

THE DEFINITE INTEGRAL OF PRODUCTS OF BESSEL FUNCTIONS

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I. Introduction

In a problem of the Ising spin glass on the Bethe lattice we encountered a nonlinear integral equation ([1], [2])

$$S(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(x, y) [S(y)]^2 dy \quad (1.1)$$

$$K(x, y) = 2\pi \delta(y) \cos x - 2j_0(y) \cos x + 2 \sum_{m=0}^{\infty} (1 + 4m) j_{2m}(x) j_{2m}(y) \quad (1.2)$$

where z is the natural number, and $j_{2m}(x)$ is the spherical Bessel function of order $2m$.

We put

$$S(x) = a + b \cos x + \sum_{l=0}^{\infty} c_{2l} j_{2l}(x) \quad (1.3)$$

Substituting (1.2) and (1.3) into (1.1), we get a system of algebraic equations for unknowns a , b , and c_{2l} of which the coefficients are given by definite integral of products of Bessel functions. Solution $S(x)$ can be solved by the solution of the simultaneous algebraic equation. We defined the following integral I :

$$I_{\ell_1 \ell_2 \dots \ell_\nu}^{(k)} \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \cos^k x \, j_{\ell_1}(x) j_{\ell_2}(x) \dots j_{\ell_\nu}(x) dx \quad (1.4)$$

Katsura [3], and Katsura and Nishihara [4] calculated $I_{\ell_1 \ell_2 \ell_3}^{(0)}$ and $I_{\ell_1 \ell_2}^{(1)}$ by using the residues and J. E. Kilpatrick, Katsura and Inoue [5] calculated

$$W_{\nu_1 \nu_2 \dots \nu_\ell}^\lambda(a_1, a_2, \dots, a_\ell) \\ \equiv \int_0^\infty J_{\nu_1}(a_1 t) J_{\nu_2}(a_2 t) \dots J_{\nu_\ell}(a_\ell t) t^{-\lambda} dt \quad (1.5)$$

In this paper we calculate the values of following integrals I, N, V, Λ , and R which appeared in the spin glass calculation as slated above by extending their method.

$$I_{n_1 n_2 \dots n_\nu}^{(h;k)} \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^h(x+k) j_{n_1}(x) j_{n_2}(x) \dots j_{n_\nu}(x) dx \quad (1.5)$$

$$N_{m_1 m_2 \dots m_\mu}^{(h;k)} \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^h(x+k) n_{m_1}(x) n_{m_2}(x) \dots n_{m_\mu}(x) dx \quad (1.6)$$

$$V_{n_1 n_2 \dots n_\nu; m_1 m_2 \dots m_\mu}^{(h_1, h_2, \dots, h_\lambda; k_1, k_2, \dots, k_\lambda)} \\ \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^{h_1}(x+k_1) \sin^{h_2}(x+k_2) \dots \sin^{h_\lambda}(x+k_\lambda) \\ \times j_{n_1}(x) j_{n_2}(x) \dots j_{n_\nu}(x) n_{m_1}(x) n_{m_2}(x) \dots n_{m_\mu}(x) dx \quad (1.7)$$

$$\Lambda(\ell, m, n) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \{j_0(x)\}^\ell \sin^m x \cos^n x dx \quad (1.8)$$

$$R_{n_1 n_2 \dots n_\nu}^{(m)} \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} e^{imx} j_{n_1}(x) j_{n_2}(x) \dots j_{n_\nu}(x) dx \quad (1.9)$$

where $n_m(x)$ is the spherical Neumann function of order m .

II. The integral for I , N , V , Λ , and R

$j_n(x)$ and $n_m(x)$ can be expressed in

$$j_n(x) = \frac{1}{2x} \sum_{r=0}^n [A_{nr} e^{ix} + A_{nr}^* e^{-ix}] x^{-r} \quad (2.1)$$

$$n_m(x) = \frac{1}{2x} \sum_{s=0}^m (-1)^{m+1} [B_{ms} e^{ix} + B_{ms}^* e^{-ix}] x^{-s} \quad (2.2)$$

$$A_{nr} = Y_{nr} i^{r-n-1}, \quad B_{ms} = Y_{ms} i^{m+s},$$

$$Y_{nr} = \frac{(n+r)!}{n!(n-r)!} 2^{-r} \quad (2.3)$$

(A_{nr}^* is the complex conjugate of A_{nr}) and

$$\begin{aligned} \sin^h(x+k) &= \sum_{p=0}^h \sum_{q=0}^p \sum_{u=0}^{h-p} \cos^p k \sin^{h-p} k (-1)^{(p+2q)/2} 2^{-h} \\ &\quad \times \binom{h}{p} \binom{p}{q} \binom{h-p}{u} \exp(i(h-2q-2u)x). \end{aligned} \quad (2.4)$$

Furthermore by using

$$P \int_{-\infty}^{\infty} \frac{\exp(iky)}{y^\ell} dy = \frac{(ik)^{\ell-1}}{(\ell-1)!} i\pi \operatorname{sgn}(k) \quad (k \neq 0) \quad (2.5)$$

where P is the principal part. Hence by using (2.1)-(2.5), we can evaluate them.

$$\begin{aligned}
I_{n_1 n_2 \dots n_\nu}^{(h;k)} &= \sum_{p=0}^h \sum_{q=0}^p \sum_{u=0}^{h-p} r_1^{\sum_{i=1}^1 n_i} r_2^{\sum_{i=2}^2 n_i} \dots r_\nu^{\sum_{i=\nu}^\nu n_i} \phi^{\sum_{i=0}^\nu} \cos^{h_k} \\
&\times \sin^{h-p_k} \binom{h}{p} \binom{p}{q} \binom{h-p}{u} \frac{1}{2^{h+\nu}} \\
&\times \frac{\prod_{w=1}^\nu Y_{n_w} r_w}{\left(\sum_{w=1}^\nu r_w + \nu - 1 \right)!} (-1)^{\frac{1}{2} \left\{ \sum_{w=1}^\nu (2r_w + n_w) + (p+2u) \right\}} \\
&\times W_\phi(\nu, 0) f^{\sum_{w=1}^\nu r_w + \nu - 1} \operatorname{sgn}(f) \quad (2.6)
\end{aligned}$$

$$\begin{aligned}
N_{m_1 m_2 \dots m_\mu}^{(h;k)} &= \sum_{p=0}^h \sum_{q=0}^p \sum_{u=0}^{h-p} s_1^{\sum_{i=1}^1 m_i} s_2^{\sum_{i=2}^2 m_i} \dots s_\nu^{\sum_{i=\nu}^\mu m_i} \phi^{\sum_{i=0}^\mu} \cos^{h_k} \\
&\times \sin^{h-p_k} \binom{h}{p} \binom{p}{q} \binom{h-p}{u} \frac{1}{2^{h+\mu}} \\
&\times \frac{\prod_{w=1}^\mu Y_{m_w} s_w}{\left(\sum_{w=1}^\mu s_w + \mu - 1 \right)!} (-1)^{\frac{1}{2} \left\{ \sum_{w=1}^\mu (2s_w + m_w + \mu) + (p+2u) \right\}} \\
&\times W_\phi(0, \mu) g^{\sum_{w=1}^\mu s_w + \mu - 1} \operatorname{sgn}(g) \quad (2.7)
\end{aligned}$$

(where if μ is odd, the integral for $N = 0$, because N is the odd function)

$$V_{n_1 n_2 \dots n_\nu; m_1 m_2 \dots m_\mu}^{(h_1, h_2, \dots, h_\lambda; k_1, k_2, \dots, k_\lambda)}$$

$$\begin{aligned}
&= p_1 \sum_{\xi=0}^{h_1} p_2 \sum_{\xi=0}^{h_2} \cdots p_\lambda \sum_{\xi=0}^{h_\lambda} q_1 \sum_{\xi=0}^{p_1} q_2 \sum_{\xi=0}^{p_2} \cdots q_\lambda \sum_{\xi=0}^{p_\lambda} u_1 \sum_{\xi=0}^{h_1-p_1} u_2 \sum_{\xi=0}^{-p_2} \cdots \\
&\quad u_\lambda \sum_{\xi=0}^{h_\lambda-p_\lambda} r_1 \sum_{\xi=0}^{n_1} r_2 \sum_{\xi=0}^{n_2} \cdots r_\nu \sum_{\xi=0}^{n_\nu} s_1 \sum_{\xi=0}^{m_1} s_2 \sum_{\xi=0}^{m_2} \cdots s_\mu \sum_{\xi=0}^{m_\mu} \phi \sum_{\xi=0}^{\nu+\mu} \\
&\quad \times \zeta \prod_{\xi=1}^{\lambda} \cos^{p_\xi} \zeta \, k_\zeta \sin^{h_\xi-p_\xi} \zeta \, k_\zeta \left\{ \begin{matrix} h_\xi \\ p_\xi \end{matrix} \right\} \left\{ \begin{matrix} p_\xi \\ q_\xi \end{matrix} \right\} \left\{ \begin{matrix} h_\xi-q_\xi \\ u_\xi \end{matrix} \right\} \\
&\quad \times \frac{1}{2^{\sum_{\xi=1}^{\lambda} h_\xi + \nu + \mu}} \frac{\prod_{\omega=1}^{\lambda} \prod_{\eta=1}^{\mu} Y_{n_\omega} r_\omega Y_{m_\eta} s_\eta}{(\sum_{\omega=1}^{\nu} r_\omega + \sum_{\eta=1}^{\mu} s_\eta + \nu + \mu - 1)!} \\
&\quad \times (-1)^{\frac{1}{2} \{ \sum_{\xi=1}^{\lambda} (p_\xi + 2u_\xi) + \sum_{\omega=1}^{\nu} (n_\omega + 2r_\omega) + \sum_{\eta=1}^{\mu} (m_\eta + s_\eta + \mu) \}} \\
&\quad \times W_{\phi}(\nu, \mu) \Phi^{\left(\sum_{\omega=1}^{\nu} r_\omega + \sum_{\eta=1}^{\mu} s_\eta + \nu + \mu - 1 \right)} \operatorname{sgn}(\Phi) \quad (2.8)
\end{aligned}$$

$$\begin{aligned}
\Lambda(\ell, m, n) &= \sum_{u=0}^{\ell} \sum_{v=0}^m \sum_{w=0}^n \left\{ \begin{matrix} \ell \\ u \end{matrix} \right\} \left\{ \begin{matrix} m \\ v \end{matrix} \right\} \left\{ \begin{matrix} n \\ w \end{matrix} \right\} \frac{(-1)^{-m/2+u+v}}{2^{\ell+m+n} (\ell-1)!} \\
&\quad \times \Gamma^{\ell-1} \operatorname{sgn}(\Gamma) \quad (2.9)
\end{aligned}$$

(where if m is odd, $\Lambda(\ell, m, n) = 0$, because $\Lambda(\ell, m, n)$ is the odd function)

$$\begin{aligned}
R_{n_1 n_2 \cdots n_\nu}^{(m)} &= r_1 \sum_{\xi=0}^{n_1} r_2 \sum_{\xi=0}^{n_2} \cdots r_\nu \sum_{\xi=0}^{n_\nu} \phi \sum_{\xi=0}^{\nu} 2^{-\nu} \\
&\quad \times (-1)^{\sum_{\omega=1}^{\nu} (2r_\omega + n_\omega)/2} \frac{\prod_{\omega=1}^{\nu} Y_{n_\omega} r_\omega}{(\sum_{\omega=1}^{\nu} r_\omega + \nu - 1)!} W_{\phi}(\nu, 0) \\
&\quad \times \Gamma^{\sum_{\omega=1}^{\nu} r_\omega + \nu - 1} \operatorname{sgn}(\Gamma) \quad (2.10)
\end{aligned}$$

where $f \equiv (h + \nu) - 2(q + u + \phi)$, $g \equiv (h + \mu) - 2(q + u + \phi)$, $\Phi \equiv \left(\sum_{\eta=1}^{\lambda} h_\eta + \nu + \mu \right)$

$-2 \left\{ \sum_{\eta=1}^{\lambda} (q_\eta + u_\eta) + \phi \right\}$, $\Gamma \equiv (\ell + m + n) - 2(u + v + w)$, and $\Upsilon \equiv m + \nu - 2\phi$. $W_{\phi}(\nu, \mu)$ is as

follows.

$$W_0(\nu, \mu) \equiv 1 \quad (2.11)$$

$$W_1(\nu, \mu) \equiv \sum_{\xi=1}^{\nu} (-1)^{r_{\xi}+n_{\xi}+1} + \sum_{\eta=1}^{\mu} (-1)^{s_{\eta}+m_{\eta}} \quad (2.12)$$

$W_{\phi}(\nu, \mu)$ ($\phi = 2, \dots, \nu$) is defined by the sum of $\binom{\nu+\mu}{\phi}$ products of ϕ pieces of $(-1)^{r_{\lambda}+n_{\lambda}+1}$ ($\lambda=1, \dots, \nu$) and $(-1)^{s_{\zeta}+m_{\zeta}}$ ($\zeta=1, \dots, \mu$).

Example. $\nu = 3, \mu = 2, \phi = 4$

$$\begin{aligned} W_4(3, 2) = & (-1)^{(r_1+r_2+r_3+s_1)+(n_1+n_2+n_3+m_1)+3} \\ & + (-1)^{(r_1+r_2+r_3+s_2)+(n_1+n_2+n_3+m_2)+3} \\ & + (-1)^{(r_1+r_2+s_1+s_2)+(n_1+n_2+m_1+m_2)+2} \\ & + (-1)^{(r_1+r_3+s_1+s_2)+(n_1+n_3+m_1+m_2)+2} \\ & + (-1)^{(r_2+r_3+s_1+s_2)+(n_2+n_3+m_1+m_2)+2} \end{aligned} \quad (2.13).$$

$W_{\phi}(\nu, 0)$ ($W_{\phi}(0, \mu)$) is defined only of $(-1)^{r+n+1}$ ($(-1)^{m+s}$).

Values of I and N for $2 \leq \nu$ (μ) ≤ 6 , n_{ν} (m_{μ}) = 0, 2, and $0 \leq h \leq 4$ are shown as tables in Appendix. Values of V for $\lambda = 1, \nu = 1, 2, \mu=1, 0 \leq n_{\nu} \leq 3, m_{\mu} = 0$, and $0 \leq h_{\lambda} \leq 1$ are also in Appendix.

Reference

- [1] S. Katsura, Physica 141A (1987) 556; 149A (1988) 371.
- [2] S. Katsura, Prog. Theor. Phys. Suppl. 87 (1986) 139;
- [3] J. E. Kilpatrick, S. Katsura, and Y. Inoue, Mathematics of Computation 21 NO.99 (1967) 407.
- [4] S. Katsura, Phys. Rev. 115 (1959) 1417.
- [5] S. Katsura and K. Nishihara, J. Chem. Phys. 50 (1967) 3579.

Appendix Tables of the integral of products of
Bessel functions

Here tables for I in Tables 1.1-1.3, N in Tables 2.1-2.3, V
in Tables 3.1-3.2 are shown.

Table 1.1 The integral for I ($h=0,1$, $\nu=2,\dots,6$, $n_\nu=0,2$,
 $k=\pi/2$)

$n_1 n_2 n_3 n_4 n_5 n_6$	h	0	1
0 0		1	$\frac{1}{2}$
0 2		0	0
2 2		$\frac{1}{5}$	$-\frac{7}{80}$
0 0 0		$\frac{3}{4}$	$\frac{1}{2}$
0 0 2		$\frac{1}{16}$	0
0 2 2		$\frac{1}{160}$	0
2 2 2		$\frac{9}{256}$	$-\frac{1}{35}$
0 0 0 0		$\frac{2}{3}$	$\frac{23}{48}$
0 0 0 2		$\frac{1}{30}$	$\frac{1}{120}$
0 0 2 2		$\frac{1}{105}$	$-\frac{31}{13440}$
0 2 2 2		0	$-\frac{3}{17920}$
2 2 2 2		$\frac{4}{385}$	$-\frac{6037}{788480}$
0 0 0 0 0		$\frac{115}{192}$	$\frac{11}{24}$
0 0 0 0 2		$\frac{49}{1920}$	$\frac{1}{120}$
0 0 0 2 2		$\frac{123}{35840}$	0
0 0 2 2 2		$\frac{121}{71680}$	$-\frac{1}{1400}$
0 2 2 2 2		$\frac{37}{1892352}$	$\frac{1}{23100}$
2 2 2 2 2		$\frac{6569}{2408448}$	$-\frac{2377}{1051050}$
0 0 0 0 0 0		$\frac{11}{20}$	$\frac{841}{1920}$
0 0 0 0 0 2		$\frac{2}{105}$	$\frac{9}{1190}$
0 0 0 0 2 2		$\frac{1}{420}$	$\frac{103}{645120}$
0 0 0 2 2 2		$\frac{1}{2200}$	$-\frac{919}{11827200}$
0 0 2 2 2 2		$\frac{1}{2730}$	$-\frac{275}{1490944}$
0 2 2 2 2 2		$\frac{151}{3503500}$	$\frac{1362967}{64576512000}$
2 2 2 2 2 2		$\frac{3331}{4254250}$	$-\frac{2165020841}{334567833600}$

Table 1.2 The integral for I ($h=2,3$, $\nu=2,\dots,6$, $n_\nu=0,2$,
 $k=\pi/2$)

$n_1 n_2 n_3 n_4 n_5 n_6$	h	2	3
0 0		$\frac{1}{2}$	$\frac{3}{8}$
0 2		0	0
2 2		$\frac{1}{10}$	$-\frac{21}{320}$
0 0 0		$\frac{7}{16}$	$\frac{3}{8}$
0 0 2		$\frac{1}{64}$	0
0 2 2		$\frac{1}{640}$	0
2 2 2		$\frac{811}{35840}$	$-\frac{3}{140}$
0 0 0 0		$\frac{5}{12}$	$\frac{35}{96}$
0 0 0 2		$\frac{1}{120}$	$\frac{1}{240}$
0 0 2 2		$\frac{1}{420}$	$-\frac{31}{26880}$
0 2 2 2		0	$-\frac{3}{35840}$
2 2 2 2		$\frac{53}{7700}$	$-\frac{6707}{1126400}$
0 0 0 0 0		$\frac{51}{128}$	$\frac{17}{48}$
0 0 0 0 2		$\frac{9}{1280}$	$\frac{1}{240}$
0 0 0 2 2		$\frac{41}{71680}$	0
0 0 2 2 2		$\frac{53}{102400}$	$-\frac{1}{2800}$
0 2 2 2 2		$-\frac{381}{31539200}$	$\frac{1}{46200}$
2 2 2 2 2		$\frac{11210223}{5740134400}$	$-\frac{1871}{1051050}$
0 0 0 0 0 0		$\frac{23}{60}$	$\frac{1761}{5120}$
0 0 0 0 0 2		$\frac{1}{168}$	$\frac{109}{26880}$
0 0 0 0 2 2		$\frac{1}{2520}$	$\frac{103}{1720320}$
0 0 0 2 2 2		$\frac{1}{13200}$	$-\frac{919}{31539200}$
0 0 2 2 2 2		$\frac{1}{15015}$	$-\frac{189269}{1968046080}$
0 2 2 2 2 2		$\frac{1091}{63063000}$	$\frac{396367}{34440806400}$
2 2 2 2 2 2		$\frac{103013}{178678500}$	$\frac{32493878367}{62452662272000}$

Table 1.3 The integral for I ($h=4$, $\nu=2, \dots, 6$, $n_\nu=0, 2$, $k=\pi/2$)

$n_1 n_2 n_3 n_4 n_5 n_6$	h	4
0 0		$\frac{3}{8}$
0 2		0
2 2		$\frac{3}{40}$
0 0 0		$\frac{11}{32}$
0 0 2		$\frac{1}{128}$
0 2 2		$\frac{1}{1280}$
2 2 2		$\frac{1307}{71680}$
0 0 0 0		$\frac{1}{3}$
0 0 0 2		$\frac{1}{240}$
0 0 2 2		$\frac{1}{840}$
0 2 2 2		0
2 2 2 2		$\frac{43}{7700}$
0 0 0 0 0		$\frac{995}{3072}$
0 0 0 0 2		$\frac{113}{30720}$
0 0 0 2 2		$\frac{123}{573440}$
0 0 2 2 2		$\frac{1621}{5734400}$
0 2 2 2 2		$\frac{5933}{756940800}$
2 2 2 2 2		$\frac{222940799}{137763225600}$
0 0 0 0 0 0		$\frac{101}{320}$
0 0 0 0 0 2		$\frac{11}{3360}$
0 0 0 0 2 2		$\frac{1}{6720}$
0 0 0 2 2 2		$\frac{1}{35200}$
0 0 2 2 2 2		$\frac{37}{480480}$
0 2 2 2 2 2		$\frac{19}{1848000}$
2 2 2 2 2 2		$\frac{17669}{36652000}$

Table 2.1 The integral for N ($h=0,1$, $\mu=2,\dots,6$, $m_\mu=0,2$, $k=\pi/2$)

$m_1 m_2 m_3 m_4 m_5 m_6$	h	0	1
0 0		-1	$-\frac{3}{2}$
0 2		0	$\frac{3}{2}$
2 2		$-\frac{1}{5}$	$-\frac{39}{80}$
0 0 0 0		2	$\frac{45}{16}$
0 0 0 2		$-\frac{11}{10}$	$-\frac{45}{16}$
0 0 2 2		$\frac{13}{35}$	$\frac{3123}{4480}$
0 2 2 2		0	$-\frac{639}{2560}$
2 2 2 2		$-\frac{12}{385}$	$\frac{811287}{3942400}$
0 0 0 0 0 0		$-\frac{15}{4}$	$-\frac{2191}{384}$
0 0 0 0 0 2		$\frac{18}{7}$	$\frac{2191}{384}$
0 0 0 0 2 2		$-\frac{107}{140}$	$-\frac{275197}{129024}$
0 0 0 2 2 2		$\frac{741}{3080}$	$\frac{1004981}{1182720}$
0 0 2 2 2 2		$-\frac{229}{2002}$	$-\frac{30954931}{1230028800}$
0 2 2 2 2 2		$\frac{24747}{700700}$	$\frac{66527233}{1986969600}$
2 2 2 2 2 2		$\frac{6393}{850850}$	$-\frac{83762843347}{4683949670400}$

Table 2.2 The integral for N ($h=2,3$, $\mu=2,\dots,6$, $m_\mu=0,2$,
 $k=\pi/2$)

$m_1 m_2 m_3 m_4 m_5 m_6$	h	2	3
0 0		$-\frac{3}{2}$	$-\frac{15}{8}$
0 2		3	$\frac{21}{4}$
2 2		$-\frac{3}{10}$	$-\frac{147}{64}$
0 0 0 0		$\frac{15}{4}$	$\frac{455}{96}$
0 0 0 2		$\frac{45}{8}$	$-\frac{1849}{192}$
0 0 2 2		$\frac{303}{140}$	$\frac{4579}{768}$
0 2 2 2		$-\frac{36}{35}$	$-\frac{41403}{17920}$
2 2 2 2		$\frac{369}{1540}$	$\frac{927659}{1576960}$
0 0 0 0 0 0		$-\frac{49}{6}$	$-\frac{11445}{1024}$
0 0 0 0 0 2		$\frac{263}{24}$	$\frac{3653}{192}$
0 0 0 0 2 2		$-\frac{2881}{504}$	$-\frac{669925}{49125}$
0 0 0 2 2 2		$\frac{8293}{3696}$	$\frac{35663743}{6307840}$
0 0 2 2 2 2		$-\frac{20407}{30030}$	$-\frac{855175591}{393609216}$
0 2 2 2 2 2		$\frac{485993}{1801800}$	$\frac{4084660117}{4920115200}$
2 2 2 2 2 2		$-\frac{132823}{1021020}$	$-\frac{2499231739269}{12490532454400}$

Table 2.3 The integral for N ($h=4$, $\mu=2, \dots, 6$, $m_\mu=0, 2$,
 $k=\pi/2$)

$m_1 m_2 m_3 m_4 m_5 m_6$	h	
0 0		$-\frac{15}{8}$
0 2		$\frac{15}{2}$
2 2		$-\frac{51}{8}$
0 0 0 0		$\frac{35}{6}$
0 0 0 2		$-\frac{721}{48}$
0 0 2 2		$\frac{329}{24}$
0 2 2 2		$-\frac{39}{7}$
2 2 2 2		$\frac{3659}{1540}$
0 0 0 0 0 0		$-\frac{945}{64}$
0 0 0 0 0 2		$\frac{981}{32}$
0 0 0 0 2 2		$-\frac{1849}{64}$
0 0 0 2 2 2		$\frac{19831}{1408}$
0 0 2 2 2 2		$-\frac{194011}{32032}$
0 2 2 2 2 2		$\frac{680133}{320320}$
2 2 2 2 2 2		$-\frac{134641}{1944800}$

Table 3.1 The integral for V ($\lambda=1$, $\nu, \mu=1$, $n_\nu=0, 1, 2, 3$, $m_\mu=0$, $h_\lambda=0,1$)

$n_1:m_1$	h_1	0		1	
		$k = 0$	$k = \frac{\pi}{2}$	$k = 0$	$k = \frac{\pi}{2}$
0 0		0	0	$-\frac{1}{2}$	0
1 0		0	0	0	$-\frac{1}{4}$
2 0		0	0	0	0
3 0		0	0	0	$-\frac{1}{16}$

Table 3.2 The integral for V ($\lambda=1$, $\nu=2$, $\mu=1$, $n_\nu=0, 1, 2, 3$, $m_\mu=0$, $h_\lambda=0$, $k=0$ or $\pi/2$)

$n_1 n_2 : m_1$	h_1	0
0 0 0		0
0 1 0		$-\frac{1}{6}$
0 2 0		0
0 3 0		0
1 1 0		0
1 2 0		$\frac{7}{240}$
1 3 0		0
2 2 0		0
2 3 0		$\frac{19}{1120}$
3 3 0		0