

Some reduced expressions of the classical Weyl groups and the Weyl groupoids of the Lie superalgebras $\mathfrak{osp}(2m|2n)$

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Abstract

We give some reduced expressions of the classical Weyl groups $W(A_{N-1})$, $W(B_N) = W(C_N)$, $W(D_N)$ and the Weyl groupoid of the Lie superalgebra $\mathfrak{osp}(2m|2(N-m))$.

1 Some reduced expressions of the classical Weyl groups

For $m, n \in \mathbb{Z}$, let $J_{n,m} := \{k \in \mathbb{Z} \mid m \leq k \leq n\}$.

Let $N \in \mathbb{N}$. Let $M_N(\mathbb{R})$ be the $\mathbb{R}$ -algebra of $N \times N$ -matrices. For $k, r \in J_{1,N}$, let $E_{k,r} := [\delta_{k,k'}\delta_{r,r'}]_{k',r' \in J_{1,N}} \in M_N(\mathbb{R})$, that is $E_{k,r}$ is the matrix unite such that its (k, r) -component is 1 and the other components is 0. Then $M_N(\mathbb{R}) = \bigoplus_{k,r \in J_{1,N}} \mathbb{R}E_{k,r}$. Let $\mathbb{R}^N$ denote the $\mathbb{R}$ -linear space of $N \times 1$ -matrices. For $k \in J_{1,N}$, let e_k is the element of $\mathbb{R}^N$ such that its $(k, 1)$ -component is 1 and the other components is 0. That is $\{e_k \mid k \in J_{1,N}\}$ is the standard basis of $\mathbb{R}^N$. The $\mathbb{R}$ -algebra $M_N(\mathbb{R})$ acts on $\mathbb{R}^N$ in the ordinal way, that is $E_{k,r}e_p = \delta_{r,p}e_k$. Let $GL_N(\mathbb{R})$ be the group of invertible $N \times N$ -matrices, that is $GL_N(\mathbb{R}) = \{X \in M_N(\mathbb{R}) \mid \det X \neq 0\}$. Let $(,) : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ be the $\mathbb{R}$ -bilinear map defined by $(e_k, e_r) := \delta_{kr}$.

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Definition 1.1. For $v \in \mathbb{R}^N \setminus \{0\}$, define $s_v \in \text{GL}_N(\mathbb{R})$ by $s_v(u) := u - \frac{2(u,v)}{(v,v)}v$ ($u \in \mathbb{R}^N$), that is s_v is the reflection with respect to v .

Note that

$$(1.1) \quad s_v^2 = 1.$$

We say that a subset R of $\mathbb{R}^N \setminus \{0\}$ is a *root system* (in $\mathbb{R}^N$) if $|R| < \infty$, $s_v(R) = R$ and $\mathbb{R}v \cap R = \{v, -v\}$ for all $v \in R$, see [Hum, 1.1].

Let R be a root system in $\mathbb{R}^N$. We say that a subset Π of R is a *root basis* of R if Π is a (set) basis of $\text{Span}_{\mathbb{R}}(\Pi)$ as an $\mathbb{R}$ -linear space and $R \subset \text{Span}_{\mathbb{R}_{\geq 0}}(\Pi) \cup -\text{Span}_{\mathbb{R}_{\geq 0}}(\Pi)$ (this is called a *simple system* in [Hum, 1.3]).

Let R be a root system in $\mathbb{R}^N$. Let Π be a root basis of R . Let $R^+(\Pi) := R \cap \text{Span}_{\mathbb{R}_{\geq 0}}(\Pi)$. We call $R^+(\Pi)$ a *positive root system* of R associated with Π (this is called a *positive system* in [Hum, 1.3]).

Definition 1.2. (See [Hum, 2.10].) Let R be a root system in $\mathbb{R}^N$. Let Π be a root basis of R .

(1) Assume $N \geq 2$. We call R the A_{N-1} -type root system if

$$R = \{e_x - e_y \mid x, y \in J_{1,N}, x \neq y\}.$$

We call Π the A_{N-1} -type standard root basis if

$$\Pi = \{e_x - e_{x+1} \mid x \in J_{1,N-1}\}.$$

(2) Assume $N \geq 2$. We call R the B_N -type standard root system if

$$R = \{ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\}\} \cup \{c''e_z \mid c'' \in \{1, -1\}\}.$$

We call Π the B_N -type standard root basis if

$$\Pi = \{e_x - e_{x+1} \mid x \in J_{1,N-1}\} \cup \{e_N\}.$$

(3) Assume $N \geq 2$. We call R the C_N -type root system if

$$R = \{ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\}\} \cup \{2c''e_z \mid c'' \in \{1, -1\}\}.$$

We call Π the C_N -type standard root basis if

$$\Pi = \{e_x - e_{x+1} \mid x \in J_{1,N-1}\} \cup \{2e_N\}.$$

(4) Assume $N \geq 4$. We call R the D_N -type root system if

$$R = \{ ce_x + c'e_y \mid x, y \in J_{1,N}, x < y, c, c' \in \{1, -1\} \}.$$

We call Π the D_N -type standard root basis if

$$\Pi = \{ e_x - e_{x+1} \mid x \in J_{1,N-1} \} \cup \{ e_{N-1} + e_N \}.$$

Let R be a root system in $\mathbb{R}^N$. Let Π be a root basis of R . Let $W(\Pi)$ be the subgroup of $GL_N(\mathbb{R})$ generated by all s_v with $v \in \Pi$. We call $W(\Pi)$ the *Coxeter group associated with (R, Π)* . Let $S(\Pi) := \{ s_v \in W(\Pi) \mid v \in \Pi \}$. We call $(W(\Pi), S(\Pi))$ the *Coxeter system associated with (R, Π)* , see [Hum, 1.9 and Theorem 1.5]. Define the map $\ell : W(\Pi) \rightarrow \mathbb{Z}_{\geq 0}$ in the following way, see [Hum, 1.6]. Let $\ell(1) := 0$, where 1 is a unit of $W(\Pi)$. Note that an arbitrary $w \in W(\Pi)$ can be written as a product of finite s_v 's with some $v \in \Pi$, say $w = \underbrace{s_{v_1} \cdots s_{v_r}}_r$ for some $r \in \mathbb{N}$ and some $v_x \in \Pi$ ($x \in J_{1,r}$). If

$w \neq 1$, let $\ell(w)$ be the smallest r for which such an expression exists, and call the expression *reduced*. For $w \in W(\Pi)$, we call $\ell(w)$ the *length of w* . Let

$$\mathfrak{L}(w) := \{ v \in R^+(\Pi) \mid w(v) \in -R^+(\Pi) \}.$$

It is well-known that

$$(1.2) \quad \ell(w) = |\mathfrak{L}(w)|$$

(see [Hum, Corollary 1.7]). It is also well-known that for $v \in \Pi$,

$$(1.3) \quad s_v(R^+(\Pi) \setminus \{v\}) = R^+(\Pi) \setminus \{v\}$$

(see [Hum, Proposition 1.4]), and

$$(1.4) \quad \ell(ws_v) = \begin{cases} \ell(w) + 1 & \text{if } w(v) \in R^+(\Pi), \\ \ell(w) - 1 & \text{if } w(v) \in -R^+(\Pi) \end{cases}$$

(see [Hum, Lemma 1.6 and Corollary 1.7]). Assume that $|R| < \infty$. By the above properties, we can see that there exists a unique $w_\circ \in W(\Pi)$ such that $w_\circ(\Pi) = -\Pi$, see [Hum, 1.8]. It is well-known that

$$(1.5) \quad \ell(w_\circ) = |R^+(\Pi)|,$$

which can easily be proved by (1.2), (1.3) and (1.4). Note that w_o is the only element $W(\Pi)$ that $\ell(w) \leq \ell(w_o)$ for all $w \in W(\Pi)$, and $\ell(w) = \ell(w_o) - \ell(w_o w^{-1})$ for all $w \in W(\Pi)$. We call w_o the longest element of the Coxeter system of $(W(\Pi), S(\Pi))$.

Let $k, r \in J_{1,N}$ be such that $k \leq r$. For $z_p \in J_{k,r} \cup (-J_{k,r})$ ($p \in J_{k,r}$) with $|u_p| \neq |u_t|$ ($p \neq t$), let

$$\left\{ \begin{array}{cccc} k & k+1 & \dots & r \\ z_k & z_{k+1} & \dots & z_r \end{array} \right\} := \sum_{p \in J_{k,r}} \frac{z_p}{|z_p|} E_{|z_p|,p} + \sum_{t \in J_{1,N} \setminus J_{k,r}} E_{t,t} \in \text{GL}_N(\mathbb{R}).$$

We have

$$(1.6) \quad s_{e_k} = \left\{ \begin{array}{c} k \\ -k \end{array} \right\} \quad (k \in J_{1,N}),$$

$$(1.7) \quad s_{e_k - e_{k+1}} = \left\{ \begin{array}{cc} k & k+1 \\ k+1 & k \end{array} \right\} \quad (k \in J_{1,N-1}),$$

and

$$(1.8) \quad s_{e_k + e_{k+1}} = \left\{ \begin{array}{cc} k & k+1 \\ -(k+1) & -k \end{array} \right\} \quad (k \in J_{1,N-1}).$$

Let $k, p, r \in J_{k,r}$ with $k < r$ and $k \leq p \leq r$, let

$$\left\{ \begin{array}{ccc} k & \dots & p \\ z_k & \dots & z_p \end{array} ; \begin{array}{ccc} p+1 & \dots & r \\ z_{p+1} & \dots & z_r \end{array} \right\} := \left\{ \begin{array}{ccc} k & \dots & p \\ z_k & \dots & z_p \end{array} \right\} \left\{ \begin{array}{ccc} p+1 & \dots & r \\ z_{p+1} & \dots & z_r \end{array} \right\}.$$

Let $k, r \in J_{1,N-1}$ with $k \leq r$. Define $s_{(k,r)}$ inductively by

$$(1.9) \quad s_{(k,r)} := \begin{cases} 1 & \text{if } k = r \\ s_{(k,r-1)} s_{e_{r-1} - e_r} & \text{if } k < r. \end{cases}$$

Then, if $r > k$, we have

$$(1.10) \quad s_{(k,r)} = \left\{ \begin{array}{cccccc} k & \dots & p & \dots & r-1 & ; & r \\ k+1 & \dots & p+1 & \dots & r & ; & k \end{array} \right\},$$

since (if $r \geq k+2$)

$$\begin{aligned} s_{(k,r)} &= s_{(k,r-1)} s_{e_{r-1} - e_r} \\ (1.11) \quad &= \left\{ \begin{array}{cccccc} k & \dots & p & \dots & r-2 & ; & r-1 \\ k+1 & \dots & p+1 & \dots & r-1 & ; & k \end{array} \right\} \left\{ \begin{array}{cc} r-1 & r \\ r & r-1 \end{array} \right\} \\ &\quad \text{(by (1.7) and an induction)} \\ &= \left\{ \begin{array}{cccccc} k & \dots & p & \dots & r-1 & ; & r \\ k+1 & \dots & p+1 & \dots & r & ; & k \end{array} \right\}. \end{aligned}$$

Define $s_{(r,k)}$ inductively by $s_{(r,k)} := s_{e_{r-1}-e_r} s_{(r-1,k)}$ if $r \geq k+1$. Clearly (if $r > k$) we have

$$(1.12) \quad s_{(r,k)} = s_{(k,r)}^{-1} = \begin{Bmatrix} k & ; & k+1 & \dots & p & \dots & r \\ r & ; & k & \dots & p-1 & \dots & r-1 \end{Bmatrix}.$$

Lemma 1.3. *Let Π be the A_{N-1} -type standard root basis. Let $w_\circ$ be the longest element of $(W(\Pi), S(\Pi))$. Let $s_k := s_{e_k - e_{k+1}} \in S(\Pi)$ for $k \in J_{1,N-1}$.*

(1) *We have*

$$(1.13) \quad w_\circ = \begin{Bmatrix} 1 & \dots & p & \dots & N \\ N & \dots & N-p+1 & \dots & 1 \end{Bmatrix}.$$

Moreover

$$(1.14) \quad w_\circ = \underbrace{(s_1 s_2 \cdots s_{N-1})}_{N-1} \underbrace{(s_1 s_2 \cdots s_{N-2})}_{N-2} \cdots \underbrace{(s_1 s_2)}_2 \underbrace{s_1}_1.$$

Furthermore RHS of (1.14) is the reduced expression of $w_\circ$.

(2) *Let $m \in J_{2,N-1}$. Then*

$$(1.15) \quad \begin{aligned} w_\circ = & \underbrace{(s_1 s_2 \cdots s_{m-1})}_{m-1} \underbrace{(s_1 s_2 \cdots s_{m-2})}_{m-2} \cdots \underbrace{(s_1 s_2)}_2 \underbrace{s_1}_1 \\ & \cdot \underbrace{(s_{m+1} s_{m+2} \cdots s_{N-1})}_{N-m-1} \underbrace{(s_{m+1} s_{m+2} \cdots s_{N-1})}_{N-m-2} \cdots \underbrace{(s_{m+1} s_{m+2})}_2 \underbrace{s_{m+1}}_1 \\ & \cdot \underbrace{(s_m s_{m+1} \cdots s_{N-1})}_{N-m} \underbrace{(s_{m-1} s_m \cdots s_{N-2})}_{N-m} \cdots \underbrace{(s_1 s_2 \cdots s_{N-m})}_{N-m}, \end{aligned}$$

and RHS of (1.15) is a reduced expression of $w_\circ$.

Proof. By (1.5), we have

$$(1.16) \quad \ell(w) = \frac{N(N-1)}{2}.$$

Let $k, r \in J_{1,n}$ with $k < r$. Let

$$x_{(k,r)} := \begin{Bmatrix} k & \dots & p & \dots & r \\ r & \dots & r-p+k & \dots & k \end{Bmatrix}.$$

Then

$$(1.17) \quad s_{(k,r)} s_{(k,r-1)} \cdots s_{(k,k+1)} = x_{(k,r)},$$

since, if $r \geq k+2$, we have

$$\begin{aligned} & s_{(k,r)} (s_{(k,r-1)} \cdots s_{(k,k+1)}) \\ &= \left\{ \begin{array}{cccccc} k & \cdots & p & \cdots & r-1 & ; & r \\ k+1 & \cdots & p+1 & \cdots & r & ; & k \end{array} \right\} \cdot x_{(k,r-1)} \\ & \quad \text{(by (1.11) and an induction)} \\ &= x_{(k,r)}. \end{aligned}$$

We have

$$(1.18) \quad x_{(k,r)} \in W(\Pi) \quad \text{and} \quad \ell(x_{(k,r)}) = \frac{(k-r+1)(k-r)}{2},$$

where the first claim follows from (1.17) and the second claim follows from by (1.2), since $\mathfrak{L}(x_{(k,r)}) = \{e_x - e_y | k \leq x < y \leq r\}$.

We obtain the claim (1) from (1.16). (1.17) and (1.18) for $k=1$ and $r=N$.

For $k, r, t \in J_{1,N-1}$ with $k < r \leq t$, let

$$(1.19) \quad \begin{aligned} & y_{(k,r-1;r,t)} \\ &:= \left\{ \begin{array}{cccccc} k & \cdots & x & \cdots & r-1 & ; & r & \cdots & y & \cdots & t \\ k+t-r+1 & \cdots & x+t-r+1 & \cdots & t & ; & k & \cdots & y+k-r & \cdots & t+k-r \end{array} \right\} \end{aligned}$$

We have

$$(1.20) \quad s_{(k+t-r,t)} s_{(k+t-r-1,t-1)} \cdots s_{(k+1,r+1)} s_{(k,r)} = y_{(k,r-1;r,t)}$$

since, if $t > r$,

$$\begin{aligned} & (s_{(k+t-r,t)} s_{(k+t-r-1,t-1)} \cdots s_{(k+1,r+1)}) s_{(k,r)} \\ &= y_{(k+1,r;r+1,t)} \cdot \left\{ \begin{array}{cccccc} k & \cdots & p & \cdots & r-1 & ; & r \\ k+1 & \cdots & p+1 & \cdots & r & ; & k \end{array} \right\} \\ & \quad \text{(by (1.11) and an induction)} \\ &= y_{(k,r-1;r,t)}. \end{aligned}$$

We have

$$(1.21) \quad y_{(k,r-1;r,t)} \in W(\Pi) \quad \text{and} \quad \ell(y_{(k,r-1;r,t)}) = (t-r+1)(r-k),$$

where the first claim follows from (1.20) and the second claim follows from by (1.2), since $\mathfrak{L}(x_{(k,r)}) = \{e_x - e_y | x \in J_{k,r-1}, x \in J_{r,t}\}$.

Let $m \in J_{2,N-1}$. By (1.13), we have

$$(1.22) \quad w_o = x_{(1,m)} x_{(m+1,N)} y_{(1,N-m;N-m+1,N)}.$$

Then we obtain the claim (2) from (1.16), (1.18), (1.21) and (1.22), since $\frac{m(m-1)}{2} + \frac{(N-m)(N-m-1)}{2} + (N-m)m = \frac{N(N-1)}{2}$. $\square$

Let $k, r \in J_{1,N}$ with $k \leq r$. Let

$$(1.23) \quad b_{(k,r)} := \underbrace{s_{e_k} \cdots s_{e_r}}_{r-k+1} = \left\{ \begin{array}{ccccc} k & \cdots & p & \cdots & r \\ -k & \cdots & -p & \cdots & -r \end{array} \right\},$$

see also (1.6). By (1.10), we have

$$(1.24) \quad (s_{(k,r)})^{r-k+1} = 1.$$

By (1.6) and (1.10), we have

$$(1.25) \quad s_{e_t} s_{(k,r)} = s_{(k,r)} s_{e_{t-1}}$$

By (1.23), (1.24) and (1.25), for $t \in J_{k+1,r}$, we have

$$(1.26) \quad (s_{(k,r)} s_{e_r})^{r-k+1} = (s_{(k,r)})^{r-k+1} s_{e_k} \cdots s_{e_r} = b_{(k,r)}.$$

By (1.6), (1.10) and (1.12), we have

$$(1.27) \quad \underbrace{s_{e_{k-e_{k+1}}} \cdots s_{e_{r-1-e_r}}}_{k-r} s_{e_r} \underbrace{s_{e_{r-1-e_r}} \cdots s_{e_{k-e_{k+1}}}}_{k-r} = s_{(k,r)} s_{e_r} s_{(r,k)} = s_{e_k}.$$

Lemma 1.4. *Let Π be the B_N -type standard root basis. Let w_o be the longest element of $(W(\Pi), S(\Pi))$. Let $s_k := s_{e_k - e_{k+1}} \in S(\Pi)$ for $k \in J_{1,N-1}$ and let $s_N := s_{e_N} \in S(\Pi)$.*

(1) *We have*

$$(1.28) \quad w_o = b_{(1,N)} = \underbrace{(s_1 s_2 \cdots s_N)}_N.$$

Moreover the rightmost hand side of (1.28) is a reduced expression of $w_\circ$.

(2) Let $k, r \in J_{1,N}$ with $k \leq r$. Then

$$(1.29) \quad b_{(k,r)} = \underbrace{(s_k s_{k+1} \cdots s_{N-1} s_N s_{N-1} \cdots s_{r+1} s_r)}_{2N-k-r+1}^{r-k+1}.$$

Moreover RHS of (1.29) is a reduced expression of $b_{(k,r)}$.

(3) Let $k_1, k_2, \dots, k_{r-1} \in J_{1,N}$ with $k_1 < k_2 < \dots < k_{r-1}$. Let $b'_y := b_{(k_{y-1}, k_y-1)}$ ($y \in J_{1,r}$), where let $k_0 := 1$ and $k_r := N+1$. Then we have $w_\circ = b'_1 b'_2 \cdots b'_r$ and $\ell(w_\circ) = \sum_{y=1}^r \ell(b'_y)$. Moreover $b'_y b'_z = b'_z b'_y$ for $y, z \in J_{1,r}$.

(4) Let $m \in J_{1,N-1}$. Then

$$(1.30) \quad w_\circ = \underbrace{(s_{N-m+1} s_{N-m+2} \cdots s_N)}_m^m \cdot \underbrace{(s_1 s_2 \cdots s_{N-1} s_N s_{N-1} \cdots s_{N-m+1} s_{N-m})}_{N+m}^{N-m}.$$

Moreover RHS of (1.30) is a reduced expression of $w_\circ$.

Proof. We can easily show (1.29) by (1.26) and (1.27).

Let $k, r \in J_{1,N}$ be such that $k \leq r$. Note that

$$\mathfrak{L}(b_{(k,r)}) = \{e_t \mid t \in J_{k,r}\} \cup \{e_t + ce_{t'} \mid c \in \{-1, 1\}, t \in J_{k,r}, t' \in J_{t',N}\}.$$

Hence by (1.2), we have

$$(1.31) \quad \begin{aligned} \ell(b_{(k,r)}) &= (r - k + 1) + 2 \sum_{t=k}^r (N - t) \\ &= (r - k + 1) + 2N(r - k + 1) - 2\left(\frac{r(r+1)}{2} - \frac{k(k-1)}{2}\right) \\ &= (r - k + 1)(1 + 2N - (r + k)) \\ &= (2N - k - r + 1)(r - k + 1). \end{aligned}$$

Hence we obtain the second claim of the claim (2). We also obtain the claim (1) since $|R^+(\Pi)| = N^2$.

Let $k, t, r \in J_{1,N}$ be such that $k \leq t < r$. By (1.23), we have

$$(1.32) \quad b_{(k,t)} b_{(t+1,r)} = b_{(k,r)}.$$

By (1.31), we have

$$\begin{aligned}
 & \ell(b_{(k,t)}) + \ell(b_{(t+1,r)}) \\
 &= (2N - k - t + 1)(t - k + 1) + (2N - t - r)(r - t) \\
 &= 2N(r - k + 1) - (k + t - 1)(t - k + 1) - (t + r)(r - t) \\
 (1.33) \quad &= 2N(r - k + 1) - (-k^2 + t^2 + 2k - 1) - (r^2 - t^2) \\
 &= 2N(r - k + 1) + (k^2 - r^2 - 2k + 1) \\
 &= 2N(r - k + 1) + (k - 1 + r)(k - 1 - r) \\
 &= (2N - r - k - 1)(r - k + 1) \\
 &= \ell(b_{(k,r)}).
 \end{aligned}$$

By (1.32), (1.32) and the claim (1), we get the claim (3).

The claim (4) follows immediately from the claims (1) and (2). $\square$

Using Lemma 1.4, we have

Lemma 1.5. *Let Π be the D_N -type standard root basis. Let $w_\circ$ be the longest element of $(W(\Pi), S(\Pi))$. Let $s_k := s_{e_k - e_{k+1}} \in S(\Pi)$ for $k \in J_{1,N-1}$ and let $s_N := s_{e_k + e_{k+1}} \in S(\Pi)$. For $k \in J_{1,N-1}$, let*

$$(1.34) \quad d_{(k)} := \underbrace{(s_k \cdots s_{N-2} s_{N-1} s_N)}_{N-k+1}^{N-k}.$$

Then

$$(1.35) \quad \ell(d_{(k)}) = (N - k)(N - k + 1)$$

and

$$(1.36) \quad d_{(k)} = \begin{cases} b_{(k,N)} & \text{if } N - k \text{ is odd,} \\ b_{(k,N-1)} & \text{if } N - k \text{ is even.} \end{cases}$$

In particular,

$$(1.37) \quad w_\circ = d_{(1)}.$$

Proof. By (1.6), (1.7) and (1.8), we have

$$(1.38) \quad s_{N-1}s_N = \begin{Bmatrix} N-1 & N \\ -(N-1) & -N \end{Bmatrix} = s_{e_{N-1}}s_{e_N}.$$

Then we have

$$(1.39) \quad \begin{aligned} & \text{RHS of (1.34)} \\ &= (s_{(k,N-1)}s_{e_{N-1}}s_{e_N})^{N-k} \quad (\text{by (1.38)}) \\ &= (s_{(k,N-1)}s_{e_{N-1}})^{N-k}s_{e_N}^{N-k} \quad (\text{by (1.6) and (1.10)}) \\ &= b_{(k,N-1)}s_{e_N}^{N-k} \quad (\text{by (1.26)}) \\ &= \text{RHS of (1.36)} \end{aligned}$$

By (1.36), we have

$$\mathfrak{L}(d_{(k)}) = \{e_t + ce_{t'} \mid c \in \{-1, 1\}, t \in J_{k,r}, t' \in J_{t',N}\}.$$

Hence by (1.2), we have (1.35) and (1.37). This completes the proof. $\square$

2 Weyl groupoids of super CD -type

Let $m \in J_{1,N-1}$. Let $\mathcal{D}_{m|N-m}$ be the set of maps $a : J_{1,n} \rightarrow J_{0,1}$ with $|a^{-1}(\{0\})| = m$.

Let $a \in \mathcal{D}_{m|N-m}$. Let $(\cdot, \cdot)^a : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ be the $\mathbb{R}$ -bilinear map defined by $(e_i, e_j)^a := \delta_{ij} \cdot (-1)^{a(i)}$. For $v \in \mathbb{R}^N$ with $(v, v)^a \neq 0$, define $s_v \in \text{GL}_N(\mathbb{R})$ by $s_v^a(u) := u - \frac{2(u,v)^a}{(v,v)^a}v$ ($u \in \mathbb{R}^N$),

Let

$$\dot{\mathcal{D}}_{m|N-m} := \{(a, d) \in \mathcal{D}_{m|N-m} \times J_{0,1} \mid d \in J_{0,a(N)}\}.$$

For $i \in J_{1,N}$, define the bijection $\tau_i : \dot{\mathcal{D}}_{m|N-m} \rightarrow \dot{\mathcal{D}}_{m|N-m}$ by

$$\tau_i(a, d) := \begin{cases} (a \circ s_{e_i - e_{i+1}}, d) & \text{if } i \in J_{1,N-2} \text{ and } a(i) \neq a(i+1), \\ (a \circ s_{e_{N-1} - e_N}, d) & \text{if } i \in N-1, d=0 \text{ and } a(N-1) \neq b(N), \\ (a \circ s_{e_{N-1} - e_N}, 1) & \text{if } i = N, a(N-1) = 1, a(N) = 0, \\ (a \circ s_{e_{N-1} - e_N}, 0) & \text{if } i = N, a(N-1) = 0, a(N) = 1 \text{ and } d = 1, \\ (a, d) & \text{otherwise.} \end{cases}$$

Then $\tau_i^2 = \text{id}_{\mathbb{R}^N}$.

Let $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$. Let

$$R_+^{(a,d)} := \{e_x + te_y \mid x, y \in J_{1,N}, x < y, t \in \{1, -1\}\} \\ \cup \{2e_z \mid z \in J_{1,N}, a(z) = 1\},$$

and $R^{(a,d)} := R_+^{(a,d)} \cup -R_+^{(a,d)}$. Then

$$(2.1) \quad |R_+^{(a,d)}| = N(N-1) + (N-m) = N^2 - m.$$

For $i \in J_{1,N}$, let

$$\alpha_i^{(a,d)} := \begin{cases} e_i - e_{i+1} & \text{if } i \in J_{1,N-2}, \\ e_{N-1} - e_N & \text{if } i = N-1 \text{ and } d = 0, \\ 2e_N & \text{if } i = N-1 \text{ and } d = 1, \\ e_{N-1} + e_N & \text{if } i = N, a(N) = 0 \text{ and } d = 0, \\ 2e_N & \text{if } i = N, a(N) = 1 \text{ and } d = 0, \\ e_{N-1} - e_N & \text{if } i = N, d = 1. \end{cases}$$

Let $\Pi^{(a,d)} := \{\alpha_i^{(a,d)} \mid i \in J_{1,N}\}$. Then $\Pi^{(a,d)}$ is an $\mathbb{R}$ -basis of $\mathbb{R}^N$. Moreover

$$\Pi^{(a,d)} \subset R_+^{(a,d)} \subset \left(\bigoplus_{i=1}^N \mathbb{Z}_{\geq 0} \alpha_i^{(a,d)} \right) \setminus \{0\}.$$

Note that

$$\tau_i(a, d) = (a, d) \quad \text{if and only if} \quad (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a \neq 0.$$

For $i \in J_{1,N}$, define $s_i^{(a,d)} \in \text{GL}_N(\mathbb{R})$ by

$$s_i^{(a,d)}(\alpha_i^{(a,d)}) := \begin{cases} -\alpha_i^{\tau_i(a,d)} & \text{if } i = j, \\ s_{\alpha_i^{\tau_i(a,d)}}^a(\alpha_j^{\tau_i(a,d)}) & \text{if } i \neq j \text{ and } (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a \neq 0, \\ \alpha_j^{\tau_i(a,d)} & \text{if } i \neq j \text{ and } (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a = (\alpha_i^{(a,d)}, \alpha_j^{(a,d)})^a = 0, \\ \alpha_j^{\tau_i(a,d)} + \alpha_i^{\tau_i(a,d)} & \text{if } i \neq j, (\alpha_i^{(a,d)}, \alpha_i^{(a,d)})^a = 0 \text{ and } (\alpha_i^{(a,d)}, \alpha_j^{(a,d)})^a \neq 0. \end{cases}$$

We can directly see

Lemma 2.1. *Let $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$, and $i \in J_{1,N}$. Assume that $d = 0$. Assume that $i \in J_{1,N-1}$ if $a(N-1) = 1$ and $a(N) = 0$. Then $s_i^{(a,d)} = s_{\alpha_i^{(a,d)}}$, where $s_{\alpha_i^{(a,d)}}$ is the one of Definition 1.1.*

Notation. Let $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$. Let Map_0^N be a set with $|\text{Map}_0^N| = 1$. For $r \in \mathbb{N}$, let Map_r^N be the set of all maps from $J_{1,r}$ to $J_{1,N}$. Let Map_∞^N be the set of all maps from $\mathbb{N}$ to $J_{1,N}$. For $r \in \mathbb{Z}_{\geq 0}$, $f \in \text{Map}_r^N \cup \text{Map}_\infty^N$ and $t \in J_{1,r}$, let

$$\begin{aligned} (a, d)_{f,0} &:= (a, d), \quad 1^{(a,d)} s_{f,0} := \text{id}_{\mathbb{R}^N} \\ (a, d)_{f,t} &:= \tau_i((a, d)_{f,t-1}), \quad 1^{(a,d)} s_{f,t} := 1^{(a,d)} s_{f,t-1} s_{f(t)}^{(a,d)_{f,t}}. \end{aligned}$$

Proposition 2.2. *Let $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$ be such that $d = 0$, $b(z) = 1$ ($z \in J_{1,N-m}$) and $b(z') = 0$ ($z' \in J_{N-m+1,N}$). Let $n := |R_+^{(a,d)}|$. Define $f \in \text{Map}_n^N$ by*

$$(2.2) \quad f(t) := \begin{cases} N - m + t & (\text{if } t \in J_{1,m}), \\ f(t - m) & (\text{if } t \in J_{m+1,m(m-1)}), \\ t - m(m-1) & (\text{if } t \in J_{m(m-1)+1,m(m-1)+N}), \\ 2N + m(m-1) - t & (\text{if } t \in J_{m(m-1)+N+1,m^2+N}), \\ f(t - (N + m)) & (\text{if } t \in J_{m^2+N+1,n}). \end{cases}$$

Then

$$(2.3) \quad 1^{(a,d)} s_{f,n} = \begin{cases} b_{(1,N)} & \text{if } m \text{ is odd,} \\ b_{(1,N-1)} & \text{if } m \text{ is even.} \end{cases}$$

Proof. For $y \in J_{1,m}$, define $a^{(y)} \in \mathcal{D}_{m|N-m}$ by

$$a^{(y)}(z) := \begin{cases} 1 & \text{if } z \in J_{1,N-m-1} \cup \{N-m+y\}, \\ 0 & \text{if } z \in J_{N-m,N-m+y-1} \cup J_{N-m+y+1,N}. \end{cases}$$

Then we can directly see that for $t \in J_{1,n}$,

$$(a, d)_{f,t} = \begin{cases} (a, d) & \text{if } t \in J_{1,m(m-1)+N-m-1}, \\ (a^{(t-(N-m-1))}, 0) & \text{if } t \in J_{m(m-1)+N-m,m(m-1)+N-1}, \\ (a^{(m-(t-(m(m-1)+N)))}, 0) & \text{if } t \in J_{m(m-1)+N,m(m-1)+N+m}, \\ (a, d)_{f,t-(N+m)} & \text{if } t \in J_{m^2+N+1,n}. \end{cases}$$

So we see that for $t \in J_{1,n}$,

$$(2.4) \quad s_{f(t)}^{(a,d)_{f,t}} = \begin{cases} s_{e_{f(t)}-e_{f(t)+1}} & \text{if } f(t) \in J_{1,N-1}, \\ s_{e_{N-1}+e_N} & \text{if } t \in J_{1,m(m-1)} \text{ and } f(t) = N, \\ s_{2e_N} (= s_{e_N}) & \text{if } t \in J_{m(m-1)+1,n} \text{ and } f(t) = N. \end{cases}$$

Define $f' \in \text{Map}_{n-m(m-1)}^N$ by $f'(t) := f(t + m(m-1))$, so

$$(2.5) \quad 1^{(a,d)} s_{f,n} = 1^{(a,d)} s_{f,m(m-1)} 1_{f',n-m(m-1)}^{(a,d)}.$$

By (1.29) and (1.36), $1^{(a,d)} s_{f,m(m-1)}$ equals $b_{(N-m+1,N)}$ (resp. $b_{(N-m+1,N-1)}$) if m is odd (resp. even). By (1.29) and (2.4), $1_{f',n-m(m-1)}^{(a,d)} = b_{(1,N-m)}$. Hence by (1.22) and (2.5), we have (2.3), as desired. $\square$

For $(a, d) \in \mathcal{D}_{m|N-m}$ and $i, j \in J_{1,N}$, define $C^{(a,d)} = [c_{ij}^{(a,d)}]_{i,j \in J_{1,N}} \in M_N(\mathbb{Z})$ by

$$s_i^{(a,d)}(\alpha_j^{(a,d)}) = \alpha_j^{\tau_i(a,d)} - c_{ij}^{(a,d)} \alpha_i^{\tau_i(a,d)}.$$

Then $C^{(a,d)}$ is a *generalized Cartan matrix*, i.e., (M1) and (M2) below hold.

- (M1) $c_{ii}^{(a,d)} = 2$ ($i \in J_{1,N}$).
(M2) $c_{jk}^{(a,d)} \leq 0$, $\delta_{c_{jk}^{(a,d)},0} = \delta_{c_{kj}^{(a,d)},0}$ ($j, k \in J_{1,N}$, $j \neq k$).

Then the data

$$\dot{\mathcal{C}}_{m|N-m} := \mathcal{C}(J_{1,N}, \dot{\mathcal{D}}_{m|N-m}, (\tau_i)_{i \in J_{1,N}}, (C^{(a,d)})_{(a,d) \in \dot{\mathcal{D}}_{m|N-m}})$$

a (*rank- N*) *Cartan scheme*, i.e., (C1) and (C2) below hold.

- (C1) $\tau_i^2 = \text{id}_{\dot{\mathcal{D}}_{m|N-m}}$ ($i \in J_{1,N}$).
(C2) $c_{ij}^{\tau_i((a,d))} = c_{ij}^{(a,d)}$ ($i \in J_{1,N}$).

Note that

$$-c_{ij}^{(a,d)} = |R_+^{(a,d)} \cap (\mathbb{Z}\alpha_i^{(a,d)} \oplus \mathbb{Z}\alpha_j^{(a,d)})| \quad (i, j \in J_{1,N}, i \neq j).$$

The data

$$\dot{\mathcal{R}}_{m|N-m} := \mathcal{R}(\dot{\mathcal{C}}_{m|N-m}, (R_+^{(a,d)})_{(a,d) \in \dot{\mathcal{D}}_{m|N-m}}).$$

is a *generalized root system of type $\mathcal{C}$* , i.e., (R1)-(R4) below hold.

- (R1) $R^{(a,d)} = R_+^{(a,d)} \cup -R_+^{(a,d)}$ ($(a,d) \in \dot{\mathcal{D}}_{m|N-m}$).
(R2) $R^{(a,d)} \cap \mathbb{Z}\alpha_i = \{\alpha_i, -\alpha_i\}$ ($(a,d) \in \dot{\mathcal{D}}_{m|N-m}$, $i \in J_{1,N}$).
(R3) $s_i^{(a,d)}(R^{(a,d)}) = R^{\tau_i(a,d)}$ ($(a,d) \in \dot{\mathcal{D}}_{m|N-m}$, $i \in J_{1,N}$).
(R4) $(\tau_i \tau_j)^{-c_{ij}^{(a,d)}}(a,d) = (a,d)$ ($(a,d) \in \dot{\mathcal{D}}_{m|N-m}$, $i, j \in J_{1,N}$).

For $(a,d) \in \dot{\mathcal{D}}_{m|N-m}$, let

$$W^{(a,d)} := \{1^{(a,d)} s_{f,r} \in \text{GL}_N(\mathbb{R}) \mid r \in \mathbb{Z}_{\geq 0}, f \in \text{Map}_r^N\},$$

and define the map $\ell^{(a,d)} : W^{(a,d)} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\ell^{(a,d)}(w) := \min\{r \in \mathbb{Z}_{\geq 0} \mid \exists f \in \text{Map}_r^N, w = 1^{(a,d)} s_{f,r}\}.$$

By [HY08, Lemma 8 (iii)], we see that

$$(2.6) \quad 1^{(a,d)} s_{f,r} = 1^{(a,d)} s_{f',r'} \text{ implies } (a,d)_{f,r} = (a,d)_{f',r'},$$

and that

$$(2.7) \quad \ell^{(a,d)}(w) = |w^{-1}(R_+^{(a,d)}) \cap -\oplus_{i=1}^N \mathbb{Z}_{\geq 0}\alpha_i|.$$

For $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$, $w \in W^{(a,d)}$ and $f \in \text{Map}_{\ell^{(a,d)}(w)}^N$, if $w = 1^{(a,d)} s_{f, \ell^{(a,d)}(w)}$, we call f a *reduced word map* of w .

By (2.6) and (2.7), we have formulas for $W^{(a,d)}$ similar to (1.3) and (1.4). In particular, for each $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$, there exists a unique $w_{\circ}^{(a,d)} \in W^{(a,d)}$ such that

$$\ell^{(a,d)}(w_{\circ}^{(a,d)}) = |R_+^{(a,d)}|,$$

and we call $w_{\circ}^{(a,d)}$ the *longest element* of $W^{(a,d)}$.

By Proposition 2.2, we have

Theorem 2.3. *Let $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$ be such that $d = 0$, $a(z) = 1$ ($z \in J_{1,N-m}$) and $a(z') = 0$ ($z' \in J_{N-m+1,N}$). Then a reduced word map of $w_{\circ}^{(a,d)}$ is given by (2.2). Moreover,*

$$(2.8) \quad w_{\circ}^{(a,d)} = \begin{cases} b_{(1,N)} & \text{if } m \text{ is odd,} \\ b_{(1,N-1)} & \text{if } m \text{ is even.} \end{cases}$$

Definition 2.4. For $(a, d), (a', d') \in \dot{\mathcal{D}}_{m|N-m}$, let $W_{(a',d')}^{(a,d)}$ be the subset of $W^{(a,d)}$ composed of all the elements $1^{(a,d)} s_{f,r}$ with $r \in \mathbb{Z}_{\geq 0}$, $f \in \text{Map}_r^N$ and $(a, d)_{f,r} = (a', d')$, and $\mathcal{H}_{(a',d')}^{(a,d)} := \{(a, d)\} \times W_{(a',d')}^{(a,d)} \times \{(a', d')\} \subset \dot{\mathcal{D}}_{m|N-m} \times \text{GL}_N(\mathbb{R}) \times \dot{\mathcal{D}}_{m|N-m}$. Let

$$(\dot{\mathcal{W}}_{m|N-m})' := \bigcup_{(a,d),(a',d') \in \dot{\mathcal{D}}_{m|N-m}} \mathcal{H}_{(a',d')}^{(a,d)},$$

and $\dot{\mathcal{W}}_{m|N-m} := (\dot{\mathcal{W}}_{m|N-m})' \cup \{o\}$, where o is an element such that $o \notin (\dot{\mathcal{W}}_{m|N-m})'$. We regard $\dot{\mathcal{W}}_{m|N-m}$ as the semigroup by $\omega\omega := \omega o := o$ ($\omega \in \dot{\mathcal{W}}_{m|N-m}$) and

$$\begin{aligned} & ((a_1, d_1), w_1, (a_2, d_2))((a_3, d_3), w_2, (a_4, d_4)) \\ & := \begin{cases} ((a_1, d_1), w_1 w_2, (a_4, d_4)) & \text{if } (a_2, d_2) = (a_3, d_3), \\ o & \text{if } (a_2, d_2) \neq (a_3, d_3). \end{cases} \end{aligned}$$

We call $\dot{\mathcal{W}}_{m|N-m}$ the *Weyl groupoid* of the Lie superalgebra $\text{osp}(2m|2(N-m))$.

For $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$, let $\varepsilon^{(a,d)} := ((a, d), \text{id}_{\mathbb{R}^N}, (a, d)) \in \mathcal{H}_{(a,d)}^{(a,d)}$. For $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$ and $i \in J_{1,N}$, let $\sigma_i^{(a,d)} := (\tau_i(a, d), s_i^{(a,d)}, (a, d)) \in \mathcal{H}_{\tau_i(a,d)}^{(a,d)}$. For $r \in \mathbb{Z}_{\geq 0}$, $t \in J_{0,r}$ and $f \in \text{Map}_r^N$, let $1^{(a,d)}\sigma_{f,r} := ((a, d), 1^{(a,d)}s_{f,r}, (a, d)_{f,r}) \in \mathcal{H}_{(a,d)_{f,r}}^{(a,d)}$. For $i, j \in J_{1,N}$, define $f_{ij} \in \text{Map}_\infty^N$ by $f_{ij}(2t-1) := i$, $f_{ij}(2t) := j$ ($t \in \mathbb{N}$).

By [HY08, Theorem 1], we have

Theorem 2.5. *The semigroup $\dot{\mathcal{W}}_{m|N-m}$ can also be defined by the generators*

$$o, \varepsilon^{(a,d)}, \sigma_i^{(a,d)} \quad ((a, d) \in \dot{\mathcal{D}}_{m|N-m}, i \in J_{1,N}),$$

and relations

$$\begin{aligned} o\omega &= \omega o = o \quad (\omega \in \dot{\mathcal{W}}_{m|N-m}), \\ \varepsilon^{(a,d)}\varepsilon^{(a,d)} &= \varepsilon^{(a,d)}, \quad \varepsilon^{(a,d)}\varepsilon^{(a',d')} = o \quad ((a, d) \neq (a', d')), \\ \varepsilon^{\tau_i(a,d)}\sigma_i^{(a,d)} &= \sigma_i^{(a,d)}\varepsilon^{(a,d)} = \sigma_i^{(a,d)}, \quad \sigma_i^{\tau_i(a,d)}\sigma_i^{(a,d)} = \varepsilon^{(a,d)}, \\ 1^{(a,d)}\sigma_{f_{ij}, -2c_{ij}^{(a,d)}} &= \varepsilon^{(a,d)} \quad (i \neq j). \end{aligned}$$

Let $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$, $r \in \mathbb{Z}_{\geq 0}$ and $f, f' \in \text{Map}_r^N$. We write $f \dot{\sim}_r^{(a,d)} f'$ if there exist $i, j \in J_{1,N}$ and $t \in J_{0,r}$ such that $i \neq j$, $t - c_{ij}^{(a,d)f,k} \leq r$, $f(k_1) = f'(k_1)$ ($k_1 \in J_{1,t} \cup J_{t-c_{ij}^{(a,d)f,k+1}, r}$), $f(k_2) = i$, $f'(k_2) = j$ ($k_2 \in J_{t+1, t-c_{ij}^{(a,d)f,k}} \cap 2\mathbb{N} - 1$) and $f(k_3) = j$, $f'(k_3) = i$ ($k_3 \in J_{t+1, t-c_{ij}^{(a,d)f,k}} \cap 2\mathbb{N}$). We write $f \sim_r^{(a,d)} f'$ if $f = f'$ or there exists $t \in \mathbb{N}$ and $f_k \in \text{Map}_r^N$ ($k \in J_{1,t}$) such that $f \dot{\sim}_r^{(a,d)} f_1$, $f_k \dot{\sim}_r^{(a,d)} f_{k+1}$ ($k \in J_{1,t-1}$) and $f_t \dot{\sim}_r^{(a,d)} f'$.

By [HY08, Theorem 5, Corollary 6], we have

Theorem 2.6. *Let $(a, d) \in \dot{\mathcal{D}}_{m|N-m}$ and $w \in W^{(a,d)}$.*

(1) *Let $f, f' \in \text{Map}_{\ell^{(a,d)}(w)}^N$ be such that $1^{(a,d)}s_{f, \ell^{(a,d)}(w)} = 1^{(a,d)}s_{f', \ell^{(a,d)}(w)} = w$. Then $f \sim_{\ell^{(a,d)}(w)}^{(a,d)} f'$.*

(2) *Let $r \in \mathbb{N}$ and $f \in \text{Map}_r^N$ be such that $r > \ell^{(a,d)}(w)$ and $1^{(a,d)}s_{f,r} = w$. Then there exist $f' \in \text{Map}_r^N$ and $t \in J_{1,r-1}$ such that $f \sim_r^{(a,d)} f'$ and $f'(t) = f'(t+1)$.*

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