

L^p well-posedness for the complex Ginzburg-Landau equation

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1 Introduction

This note is based on a joint work with Prof. Naoki Tanaka (Shizuoka University).

We consider the initial-boundary value problem for the complex Ginzburg-Landau equation:

$$(CGL) \quad \begin{cases} \frac{\partial u}{\partial t} - (\lambda + i\mu)\Delta u + (\kappa + i\nu)|u|^{q-2}u - \gamma u = 0, & (x, t) \in \Omega \times (0, \infty), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \partial\Omega. \end{cases}$$

Here Ω is a general domain in $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial\Omega$, $\lambda > 0$, $\kappa > 0$, $\mu, \nu, \gamma \in \mathbb{R}$, and $i = \sqrt{-1}$. The complex Ginzburg-Landau equation with $q = 4$ was derived to describe the destabilization of plane shear flow and the instability problem in nonlinear chemical kinetics. The general case ($q \geq 2$) has been studied as a model for turbulent dynamics. For details we refer to [1].

Our aim in this note is to show the time global well-posedness for (CGL) in $L^p(\Omega)$. The weak or strong global well-posedness for (CGL) has been studied by many authors. We refer to [2, 3, 4, 6, 7, 8, 9, 10, 11] and references therein. Among them, Ginibre-Velo [4] established the global well-posedness for (CGL) in $L^p(\mathbb{R}^N)$ and locally uniform $L^p(\mathbb{R}^N)$ spaces under the assumptions that $p \geq 2$,

$$|\mu|/\lambda < 2\sqrt{p-1}/|p-2|, \quad (1.1)$$

$$2 \leq q \leq 2 + 2p/N. \quad (1.2)$$

It should be remarked that Yokota and Okazawa [11] studied (CGL) for $u_0 \in L^2(\Omega) \cap L^p(\Omega)$. They proved the unique existence of strong solution to (CGL) and the continuous dependence on its initial data in $L^2(\Omega)$ under the assumptions that $p \geq 2$,

$$|\mu|/\lambda \leq 2\sqrt{p-1}/|p-2|,$$

$$2 \leq q < 2 + 2p/N.$$

Our main result is an extension of one of the results of [4].

Main Theorem. Assume $p \in (1, \infty)$, (1) and (2). Then for each $u_0 \in L^p(\Omega)$ there exists a unique solution $u(\cdot; u_0)$ to (CGL) in the class

$$C([0, \infty); L^p(\Omega)) \cap C^1((0, \infty); L^p(\Omega)) \cap C((0, \infty); W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)).$$

Moreover, the following continuous dependence of solutions on their initial data holds: for each $\tau > 0$ and $r > 0$ there exists $M(\tau, r) > 0$ such that

$$\|u(t; u_0) - u(t; \hat{u}_0)\|_{L^p} \leq M(\tau, r) \|u_0 - \hat{u}_0\|_{L^p}$$

for $t \in [0, \tau]$ and $u_0, \hat{u}_0 \in L^p(\Omega)$ with $\|u_0\|_{L^p} \leq r$ and $\|\hat{u}_0\|_{L^p} \leq r$.

Our approach to (CGL) is quite different from that of [4]. First we rewrite (CGL) as an abstract semilinear Cauchy problem

$$u'(t) = Au(t) + Bu(t) \quad (t > 0), \quad u(0) = u_0$$

in a Banach space $X = L^p(\Omega)$. Then we apply a characterization theorem for semigroups of locally Lipschitz operators associated with the above Cauchy problem obtained by [7] to prove our main theorem.

2 Characterization theorem for semigroups of locally Lipschitz operators

In this section we introduce semigroups of locally Lipschitz operators and recall the characterization theorem for them.

We consider a semilinear Cauchy problem

$$u'(t) = Au(t) + Bu(t) \quad (t > 0), \quad u(0) = u_0 \in D \quad (\text{SP}; u_0)$$

in a Banach space $(X, \|\cdot\|)$. Here A is the infinitesimal generator of an analytic (C_0) semigroup $\{T(t) \mid t \geq 0\}$ on X satisfying the condition below:

(A) There exist constants $M \geq 1$ and $\omega_A < 0$ such that $\|T(t)\| \leq Me^{\omega_A t}$ for $t \geq 0$.

Let $\alpha \in (0, 1)$ and let Y be the Banach space $D((-A)^\alpha)$ equipped with the norm $\|x\|_Y = \|(-A)^\alpha x\|_X$ for $x \in D((-A)^\alpha)$. Let D be a subset of X and let φ be a proper lower semicontinuous functional from X into $[0, \infty]$ such that $D(\varphi) = D$ and for each $r > 0$, $C_r := D_r \cap Y$ is dense in D_r , where $D_r = \{x \in X; \varphi(x) \leq r\}$ for $r > 0$. Let $C = D \cap Y$. The operator B from C into X is assumed to satisfy the following conditions:

(B1) For each $r > 0$ the operator B is continuous from C_r into X .

(B2) For each $r > 0$ there exists $M_B(r) > 0$ such that

$$\|Bx\|_X \leq M_B(r)(1 + \|x\|_Y) \quad \text{for } x \in C_r. \quad (2.1)$$

Now we give the definition of semigroups of locally Lipschitz operators.

Definition 2.1. A one-parameter family $\{S(t); t \geq 0\}$ of locally Lipschitz operators from D into itself is called a *semigroup of locally Lipschitz operators on D with respect to the functional φ* if the following three conditions are satisfied:

- (S1) $S(0)x = x$ for $x \in D$, and $S(t+s)x = S(t)S(s)x$ for $s, t \geq 0$ and $x \in D$.
- (S2) For each $x \in D$, $S(\cdot)x : [0, \infty) \rightarrow X$ is continuous.
- (S3) For each $\tau \geq 0$ and $r \geq 0$ there exists $M(\tau, r) > 0$ such that

$$\|S(t)x - S(t)y\| \leq M(\tau, r)\|x - y\| \quad \text{for } x, y \in D_r \text{ and } t \in [0, \tau].$$

The characterization theorem is stated as follows:

Theorem 2.2. ([7, Proposition 2.4 and Theorem 3.5]) *Let $a_0 \geq 0$. The following two statements (i) and (ii) are equivalent:*

- (i) *There exists a semigroup $\{S(t); t \geq 0\}$ of locally Lipschitz operators on D with respect to φ such that for $x \in D$, $S(t)x \in C$ for $t > 0$, $BS(t)x \in C((0, \infty); X) \cap L^1_{loc}(0, \infty; X)$ and $S(t)x$ satisfies the integral equation*

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x ds \quad \text{for } t \geq 0,$$

and the growth condition

$$\varphi(S(t)x) \leq e^{a_0 t} \varphi(x) \quad \text{for } x \in D \text{ and } t \geq 0.$$

- (ii) *The following three conditions are satisfied:*

- (ii-1) *There exist $\tau > 0$ and a family $\{V_r(\cdot, \cdot, \cdot); r > 0\}$ of nonnegative functionals on $[0, \tau] \times X \times X$ such that*

(V1) *for each $r > 0$ and $x, y \in D_r$, $V_r(\cdot, x, y) : [0, \tau] \rightarrow [0, \infty)$ is continuous,*

(V2) *for each $r > 0$ there exists $L(r) > 0$ such that*

$$|V_r(t, x, y) - V_r(t, \hat{x}, \hat{y})| \leq L(r)(\|x - \hat{x}\|_X + \|y - \hat{y}\|_X)$$

for $(t, x, y), (t, \hat{x}, \hat{y}) \in [0, \tau] \times X \times X$,

- (V3) *for each $r > 0$ there exist $M(r) \geq m(r) > 0$ such that*

$$m(r)\|x - y\|_X \leq V_r(t, x, y) \leq M(r)\|x - y\|_X$$

for $(t, x, y) \in [0, \tau] \times D_r \times D_r$.

(ii-2) For each $r > 0$ there exist $R \geq r$ and $\omega \geq 0$ such that

$$\liminf_{h \downarrow 0} (V_R(t+h, J(h)x, J(h)y) - V_R(t, x, y))/h \leq \omega V_R(t, x, y)$$

for $(t, x, y) \in [0, \tau) \times C_r \times C_r$, where

$$J(t)w = T(t)w + \int_0^t T(s)Bw ds \quad \text{for } w \in C \text{ and } t \geq 0.$$

(ii-3) There exists $b_0 \in (0, 1)$ such that to each $x \in C$ and $\varepsilon > 0$ there correspond $\delta \in (0, \varepsilon]$, $x_\delta \in C$ and $z_\delta \in Y$ satisfying

$$\begin{aligned} x_\delta &= J(\delta)x + z_\delta, \quad \|z_\delta\|_X \leq \varepsilon\delta, \quad \|z_\delta\|_Y \leq \varepsilon\delta^{b_0}, \\ (\varphi(x_\delta) - \varphi(x))/\delta &\leq a_0\varphi(x) + \varepsilon. \end{aligned}$$

Moreover, for each $x \in D$, $S(t)x$ is continuously differentiable on $(0, \infty)$ in X , $AS(t)x$ is continuous on $(0, \infty)$ in X , and $S(t)x$ satisfies (SE; x) for $t > 0$, if the operator B satisfies that for each $\rho > 0$ there exists $L_B(\rho) > 0$ such that

$$\|Bu - Bv\|_X \leq L_B(\rho)\|u - v\|_Y \quad \text{for } u, v \in C \text{ and } \|u\|_Y \leq \rho, \|v\|_Y \leq \rho. \quad (2.2)$$

3 Outline of the Proof of Main Theorem

In order to prove our main theorem, we first rewrite (CGL) as an abstract Cauchy problem (SP; u_0) in $X = L^p(\Omega)$ and then apply the characterization theorem. In what follows K stands for various constants.

Let $X = L^p(\Omega)$ and $\|u\|_X = \|u\|_{L^p}$ for $u \in X$. Then, the operator A defined by

$$Au = (\lambda + i\mu)(\Delta u - u) \quad \text{for } u \in D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

is the generator of an analytic (C_0) semigroup $\{T(z)\}$ on X such that $T(z)$ is analytic in the sector $|\arg z| < \omega_p$ and $\|T(t)\|_X \leq e^{-\lambda t}$ for $t \geq 0$, where $\omega_p = \tan^{-1}(2\sqrt{p-1}/|p-2|)$.

In what follows we assume that $q > 2$. By (1.1) we can choose $\tilde{p}$ such that $p < \tilde{p} < p + q - 2$,

$$|\mu|/\lambda < 2\sqrt{\tilde{p}-1}/|\tilde{p}-2|, \quad (3.1)$$

$$\tilde{\theta} := (N/2)(1/p - 1/(\tilde{p}(q-1))) < 1. \quad (3.2)$$

Let $\beta = (N/2)(1/p - 1/\tilde{p})$ and $\theta = (1/\tilde{p} - 1/p + 2/N)^{-1}(1/\tilde{p} - 1/(p(q-1)))$. Then it is easily seen that $\beta, \theta, \beta + (q-1)\theta(1-\beta) \in (0, 1)$. The Gagliardo-Nirenberg inequality implies that $D(A) \subset L^{\tilde{p}}(\Omega) \cap L^{p(q-1)}(\Omega) \cap L^{\tilde{p}(q-1)}(\Omega)$. Choose $\alpha \in (0, 1)$ satisfying

$$\tilde{\theta} < \alpha < 1 \quad \text{and} \quad \beta + (q-1)\theta(1-\beta) < \alpha < 1. \quad (3.3)$$

Let $D = L^p(\Omega) \cap L^{\bar{p}}(\Omega)$. We define a lower semicontinuous functional $\varphi : X \rightarrow [0, \infty]$ by $\varphi(u) = \|u\|_{L^p}^p + \|u\|_{L^{\bar{p}}}^{\bar{p}}$ if $u \in D$ and $\varphi(u) = \infty$ otherwise. Let $Y = D((-A)^\alpha)$. We introduce a nonlinear operator B defined by

$$Bu = -(\kappa + i\nu)|u|^{q-2}u + (\lambda + i\mu + \gamma)u \quad \text{for } u \in D(B) = C (= D \cap Y). \quad (3.4)$$

Then (CGL) is rewritten as a semilinear Cauchy problem (SP; u_0). The operators A and B have the properties below.

Lemma 3.1. *For $s \in \{p, \bar{p}\}$, the following are valid.*

(i) *The operator A generates an analytic semigroup $\{T(t); t \geq 0\}$ on X such that*

$$\|T(t)v\|_{L^s} \leq e^{-\lambda t} \|v\|_{L^s} \quad \text{for } v \in D, \quad t \geq 0.$$

(ii) *The Banach space Y is continuously embedded in $D \cap L^{p(q-1)}(\Omega) \cap L^{\bar{p}(q-1)}(\Omega)$.*

(iii) *For each $r > 0$ there exists $M_B(r) > 0$ such that*

$$\|Bv\|_{L^s} \leq M_B(r)(1 + \|v\|_Y) \quad \text{for } v \in C_r.$$

(iv) *For each $\rho > 0$ there exists $L_B(\rho) > 0$ such that*

$$\|Bv - B\hat{v}\|_{L^s} \leq L_B(\rho)\|v - \hat{v}\|_Y \quad \text{for } v, \hat{v} \in C \text{ with } \|v\|_Y \leq \rho, \|\hat{v}\|_Y \leq \rho.$$

(v) *The domain of A is continuously embedded in $L^{s+q-2}(\Omega)$ and it holds that*

$$\operatorname{Re} \langle Av + Bv, |v|^{s-2}v \rangle + \kappa \|v\|_{L^{s+q-2}}^{s+q-2} - \gamma \|v\|_{L^s}^s \leq 0$$

for $v \in D(A) \cap W^{2,\bar{p}}(\Omega) \cap W_0^{1,\bar{p}}(\Omega)$, where $\langle w, z \rangle = \int_\Omega w(x) \overline{z(x)} dx$.

(vi) *There exist constants $a > 0$ and $b > 0$ such that*

$$\begin{aligned} & \operatorname{Re} \langle Au + Bu - (A\hat{u} + B\hat{u}), |u - \hat{u}|^{p-2}(u - \hat{u}) \rangle \\ & \leq (a + b(\|u\|_{L^{p+q-2}}^{p+q-2} + \|\hat{u}\|_{L^{p+q-2}}^{p+q-2}))\|u - \hat{u}\|_{L^p}^p \quad \text{for } u, \hat{u} \in D(A). \end{aligned}$$

Let $\tau > 0$ and define a family $\{V_r(\cdot, \cdot, \cdot); r > 0\}$ of nonnegative functions on $[0, \tau] \times X \times X$ by

$$V_r(t, u, v) = \exp((b/\kappa p)((\|u\|_X \wedge r^{1/p})^p + (\|v\|_X \wedge r^{1/p})^p))(\|u - v\|_X \wedge (2r^{1/p}))$$

for $(t, u, v) \in [0, \tau] \times X \times X$, where b is a positive number satisfying Lemma 3.1 (vi) and $\xi \wedge \eta = \min(\xi, \eta)$ for $\xi, \eta \in \mathbb{R}$.

By applying Theorem 2.2, we obtain the following proposition.

Proposition 3.2. *There exists a semigroup $\{S(t); t \geq 0\}$ of locally Lipschitz operators on D with respect to φ satisfying the following:*

- (i) *For $u_0 \in D$, $S(\cdot)u_0 \in C([0, \infty); X) \cap C^1((0, \infty); X) \cap C((0, \infty); D(A))$.*
- (ii) *For $u_0 \in D$, $u(t) \equiv S(t)u_0$ gives a C^1 solution to $(SP; u_0)$ satisfying*

$$\|u(t)\|^p + p\kappa \int_0^t e^{p\gamma(t-s)} \|u(s)\|_{L^{p+q-2}}^{p+q-2} ds \leq e^{p\gamma t} \|u_0\|^p \quad \text{for } t \geq 0. \quad (3.5)$$

- (iii) *For $u_0 \in D$ and $t \geq 0$, $\|S(t)u_0\| \leq e^{\gamma t} \|u_0\|$ and $\|S(t)u_0\|_{L^{\tilde{p}}} \leq e^{\gamma t} \|u_0\|_{L^{\tilde{p}}}$.*
- (iv) *For $\tau > 0$ and $r \geq 0$ there exists $M_{\tau, r} > 0$ such that*

$$\|S(t)u_0 - S(t)\hat{u}_0\| \leq M_{\tau, r} \|u_0 - \hat{u}_0\|$$

for $t \in [0, \tau]$ and $u_0, \hat{u}_0 \in D$ with $\|u_0\| \leq r$ and $\|\hat{u}_0\| \leq r$.

By Proposition 3.2 (iv), we observe that the family $\{S(t); t \geq 0\}$ can be uniquely extended to a semigroup $\{\tilde{S}(t); t \geq 0\}$ of locally Lipschitz operators on X . We have only to show that for each $u_0 \in X = L^p(\Omega)$, $\tilde{S}(t)u_0$ gives a C^1 solution to $(SP; u_0)$. Let $u_0 \in X$. Since $u(t) = \tilde{S}(t)u_0$ also satisfies the inequality (3.5), we see that $\tilde{S}(t)u_0 \in L^p(\Omega) \cap L^{p+q-2}(\Omega)$ for a.e. $t > 0$. This fact ensures that there exists a decreasing sequence $\{t_n\}$ of positive numbers such that $\lim_{n \rightarrow \infty} t_n = 0$ and $\tilde{S}(t_n)u_0 \in L^p(\Omega) \cap L^{\tilde{p}}(\Omega)$ for $n \geq 1$. Since $\tilde{S}(t)u_0 = S(t - t_n)\tilde{S}(t_n)u_0$ for $t > t_n$, $\tilde{S}(t)u_0$ gives a global C^1 solution. $\square$

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