

SOME RESULTS ON SET-VALUED STOCHASTIC INTEGRALS WITH RESPECT TO POISSON JUMP IN AN M-TYPE 2 BANACH SPACE

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1. INTRODUCTION

Probability theory is an important tool of modeling randomness in a practical problem. But besides randomness, in the real world, there exists other kind of uncertainties such as impreciseness or vagueness. Set-valued functions are employed to model the impreciseness in applied field such as in Economics, control theory (see for example [1]). Integrals of set-valued functions have been received much attention with widespread applications, see for example [2, 7, 9, 10] etc. Recently, stochastic integrals for set-valued stochastic processes with respect to the Brownian motion and martingales have been received much attention, e.g. see [12, 13, 18, 23, 32, 37]. Correspondingly, the set-valued stochastic differential equations are studied, e.g. see [23, 25, 33, 34, 35, 36]. Michta (2011) [22] extended the integrator to a larger class: semimartingales. But the integrably boundedness of the corresponding set-valued stochastic integrals are not obtained since the semimartingales may not be of finite variation. In such cases, the set-valued stochastic integrals may not be well defined as Ogura pointed out [25].

The Poisson stochastic processes are special. They play important roles both in the random mathematics (c.f. [11, 8, 17]) and in applied fields, for example, in the financial mathematics [17]. If the characteristic measure ν of a stationary Poisson process $\mathbf{p}$ is finite, then both of the Poisson random measure $N(ds dz)$ (where $z \in Z$, the state space of $\mathbf{p}$) and the compensated Poisson random measure $\tilde{N}(ds dz)$ are of finite variation a.s. We will give some results (without giving proof since the page limitation) on the set-valued stochastic integrals with respect to the Poisson random measure $N(ds dz)$, $\tilde{N}(ds dz)$. For the detail proof, the reader can refer to [31, 38]. For example, the stochastic integrals for set-valued $\mathcal{S}$ -predictable (see Definition 3.2) processes with respect to $N(ds dz)$ and $\tilde{N}(ds dz)$ are L^2 -integrably bounded. For Brownian or Martingale integrator with continuous part, the integrable boundedness are not obtained until now. Furthermore, if the σ -algebra $\mathcal{F}$ is separable, then the integral $\{I_t(F)\}$ of convex set-valued stochastic process will not become a set-valued martingale, which is very different from single valued case. We would like to pointed out that there is a gap in the proof of Theorem 3.7 in [31] about the set-valued martingale property of set-valued stochastic integral with respect to the compensated Poisson measure, which is corrected and proven in [38].

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This paper is organized as follows: In Section 2 we give the notations and the preliminaries in the set-valued theory. Section 3 is on the definitions and results of stochastic integrals for set-valued $\mathcal{S}$ -predictable processes with respect to $N(ds dz)$ and $\tilde{N}(ds dz)$.

2. PRELIMINARIES

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t\}_{t \geq 0}$ a filtration satisfying the usual conditions, that is: $\mathcal{F}_0$ includes all P -null sets in $\mathcal{F}$, the filtration is non-decreasing and right continuous. Let $\mathcal{B}(E)$ be the Borel field of a topological space E , $(X, \|\cdot\|)$ a separable Banach space equipped with the norm $\|\cdot\|$ and $\mathbf{K}(\mathbf{X})$ (resp. $\mathbf{K}_b(\mathbf{X})$, $\mathbf{K}_c(\mathbf{X})$) the family of all nonempty closed (resp. closed bounded, closed convex) subsets of X . Let $1 \leq p < +\infty$ and $L^p(\Omega, \mathcal{F}, P; X)$ (denoted briefly by $L^p(\Omega; X)$) be the Banach space of equivalence classes of X -valued $\mathcal{F}$ -measurable functions $f : \Omega \rightarrow X$ such that the norm

$$\|f\|_p = \left\{ \int_{\Omega} \|f(\omega)\|^p dP \right\}^{1/p}$$

is finite. An X -valued function f is called *L^p -integrable* if $f \in L^p(\Omega; X)$.

A set-valued function $F : \Omega \rightarrow \mathbf{K}(X)$ is said to be *measurable* if for any open set $O \subset X$, the inverse $F^{-1}(O) := \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\}$ belongs to $\mathcal{F}$. Such a function F is called a *set-valued random variable*. Let $\mathcal{M}(\Omega, \mathcal{F}, P; \mathbf{K}(X))$ be the family of all set-valued random variables, which is briefly denoted by $\mathcal{M}(\Omega; \mathbf{K}(X))$.

For any open subset $O \subset X$, set

$$Z_O := \{E \in \mathbf{K}(X) : E \cap O \neq \emptyset\},$$

$$\mathcal{C} := \{Z_O : O \subset X, O \text{ is open}\},$$

and let $\sigma(\mathcal{C})$ be the σ -algebra generated by $\mathcal{C}$.

A set-valued function $F : \Omega \rightarrow \mathbf{K}(X)$ is measurable if and only if F is $\mathcal{F}/\sigma(\mathcal{C})$ -measurable.

For $A, B \in 2^X$ (the power set of X), $H(A, B) \geq 0$ is defined by

$$H(A, B) := \max\left\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\right\}.$$

$H(A, B)$ for $A, B \in \mathbf{K}_b(X)$ is called the *Hausdorff metric*. It is well-known that $\mathbf{K}_b(X)$ equipped with the H -metric denoted by $(\mathbf{K}_b(X), H)$ is a complete metric space.

The following results are also well-known. (see e.g. [9], [19], [24]).

PROPOSITION 2.1. (i) For $A, B, C, D \in \mathbf{K}(X)$, we have

$$H(A + B, C + D) \leq H(A, C) + H(B, D),$$

$$H(A \oplus B, C \oplus D) = H(A + B, C + D),$$

where $A \oplus B := cl\{a + b; a \in A, b \in B\}$.

(ii) For $A, B \in \mathbf{K}(X)$, $\mu \in \mathbb{R}$, we have

$$H(\mu A, \mu B) = |\mu| H(A, B).$$

For $F \in \mathcal{M}(\Omega, \mathbf{K}(X))$, the family of all L^p -integrable selections is defined by

$$S_F^p(\mathcal{F}) := \{f \in L^p(\Omega, \mathcal{F}, P; X) : f(\omega) \in F(\omega) \text{ a.s.}\}.$$

In the following, $S_F^p(\mathcal{F})$ is denoted briefly by S_F^p . If S_F^p is nonempty, F is said to be L^p -integrable. F is called L^p -integrably bounded if there exists a function $h \in L^p(\Omega, \mathcal{F}, P; \mathbb{R})$ such that $\|x\| \leq h(\omega)$ for any x and ω with $x \in F(\omega)$. It is equivalent to that $\|F\|_{\mathbf{K}} \in L^p(\Omega; \mathbb{R})$, where $\|F(\omega)\|_{\mathbf{K}} := \sup_{a \in F(\omega)} \|a\|$. The family of all measurable $\mathbf{K}(X)$ -valued L^p -integrably bounded functions is denoted by $L^p(\Omega, \mathcal{F}, P; \mathbf{K}(X))$. Write it for brevity as $L^p(\Omega; \mathbf{K}(X))$.

The *integral (or expectation)* of a set-valued random variable F was defined by Aumann in 1965 ([2]):

$$E[F] := \{E[f] : f \in S_F^1\}.$$

PROPOSITION 2.2. ([35]) *Let $F \in \mathcal{M}(\Omega; X)$, $1 \leq p < +\infty$. Then F is L^p -integrably bounded if and only if S_F^p is nonempty and bounded in $L^p(\Omega; X)$.*

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers and $\mathcal{B}_+ := \mathcal{B}(\mathbb{R}_+)$. $\mathbb{N}$ denotes the set of natural numbers. An X -valued stochastic process $f = \{f_t : t \geq 0\}$ (or denoted by $f = \{f(t) : t \geq 0\}$) is defined as a function $f : \mathbb{R}_+ \times \Omega \rightarrow X$ with the $\mathcal{F}$ -measurable section f_t , for $t \geq 0$. We say f is *measurable* if f is $\mathcal{B}_+ \otimes \mathcal{F}$ -measurable. The process $f = \{f_t : t \geq 0\}$ is called $\mathcal{F}_t$ -adapted if f_t is $\mathcal{F}_t$ -measurable for every $t \geq 0$. $f = \{f_t : t \geq 0\}$ is called *predictable* if it is $\mathcal{P}$ -measurable, where $\mathcal{P}$ is the σ -algebra generated by all left continuous and $\mathcal{F}_t$ -adapted stochastic processes.

In a fashion similar to the X -valued stochastic process, a *set-valued stochastic process* $F = \{F_t : t \geq 0\}$ is defined as a set-valued function $F : \mathbb{R}_+ \times \Omega \rightarrow \mathbf{K}(X)$ with $\mathcal{F}$ -measurable section F_t for $t \geq 0$. It is called *measurable* if it is $\mathcal{B}_+ \otimes \mathcal{F}$ -measurable, and $\mathcal{F}_t$ -adapted if for any fixed t , $F_t(\cdot)$ is $\mathcal{F}_t$ -measurable. $F = \{F_t : t \geq 0\}$ is called *predictable* if it is $\mathcal{P}$ -measurable.

DEFINITION 2.3. (see [9]) An integrable bounded convex set-valued $\mathcal{F}_t$ -adapted stochastic process $\{F_t, \mathcal{F}_t : t \geq 0\}$ is called a *set-valued $\mathcal{F}_t$ -martingale* if for any $0 \leq s \leq t$ it holds that $E[F_t | \mathcal{F}_s] = F_s$ in the sense of $S_{E[F_t | \mathcal{F}_s]}^1(\mathcal{F}_s) = S_{F_s}^1(\mathcal{F}_s)$.

It is called a *set-valued submartingale (supermartingale)* if for any $0 \leq s \leq t$, $E[F_t | \mathcal{F}_s] \supset F_s$ (resp. $E[F_t | \mathcal{F}_s] \subset F_s$) in the sense of $S_{E[F_t | \mathcal{F}_s]}^1(\mathcal{F}_s) \supset S_{F_s}^1(\mathcal{F}_s)$ (resp. $S_{E[F_t | \mathcal{F}_s]}^1(\mathcal{F}_s) \subset S_{F_s}^1(\mathcal{F}_s)$).

3. STOCHASTIC INTEGRALS WITH RESPECT TO POISSON POINT PROCESSES

3.1. Single Valued Stochastic Integrals w.r.t. Poisson Point Processes. Let X be a separable Banach space and Z be another separable Banach space with σ -algebra $\mathcal{B}(Z)$. A *point function* $\mathbf{p}$ on Z means a mapping $\mathbf{p} : \mathbf{D}_{\mathbf{p}} \rightarrow Z$, where the domain $\mathbf{D}_{\mathbf{p}}$ is a countable subset of $[0, T]$. $\mathbf{p}$ defines a counting measure $N_{\mathbf{p}}(dtdz)$ on $[0, T] \times Z$ (with the product σ -algebra $\mathcal{B}([0, T]) \otimes \mathcal{B}(Z)$) by

$$(3.1) \quad \begin{aligned} N_{\mathbf{p}}((0, t], U) &:= \#\{\tau \in \mathbf{D}_{\mathbf{p}} : \tau \leq t, \mathbf{p}(\tau) \in U\}, \\ t &\in (0, T], U \in \mathcal{B}(Z). \end{aligned}$$

For $0 \leq s < t \leq T$,

$$(3.2) \quad N_{\mathbf{p}}((s, t], U) := N_{\mathbf{p}}((0, t], U) - N_{\mathbf{p}}((0, s], U).$$

In the following, we also write $N_{\mathbf{p}}((0, t], U)$ as $N_{\mathbf{p}}(t, U)$.

A *point process* is obtained by randomizing the notion of point functions. If there is a continuous $\mathcal{F}_t$ -adapted increasing process $\hat{N}_{\mathbf{p}}$ such that for $U \in \mathcal{B}(Z)$ and $t \in [0, T]$, $\tilde{N}_{\mathbf{p}}(t, U) := N_{\mathbf{p}}(t, U) - \hat{N}_{\mathbf{p}}(t, U)$ is an $\mathcal{F}_t$ -martingale, then the random measure $\{\tilde{N}_{\mathbf{p}}(t, U)\}$ is called the *compensator* of the point process $\mathbf{p}$ (or $\{N_{\mathbf{p}}(t, U)\}$) and the process $\{\tilde{N}_{\mathbf{p}}(t, U)\}$ is called the *compensated point process*.

A point process $\mathbf{p}$ is called the *Poisson Point Process* if $N_{\mathbf{p}}(dtdz)$ is a Poisson random measure on $[0, T] \times Z$. A Poisson point process is stationary if and only if its intensity measure $\nu_{\mathbf{p}}(dtdz) = E[N_{\mathbf{p}}(dtdz)]$ is of the form

$$(3.3) \quad \nu_{\mathbf{p}}(dtdz) = dt\nu(dz)$$

for some measure $\nu(dz)$ on $(Z, \mathcal{B}(Z))$. $\nu(dz)$ is called the *characteristic measure* of $\mathbf{p}$.

Let ν be a σ -finite measure on $(Z, \mathcal{B}(Z))$, (i.e. there exists $U_i \in \mathcal{B}(Z)$, $i \in \mathbb{N}$, pairwise disjoint such that $\nu(U_i) < \infty$ for all $i \in \mathbb{N}$ and $Z = \bigcup_{i=1}^{\infty} U_i$), $\mathbf{p} = (\mathbf{p}_t)$ be the $\mathcal{F}_t$ -adapted stationary Poisson point process on Z with the characteristic measure ν such that the compensator $\hat{N}_{\mathbf{p}}(t, U) = E[N_{\mathbf{p}}(t, U)] = t\nu(U)$ (non-random).

The above definitions and notations of Poisson point processes come from [11] and [30].

For convenience, we will omit the subscript $\mathbf{p}$ in the above notations.

PROPOSITION 3.1. ([31]) *Assume $\nu(Z)$ is finite. Then for any $U \in \mathcal{B}(Z)$, both $\{N(t, U), t \in [0, T]\}$ and $\{\tilde{N}(t, U), t \in [0, T]\}$ are stochastic processes with finite variation a.s.*

For convenience, from now on, we suppose ν is a finite measure in the measurable space $(Z, \mathcal{B}(Z))$.

DEFINITION 3.2. An X -valued mapping $f(t, z, \omega)$ defined on $[0, T] \times Z \times \Omega$ is called $\mathcal{S}$ -predictable if the mapping $(t, z, \omega) \rightarrow f(t, z, \omega)$ is $\mathcal{S}/\mathcal{B}(X)$ -measurable, where $\mathcal{S}$ is the smallest σ -algebra on $[0, T] \times Z \times \Omega$ with respect to which all mappings $g : [0, T] \times Z \times \Omega \rightarrow X$ satisfying (i) and (ii) below are measurable:

- (i) for each $t \in [0, T]$, the mapping $(z, \omega) \rightarrow g(t, z, \omega)$ is $\mathcal{B}(Z) \otimes \mathcal{F}_t$ -measurable;
- (ii) for each $(z, \omega) \in Z \times \Omega$, the mapping $t \rightarrow g(t, z, \omega)$ is left continuous.

REMARK 3.3. (see e.g. [30]) $\mathcal{S} = \mathcal{P} \otimes \mathcal{B}(Z)$, where $\mathcal{P}$ denotes the σ -field on $[0, t] \times \Omega$ generated by all left continuous and $\mathcal{F}_t$ -adapted processes.

Set

$$\mathcal{L} = \left\{ f(t, z, \omega) : f \text{ is } \mathcal{S}\text{-predictable and} \right. \\ \left. E \left[\int_0^T \int_Z \|f(t, z, \omega)\|^2 \nu(dz) dt \right] < \infty \right\}$$

equipped with the norm

$$\|f\|_{\mathcal{L}} := \left(E \left[\int_0^T \int_Z \|f(t, z, \omega)\|^2 \nu(dz) dt \right] \right)^{1/2}.$$

Let $\mathbb{S}$ be the subspace of those $f \in \mathcal{L}$ for which there exists a partition $0 = t_0 < t_1 < \dots < t_n = T$ of $[0, T]$ such that

$$f(t, z, \omega) = f(0, z, \omega)\chi_{\{0\}}(t) + \sum_{i=1}^n \chi_{(t_{i-1}, t_i]}(t) f(t_{i-1}, z, \omega).$$

Let f be in $\mathbb{S}$ and

$$(3.4) \quad f(t, z, \omega) = f(0, z, \omega)\chi_{\{0\}}(t) + \sum_{i=1}^n \chi_{(t_{i-1}, t_i]}(t) f(t_{i-1}, z, \omega),$$

where $0 = t_0 < t_1 < \dots < t_n = T$ is a partition of $[0, T]$. Define

$$(3.5) \quad \begin{aligned} J_T(f) &= \int_0^{T+} \int_Z f(s-, z, \omega) N(dt dz) \\ &:= \sum_{i=1}^n \int_Z f(t_{i-1}, z, \omega) N((t_{i-1}, t_i], dz), \end{aligned}$$

and

$$(3.6) \quad \begin{aligned} I_T(f) &= \int_0^{T+} \int_Z f(s-, z, \omega) \tilde{N}(dt dz) \\ &:= \sum_{i=1}^n \int_Z f(t_{i-1}, z, \omega) \tilde{N}((t_{i-1}, t_i], dz), \end{aligned}$$

where $\int_Z f(t_{i-1}, z, \omega) N((t_{i-1}, t_i], dz)$ and $\int_Z f(t_{i-1}, z, \omega) \tilde{N}((t_{i-1}, t_i], dz)$ are the Bochner integrals. The notation ' $\int_0^{T+}$ ' means ' $\int_{(0, T]}$ '.

For any integer $0 \leq k \leq n$, let

$$M_k = \sum_{i=1}^k \int_Z f(t_{i-1}, z, \omega) \tilde{N}((t_{i-1}, t_i], dz)$$

then M_k is $\mathcal{F}_{t_k}$ -measurable, $E[M_k] = 0$, $E[I_T(f)] = E[M_n] = 0$ and

$$(3.7) \quad \begin{aligned} E[M_k | \mathcal{F}_{t_{k-1}}] &= E[(M_{k-1} + \int_Z f(t_{k-1}, z, \omega) \tilde{N}((t_{k-1}, t_k], dz) | \mathcal{F}_{t_{k-1}}] \\ &= M_{k-1} + E[\int_Z f(t_{k-1}, z, \omega) \tilde{N}((t_{k-1}, t_k], dz) | \mathcal{F}_{t_{k-1}}] \\ &= M_{k-1} + \int_Z f(t_{k-1}, z, \omega) E[\tilde{N}((t_{k-1}, t_k], dz)] = M_{k-1}. \end{aligned}$$

For any $t \in (0, T]$, define

$$(3.8) \quad \begin{aligned} J_t(f) &= \int_0^{t+} \int_Z f(s-, z, \omega) N(dz ds) \\ &:= \sum_{i=1}^n \int_Z f(t_{i-1}, z, \omega) N((t_{i-1} \wedge t, t_i \wedge t], dz), \end{aligned}$$

and

$$(3.9) \quad \begin{aligned} I_t(f) &= \int_0^{t+} \int_Z f(s-, z, \omega) \tilde{N}(dz ds) \\ &:= \sum_{i=1}^n \int_Z f(t_{i-1}, z, \omega) \tilde{N}((t_{i-1} \wedge t, t_i \wedge t], dz). \end{aligned}$$

LEMMA 3.4. ([31]) For any $f \in \mathbb{S}$, both $\{I_t(f)\}$ and $\{J_t(f)\}$ are $\mathcal{F}_t$ -adapted integrable processes. Moreover, $\{I_t(f)\}$ is an X -valued right continuous martingale. And for any $t \in (0, T]$,

$$(3.10) \quad E\left[\int_0^{t+} \int_Z f(s-, z, \omega) \tilde{N}(dsdz)\right] = 0,$$

$$(3.11) \quad E\left[\int_0^{t+} \int_Z f(s-, z, \omega) N(ds dz)\right] = \int_0^{t+} \int_Z E[f(s-, z, \omega)] d\nu(dz),$$

In order to extend the integrand from the step function which belongs to $\mathbb{S}$ to a more general case (belongs to $\mathcal{L}$), it is necessary to add some assumption in the Banach space X . Now we assume X is of M-type 2 below.

DEFINITION 3.5. ([5]) A Banach space $(X, \|\cdot\|)$ is called M-type 2 if and only if there exists a constant $C_X > 0$ such that for any X -valued martingale $\{\mathbf{M}_k\}$, it holds that

$$(3.12) \quad \sup_k E[\|\mathbf{M}_k\|^2] \leq C_X \sum_k E[\|\mathbf{M}_k - \mathbf{M}_{k-1}\|^2].$$

THEOREM 3.6. ([31]) Let X be of M-type 2 and $(Z, \mathcal{B}(Z))$ a separable Banach space with finite measure ν . Let $\mathbf{p}$ be a stationary Poisson process with the characteristic measure ν and let f be in $\mathbb{S}$. Then there exists a constant C such that

$$(3.13) \quad \begin{aligned} & E\left[\sup_{0 < s \leq t} \left\| \int_0^{s+} \int_Z f(\tau-, z, \omega) \tilde{N}(d\tau dz) \right\|^2\right] \\ & \leq C \int_0^t \int_Z E[\|f(s, z, \omega)\|^2] d\nu(dz), \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} & E\left[\sup_{0 < s \leq t} \left\| \int_0^{s+} \int_Z f(\tau-, z, \omega) N(d\tau dz) \right\|^2\right] \\ & \leq C \int_0^t \int_Z E[\|f(s, z, \omega)\|^2] d\nu(dz), \end{aligned}$$

where C depends on the constant C_X in Definition 3.5.

LEMMA 3.7. ([31]) $\mathbb{S}$ is dense in $\mathcal{L}$ with respect to the norm $\|\cdot\|_{\mathcal{L}}$.

By Lemma 3.7, for any $f \in \mathcal{L}$, there exist a sequence $\{f^n : n \in \mathbb{N}\}$ in $\mathbb{S}$ such that $\{f^n\}$ converges to f with respect to $\|\cdot\|_{\mathcal{L}}$ and the sequence

$$\left\{ \int_0^{t+} \int_Z f^n(s-, z, \omega) \tilde{N}(dsdz), n \in \mathbb{N} \right\}$$

converges to a limit in L^2 -sense. We denote the limit by

$$I_t(f) = \int_0^{t+} \int_Z f(s-, z, \omega) \tilde{N}(dsdz),$$

which is called the *stochastic integral of f* with respect to the compensated Poisson random measure $\tilde{N}(dsdz)$. Similarly, we can define the *stochastic integral of f* with respect to the Poisson random measure $N(ds dz)$, denoted by

$$J_t(f) = \int_0^{t+} \int_Z f(s-, z, \omega) N(ds dz).$$

Similarly, for any $0 < s < t < T$,

$$\int_s^t \int_Z f(\tau-, z, \omega) \tilde{N}_{\mathbf{p}}(d\tau dz)$$

and

$$\int_s^t \int_Z f(\tau-, z, \omega) N(d\tau dz)$$

can be well defined.

REMARK 3.8. When the measure ν is finite, for any $U \in \mathcal{B}(Z)$, the processes $\{N(t, U)\}$ and $\{\tilde{N}(t, U)\}$ are both of finite variation a.s. Then the stochastic integrals coincide with the Lebesgue-Stieltjes integrals.

COROLLARY 3.9. ([31]) *Let X be of M -type 2 and $(Z, \mathcal{B}(Z))$ a separable Banach space with finite measure ν . Let $\mathbf{p}$ be a stationary Poisson process with the characteristic measure ν and let f be in $\mathcal{L}$. Then there exists a constant C such that*

$$\begin{aligned} (3.15) \quad & E \left[\sup_{0 < s \leq t} \left\| \int_0^{s+} \int_Z f(\tau-, z, \omega) \tilde{N}(d\tau dz) \right\|^2 \right] \\ & \leq C \int_0^t \int_Z E[\|f(s, z, \omega)\|^2] d\nu(dz), \end{aligned}$$

and

$$\begin{aligned} (3.16) \quad & E \left[\sup_{0 < s \leq t} \left\| \int_0^{s+} \int_Z f(\tau-, z, \omega) N(d\tau dz) \right\|^2 \right] \\ & \leq C \int_0^t \int_Z E[\|f(s, z, \omega)\|^2] d\nu(dz), \end{aligned}$$

where C depends on the constant C_X in Definition 3.5.

COROLLARY 3.10. ([31]) *For any $f \in \mathcal{L}$, both $\{I_t(f)\}$ and $\{J_t(f)\}$ are $\mathcal{F}_t$ -adapted square-integrable processes. Moreover, $\{I_t(f)\}$ is an X -valued right continuous martingale. And for any $t \in (0, T]$,*

$$(3.17) \quad E \left[\int_0^{t+} \int_Z f(s-, z, \omega) \tilde{N}(ds dz) \right] = 0,$$

$$(3.18) \quad E \left[\int_0^{t+} \int_Z f(s-, z, \omega) N(ds dz) \right] = \int_0^t \int_Z E[f(s, z, \omega)] d\nu(dz),$$

3.2. Set-Valued Stochastic Integrals w.r.t. Poisson Point Processes. A set-valued stochastic process $F = \{F_t\} : [0, T] \times Z \times \Omega \rightarrow \mathbf{K}(X)$ is called $\mathcal{S}$ -predictable if $F(z, t, \omega)$ is $\mathcal{S}/\sigma(\mathcal{C})$ -measurable.

Set

$$\begin{aligned} \mathcal{M} = & \left\{ F(t, z, \omega) : F \text{ is } \mathcal{S}\text{-predictable and} \right. \\ & \left. E \left[\int_0^T \int_Z \|F(t, z, \omega)\|_{\mathbf{K}}^2 dt \nu(dz) \right] < \infty \right\} \end{aligned}$$

Given a set-valued stochastic process $\{F(t, z, \omega)\}$, the X -valued stochastic process $\{f(t, z, \omega)\}$ is called an $\mathcal{S}$ -selection if $f(t, z, \omega) \in F(t, z, \omega)$ for all (t, z, ω) and $f \in \mathcal{S}$. By Proposition ??, for $F \in \mathcal{M}$, the $\mathcal{S}$ -selection exists and satisfies $E\left[\int_0^T \int_Z \|f(t, z, \omega)\|^2 dt \nu(dz)\right] < \infty$ since

$$E\left[\int_0^T \int_Z \|f(t, z, \omega)\|^2 dt \nu(dz)\right] \leq E\left[\int_0^T \int_Z \|F(t, z, \omega)\|_{\mathbf{K}}^2 dt \nu(dz)\right] < \infty,$$

which means $f \in \mathcal{S}$. The family of all f which belongs to $\mathcal{S}$ and satisfies $f(t, z, \omega) \in F(t, z, \omega)$ for a.e. (t, z, ω) is denoted by $S(F)$, that is

$$S(F) = \{f \in \mathcal{S} : f(t, z, \omega) \in F(t, z, \omega) \text{ for a.e. } (t, z, \omega)\}.$$

Set

$$\begin{aligned} \tilde{\Gamma}_t &:= \left\{ \int_0^{t+} \int_Z f(s-, z, \omega) \tilde{N}(ds dz) : (f(t))_{t \in [0, T]} \in S(F) \right\}, \\ \Gamma_t &:= \left\{ \int_0^t \int_Z f(s-, z, \omega) N(ds dz) : (f(t))_{t \in [0, T]} \in S(F) \right\}. \end{aligned}$$

REMARK 3.11. It is easy to see for any $t \in [0, T]$, $\tilde{\Gamma}_t$ and Γ_t are the subsets of $L^2[\Omega, \mathcal{F}_t, P; X]$. Furthermore, if $\{F_t, \mathcal{F}_t : t \in [0, T]\}$ is convex, then so are $\tilde{\Gamma}_t$ and Γ_t .

Let $de\tilde{\Gamma}_t$ (resp. $de\Gamma_t$) denote the decomposable set of $\tilde{\Gamma}_t$ (resp. Γ_t) with respect to $\mathcal{F}_t$, $\overline{de\tilde{\Gamma}_t}$ (resp. $\overline{de\Gamma_t}$) the decomposable closed hull of $\tilde{\Gamma}_t$ (resp. Γ_t) with respect to $\mathcal{F}_t$, where the closure is taken in $L^1(\Omega, X)$. That is to say, for any $g \in \overline{de\tilde{\Gamma}_t}$ (resp. $\overline{de\Gamma_t}$) and any given $\epsilon > 0$, there exists a finite $\mathcal{F}_t$ -measurable partition $\{A_1, \dots, A_m\}$ of Ω and $(f^1(t))_{t \in [0, T]}, \dots, (f^m(t))_{t \in [0, T]} \in S(F)$ such that

$$\begin{aligned} \|g - \sum_{k=1}^m \chi_{A_k} \int_0^{t+} \int_Z f^k(s-, z, \omega) \tilde{N}(ds dz)\|_{L^1} &< \epsilon. \\ (\text{resp. } \|g - \sum_{k=1}^m \chi_{A_k} \int_0^t \int_Z f^k(s-, z, \omega) N(ds dz)\|_{L^1} &< \epsilon) \end{aligned}$$

Similar to Theorem 4.1 in [32], we have

THEOREM 3.12. Let $\{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$, then for any $t \in [0, T]$, $\overline{de\Gamma}_t \subset L^1(\Omega, \mathcal{F}_t, P; X)$. Moreover, there exists a set-valued random variable $J_t(F) \in \mathcal{M}(\Omega, \mathcal{F}_t, P; \mathbf{K}(X))$ such that $S_{J_t(F)}^1(\mathcal{F}_t) = \overline{de\Gamma}_t$. Similarly, there exists a set-valued random variable $I_t(F) \in \mathcal{M}(\Omega, \mathcal{F}_t, P; \mathbf{K}(X))$ such that $S_{I_t(F)}^1(\mathcal{F}_t) = \overline{de\tilde{\Gamma}_t}$. If F is convex, then so are $S_{I_t(F)}^1(\mathcal{F}_t)$ and $S_{J_t(F)}^1(\mathcal{F}_t)$.

DEFINITION 3.13. The set-valued stochastic processes $(J_t(F))_{t \in [0, T]}$ and $(I_t(F))_{t \in [0, T]}$ defined as above are called the stochastic integrals of $\{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$ with respect to the Poisson random measure $N(ds, dz)$ and the compensated random measure $\tilde{N}(ds dz)$ respectively. For each t , we denote $I_t(F) = \int_0^{t+} \int_Z F(s-, z, \omega) \tilde{N}(ds dz)$, $J_t(F) = \int_0^{t+} \int_Z F(s-, z, \omega) N(ds dz)$. Similarly, for $0 < s < t$, we also can define the set-valued random variable $I_{s,t}(F) = \int_s^t \int_Z F(\tau-, z, \omega) \tilde{N}(d\tau dz)$, $J_{s,t}(F) = \int_s^t \int_Z F(\tau-, z, \omega) N(d\tau dz)$.

For brevity, the integral $\int_0^{t+} \int_Z h(s-, z, \omega) \tilde{N}(ds dz)$ ($\int_0^{t+} \int_Z h(s-, z, \omega) N(ds dz)$) also will be denoted by $\int_0^{t+} \int_Z h_{s-} \tilde{N}(ds dz)$ ($\int_0^{t+} \int_Z h_{s-} N(ds dz)$ resp.), where h is an X -valued or $\mathbf{K}(X)$ -valued integrand.

PROPOSITION 3.14. ([38]) Assume set-valued stochastic processes $\{F_t, \mathcal{F}_t : t \in [0, T]\}$ and $\{G_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$. Then

$$J_t(F + G) = \text{cl}\{J_t(F) + J_t(G)\} \text{ a.s and } I_t(F + G) = \text{cl}\{I_t(F) + I_t(G)\} \text{ a.s.,}$$

where the cl stands for the closure in X .

THEOREM 3.15. ([31]) Assume a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$. Then $\{J_t(F)\}$ and $\{I_t(F)\}$ are integrably bounded.

THEOREM 3.16. ([31, 38]) Let a convex set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$, then the stochastic integral $\{I_t(F), \mathcal{F}_t : t \in [0, T]\}$ is a set-valued submartingale but not a set-valued martingale.

REMARK 3.17. With the assumption of $\mathcal{F}$ being separable with respect to the probability measure P , Theorem 3.7 in [31] pointed out that the integral $\{I_t(F)\}$ is a set-valued martingale. But unfortunately, now we found there is a gap in the proof. In fact, $\{I_t(F)\}$ is not a set-valued martingale except for special case (the singletons). The counterexample and rigorous proof are given in [38].

THEOREM 3.18. ([38]) Assume a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$. Then both $\{J_t(F)\}$ and $\{I_t(F)\}$ are L^2 -integrably bounded.

THEOREM 3.19. ([31])(Castaing representation of set-valued stochastic integral)
Assume $\mathcal{F}$ is separable with respect to the probability measure P . Then for a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0, T]\} \in \mathcal{M}$, there exists a sequence $\{(f_t^i)_{t \in [0, T]} : i = 1, 2, \dots\} \subset S(F)$ such that for each $t \in [0, T], z \in Z$, $F(t, z, \omega) = \text{cl}\{(f_t^i(z, \omega)) : i = 1, 2, \dots\}$ a.s., and

$$I_t(F)(\omega) = \text{cl}\left\{\int_0^{t+} \int_Z f_{s-}^i(z, \omega) \tilde{N}(dsdz)(\omega) : i = 1, 2, \dots\right\} \text{ a.s.}$$

and

$$J_t(F)(\omega) = \text{cl}\left\{\int_0^{t+} \int_Z f_{s-}^i(z, \omega) N(ds dz)(\omega) : i = 1, 2, \dots\right\} \text{ a.s.}$$

THEOREM 3.20. ([38]) Assume $\mathcal{F}$ is separable with respect to P . Let $\{F_t\}_{t \in [0, T]}$ and $\{G_t\}_{t \in [0, T]}$ be set-valued stochastic processes in $\mathcal{M}$. Then for all t , it follows that

$$\begin{aligned} & E\left[H\left(\int_0^{t+} \int_Z F(s-, z, \omega) N(ds dz), \int_0^{t+} \int_Z G(s-, z, \omega) N(ds dz)\right)\right] \\ (3.19) \quad & \leq E\left[\int_0^{t+} \int_Z H(F(s-, z, \omega), G(s-, z, \omega)) N(ds dz)\right] \\ & = E\left[\int_0^t \int_Z H(F(s, z, \omega), G(s, z, \omega)) ds \nu dz\right] \end{aligned}$$

and

$$\begin{aligned} & E\left[H^2\left(\int_0^{t+} \int_Z F(s-, z, \omega) N(ds dz), \int_0^{t+} \int_Z G(s-, z, \omega) N(ds dz)\right)\right] \\ (3.20) \quad & \leq CE\left[\int_0^{t+} \int_Z H^2(F(s-, z, \omega), G(s-, z, \omega)) N(ds dz)\right] \\ & = CE\left[\int_0^t \int_Z H^2(F(s, z, \omega), G(s, z, \omega)) ds \nu(dz)\right] \end{aligned}$$

where C is the constant appearing in Corollary 3.9.

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