

SOME RESULTS AND PROBLEMS ON ANR'S FOR STRATIFIABLE SPACES

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Stratifiable spaces are also called M_3 -spaces, which were introduced by Ceder [Ce] and renamed by Borges [Bo]. The class $\mathcal{S}$ of stratifiable spaces contains both metrizable spaces and CW-complexes and has many desirable properties (cf. [Bo]). And CW-complexes are ANR for the class $\mathcal{S}$ [Ca₁]. Hence it has been expected that ANR theory for the class $\mathcal{S}$ is established so successfully as the class $\mathcal{M}$ of metrizable spaces. An absolute (neighborhood) retract for a class $\mathcal{C}$ is simply called an AR($\mathcal{C}$) (resp. ANR($\mathcal{C}$)). Although ANR($\mathcal{S}$)'s have been studied by Borges, Cauty and Miwa, etc., many problems are still left. In this note, we present the result of [GS] and some related problems.

The *join* of spaces X and Y is defined as the space

$$X * Y = X \cup X \times Y \times (0, 1) \cup Y$$

admitting the topology generated by all open sets in the product space $X \times Y \times (0, 1)$ and the following sets:

$$U \cup U \times Y \times (0, t) \quad \text{and} \quad X \times V \times (t, 1) \cup V,$$

where U and V are open in X and Y , respectively, and $0 < t < 1$. In [Ca₃], this join is denoted by $X \tilde{*} Y$ in order to distinguish from the join as the quotient space of $X \times Y \times \mathbf{I}$.

The *mapping cylinder* of a map $f: X \rightarrow Y$ is defined as the space

$$M(f) = X \times [0, 1] \cup Y$$

admitting the topology generated by all open sets in the product space $X \times [0, 1]$ and the following sets:

$$f^{-1}(V) \times (t, 1) \cup V,$$

where V is open in Y and $0 < t < 1$. Notice that $M(f)$ is not a quotient space of $X \times \mathbf{I} \oplus Y$. It is easily observed that $X * Y$ is homeomorphic to

$$M(\text{pr}_X) \cup_{X \times Y \times \{0\}} M(\text{pr}_Y),$$

where $\text{pr}_X: X \times Y \rightarrow X$ and $\text{pr}_Y: X \times Y \rightarrow Y$ are the projections. By using the Bing Metrization Theorem, it is easy to see that $M(f)$ (hence $X * Y$) is metrizable if so are X and Y . Extending [Ca₃, Lemma 6.3], we can show the following:

LEMMA. For any map $f: X \rightarrow Y$, the mapping cylinder $M(f)$ is stratifiable if so are X and Y .

By [Hy] (cf. [KL]), $M(f)$ (hence $X * Y$) is an $\text{ANR}(\mathcal{M})$ if so are X and Y . This is expected to be true for $\text{ANR}(\mathcal{S})$'s. However we cannot apply this method to stratifiable spaces (cf. [Ca₁]). In fact, San-ou [Sa] constructed a stratifiable space X with A a closed set such that (X, A) is not semi-canonical. (For the definition of semi-canonical pairs, refer to [Hy].) In his construction, by replacing $\mathbf{N}$ and $\mathbf{Q}$ by $\mathbf{R}$, we have a stratifiable locally convex linear topological space X , hence X is an $\text{AR}(\mathcal{S})$, such that (X, A) is not semi-canonical, where $A = \{0\}$. Consider the mapping cylinder $M(i)$ of the inclusion $i: X \setminus A \subset X$. Then $(M(i), X)$ is not semi-canonical. And $((X \setminus A) * X, X)$ is not semi-canonical. Thus we need another approach.

To characterize AR 's, Borges [Bo] introduced the concept of hyperconnectedness. For a space X , let $F(X)$ be the full simplicial complex with X the set of vertices, i.e., $X = F(X)^{(0)}$. Introducing a topology on $|F(X)|$, Cauty [Ca₄] constructed a test space $Z(X)$ such that a stratifiable space X is an $\text{ANR}(\mathcal{S})$ if and only if X is a neighborhood retract of $Z(X)$. Improving the construction of $Z(X)$, Miwa [Mi] constructed a hyperconnected space $E(X)$ containing X as a closed set and proved that $E(X)$ is stratifiable if so is X . Then any stratifiable space X can be embedded in an $\text{AR}(\mathcal{S})$ $E(X)$ as a closed set. By his construction, any map $f: X \rightarrow Y$ extends to the map $\tilde{f}: E(X) \rightarrow E(Y)$ which is a simplicial map from $F(X)$ to $F(Y)$. For this extension $\tilde{f}$, we have the following:

THEOREM 1. Let $\tilde{f}: E(X) \rightarrow E(Y)$ be the extension of a map $f: X \rightarrow Y$. Then $M(\tilde{f})$ is hyperconnected. Hence $M(\tilde{f})$ is an $\text{AR}(\mathcal{S})$ in case X and Y are stratifiable.

Since $M(f)$ is a closed subset of $M(\tilde{f})$, the following problem reduces to prove that $M(f)$ is a neighborhood retract of $M(\tilde{f})$.

PROBLEM 1. Let $f: X \rightarrow Y$ be a map between $\text{ANR}(\mathcal{S})$'s. Is the mapping cylinder $M(f)$ an $\text{ANR}(\mathcal{S})$?

Although this has not yet been succeeded, the following holds:

THEOREM 2. Let X and Y be $\text{ANR}(\mathcal{S})$'s and $f: X \rightarrow Y$ a Hurewicz fibration. Then the mapping cylinder $M(f)$ is an $\text{ANR}(\mathcal{S})$.

Since the projection $\text{pr}_X: X \times Y \rightarrow X$ is a Hurewicz fibration, we have the following generalization of [Ca₃, Corollary 6.2]:

THEOREM 3. If X and Y are $\text{ANR}(\mathcal{S})$'s then so is the join $X * Y$.

Remark. We can also prove Theorem 3 by showing that $E(X) * E(Y)$ is hyperconnected and that $X * Y$ is a neighborhood retract of $E(X) * E(Y)$. This approach is easier than the above approach.

In [Ca₂], Cauty asserted that the adjunction space of $\text{ANR}(S)$'s is also an $\text{ANR}(S)$, but his key lemma is false [Sa] (even if (X, A) is a pair of $\text{ANR}(S)$'s as shown in the above). Thus his assertion is still a conjecture and Theorem 3 is still open for the quotient topology:

PROBLEM 2. *Let X and Y be $\text{ANR}(S)$'s. Is the join $X * Y$ with the quotient topology an $\text{ANR}(S)$? For any map $f: X \rightarrow Y$, is the mapping cylinder $M(f)$ with the quotient topology an $\text{ANR}(S)$?*

In [Ca₃], Cauty proved that the direct limit of the tower of compact $\text{ANR}(\mathcal{M})$'s is an $\text{ANR}(S)$. It is natural to ask the following:

PROBLEM 3. *Let $X_1 \subset X_2 \subset \cdots$ be a tower of $\text{ANR}(S)$'s such that each X_{n+1} is a closed subspace of X_n . Is the direct limit $\text{dir lim } X_n$ an $\text{ANR}(S)$?*

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