

A characterization of conjugate functions on parabolic Bergman spaces

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1. Introduction

Let H be the upper half space of $\mathbb{R}^{n+1}$ ($n \geq 1$), that is, $H = \{X = (x, t) ; x \in \mathbb{R}^n, t > 0\}$. For $0 < \alpha \leq 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} := \frac{\partial}{\partial t} + (-\Delta_x)^\alpha,$$

where $\Delta_x := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ is the Laplacian on the x -space $\mathbb{R}^n$. A real-valued continuous function u on H is said to be $L^{(\alpha)}$ -harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions. (The explicit definition of the $L^{(\alpha)}$ -harmonic function is described in section 3.) For $\lambda > -1$ and $1 \leq p < \infty$, the α -parabolic Bergman space $b_\alpha^p(\lambda)$ is the set of all $L^{(\alpha)}$ -harmonic functions u on H with

$$\| u \|_{L^p(\lambda)} := \left(\int_H |u(x, t)|^p t^\lambda dV(x, t) \right)^{1/p} < \infty,$$

where V is the Lebesgue volume measure on H and $L^p(\lambda) := L^p(H, t^\lambda dV)$. In particular, we may write $L^p = L^p(0)$ and $b_\alpha^p = b_\alpha^p(0)$, respectively.

Our aim of this paper is to study conjugate systems on α -parabolic Bergman spaces. The α -parabolic Bergman spaces b_α^p were introduced and studied by Nishio, Shimomura, and Suzuki [7]. It was shown in [7] that $b_{1/2}^p$ coincide with the usual harmonic Bergman spaces of Ramey and Yi [11]. Accordingly, usual harmonic Bergman spaces are the classes of L^p -solutions of the parabolic equation $L^{(\alpha)}u = 0$ with $\alpha = 1/2$. In [12], the Cauchy-Riemann equations on a region of the two-dimensional Euclidean space are extended to higher dimensions, and properties of systems of conjugate harmonic functions on Hardy spaces were studied (see also [3]). In the theory of harmonic Bergman spaces, properties of conjugate functions were also studied, and as an application, estimates of tangential derivative norms of harmonic Bergman functions were given (see section 6 in [11]). On the other hand, Yamada [13] studied conjugate functions of parabolic Bergman functions. However, the suitable notion of conjugacy were not extended to α -parabolic Bergman spaces. In this paper, we introduce a suitable extension of conjugacy to α -parabolic Bergman spaces and study their properties. We also give estimates of tangential derivative norms of α -parabolic Bergman functions.

Now, we introduce the extension of conjugacy to α -parabolic Bergman spaces. Let $\partial_j = \partial/\partial x_j$ ($1 \leq j \leq n$) and $\partial_t = \partial/\partial t$. Let $C(\Omega)$ be the set of all real-valued continuous functions on a region Ω , and for a positive integer k , $C^k(\Omega) \subset C(\Omega)$ denotes the set of all k times

continuously differentiable functions on Ω , and put $C^\infty(\Omega) = \cap_k C^k(\Omega)$. Furthermore, for a real number κ , let $\mathcal{D}_t^\kappa = (-\partial_t)^\kappa$ be the fractional differential operator with respect to t . (The definition of the fractional differential operator and the fundamental properties of fractional calculus for α -parabolic Bergman functions are described in section 2.)

DEFINITION 1. For a function $u \in \mathbf{b}_\alpha^p(\lambda)$, we shall say that a vector-valued function $V = (v_1, \dots, v_n)$ on H is an α -parabolic conjugate function of u if $v_j \in C^1(H)$ and V satisfies the equations

$$(C.1) \quad \nabla_x u = -\mathcal{D}_t V, \quad \nabla_x v_j = \partial_j V \quad (1 \leq j \leq n),$$

and

$$(C.2) \quad \mathcal{D}_t^{\frac{1}{\alpha}-1} u = \nabla_x \cdot V,$$

where $\nabla_x = (\partial_1, \dots, \partial_n)$ and $\nabla_x \cdot V$ is the divergence of V .

We remark that the fractional derivative $\mathcal{D}_t^{\frac{1}{\alpha}-1} u$ is well defined whenever $u \in \mathbf{b}_\alpha^p(\lambda)$ with $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$ (see section 2). Our formulation of the extension of conjugacy is based on the Cauchy-Riemann equations $u_x = v_t$ and $-u_t = v_x$ on a region of the two-dimensional Euclidean space. Evidently, when $\alpha = 1/2$, the equations (C.1) and (C.2) coincide with the generalized Cauchy-Riemann equations for harmonic functions in [12];

$$(1.1) \quad \partial_j u = \partial_t v_j, \quad \partial_k v_j = \partial_j v_k, \quad 1 \leq j, k \leq n,$$

and

$$(1.2) \quad \partial_t u + \sum_{j=1}^n \partial_j v_j = 0.$$

Particularly, an $(n+1)$ -tuple $(v_1, \dots, v_n, u)$ which satisfies (1.1) and (1.2) is said to be a system of conjugate harmonic functions on H . We present results of Ramey and Yi [11] concerning with conjugate functions of harmonic Bergman functions.

THEOREM A. (Theorem 6.1 of [11]) *Let $1 \leq p < \infty$ and $u \in \mathbf{b}_{1/2}^p$. Then, there exists a unique $1/2$ -parabolic conjugate function $V = (v_1, \dots, v_n)$ of u such that $v_j \in \mathbf{b}_{1/2}^p$. Also, there exists a constant $C = C(n, p) > 0$ independent of u such that*

$$C^{-1} \|u\|_{L^p} \leq \| |V| \|_{L^p} \leq C \|u\|_{L^p},$$

where $|V| := \{v_1^2 + \dots + v_n^2\}^{1/2}$.

For a multi-index $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$, let $\partial_x^\gamma := \partial_1^{\gamma_1} \dots \partial_n^{\gamma_n}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The following theorem gives estimates of tangential derivative norms of harmonic Bergman functions.

THEOREM B. (Theorem 6.2 of [11]) *Let $1 \leq p < \infty$ and $u \in \mathbf{b}_{1/2}^p$. Then, for each $m \in \mathbb{N}_0$,*

there exists a constant $C = C(n, p, m) > 0$ independent of u such that

$$C^{-1} \|u\|_{L^p} \leq \sum_{|\gamma|=m} \|t^m \partial_x^\gamma u\|_{L^p} \leq C \|u\|_{L^p}.$$

We describe the main results of this paper. We remark that the condition $p(\frac{1}{2\alpha} - 1) + \lambda > -1$ in Theorem 1 below holds for all $1 \leq p < \infty$ and $\lambda > -1$ whenever $0 < \alpha \leq 1/2$.

THEOREM 1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. If α , p , and λ satisfy the condition $\eta = p(\frac{1}{2\alpha} - 1) + \lambda > -1$, then there exists a unique α -parabolic conjugate function $V = (v_1, \dots, v_n)$ of u such that $v_j \in \mathbf{b}_\alpha^p(\eta)$. Also, there exists a constant $C = C(n, p, \alpha, \lambda) > 0$ independent of u such that*

$$(1.3) \quad C^{-1} \|u\|_{L^p(\lambda)} \leq \| |V| \|_{L^p(\eta)} \leq C \|u\|_{L^p(\lambda)}.$$

We remark that similar statements in Theorem 1 can not hold for the case $\eta = p(\frac{1}{2\alpha} - 1) + \lambda \leq -1$. In fact, we can show that $\mathbf{b}_\alpha^p(\lambda) = \{0\}$ when $\lambda \leq -1$. We do not know whether Theorem A is extended to the full range $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. However, we can give estimates of tangential derivative norms of $\mathbf{b}_\alpha^p(\lambda)$ -functions.

THEOREM 2. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Then, for each $m \in \mathbb{N}_0$, there exists a constant $C = C(n, p, \alpha, \lambda, m) > 0$ independent of u such that*

$$(1.4) \quad C^{-1} \|u\|_{L^p(\lambda)} \leq \sum_{|\gamma|=m} \|t^{\frac{m}{2\alpha}} \partial_x^\gamma u\|_{L^p(\lambda)} \leq C \|u\|_{L^p(\lambda)}.$$

This paper is constructed as follows. In section 2, we describe properties of fractional calculus on $\mathbf{b}_\alpha^p(\lambda)$. In section 3, we define integral operators induced by the fundamental solution of the parabolic operator $L^{(\alpha)}$ and investigate their properties, which are useful for studying α -parabolic conjugate functions. In section 4, we present more properties of α -parabolic conjugate functions.

Throughout this paper, C will denote a positive constant whose value is not necessary the same at each occurrence; it may vary even within a line.

2. Fractional calculus on $\mathbf{b}_\alpha^p(\lambda)$

In order to extend conjugacy to α -parabolic Bergman spaces, we need fractional calculus on $\mathbf{b}_\alpha^p(\lambda)$. First, we describe fractional differential operators for functions on $\mathbb{R}_+ = (0, \infty)$. For a real number $\kappa > 0$, let

$$\mathcal{FC}^{-\kappa} := \{\varphi \in C(\mathbb{R}_+) ; \exists \varepsilon > 0, \exists C > 0 \text{ s.t. } |\varphi(t)| \leq Ct^{-\kappa-\varepsilon}, \forall t \in \mathbb{R}_+\}.$$

For a function $\varphi \in \mathcal{FC}^{-\kappa}$, we can define the fractional integral $\mathcal{D}_t^{-\kappa}\varphi$ of φ by

$$(2.1) \quad \mathcal{D}_t^{-\kappa}\varphi(t) := \frac{1}{\Gamma(\kappa)} \int_0^\infty \tau^{\kappa-1} \varphi(\tau+t) d\tau = \frac{1}{\Gamma(\kappa)} \int_t^\infty (\tau-t)^{\kappa-1} \varphi(\tau) d\tau, \quad t \in \mathbb{R}_+,$$

where Γ is the gamma function. Moreover, let

$$\mathcal{FC}^\kappa := \{\varphi; d_t^{[\kappa]} \varphi \in \mathcal{FC}^{-(\lceil \kappa \rceil - \kappa)}\},$$

where $d_t = d/dt$, $\lceil \kappa \rceil$ is the smallest integer greater than or equal to κ , and we will write $\mathcal{FC}^0 := C(\mathbb{R}_+)$. We can also define the fractional derivative $\mathcal{D}_t^\kappa \varphi$ of $\varphi \in \mathcal{FC}^\kappa$ by

$$(2.2) \quad \mathcal{D}_t^\kappa \varphi(t) := \mathcal{D}_t^{-(\lceil \kappa \rceil - \kappa)} \left((-d_t)^{\lceil \kappa \rceil} \varphi \right)(t), \quad t \in \mathbb{R}_+.$$

In particular, we will write $\mathcal{D}_t^0 \varphi = \varphi$. For a real number κ , we may call both (2.1) and (2.2) *the fractional derivatives of φ with order κ* . And, we call $\mathcal{D}_t^\kappa$ *the fractional differential operator with order κ* . Some basic properties of the fractional differential operators are the following.

LEMMA 2.1. (Proposition 2.1 of [4]) *For real numbers $\kappa, \nu > 0$, the following statements hold.*

(1) *If $\varphi \in \mathcal{FC}^{-\kappa}$, then $\mathcal{D}_t^{-\kappa} \varphi \in C(\mathbb{R}_+)$.*

(2) *If $\varphi \in \mathcal{FC}^{-\kappa-\nu}$, then $\mathcal{D}_t^{-\kappa} \mathcal{D}_t^{-\nu} \varphi = \mathcal{D}_t^{-\kappa-\nu} \varphi$.*

(3) *If $d_t^k \varphi \in \mathcal{FC}^{-\nu}$ for all integers $0 \leq k \leq \lceil \kappa \rceil - 1$ and $d_t^{\lceil \kappa \rceil} \varphi \in \mathcal{FC}^{-(\lceil \kappa \rceil - \kappa) - \nu}$, then $\mathcal{D}_t^\kappa \mathcal{D}_t^{-\nu} \varphi = \mathcal{D}_t^{-\nu} \mathcal{D}_t^\kappa \varphi = \mathcal{D}_t^{\kappa-\nu} \varphi$.*

(4) *If $d_t^{k+\lceil \nu \rceil} \varphi \in \mathcal{FC}^{-(\lceil \nu \rceil - \nu)}$ for all integers $0 \leq k \leq \lceil \kappa \rceil - 1$, $d_t^{\lceil \kappa \rceil + \ell} \varphi \in \mathcal{FC}^{-(\lceil \kappa \rceil - \kappa)}$ for all integers $0 \leq \ell \leq \lceil \nu \rceil - 1$, and $d_t^{\lceil \kappa \rceil + \lceil \nu \rceil} \varphi \in \mathcal{FC}^{-(\lceil \kappa \rceil - \kappa) - (\lceil \nu \rceil - \nu)}$, then $\mathcal{D}_t^\kappa \mathcal{D}_t^\nu \varphi = \mathcal{D}_t^{\kappa+\nu} \varphi$.*

Next, we also describe some basic results concerning with the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, let

$$W^{(\alpha)}(x, t) := \begin{cases} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i x \cdot \xi) d\xi & (t > 0) \\ 0 & (t \leq 0), \end{cases}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi \cdot \xi)^{1/2}$. The function $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ and $L^{(\alpha)}$ -harmonic on H . We note that $W^{(\alpha)} \geq 0$ on H and $\int_{\mathbb{R}^n} W^{(\alpha)}(x, t) dx = 1$ for all $0 < t < \infty$. Furthermore, $W^{(\alpha)} \in C^\infty(H)$. Let $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}_0^n$ be a multi-index and $k \in \mathbb{N}_0$. The following estimate is Lemma 1 of [9]: there exists a constant $C = C(n, \alpha, \gamma, k) > 0$ such that

$$(2.3) \quad |\partial_x^\gamma \partial_t^k W^{(\alpha)}(x, t)| \leq C(t + |x|^{2\alpha})^{-(\frac{n+|\gamma|}{2\alpha} + k)}$$

for all $(x, t) \in H$. In particular, by (2.3), we note that for each $x \in \mathbb{R}^n$, the function $\varphi(\cdot) = W^{(\alpha)}(x, \cdot)$ belongs to $\mathcal{FC}^\kappa$ for $\kappa > -\frac{n}{2\alpha}$. The statements in the following lemma are consequences of [4].

LEMMA 2.2. (Theorem 3.1 of [4]) *Let $0 < \alpha \leq 1$, $\gamma \in \mathbb{N}_0^n$ be a multi-index, and κ be a real number such that $\kappa > -\frac{n}{2\alpha}$. Then, the following statements hold.*

(1) *The derivatives $\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t)$ and $\mathcal{D}_t^\kappa \partial_x^\gamma W^{(\alpha)}(x, t)$ are well defined, and*

$$\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t) = \mathcal{D}_t^\kappa \partial_x^\gamma W^{(\alpha)}(x, t).$$

Furthermore, there exists a constant $C = C(n, \alpha, \gamma, \kappa) > 0$ such that

$$|\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t)| \leq C(t + |x|^{2\alpha})^{-\left(\frac{n+|\gamma|}{2\alpha} + \kappa\right)}$$

for all $(x, t) \in H$.

(2) If a real number ν satisfies the condition $\nu + \kappa > -\frac{n}{2\alpha}$, then the derivative $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}$ is well defined, and

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t) = \partial_x^\gamma \mathcal{D}_t^{\nu+\kappa} W^{(\alpha)}(x, t)$$

for all $(x, t) \in H$.

(3) The derivative $\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x, t)$ is $L^{(\alpha)}$ -harmonic on H .

By the elementary calculation, we also give the following lemma. This lemma plays an important role for the study of conjugate functions on parabolic Bergman spaces.

LEMMA 2.3. Let $0 < \alpha \leq 1$. Then,

$$\left(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x\right) W^{(\alpha)}(x, t) = 0$$

for all $(x, t) \in H$.

We present basic properties of fractional derivatives of $\mathbf{b}_\alpha^p(\lambda)$ -functions. We begin with describing estimates of ordinary derivatives of $\mathbf{b}_\alpha^p(\lambda)$ -functions. Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, $\gamma \in \mathbb{N}_0^n$ be a multi-index, and $k \in \mathbb{N}_0$. Then, it is known that $\mathbf{b}_\alpha^p(\lambda) \subset C^\infty(H)$ (see [13]) and the following estimate is given by Lemma 3.4 of [13]: there exists a constant $C = C(n, \alpha, p, \lambda, \gamma, k) > 0$ such that

$$(2.4) \quad |\partial_x^\gamma \partial_t^k u(x, t)| \leq C t^{-\left(\frac{|\gamma|}{2\alpha} + k\right) - \left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}} \|u\|_{L^p(\lambda)}$$

for all $u \in \mathbf{b}_\alpha^p(\lambda)$ and $(x, t) \in H$. The estimate (2.4) implies that the point evaluation is a bounded linear functional on $\mathbf{b}_\alpha^p(\lambda)$. Furthermore, the estimate (2.4) also shows that a function $\varphi(\cdot) = u(x, \cdot)$ belongs to $\mathcal{FC}^\kappa$ for $u \in \mathbf{b}_\alpha^p(\lambda)$ and $\kappa > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$, so we can define fractional derivatives of $\mathbf{b}_\alpha^p(\lambda)$ -functions. Some properties of fractional derivatives of $\mathbf{b}_\alpha^p(\lambda)$ -functions are given in the following.

LEMMA 2.4. (Proposition 4.1 of [4]) Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, $\gamma \in \mathbb{N}_0^n$ be a multi-index, and κ be a real number such that $\kappa > -\left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}$. If $u \in \mathbf{b}_\alpha^p(\lambda)$, then the following statements hold.

(1) The derivatives $\partial_x^\gamma \mathcal{D}_t^\kappa u(x, t)$ and $\mathcal{D}_t^\kappa \partial_x^\gamma u(x, t)$ are well defined, and

$$\partial_x^\gamma \mathcal{D}_t^\kappa u(x, t) = \mathcal{D}_t^\kappa \partial_x^\gamma u(x, t).$$

Furthermore, there exists a constant $C = C(n, \alpha, p, \lambda, \gamma, \kappa) > 0$ independent of u such that

$$|\partial_x^\gamma \partial_t^\kappa u(x, t)| \leq C t^{-\left(\frac{|\gamma|}{2\alpha} + \kappa\right) - \left(\frac{n}{2\alpha} + \lambda + 1\right)\frac{1}{p}} \|u\|_{L^p(\lambda)}$$

for all $(x, t) \in H$.

(2) If a real number ν satisfies the condition $\nu + \kappa > -(\frac{n}{2\alpha} + \lambda + 1)\frac{1}{p}$, then the derivative $\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa u(x, t)$ is well defined, and

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa u(x, t) = \partial_x^\gamma \mathcal{D}_t^{\nu+\kappa} u(x, t)$$

for all $(x, t) \in H$.

(3) The derivative $\partial_x^\gamma \mathcal{D}_t^\kappa u(x, t)$ is $L^{(\alpha)}$ -harmonic on H .

For a real number $\lambda > -1$, let $c_\lambda = 2^{\lambda+1}/\Gamma(\lambda + 1)$. The following lemma is also a consequence of [4], and (2.5) is the reproducing formula for $\mathbf{b}_\alpha^p(\lambda)$ -functions.

LEMMA 2.5. (Theorem 5.2 of [4]) Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. Suppose that ν and κ are real numbers such that $\nu > -\frac{\lambda+1}{p}$ and $\kappa > \frac{\lambda+1}{p}$. Then,

$$(2.5) \quad u(x, t) = c_{\nu+\kappa-1} \int_H \mathcal{D}_t^\nu u(y, s) \mathcal{D}_t^\kappa W^{(\alpha)}(x - y, t + s) s^{\nu+\kappa-1} dV(y, s)$$

holds for all $u \in \mathbf{b}_\alpha^p(\lambda)$ and $(x, t) \in H$. Furthermore, (2.5) also holds whenever $p = 1$ and $\kappa = \lambda + 1$.

Finally, we present the following lemma. This lemma plays an important role for proving Theorem 2.

LEMMA 2.6. Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Then,

$$\left(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x \right) u(x, t) = 0$$

for all $(x, t) \in H$.

3. Integral operators induced by the fundamental solution

In this section, we define integral operators induced by the fundamental solution $W^{(\alpha)}$ and investigate their properties. These investigations are useful for studying α -parabolic conjugate functions of $\mathbf{b}_\alpha^p(\lambda)$ -functions.

First, we recall the definition of $L^{(\alpha)}$ -harmonic functions. (For details, see section 2 of [7].) We describe about the operator $(-\Delta_x)^\alpha$. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. Let $C_c^\infty(H) \subset C(H)$ be the set of all infinitely differentiable functions on H with compact support. Then, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(3.1) \quad (-\Delta_x)^\alpha \psi(x, t) := -C_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y|>\delta} (\psi(x + y, t) - \psi(x, t)) |y|^{-n-2\alpha} dy$$

for all $\psi \in C_c^\infty(H)$ and $(x, t) \in H$, where $C_{n,\alpha} = -4^\alpha \pi^{-n/2} \Gamma((n + 2\alpha)/2) / \Gamma(\alpha) > 0$. Let $\tilde{L}^{(\alpha)} := -\partial_t + (-\Delta_x)^\alpha$ be the adjoint operator of $L^{(\alpha)}$. Then, a function $u \in C(H)$ is said to be

$L^{(\alpha)}$ -harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions, that is, $\int_H |u\tilde{L}^{(\alpha)}\psi|dV < \infty$ and $\int_H u\tilde{L}^{(\alpha)}\psi dV = 0$ for all $\psi \in C_c^\infty(H)$. By (3.1) and the compactness of $\text{supp}(\psi)$ (the support of ψ), there exist $0 < t_1 < t_2 < \infty$ and a constant $C > 0$ such that

$$\text{supp}(\tilde{L}^{(\alpha)}\psi) \subset S = \mathbb{R}^n \times [t_1, t_2] \text{ and } |\tilde{L}^{(\alpha)}\psi(x, t)| \leq C(1 + |x|)^{-n-2\alpha} \text{ for } (x, t) \in S.$$

Hence, the condition $\int_H |u\tilde{L}^{(\alpha)}\psi|dV < \infty$ for all $\psi \in C_c^\infty(H)$ is equivalent to the following: for any $0 < t_1 < t_2 < \infty$,

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u(x, t)|(1 + |x|)^{-n-2\alpha} dx dt < \infty.$$

Next, we define integral operators induced by the fundamental solution $W^{(\alpha)}$. Let $\gamma \in \mathbb{N}_0^n$ be a multi-index and $\kappa, \rho \in \mathbb{R}$ with $\kappa > -\frac{n}{2\alpha}$. Then, we define the integral operator $P_\alpha^{\gamma, \kappa, \rho}$ by

$$P_\alpha^{\gamma, \kappa, \rho} f(x, t) := \int_H f(y, s) \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x - y, t + s) s^\rho dV(y, s),$$

whenever the integral is well defined. Some properties of $P_\alpha^{\gamma, \kappa, \rho}$ are given in the following theorem.

THEOREM 3.1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\sigma \in \mathbb{R}$. Suppose that a multi-index $\gamma \in \mathbb{N}_0^n$ and $\kappa, \rho \in \mathbb{R}$ with $\kappa > -\frac{n}{2\alpha}$ satisfy*

$$\sigma - \rho p < p - 1 < \left(\frac{|\gamma|}{2\alpha} + \kappa \right) p + \sigma - \rho p.$$

Then, for every $f \in L^p(\sigma)$, the following assertions hold.

(1) The function $P_\alpha^{\gamma, \kappa, \rho} f(x, t)$ is well defined for every $(x, t) \in H$ and there exists a constant $C > 0$ independent of f such that

$$\| P_\alpha^{\gamma, \kappa, \rho} f \|_{L^p(\eta)} \leq C \| f \|_{L^p(\sigma)},$$

where $\eta = \left(\frac{|\gamma|}{2\alpha} + \kappa - \rho - 1 \right) p + \sigma$. Moreover, $P_\alpha^{\gamma, \kappa, \rho} f$ is $L^{(\alpha)}$ -harmonic on H . Consequently, $P_\alpha^{\gamma, \kappa, \rho} f \in \mathfrak{b}_\alpha^p(\eta)$.

(2) Furthermore, let $\beta \in \mathbb{N}_0^n$ be a multi-index and $\nu \in \mathbb{R}$. If ν satisfies

$$\nu + \kappa > -\frac{n}{2\alpha} \text{ and } p - 1 < \left(\frac{|\gamma|}{2\alpha} + \nu + \kappa \right) p + \sigma - \rho p,$$

then the derivative $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f(x, t)$ is well defined for every $(x, t) \in H$ and $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f = P_\alpha^{\beta+\gamma, \nu+\kappa, \rho} f$, that is,

$$\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f(x, t) = \int_H f(y, s) \partial_x^{\beta+\gamma} \mathcal{D}_t^{\nu+\kappa} W^{(\alpha)}(x - y, t + s) s^\rho dV(y, s).$$

Consequently, put $\eta = \left(\frac{|\beta|+|\gamma|}{2\alpha} + \nu + \kappa - \rho - 1 \right) p + \sigma$, then there exists a constant $C > 0$ independent of f such that

$$\| \partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f \|_{L^p(\eta)} \leq C \| f \|_{L^p(\sigma)}$$

and $\partial_x^\beta \mathcal{D}_t^\nu P_\alpha^{\gamma, \kappa, \rho} f \in \mathbf{b}_\alpha^p(\eta)$.

By the above theorem, we have the following corollary.

COROLLARY 3.2. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\lambda > -1$. Then, the following assertions hold.*

(1) *If a real number κ satisfies $\kappa > \frac{\lambda+1}{p}$, then the operator $R_\alpha^{\kappa-1} = c_{\kappa-1} P_\alpha^{0, \kappa, \kappa-1}$ is a bounded projection from $L^p(\lambda)$ onto $\mathbf{b}_\alpha^p(\lambda)$.*

(2) *For a real number $\nu > -\frac{\lambda+1}{p}$, there exists a constant $C = C(n, p, \alpha, \lambda, \nu) > 0$ such that*

$$C^{-1} \| u \|_{L^p(\lambda)} \leq \| t^\nu \mathcal{D}_t^\nu u \|_{L^p(\lambda)} \leq \sum_{|\gamma| < \nu + \frac{\lambda+1}{p}} \| t^{\frac{|\gamma|}{2\alpha} + \nu - |\gamma|} \partial_x^\gamma \mathcal{D}_t^{\nu - |\gamma|} u \|_{L^p(\lambda)} \leq C \| u \|_{L^p(\lambda)}$$

for all $u \in \mathbf{b}_\alpha^p(\lambda)$, where $\gamma \in \mathbb{N}_0^n$ denotes a multi-index.

4. More properties of α -parabolic conjugate functions

In this section, we present more properties of α -parabolic conjugate functions. Given a harmonic function u on H , it is well known that a vector-valued function $V = (v_1, \dots, v_n)$ on H with $v_j \in C^1(H)$ satisfies the equations (1.1) and (1.2) if and only if there exists a function $g \in C^2(H)$ such that

$$(4.1) \quad g \text{ is harmonic on } H \text{ and } \nabla_{(x,t)} g = (v_1, \dots, v_n, u),$$

where $\nabla_{(x,t)} = (\partial_1, \dots, \partial_n, \partial_t)$. The following theorem is an analogous result of (4.1) for our case.

THEOREM 4.1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Then, a vector-valued function $V = (v_1, \dots, v_n)$ on H is an α -parabolic conjugate function of u if and only if there exists a function $g \in C^2(H) \cap \mathcal{FC}^{\frac{1}{\alpha}}$ such that*

$$(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x)g = 0 \text{ on } H \text{ and } \nabla_{(x,t)} g = (v_1, \dots, v_n, u).$$

Next, we give an inversion theorem, that is, for a vector-valued function $V = (v_1, \dots, v_n)$ on H we construct a function $u \in \mathbf{b}_\alpha^p(\lambda)$ such that V is an α -parabolic conjugate function of u .

THEOREM 4.2. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, and $\eta > -1$. Suppose that a vector-valued function $V = (v_1, \dots, v_n)$ on H satisfies $v_j \in \mathbf{b}_\alpha^p(\eta)$ and $\nabla_x v_j = \partial_j V$ for all $1 \leq j \leq n$. If α , p , and η satisfy the condition $\lambda = p(1 - \frac{1}{2\alpha}) + \eta > -1$, then there exists a unique function u*

on H such that $u \in \mathbf{b}_\alpha^p(\lambda)$ and V is an α -parabolic conjugate function of u . Also, there exists a constant $C = C(n, p, \alpha, \eta) > 0$ independent of V such that

$$C^{-1} \| |V| \|_{L^p(\eta)} \leq \| u \|_{L^p(\lambda)} \leq C \| |V| \|_{L^p(\eta)}.$$

We also have the following proposition.

PROPOSITION 4.3. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Let $1 \leq j \leq n$ be fixed. Suppose that a vector-valued function $V = (v_1, \dots, v_n)$ on H is an α -parabolic conjugate function of u . Then, $v_j \in \mathcal{FC}^{\frac{1}{\alpha}}$. Furthermore, if $v_k \in C^2(H)$ for all $1 \leq k \leq n$, then $(\mathcal{D}_t^{\frac{1}{\alpha}} + \Delta_x)v_j = 0$ on H .*

Finally, we present a decomposition theorem for α -parabolic conjugate functions. We begin with presenting the following lemma.

LEMMA 4.4. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Suppose α , p , and λ satisfy the condition $\eta = p \left(\frac{1}{2\alpha} - 1 \right) + \lambda > -1$. Then, for every α -parabolic conjugate function $U = (u_1, \dots, u_n)$ of u , the function $\mathcal{D}_t^{-1} \mathcal{D}_t u_j$ on H is well defined and belongs to $\mathbf{b}_\alpha^p(\eta)$ for all $1 \leq j \leq n$.*

The following theorem is a decomposition theorem for α -parabolic conjugate functions.

THEOREM 4.5. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$, $\lambda > -1$, and $u \in \mathbf{b}_\alpha^p(\lambda)$. Suppose α , p , and λ satisfy the condition $\eta = p \left(\frac{1}{2\alpha} - 1 \right) + \lambda > -1$. Then, every α -parabolic conjugate function $U = (u_1, \dots, u_n)$ of u can be uniquely expressed in the form*

$$(4.2) \quad U(x, t) = V(x, t) + F(x), \quad (x, t) \in H,$$

where $V = (v_1, \dots, v_n)$ is the unique α -parabolic conjugate function of u with $v_j \in \mathbf{b}_\alpha^p(\eta)$ in Theorem 1 and $F = (f_1, \dots, f_n)$ is an n -tuple of harmonic functions on $\mathbb{R}^n$ with $\partial_k f_j = \partial_j f_k$, $1 \leq j, k \leq n$ and $\sum_{j=1}^n \partial_j f_j = 0$ (that is, $F = (f_1, \dots, f_n)$ is a system of conjugate harmonic functions on $\mathbb{R}^n$, consequently each u_j belongs to $C^\infty(H)$). Conversely, every function U of the form (4.2) is an α -parabolic conjugate function of u .

References

- [1] S. Axler, P. Bourdon and W. Ramey, Harmonic Function Theory, Springer-Verlag, New York, 1992.
- [2] B. R. Choe and H. Yi, Representations and interpolations of harmonic Bergman functions on half-spaces, Nagoya Math. J. **151**(1998), 51–89.
- [3] C. Fefferman and E. M. Stein, H^p -spaces of several variables, Acta Math. **129**(1972), 137–193.

- [4] Y. Hishikawa, Fractional calculus on parabolic Bergman spaces, Hiroshima Math. J. **38**(2008), 471–488.
- [5] H. Koo, K. Nam, and H. Yi, Weighted harmonic Bergman kernel on half-spaces, J. Math. Soc. Japan **58**(2006), 351–362.
- [6] H. Koo, K. Nam, and H. Yi, Weighted harmonic Bergman functions of half-spaces, J. Korean Math. Soc. **42**(2005), 975–1002.
- [7] M. Nishio, K. Shimomura and N. Suzuki, α -Parabolic Bergman spaces, Osaka J. Math. **42**(2005), 133–162.
- [8] M. Nishio K. Shimomura and N. Suzuki, L^p -boundedness of Bergman projections for α -parabolic operators, Adv. Stud. Pure Math., **44**, 305-318, Math. Soc. of Japan, Tokyo, 2006.
- [9] M. Nishio, N. Suzuki and M. Yamada, Toeplitz operators and Carleson measures on parabolic Bergman spaces, Hokkaido Math. J. **36**(2007), 563–583.
- [10] M. Nishio and M. Yamada, Carleson type measures on parabolic Bergman spaces, J. Math. Soc. Japan **58**(2006), 83–96.
- [11] W. Ramey and H. Yi, Harmonic Bergman functions on half-spaces, Trans. Amer. Math. Soc. **348**(1996), 633–660.
- [12] E. M. Stein and G. Weiss, On the theory of harmonic functions of several variables I. The theory of H^p -spaces, Acta Math. **103**(1960), 25–62.
- [13] M. Yamada, Harmonic conjugates of parabolic Bergman functions, Potential theory in Matsue, 391–402, Adv. Stud. Pure Math., **44**, Math. Soc. Japan, Tokyo, 2006.