

A survey on Shapovalov determinants of (generalized) quantum groups at roots of 1

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Abstract

This is an informal survey on a joint work [HY08b] with Istvan Heckenberger.

1 A quantum group $U(\chi)$ defined for any bi-character χ

Recently study of Nichols algebras has been achieved very actively for the viewpoint of classification of Hopf algebras, see [AS98], [AS02], [Hec06]. One of their examples is the positive part $U^+(\chi)$ of a generalized quantum group $U(\chi)$ defined below.

Let $\mathbb{k}$ be a field and $\mathbb{k}^\times = \mathbb{k} \setminus \{0\}$. For $n \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{k}$, let

$$(1) \quad [n]_x = \sum_{m=1}^n x^{m-1}, \quad [n]_x! = \prod_{m=1}^n [m]_x.$$

For two elements X_1 and X_2 of a $\mathbb{k}$ -algebra we use the convention:

$$(2) \quad X_1 \quad X_2 \quad \begin{array}{c} \Longleftrightarrow \\ \text{def} \end{array} \quad \exists x \in \mathbb{k}^\times \quad X_1 = xX_2.$$

Let I be a finite index set. Let $\mathbb{Z}\Pi = \sum_{i \in I} \mathbb{Z}\alpha_i$ be a rank $|I|$ free $\mathbb{Z}$ -module with a basis $\Pi = \{\alpha_i | i \in I\}$. We say that a map $\chi : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \rightarrow \mathbb{k}^\times$ is a *bi-character* if $\chi(a+b, c) = \chi(a, c)\chi(b, c)$, and $\chi(a, b+c) = \chi(a, b)\chi(a, c)$ for all $a, b, c \in \mathbb{Z}\Pi$.

Let χ be *any* bi-character. Then, as we explain more precisely in Section 2, Lusztig's definition [L, 3.1.1] of the quantum groups can be applied to define the Hopf $\mathbb{k}$ -algebra $U(\chi)$ with the generators

$$(3) \quad K, L \quad (\lambda \in \mathbb{Z}\Pi), E_i, F_i \quad (i \in I),$$

for which K, L ($\lambda, \mu \in \mathbb{Z}\Pi$) are linearly independent and the following equations hold:

$$(4) \quad K_0 = L_0 = 1, \quad K_+ = K K, \quad L_+ = L L, \quad K L = L K,$$

$$(5) \quad K L E_j (K L)^{-1} = \frac{\chi(\lambda, \alpha_j)}{\chi(\alpha_j, \mu)} E_j, \quad K L F_j (K L)^{-1} = \frac{\chi(\alpha_j, \mu)}{\chi(\lambda, \alpha_j)} F_j,$$

$$(6) \quad E_i F_j - F_j E_i = \delta_{ij} (K_i - L_i).$$

$$(7) \quad \Delta(K L) = K L - K L, \quad \varepsilon(K L) = 1, \quad S(K L) = (K L)^{-1},$$

$$(8) \quad \Delta(E_i) = E_i - 1 + K_i - E_i, \quad \Delta(F_i) = F_i - L_i + 1 - F_i,$$

$$(9) \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i L_i^{-1}.$$

Let $U^0(\chi) := \sum_{\lambda, \mu \in \mathbb{Z}\Pi} \mathbb{k} K L$. Let $U^+(\chi)$ and $U^-(\chi)$ be the subalgebra of $U(\chi)$ generated by E_i and F_i with all $i \in I$ respectively. Then $U(\chi) = U^+(\chi) U^0(\chi) U^-(\chi)$, as a $\mathbb{k}$ -linear space. We have the $\mathbb{Z}_{\geq 0}\Pi$ -grading $U^\pm(\chi) = \sum_{\lambda \in \mathbb{Z}_{\geq 0}\Pi} U^\pm(\chi)_\lambda$ defined by $U^+(\chi)_\lambda = \mathbb{k} E_i$, $U^-(\chi)_{-\lambda} = \mathbb{k} F_i$, and $U^\pm(\chi) U^\pm(\chi) \subset U^\pm(\chi)_{+\lambda}$. We also have $\dim U^-(\chi)_{-\lambda} = \dim U^+(\chi)_\lambda$ for all $\lambda \in \mathbb{Z}_{\geq 0}\Pi$.

2 Drinfeld pairing of $U(\chi)$

Here we will explain how to define $U(\chi)$ more precisely. By abuse of notation, we use the same symbols as above for the generators of the algebras introduced in this paragraph. Let $\tilde{U}^+(\chi)$ and $\tilde{U}^-(\chi)$ be the free $\mathbb{k}$ -algebras (with 1) with the generators $\{E_i | i \in I\}$ and $\{F_i | i \in I\}$ respectively. Let $\tilde{U}^0(\chi)$ be the $\mathbb{k}$ -linear space with the basis $\{K L | \lambda, \mu \in \mathbb{Z}\Pi\}$. Let $\tilde{U}(\chi) = \tilde{U}^+(\chi) \mathbb{k} \tilde{U}^0(\chi) \mathbb{k} \tilde{U}^-(\chi)$. Identify $X \in \tilde{U}^+(\chi)$, $Z \in \tilde{U}^0(\chi)$ and $Y \in \tilde{U}^-(\chi)$ with $X - 1 - 1, 1 - Z - 1$ and $1 - 1 - Y$ respectively, and regard $\tilde{U}^+(\chi)$, $\tilde{U}^0(\chi)$ and $\tilde{U}^-(\chi)$ as subspaces of $\tilde{U}(\chi)$ in this way. Then $\tilde{U}(\chi)$ can be regarded as the $\mathbb{k}$ -algebra (with 1) presented by the same generators as the ones for $U(\chi)$ and the relations (4), (5) and (6) (cf. [L, Prop. 3.2.4]). Further $\tilde{U}(\chi)$ can be regarded as the Hopf $\mathbb{k}$ -algebra with the same equalities as (7), (8) and (9). Let $\tilde{U}^{+,K}(\chi)$ be the subalgebra of $\tilde{U}(\chi)$ generated by E_i 's and K 's. Let $\tilde{U}^{L,-}(\chi)$ be the subalgebra of $\tilde{U}(\chi)$ generated by F_i 's and L 's. Then there exists a unique $\mathbb{k}$ -bilinear form

$$(10) \quad \langle, \rangle : \tilde{U}^{+,K}(\chi) \times \tilde{U}^{L,-}(\chi) \rightarrow \mathbb{k}$$

with

$$(11) \quad \langle 1, Y \rangle = \varepsilon(Y), \quad \langle X, 1 \rangle = \varepsilon(X), \quad \langle S(X), Y \rangle = \langle X, S^{-1}(Y) \rangle,$$

$$(12) \quad \langle X_1 X_2, Y \rangle = \sum_g \langle X_2, Y_g^{(1)} \rangle \langle X_1, Y_g^{(2)} \rangle,$$

$$(13) \quad \langle X, Y_1 Y_2 \rangle = \sum_h \langle X_h^{(1)}, Y_1 \rangle \langle X_h^{(2)}, Y_2 \rangle,$$

$$(14) \quad \langle E_i, F_j \rangle = \delta_{ij}, \quad \langle K, L \rangle = \chi(\lambda, \mu), \quad \langle E_i, L \rangle = \langle K, F_j \rangle = 0$$

for $X, X_1, X_2 \in \tilde{U}^{+,K}(\chi)$ with $\Delta(X) = \sum_h X_h^{(1)} X_h^{(2)}$, and $Y, Y_1, Y_2 \in \tilde{U}^{L,-}(\chi)$ with $\Delta(Y) = \sum_g Y_g^{(1)} Y_g^{(2)}$ and for $i, j \in I$ and $\lambda, \mu \in \mathbb{Z}\Pi$. We see

$$(15) \quad \langle \tilde{E}K, \tilde{F}L \rangle = \langle \tilde{E}, \tilde{F} \rangle \langle K, L \rangle$$

for $\tilde{E} \in \tilde{U}^+(\chi)$ and $\tilde{F} \in \tilde{U}^-(\chi)$. Further, letting $\tilde{U}^\pm(\chi) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}\Pi} \tilde{U}^\pm(\chi)_\alpha$ be the $\mathbb{Z}_{\geq 0}\Pi$ -grading on $\tilde{U}^\pm(\chi)$ defined in a way similar to the one on $U^\pm(\chi)$, we have $\langle \tilde{U}^+(\chi), \tilde{U}^-(\chi)_- \rangle = \{0\}$ if $\lambda \neq \mu$. Let

$$(16) \quad \tilde{J}^+(\chi) = \{\tilde{E} \in \tilde{U}^+(\chi) \mid \langle \tilde{E}, \tilde{U}^-(\chi) \rangle = \{0\}\},$$

$$(17) \quad \tilde{J}^-(\chi) = \{\tilde{F} \in \tilde{U}^-(\chi) \mid \langle \tilde{U}^+(\chi), \tilde{F} \rangle = \{0\}\},$$

$$(18) \quad \tilde{J}(\chi) = \text{Span}_{\mathbf{k}}(\tilde{J}^+(\chi)\tilde{U}^0(\chi)\tilde{U}^-(\chi) + \tilde{U}^+(\chi)\tilde{U}^0(\chi)\tilde{J}^-(\chi)).$$

Then $\tilde{J}(\chi)$ is the kernel of the Hopf algebra epimorphism from $\tilde{U}(\chi)$ to $U(\chi)$ sending the generators to the ones denoted by the same symbols.

Theorem 1. (Kharchenko [Kha99]) *There exist $M \in \mathbb{N} \cup \{\infty\}$ and elements $\hat{E}_i \in U^+(\chi)$, $(1 \leq i \leq M)$ for some $\beta_i \in \mathbb{Z}_{\geq 0}\Pi \setminus \{0\}$ such that we have the $\mathbf{k}$ -basis of $U^+(\chi)$ formed by the elements*

$$(19) \quad \begin{cases} \hat{E}_1^{m_1} \hat{E}_2^{m_2} \cdots \hat{E}_M^{m_M} & \text{if } M \text{ is finite, that is } M \in \mathbb{N}, \\ \hat{E}_1^{m_1} \hat{E}_2^{m_2} \cdots \hat{E}_{M'}^{m_{M'}} & \text{for some } M' \in \mathbb{N} \text{ if } M = \infty \end{cases}$$

with $0 \leq m_i \leq h_i$, and $h_i := \text{Max}\{n \mid [n]_{\beta_i}! \neq 0\} \in \mathbb{N} \cup \{+\infty\}$.

Let

$$(20) \quad R_+ := \{\beta_i \mid 1 \leq i \leq M\}.$$

Note that $|R_+| \leq M$, that is, β_i and β_j may be the same for some $i \neq j$.

We say that χ is *finite-type* if $|R_+| < +\infty$. See [H09] for the classification.

Note that if $\dim U(\chi) < \infty$, then χ is finite-type.

Theorem 2. (see [HY08b, Theorems 4.8, 4.9]) Assume that χ is finite-type. Then $|R_+| = M$ as for (20). We write $E_i = \widehat{E}_i$ if $\widehat{E}_i \in U(\chi)_i$. Then after re-choosing E_i (as in (51)), we may assume that $E_i^{h_{\beta}^{\chi}+1} = 0$ if $h_i < +\infty$ and that $E_i E_j = \chi(\beta_i, \beta_j) E_j E_i \in \langle E_r | i < r < j \rangle$ for any $i < j$, so

$$(21) \quad \{E_{f(1)}^{m_{f(1)}} E_{f(2)}^{m_{f(2)}} \cdots E_{f(M)}^{m_{f(M)}} | 0 \leq m_i \leq h_i\}$$

is a $\mathbb{k}$ -basis of $U(\chi)$ for any bijective map $f : \{1, 2, \dots, M\} \rightarrow \{1, 2, \dots, M\}$.

Convention. Let $\chi_1, \chi_2 : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \rightarrow \mathbb{k}^\times$ be two bi-characters. Let $f_1, f_2 : U(\chi_1) \rightarrow U(\chi_2)$ be two $\mathbb{k}$ -algebra homomorphisms. Then we write

$$(22) \quad f_1 = f_2$$

if

$$(23) \quad f_1(K L) = f_2(K L), f_1(E_i) = f_2(E_i), f_1(F_i) = f_2(F_i)$$

for all $\lambda, \mu \in \mathbb{Z}\Pi$ and $i \in I$.

3 Heckenberger's Lusztig-type isomorphisms

Here we explain a generalization [H07] of Lusztig-type isomorphisms [L].

Assume χ to be any bi-character. Let

$$(24) \quad \llbracket X, Y \rrbracket^+ = X Y = \chi(\lambda, \mu) Y X,$$

$$(25) \quad \llbracket X, Y \rrbracket^- = X Y = \chi(\lambda, \mu)^{-1} Y X,$$

$$(26) \quad \llbracket X, Y \rrbracket^{\vee,+} = X Y = \chi(\mu, \lambda) Y X,$$

$$(27) \quad \llbracket X, Y \rrbracket^{\vee,-} = X Y = \chi(\mu, \lambda)^{-1} Y X$$

for $X \in U(\chi)$ and $Y \in U(\chi)$ with $\lambda, \mu \in \mathbb{Z}\Pi$. Let $i, j \in I$ be such that $i \neq j$. Let

$$(28) \quad \begin{aligned} E_j^+ &= E_j, \quad E_j^- = E_j, \\ E_{j+m}^+ &= \llbracket E_i, E_{j+(m-1)}^+ \rrbracket, \quad E_{j+m}^- = \llbracket E_i, E_{j+(m-1)}^- \rrbracket^{\vee,-}, \\ F_{j+m}^+ &= \llbracket F_i, F_{j+(m-1)}^+ \rrbracket^{\vee,+}, \quad F_{j+m}^- = \llbracket F_i, F_{j+(m-1)}^- \rrbracket^- \end{aligned}$$

for $m \in \mathbb{N}$. For $m \in \mathbb{Z}_{\geq 0}$, we have

$$(29) \quad \begin{aligned} [m]_{(\alpha_i, \alpha_i)}! \prod_{s=1}^m (1 - \chi(\alpha_i, \alpha_i)^{s-1} \chi(\alpha_i, \alpha_j) \chi(\alpha_j, \alpha_i)) &\neq 0 \\ \iff E_{j+m}^+ &\neq 0 \iff E_{j+m}^- \neq 0 \iff F_{j+m}^+ \neq 0 \iff F_{j+m}^- \neq 0 \\ &\iff \alpha_j + m\alpha_i \in R_+. \end{aligned}$$

We also have

$$(30) \quad [E_{j+m_i}^+, F_{j+m_i}^+] = (\chi(\alpha_i, \alpha_i)^{m-1} \chi(\alpha_i, \alpha_j) \chi(\alpha_j, \alpha_i))^m [E_{j+m_i}^-, F_{j+m_i}^-] \\ = (-1)^m ([m]_{(-\alpha_i, -\alpha_i)}!) \prod_{s=1}^m (1 - \chi(\alpha_i, \alpha_i)^{s-1} \chi(\alpha_i, \alpha_j) \chi(\alpha_j, \alpha_i)) (K_{j+m_i} - L_{j+m_i}).$$

Theorem 3. ([H07]) *Let $i \in I$. Assume that for all $j \in I \setminus \{i\}$, there exist $m_{ij} \in \mathbb{Z}_{\geq 0}$ such that $E_{j+m_{ij}^\chi}^+ \neq 0$ and $E_{j+(m_{ij}^\chi+1)^\chi}^+ = 0$.*

(1) *There exist a bi-character $r_i(\chi) : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \rightarrow \mathbb{k}^\times$ and $\mathbb{k}$ -algebra isomorphisms*

$$(31) \quad T_i = T_i^+ : U(r_i(\chi)) \rightarrow U(\chi), \quad T_i^- : U(r_i(\chi)) \rightarrow U(\chi)$$

such that

$$(32) \quad T_i^\pm(K_{-i}) = K_{-i}, \quad T_i^\pm(L_{-i}) = L_{-i},$$

$$(33) \quad T_i^\pm(K_j) = K_{j+m_{ij}^\chi}, \quad T_i^\pm(L_j) = L_{j+m_{ij}^\chi},$$

$$(34) \quad T_i(E_i) = F_i L_{-i}, \quad T_i(F_i) = K_{-i} E_i,$$

$$(35) \quad T_i^-(E_i) = K_{-i} F_i, \quad T_i^-(F_i) = E_i L_{-i},$$

$$(36) \quad T_i^\pm(E_j) = E_{j+m_{ij}^\chi}^\pm, \quad T_i^\pm(F_j) = F_{j+m_{ij}^\chi}^\pm,$$

where $j \in I \setminus \{i\}$.

(2) $r_i(r_i(\chi))$ exists in the same way as above with $r_i(\chi)$ in place of χ . Further $r_i(r_i(\chi)) = \chi$, $m_{ij}^{r_i(\chi)} = m_{ij}$ for all $j \in I \setminus \{i\}$.

(3) Let $T_i : U(r_i(\chi)) \rightarrow U(\chi)$ be as in (31). Let $T_i^- : U(\chi) \rightarrow U(r_i(\chi))$ be the one as in (31) defined with $r_i(\chi)$ in place of χ . Then $T_i^- T_i = \text{id}_{U(r_i(\chi))}$ and $T_i T_i^- = \text{id}_{U(\chi)}$.

(4) Define the $\mathbb{Z}$ -module isomorphism $\sigma_i^{r_i(\chi)} = \sigma_i : \mathbb{Z}\Pi \rightarrow \mathbb{Z}\Pi$ by $T_i^\pm(U(r_i(\chi))^\lambda) = U(\chi)^{\sigma_i(\lambda)}$ for all $\lambda \in \mathbb{Z}\Pi$. Then

$$(37) \quad \sigma_i^{r_i(\chi)} = \sigma_i, \quad \sigma_i \sigma_i^{r_i(\chi)} = \text{id}_{\mathbb{Z}\Pi}$$

and

$$(38) \quad \sigma_i^{r_i(\chi)}(R_+^{r_i(\chi)} \setminus \{\alpha_i\}) = R_+ \setminus \{\alpha_i\}, \quad \sigma_i^{r_i(\chi)}(\alpha_i) = -\alpha_i.$$

Theorem 4. ([H07]) *Assume χ to be finite-type. Let $i, j \in I$ to be such that $i \neq j$. Let $M = |R_+ \cap (\mathbb{Z}_{\geq 0}\alpha_i - \mathbb{Z}_{\geq 0}\alpha_j)|$. For $n \in \{1, 2, \dots, M\}$, define two bi-characters χ_n, χ'_n , two $\mathbb{Z}$ -module automorphism $\bar{\sigma}_n, \bar{\sigma}'_n$ of $\mathbb{Z}\Pi$ and two $\mathbb{k}$ -algebra*

isomorphisms $\bar{T}_n : U(\chi_n) \rightarrow U(\chi)$, $\bar{T}'_n : U(\chi'_n) \rightarrow U(\chi)$ in the way that $\chi_1 = \chi'_1 = \chi$, $\bar{\sigma}_1 = \bar{\sigma}'_1 = \text{id}_{Z\Pi}$, $\bar{T}_1 = \bar{T}'_1 = \text{id}_{U(\chi)}$, and

$$(39) \quad \chi_{2n} = r_i(\chi_{2n-1}), \chi_{2n+1} = r_j(\chi_{2n}), \chi'_{2n} = r_j(\chi'_{2n-1}), \chi'_{2n+1} = r_i(\chi'_{2n}),$$

$$(40) \quad \bar{\sigma}_{2n} = \bar{\sigma}_{2n-1} \sigma_i^{2n}, \bar{\sigma}_{2n+1} = \bar{\sigma}_{2n} \sigma_j^{2n+1}, \bar{\sigma}'_{2n} = \bar{\sigma}'_{2n-1} \sigma_j^{2n}, \bar{\sigma}'_{2n+1} = \bar{\sigma}'_{2n} \sigma_i^{2n+1},$$

$$(41) \quad \bar{T}_{2n} = \bar{T}_{2n-1} T_i, \bar{T}_{2n+1} = \bar{T}_{2n} T_j, \bar{T}'_{2n} = \bar{T}'_{2n-1} T_j, \bar{T}'_{2n+1} = \bar{T}'_{2n} T_i.$$

Then we have

$$(42) \quad \chi_M = \chi'_M,$$

$$(43) \quad \bar{\sigma}_M = \bar{\sigma}'_M$$

and

$$(44) \quad \bar{T}_M = \bar{T}'_M.$$

4 Longest elements of Weyl groupoids

In this section we always assume χ to be finite-type, and refer to [CH08] for categorical definitions of Weyl groupoids.

Convention. For a category $\mathcal{C}$, we denote the product of the morphisms by $\cdot$. That is, for two morphism $f_1 \in \text{Mor}(a_1, b_1)$ and $f_2 \in \text{Mor}(a_2, b_2)$ with a_1, b_1, a_2 and $b_2 \in \text{Ob}(\mathcal{C})$, we denote their product by

$$(45) \quad f_1 \cdot f_2 \quad \text{if } b_2 = a_1.$$

Set

$$(46) \quad \mathcal{C}(\chi) = \{\chi\} \cup \bigcup_{n=1}^{\infty} \{r_{i_1} \cdots r_{i_n}(\chi) | i_1, \dots, i_n \in I\}.$$

Let $W = W(\chi)$ be the category with $\text{Ob}(W) = \mathcal{C}(\chi)$ and generated by the maps $\sigma_i' \in \text{Mor}_W(\chi', r_i(\chi'))$ with $\chi' \in \text{Ob}(W)$ and $i \in I$. Let $\mathcal{W} = \mathcal{W}(\chi)$ be the (abstract) category with $\text{Ob}(\mathcal{W}) = \mathcal{C}(\chi)$ defined by generators $s_i' \in \text{Mor}_{\mathcal{W}}(\chi', r_i(\chi'))$ with $\chi' \in \text{Ob}(W)$ and $i \in I$ and relations

$$(47) \quad s_i' \cdot s_i^{r_i(\chi')} = 1_{r_i(\chi')},$$

$$(48) \quad s_i' s_j^{r_i(\cdot)} s_i^{r_j r_i(\cdot)} = s_j' s_i^{r_j(\cdot)} s_j^{r_i r_j(\cdot)}$$

(both sides are composed of $|R_+ \cap (\mathbb{Z}\alpha_i - \mathbb{Z}\alpha_j)|$ -factors).

We call $\mathcal{W}$ the *Weyl groupoid*. Define the morphism $\phi: \mathcal{W} \rightarrow W$ by $\phi(s_i') = \sigma_i'$. Then ϕ is bijective, see [HY08a, Theorem 1]. Let $\ell(1') = 0$ for $\chi' \in \mathcal{C}(\chi)$. Let $\ell(s_i') = 1$. For $w \in \text{Mor}_{\mathcal{W}}(\chi_1, \chi_2)$, let $\ell(w)$ be the least number $\ell(w') + \ell(w'')$ with $w = w' w''$ for some $\chi_3 \in \mathcal{C}(\chi)$, and some $w' \in \text{Mor}_{\mathcal{W}}(\chi_3, \chi_2)$, some $w'' \in \text{Mor}_{\mathcal{W}}(\chi_1, \chi_3)$. By [HY08a, Lemma 8(iii)], we have

$$(49) \quad \ell(w) = |\{\alpha \in R_+^1 | \phi(w)(\alpha) \in -R_+^2\}|.$$

Moreover for each $\chi_1 \in \mathcal{C}(\chi)$, there exists unique $\chi_2 \in \mathcal{C}(\chi)$ and ${}^1w_0 \in \text{Mor}_{\mathcal{W}}(\chi_2, \chi_1)$ such that $\phi({}^1w_0)(R_+^2) = -R_+^1$. We call 1w_0 the *longest element* since $\ell({}^1w_0) = \ell(w')$ for any $w' \in \text{Mor}_{\mathcal{W}}(\chi_3, \chi_4)$ for any $\chi_3, \chi_4 \in \mathcal{C}(\chi)$.

Let $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(\chi)$ be the (abstract) category with $\text{Ob}(\widetilde{\mathcal{W}}) = \mathcal{C}(\chi)$ defined by generators $\widetilde{s}_i' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi', r_i(\chi'))$ with $\chi' \in \text{Ob}(W)$ and $i \in I$ and relations

$$(50) \quad \widetilde{s}_i' \widetilde{s}_j^{r_i(\cdot)} \widetilde{s}_i^{r_j r_i(\cdot)} = \widetilde{s}_j' \widetilde{s}_i^{r_j(\cdot)} \widetilde{s}_j^{r_i r_j(\cdot)}$$

(both sides are composed of $|R_+ \cap (\mathbb{Z}\alpha_i - \mathbb{Z}\alpha_j)|$ -factors).

Let $\widetilde{1}' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi', \chi')$ denote the identity morphism. Define the morphism $\widetilde{\phi}: \widetilde{\mathcal{W}} \rightarrow \mathcal{W}$ by $\widetilde{\phi}(\widetilde{s}_i') = s_i'$.

Let $\widetilde{\ell}(\widetilde{1}') = 0$ for $\chi' \in \mathcal{C}(\chi)$. Let $\widetilde{\ell}(\widetilde{s}_i') = 1$. For $\widetilde{w} \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_2)$, let $\widetilde{\ell}(\widetilde{w})$ be the least number $\widetilde{\ell}(\widetilde{w}') + \widetilde{\ell}(\widetilde{w}'')$ with $\widetilde{w} = \widetilde{w}' \widetilde{w}''$ for some $\chi_3 \in \mathcal{C}(\chi)$, and some $\widetilde{w}' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_3, \chi_2)$, some $\widetilde{w}'' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_3)$.

Theorem 5. ([HY08a, Theorem 5, Corollary 6]) *Let $\chi_1, \chi_2 \in \mathcal{C}(\chi)$. For $w \in \text{Mor}_{\mathcal{W}}(\chi_1, \chi_2)$ and $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{w} \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_2)$ with $\widetilde{\phi}(\widetilde{w}_1) = \widetilde{\phi}(\widetilde{w}_2) = w$ and $\ell(w) = \widetilde{\ell}(\widetilde{w}_1) = \widetilde{\ell}(\widetilde{w}_2)$, we have $\widetilde{w}_1 = \widetilde{w}_2$. Further, if $\widetilde{w} \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_2)$ is such that $\widetilde{\ell}(\widetilde{w}) > \ell(\widetilde{\phi}(\widetilde{w}))$, then $\widetilde{w} = \widetilde{w}' \widetilde{s}_i^{r_i(\cdot)} \widetilde{s}_i^{r_i(\cdot)} \widetilde{w}''$ for some $i \in I$, $\widetilde{w}' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_1, \chi_3)$ and $\widetilde{w}'' \in \text{Mor}_{\widetilde{\mathcal{W}}}(\chi_3, \chi_2)$ with $\widetilde{\ell}(\widetilde{w}') + \widetilde{\ell}(\widetilde{w}'') = \widetilde{\ell}(\widetilde{w}) - 2$.*

Assume w_0 to be $s_{j_1}^{-1} s_{j_2}^{-2} \dots s_{j_M}^{-M}$, where $M = |R_+|$, $r_1(\chi_1) = \chi$, and $r_j(\chi_j) = \chi_{j-1}$. Let $\bar{T}_1 = \text{id}_{U(\cdot)}$. For $2 \leq n \leq M$, define the $\mathbb{k}$ -algebra isomorphism $\bar{T}_n: U(\chi_{n-1}) \rightarrow U(\chi)$ by $\bar{T}_n = \bar{T}_{n-1} T_{j_{n-1}}$. Then as for E_i of Theorem 2, we may put

$$(51) \quad E_i = \bar{T}_i(E_{j_i})$$

for $1 \leq j \leq M$.

5 Shapovalov determinants

Let χ be a bi-character. We define the *Shapovalov matrix* Sh in the natural way for each $\alpha \in \mathbb{Z}_{\geq 0}\Pi$. More precisely, Sh is a $\dim U^+(\chi) \times \dim U^+(\chi)$ -matrix whose components are elements of $U^0(\chi)$. Let $\rho : \mathbb{Z}\Pi \rightarrow \mathbb{k}^\times$ be the (abelian) group homomorphism defined by $\rho(\alpha_i) = \chi(\alpha_i, \alpha_i)$. We use the *Kostant partition function* $P(\alpha, \beta, t) := \dim E^t U^+(\chi)_{-\alpha - \beta}$, where we define $P(\alpha, \beta, t) = 0$ in case $\alpha - t\beta \notin \mathbb{Z}_{\geq 0}\Pi$.

Theorem 6. ([HY08b, Theorem 7.3]) *Let χ be finite-type. Assume that $\chi(\beta, \beta) \neq 1$ for all $\beta \in R_+$. Then for $\alpha \in \mathbb{Z}_{\geq 0}\Pi$, we have*

$$\det \text{Sh} = c \prod_{\beta \in R_+} \prod_{t=1}^{h_\beta^\chi} (\rho(\beta) K^{-\chi(\beta, \beta)t} L)^{P^\chi(\alpha, t)}$$

for some $c \in \mathbb{k}^\times$.

As stated below, for $U(\chi)$ which is the (ordinary or small) quantum group of a finite dimensional Lie algebra $\mathfrak{g}$, we have the generalization of (1) the one [dDK90] for $q \in \mathbb{C}^\times$ which is not a root of unity, and (2) the one [KL97] for $q \in \mathbb{C}^\times$ which is a primitive p -th root of unity for some prime number p .

Corollary 7. *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of type A-G or a finite dimensional simple Lie superalgebra of type A-G. Then the Shapovalov determinant of the quantum group $U_q(\mathfrak{g})$ when q is not root of unity or the small quantum group $u_q(\mathfrak{g})$ when q is a primitive r -th root of unity for some positive integer $r \geq 2$ is given by*

$$c \prod_{\beta \in R_+} \prod_{t=1}^{h_\beta^\chi} (q^{2(\beta, \beta)} K^{-q(\beta, \beta)t} K^{-1})^{P^\chi(\alpha, t)}$$

for some $c \in \mathbb{C}^\times$.

We even recover the original ones due to Shapovalov [Sha72], and Kac [Kac77] (super cases):

Corollary 8. *Let $\mathfrak{g}$ be as above. Then the Shapovalov determinant of the enveloping algebra $U(\mathfrak{g})$ is given by*

$$c \prod_{\beta \in R_+} \prod_{t=1}^{\infty} (H + (\rho, \beta) - \frac{(\beta, \beta)t}{2})^{P^\chi(\alpha, t)}$$

for some $c \in \mathbb{C}^\times$.

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References

- [AS98] N. Andruskiewitsch and H.-J. Schneider, Lifting of quantum linear spaces and pointed Hopf algebras of order p^3 , *J. Algebra* 209 (1998), 658-691.
- [AS02] ———, Pointed Hopf algebras, *New Directions in Hopf Algebras*, MSRI Publications, vol. 43, Cambridge University Press, 2002.
- [CH08] M. Cuntz, I. Heckenberger, *Weyl groupoids and at most three objects* *J. Pure Appl. Algebra* 213 (2009) 1112-1128
- [Hec06] I. Heckenberger, The Weyl groupoid of a Nichols algebra of diagonal type, *Invent. Math.* 164 (2006), 175-188.
- [H09] ———, *Classification of arithmetic root systems*, *Adv. Math.* 220 (2009) 59-124.
- [H07] ———, *Lusztig isomorphisms for Drinfel'd doubles of Nichols algebras of diagonal type*, Preprint arXiv:0710.4521v1 (2007)
- [HY08a] I. Heckenberger, H. Yamane, *A generalization of Coxeter groups, root systems, and Matsumoto's theorem*, *Math. Z.* 259 (2008) 255-276.
- [HY08b] ———, *Drinfel'd doubles and Shapovalov determinants*, preprint arXiv:0810.1621 (2008)
- [Kac77] V.G. Kac, Representation of classical Lie superalgebras, *Lecture Notes in Physics*, vol. 676, pp. 597-626, Springer-Verlag, 1977.
- [Kha99] V. Kharchenko, *A quantum analogue of the Poincaré-Birkhoff-Witt theorem*, *Algebra and Logic* 38 (1999), 259-276
- [dDK90] C. de Concini, V.G. Kac, *Representations of quantum groups at roots of 1*, *Progress in Math.* vol. 92, Birkhäuser, 1990, pp. 471-506
- [KL97] S. Kummer, G. Letzter, *Shapovalov determinant for restricted and quantized restricted enveloping algebras*, *Pacific. J. Math.* 179 (1997) no. 1, 123-161
- [L] G. Lusztig, *Introduction to quantum groups*, Birkhäuser, Boston, MA 1993

[Sha72] N.N. Shapovalov, On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra, *Funct. Anal. Appl* 6 (1972), 307-312.

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