^2_{L^2(w_1)}} &\geq -\frac{\|v_x^\epsilon\|^2_{L^2(w)}}{\|v^\epsilon\|^2_{L^2(w_1)}} + c_1(\sigma) - \delta \\ &\geq -(\sigma^2/4 + \delta) + c_1(\sigma) - \delta \geq \delta. \end{aligned}$$

This completes the proof of the non-dissipative case (2). Thus the proof of Theorem 3.2 is complete. □

In the special case where $g(u) \equiv 0$ so that $f(u) = \frac{1}{q}(-u)^{q+1}$, we have $\phi = -(1+x)^{-1/q}$ and the operator $L$ in (1.5) is reduced to

$$L_0 v = v_{xx} + \frac{q+1}{q}\Big(\frac{v}{1+x}\Big)_x. \tag{3.13}$$

In this simplest case, we have the complete characterization of the dissipativity of the operator $L_0$.

**Theorem 3.3.** *Let $\alpha_c(q) := 3 + 2/q$. Then we have the complete characterization of the dissipativity of the operator $L_0$ given in* (3.13)*:*
(1) *Let $-1 < \alpha < \alpha_c(q)$. Then $L_0$ is uniformly dissipative in $L^2_\alpha$. Namely, there is a positive constant $\delta$ such that*

$$\langle L_0 v, v\rangle_{L^2_\alpha} \le -\delta(\|v_x\|^2_{L^2_\alpha} + \|v\|^2_{L^2_{\alpha-2}}) \quad \textit{for } v \in H^1_{\alpha,0}. \tag{3.14}$$

(2) *Let $\alpha = \alpha_c(q)$ or $\alpha = -1$. Then $L_0$ is strictly dissipative in $L^2_\alpha$. Namely, we have $\langle L_0 v, v\rangle_{L^2_\alpha} < 0$ for $v \in H^1_{\alpha,0}$ with $v \neq 0$.*
(3) *Let $\alpha > \alpha_c(q)$ or $\alpha < -1$. Then $L_0$ can not be dissipative in $L^2_\alpha$. Namely, we have $\langle L_0 v, v\rangle_{L^2_\alpha} > 0$ for some $v \in H^1_{\alpha,0}$ with $v \neq 0$.*

*Proof.* In this case, we have $\phi = -(1+x)^{-1/q}$, $L = L_0$ and $r(u) \equiv 0$. Therefore, (3.7) is reduced to

$$\langle L_0 v, v\rangle_{L^2(w)} = -\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)}, \tag{3.15}$$

where $w$ and $w_1$ are the weight functions defined in (2.8) with $\phi = -(1+x)^{-1/q}$ and $\sigma = (\alpha-1)q$. The desired conclusions easily follow from (3.15) by applying the same argument as in Theorem 3.2. We omit the details. □

# 4 Nonlinear stability

The following stability result for the nonlinear problem (1.4) was obtained in [2] as a refinement of the result in [14].

**Theorem 4.1.** *Assume* (1.2). *Suppose that $v_0 \in L^2_\alpha \cap L^\infty$ for some $\alpha$ with $1 \le \alpha < \alpha_c(q) := 3 + q/2$. Then there is a positive constant $\delta_1$ such that if $\|v_0\|_{L^2_1} \le \delta_1$, then the problem* (1.4) *has a unique global solution $v \in C^0([0,\infty); L^2_\alpha \cap L^p)$ for each $p$ with $2 \le p < \infty$. Moreover, the solution verifies the decay estimate*

$$\|v(t)\|_{L^p} \le C(\|v_0\|_{L^2_\alpha} + \|v_0\|_{L^\infty})(1+t)^{-\alpha/4-\nu} \tag{4.1}$$

*for $t \ge 0$, where $2 \le p < \infty$, $\nu = (1/2)(1/2 - 1/p)$, and $C$ is a positive constant.*

*Proof.* A key to the proof of this theorem is to show the following space-time weighted energy inequality:

$$\begin{aligned}&(1+t)^\gamma\|v(t)\|^2_{L^2_\beta}+\int_0^t(1+\tau)^\gamma(\|v_x(\tau)\|^2_{L^2_\beta}+\|v(\tau)\|^2_{L^2_{\beta-2}})\,d\tau\\&\le C\|v_0\|^2_{L^2_\beta}+\gamma C\int_0^t(1+\tau)^{\gamma-1}\|v(\tau)\|^2_{L^2_\beta}\,d\tau+CS^\gamma_\beta(t)\end{aligned}\tag{4.2}$$

for any $\gamma\ge 0$ and $\beta$ with $0\le\beta\le\alpha$, where $1\le\alpha<\alpha_c(q):=3+2/q$, $C$ is a constant independent of $\gamma$ and $\beta$, and

$$S^\gamma_\beta(t)=\int_0^t(1+\tau)^\gamma\|v(\tau)\|^3_{L^3_{\beta-1}}\,d\tau.\tag{4.3}$$

Here we give an outline of the proof of (4.2) and omit the other discussions. We refer to [2, 14] for the complete proof of Theorem 4.1.

*Proof of* (4.2) *for* $\beta=0$. The proof is based on the time weighted $L^2$ energy method. First we note that

$$\|v(t)\|_{L^\infty}\le M_\infty,\tag{4.4}$$

where $M_\infty=\|v_0\|_{L^\infty}+2$. This is an easy consequence of the maximum principle (see [5] for the details). Now we multiply the equation (1.4) by $v$. This yields

$$\Big(\frac{1}{2}v^2\Big)_t+(F-vv_x)_x+v_x^2+G=0,\tag{4.5}$$

where

$$\begin{aligned}F&=(f(\phi+v)-f(\phi))v-\int_0^v(f(\phi+\eta)-f(\phi))d\eta,\\G&=\int_0^v(f'(\phi+\eta)-f'(\phi))d\eta\cdot\phi_x.\end{aligned}\tag{4.6}$$

We note that

$$F=\frac{1}{2}f'(\phi)v^2+O(|v|^3),\quad G=\frac{1}{2}f''(\phi)\phi_xv^2+\phi_xO(|v|^3)\tag{4.7}$$

for $v\to 0$. Here, a careful computation, using (3.2) and (4.4), shows that

$$G\ge c(1+x)^{-2}v^2-C(1+x)^{-1-1/q}|v|^3\tag{4.8}$$

for any $x\in\mathbb{R}_+$. We integrate (4.5) over $\mathbb{R}_+$ and substitute (4.8) into the resulting equality, obtaining

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2}+\|v_x\|^2_{L^2}+c\|v\|^2_{L^2_{-2}}\le C\|v\|^3_{L^3_{-1}}.$$

We multiply this inequality by $(1+t)^\gamma$ and integrate with respect $t$. This yields the desired inequality (4.2) for $\beta = 0$.

*Proof of* (4.2) *for* $\beta > 0$. We apply the space-time weighted energy method employed in [14, 2] (see also [3]). Let $w > 0$ be a smooth weight function depending only on $x$, which will be specified later. We multiply (4.5) by $w$, obtaining

$$\begin{aligned}\Big(\frac{1}{2}v^2 w\Big)_t &+ \Big\{(F - \mu v v_x)w + \frac{1}{2}v^2 w_x\Big\}_x \\ &+ v_x^2 w - \Big(\frac{1}{2}v^2 w_{xx} + F w_x - G w\Big) = 0.\end{aligned} \tag{4.9}$$

Here, using (4.7), we have

$$\frac{1}{2}v^2 w_{xx} + F w_x - G w = \frac{1}{2}v^2(w_{xx} + w_x f'(\phi) - w f''(\phi)\phi_x) + R, \tag{4.10}$$

where $R = w_x O(|v|^3) - w\phi_x O(|v|^3)$ for $v \to 0$. Notice that the coefficient $w_{xx} + w_x f'(\phi) - w f''(\phi)\phi_x$ in (4.10) is just the same as that appeared in (3.4). Now we choose the weight function $w$ and the corresponding $w_1$ by (2.8) with $\sigma = (\beta-1)q$, where $0 \le \beta \le \alpha$ and $1 \le \alpha < \alpha_c(q) := 3 + 2/q$. Then we have (3.5) with $\sigma = (\beta - 1)q$. Substituting these expressions into (4.9) and integrating over $\mathbb{R}_+$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(w)} &+ \|v_x\|^2_{L^2(w)} - c_1(\sigma)\|v\|^2_{L^2(w_1)} \\ &+ \int_0^\infty v^2 r(\phi) w_1\,dx = \int_0^\infty R\,dx,\end{aligned} \tag{4.11}$$

where $c_1(\sigma)$ and $r(\phi)$ are given in (3.6) with $\sigma = (\beta-1)q$. Here our weight functions verify

$$w \sim (1+x)^\beta, \quad w_1 \sim (1+x)^{\beta-2}, \tag{4.12}$$

where the symbol $\sim$ means the equivalence. This implies that the norms $\|\cdot\|_{L^2(w)}$ and $\|\cdot\|_{L^2(w_1)}$ are equivalent to $\|\cdot\|_{L^2_\beta}$ and $\|\cdot\|_{L^2_{\beta-2}}$, respectively.

We estimate (4.11) similarly as in (1) of Theorem 3.2. To this end, we note that $\sigma_1 \le \sigma \le \sigma_2$, where $\sigma_1 = -q$ and $\sigma_2 = (\alpha-1)q$. Since $c_1(\sigma) < \sigma^2/4$ for $-2q < \sigma < 2(q+1)$ and since $-2q < \sigma_1 < \sigma_2 < 2(q+1)$, we can choose $\delta > 0$ so small that

$$\delta \le \min_{\sigma_1 \le \sigma \le \sigma_2} \frac{\sigma^2/4 - c_1(\sigma)}{2 + \sigma^2/4}.$$

Notice that this $\delta$ is independent of $\beta$. For this choice of $\delta$, we take $a = a(\delta) > 0$ so large that $|r(\phi)| \le \delta$ for $x \ge a$. Then we have

$$\Big|\int_0^\infty v^2 r(\phi) w_1\,dx\Big| \le \delta\|v\|^2_{L^2(w_1)} + C\|v\|^2_{L^2_{-2}},$$

where $C$ is a constant satisfying $C \geq (1+x)^2|r(\phi)|w_1$ for $0 \leq x \leq a$. Also, using the Hardy type inequality $(\sigma^2/4)\|v\|^2_{L^2(w_1)} \leq \|v_x\|^2_{L^2(w)}$ in (2.9) and estimating similarly as in (3.10), we have

$$\|v_x\|^2_{L^2(w)} - c_1(\sigma)\|v\|^2_{L^2(w_1)} \geq \delta\|v_x\|^2_{L^2(w)} + 2\delta\|v\|^2_{L^2(w_1)},$$

where we have used the fact that $(1-\delta)\sigma^2/4 - c_1(\sigma) \geq 2\delta$. On the other hand, using (4.4), we see that $|R| \leq C(|w_x|+w\phi_x)|v|^3$. Moreover, a straightforward computation shows that $|w_x| + w\phi_x \leq C(1+x)^{\beta-1}$. Substituting all these estimates into (4.11), we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(w)} + \delta(\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)}) \leq C\|v\|^2_{L^2_{-2}} + C\|v\|^3_{L^3_{\beta-1}}, \tag{4.13}$$

where $\delta$ and $C$ are independent of $\beta$. We multiply this inequality by $(1+t)^\gamma$ and integrate with respect to $t$. By virtue of (4.12), we have

$$\begin{aligned}
&(1+t)^\gamma\|v(t)\|^2_{L^2_\beta} + \int_0^t (1+\tau)^\gamma(\|v_x(\tau)\|^2_{L^2_\beta} + \|v(\tau)\|^2_{L^2_{\beta-2}})\,d\tau \\
&\leq C\|v_0\|^2_{L^2_\beta} + \gamma C\int_0^t (1+\tau)^{\gamma-1}\|v(\tau)\|^2_{L^2_\beta}d\tau \\
&\qquad + C\int_0^t (1+\tau)^\gamma\|v(\tau)\|^2_{L^2_{-2}}d\tau + CS^\gamma_\beta(t),
\end{aligned} \tag{4.14}$$

where the constant $C$ is independent of $\gamma$ and $\beta$. Here the third term on the right hand side of (4.14) was already estimated by (4.2) with $\beta = 0$. Hence we have proved (4.2) also for $0 < \beta \leq \alpha$. This completes the proof. □

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