

Surfaces with big anti-canonical divisors and some related problems

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1 Background

- $\mathbf{k}$: field of characteristic $p \geq 0$
- X : smooth projective surface over $\mathbf{k}$ such that $-K_X$ is big

Theorem 1.1 (= [4, Theorem 1]).

If X is rational, X is a Mori dream space.

Question 1.2.

What if X is NOT rational?

$$\Rightarrow \kappa(X) = -\infty$$

$\Rightarrow X$ is obtained by repeatedly blowing up a geometrically ruled surface $\mathbb{P}_C(E) \rightarrow C$ with big anti-canonical line bundle.

— AIM —

- (1) Classify geometrically ruled surfaces with big anti-canonical bundle (**Theorem 3.1**).
- (2) Apply (1) to show that the Picard group of (weak) del Pezzo pairs (X, Δ) is finitely generated and torsion free (**Corollary 4.2**). This is a version of the base point free theorem.

2 Stability of vector bundles on curves

C : smooth projective curve

E : vector bundle on C

$F: C \rightarrow C$: the Frobenius map

Definition 2.1.

E is *strongly semi-stable* if the vector bundle $(F^e)^*E$ over C is semi-stable for all $e \geq 0$.

Remark 2.2.

$\text{char}(\mathbf{k}) = 0 \Rightarrow \text{semi-stable} = \text{strongly semi-stable}$.

We use the following standard facts.

Proposition 2.3.

- (1) E : strongly semi-stable $\Rightarrow$ the n -th symmetric power $S^n(E)$ is also strongly semi-stable.
- (2) If $g(C) \leq 1$, semi-stable = strongly semi-stable.

Proposition 2.4 (O-Okawa, [1]).

Suppose $g(C) \geq 1$, $\text{rank } E = 2$. If $-K_{\mathbb{P}_C(E)}$ is big, then E is not strongly semi-stable.

3 Geometrically ruled surface with big anti-canonical divisor

Theorem 3.1 (O-Okawa, [1], Main theorem I).

Suppose $g(C) \geq 1$, $\text{rank } E = 2$, and E : unstable. Then $-K_X$ is big $\iff$ there are line bundles L and M such that $E \simeq L \oplus M$ and $\deg L - \deg M > 2g - 2$.

Remark 3.2.

If $\text{char}(\mathbf{k}) = 0$ or $g(C) = 1$, we can omit the assumption that E is unstable by Proposition 2.3 and Proposition 2.4.

Corollary 3.3.

Suppose $g(C) \geq 1$ and $\text{rank } E = 2$. If $-K_{\mathbb{P}_C(E)}$ is big, there exists an integer $e \geq 0$ and line bundles L', M' such that $(F^e)^*E \simeq L' \oplus M'$ and $\deg L' - \deg M' > 2g - 2$.

The question below remains open.

Question 3.4.

Is E itself unstable under the same assumption?

4 Picard group of log del Pezzo surfaces via Theorem 3.1

Theorem 4.1 (O-Okawa, [1]).

Let (X, Δ) be a pair of a normal projective surface and an effective $\mathbb{R}$ -divisor such that $-(K_X + \Delta)$ is $\mathbb{R}$ -Cartier. If either

- $[\Delta] = 0$ and $-(K_X + \Delta)$ is nef and big, or

- $\Delta \leq 1$ and $-(K_X + \Delta)$ is ample,

then there is a positive integer $e = e(X)$ such that $L^{\otimes e} \simeq \mathcal{O}_X$ for any numerically trivial line bundle L on X .

— Rough sketch of the proof —

Take the minimal resolution $\tilde{X} \rightarrow X$. One can easily check that $-K_{\tilde{X}}$ is big, so that there is a birational morphism $\varepsilon: \tilde{X} \rightarrow Y = \mathbb{P}_C(E)$ such that $-K_Y$ is also big. Then we can apply Corollary 3.3 to E and obtain the following diagram.

$$\begin{array}{ccc} & \tilde{X} & \xrightarrow{\varphi} X \\ & \downarrow \varepsilon & \\ Y' & \xrightarrow{f'} Y & \\ \downarrow \pi' & & \downarrow \pi \\ C & \xrightarrow{F^e} C & \end{array} \quad (4.1)$$

When $\tilde{X}$ is not rational, consider the section C' of π' corresponding to $(F^e)^*E \twoheadrightarrow M'$. It follows that $\varepsilon_*^{-1}f'(C')$ is contracted by φ . One easily sees that $\varphi^*L \simeq \varepsilon^*\pi^*L_C$ for some line bundle L_C on C , but the contractibility implies that $((f')^*\pi^*L_C)|_{C'} \simeq \mathcal{O}_{C'}$. Hence $L^{\otimes p^e} \simeq \mathcal{O}_X$.

Corollary 4.2 (O-Okawa, [1], Main theorem II).

Let (X, Δ) be as in Theorem 4.1. Then $\text{Pic}(X)$ is a free abelian group of finite rank.

Remark 4.3.

Corollary 4.2 is shown in [3, Corollary 3.6] when X has only rational singularities. In fact, the rest is covered by [2]. Our proof as an application of Theorem 3.1 is more straightforward.

Remark 4.4.

The assumptions of Theorem 4.1 are optimal. In fact, consider a smooth cubic curve $C \subset \mathbb{P}^2$ and $X = \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(1))$. The projective cone over C is obtained from X by contracting the negative section E . Then $(X, \Delta = E)$ is a pair such that $-(K_X + \Delta)$ is nef and big, but $\text{Pic}(X)$ is not a free abelian group.

References

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