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Shintani Functions and Rankin-Selberg Convolution

I. Local Theory

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In this note, we report a recent progress of the local theory of Shintani functions on split orthogonal groups (joint work with Shin-ichi Kato).

§1. Notation

Let F be a non-archimedean local field with $\mathrm{char}(F) \neq 2$ and denote by o the integer ring of F. Fix a prime element π of F and put $q = \#(o/\pi o)$. For a positive integer n, we put

$$S_n = \begin{cases} \begin{bmatrix} 0 & J_\nu \\ J_\nu & 0 \end{bmatrix} & \text{if } n \text{ is even} \\ \begin{bmatrix} 0 & 0 & J_\nu \\ 0 & 2 & 0 \\ J_\nu & 0 & 0 \end{bmatrix} & \text{if } n \text{ is odd} \end{cases}$$

where $\nu = \left[\frac{n}{2}\right]$ and $J_\nu = \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix} \in GL_\nu(F)$. Let G_n be the orthogonal group of S_n over F: $G_n = \{ g \in GL_n(F) \mid {}^t g S_n g = S_n \}$. We define an embedding ι_n of G_{n-1} into G_n as follows (we put $\nu' = \left[\frac{n-1}{2}\right]$):

(a) If n is even,

$$\iota_n\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 & \frac{a_2}{2} & \frac{a_2}{2} & a_3 \\ b_1 & \frac{b_2+1}{2} & \frac{b_2-1}{2} & b_3 \\ b_1 & \frac{b_2-1}{2} & \frac{b_2+1}{2} & b_3 \\ c_1 & \frac{c_2}{2} & \frac{c_2}{2} & c_3 \end{bmatrix}$$

where $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \in G_{n-1}$ is the block decomposition according to the partition $n-1 = \nu' + 1 + \nu'$.

(b) If n is odd,

$$\iota_n\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_{n-1}$ is the block decomposition according to the partition $n-1 = \nu' + \nu'$.

In what follows, we write G and G' for G_n and G_{n-1} respectively. Let $\nu = [n/2]$ (resp. $\nu' = [(n-1)/2]$ be the Witt index of S_n (resp. S_{n-1}). We identify G' with a subgroup of G via ι_n. Put $K = G \cap GL_n(o)$ (resp. $K' = G' \cap GL_{n-1}(o)$), and let $\mathcal{H} = \mathcal{H}(G, K)$ (resp. $\mathcal{H}' = \mathcal{H}(G', K')$) be the Hecke algebra of (G, K) (resp. (G', K')). To parametrize $\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathbf{C})$, let $X_{unr}(F^\times)^\nu$ be the group of ν-tuples of unramified characters of $F^\times$. Let P be the subgroup of upper triangular matrices in G (the standard minimal parabolic subgroup of G). Then $\chi \in X_{unr}(F^\times)^\nu$ is regarded as a character of P in a natural manner. Define a function Φ_χ on G to be $\Phi_\chi(pk) = (\chi\delta_P^{1/2})(p)$ for $p \in P$ and $k \in K$, where δ_P is the module of P. For $\varphi \in \mathcal{H}$, put

$$\chi^\wedge(\varphi) = \int_G \varphi(g)\, \Phi_\chi(g)\, dg.$$

Then $\varphi \mapsto \chi^\wedge(\varphi)$ defines an element of $\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathbf{C})$. The correspondence $\chi \mapsto \chi^\wedge$ gives rise to a bijection from $X_{unr}(F^\times)^\nu / W_G$ onto $\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}, \mathbf{C})$, where W_G is the Weyl group of G (cf. [Sa]). Similarly we can identify $\mathrm{Hom}_{\mathbf{C}}(\mathcal{H}', \mathbf{C})$ with $X_{unr}(F^\times)^{\nu'} / W_{G'}$, where $W_{G'}$ is the Weyl group of G'.

§2. Main results

As in [MS1], we define the space $\mathrm{Sh}(\chi, \chi')$ of local Shintani functions on G attached to $(\chi, \chi') \in X_{unr}(F^\times)^\nu \times X_{unr}(F^\times)^{\nu'}$ by

$$\begin{aligned}\mathrm{Sh}(\chi, \chi') = \{ W: G \to \mathbf{C} \mid & \text{(i) } W(k'gk) = W(g) \quad (k' \in K', k \in K, g \in G) \\ & \text{(ii) } \varphi' * W * \varphi = \chi'^\wedge(\varphi')\, \chi^\wedge(\varphi)\, W \quad (\varphi' \in \mathcal{H}', \varphi \in \mathcal{H}) \}.\end{aligned}$$

Here we put

$$(\varphi' * W * \varphi)(g) = \int_{G'} dx' \int_{G} dx\, \varphi'(x')\, W(x'^{-1} g\, x)\, \varphi(x).$$

Note that Shintani functions can be regarded as spherical functions on a spherical homogeneous space $\mathcal{X} = G'^{diag} \backslash G' \times G$, where $G'^{diag} = \{ (g', g') \mid g' \in G' \}$ is a spherical subgroup of $G' \times G$ in the sense of [Br]. The following has been conjectured in [MS1].

Theorem 1 *Let* $(\chi, \chi') \in X_{unr}(F^\times)^\nu \times X_{unr}(F^\times)^{\nu'}$. *Then we have* $\dim_{\mathbf{C}} \mathrm{Sh}(\chi, \chi') = 1$. *Moreover, there exists a* $W_{\chi,\chi'} \in \mathrm{Sh}(\chi, \chi')$ *with* $W_{\chi,\chi'}(1) = 1$.

To state an explicit formula for $W_{\chi,\chi'}$, we need several preparations. Let $\Lambda_\nu = \{ (m_1, \ldots, m_\nu) \in \mathbf{Z}^\nu \mid m_1 \geq \ldots \geq m_\nu \geq 0 \}$. For $m = (m_1, \ldots, m_\nu) \in \Lambda_\nu$, we put $\Pi_m = \mathbf{d}_n\left(\begin{bmatrix} \pi^{m_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{m_\nu} \end{bmatrix}\right) \in G$, where

$$\mathbf{d}_n(A) = \begin{cases} \begin{bmatrix} A & 0 \\ 0 & J_v {}^tA^{-1} J_v \end{bmatrix} & \text{if } n \text{ is even} \\ \begin{bmatrix} A & & 0 \\ & 1 & \\ 0 & & J_v {}^tA^{-1} J_v \end{bmatrix} & \text{if } n \text{ is odd} \end{cases} \qquad \text{for } A \in GL_v(F).$$

Similarly we define $\Pi'_{m'} \in G'$ for $m' \in \Lambda_{v'}$. Let g_o be an element of G given by

$$g_o = \begin{cases} \mathbf{d}_n(A_o) & \text{if } n \text{ is even} \\ \begin{bmatrix} 1_v & -2\eta & -\eta {}^t\eta J_v \\ 0 & 1 & {}^t\eta J_v \\ 0 & 0 & 1_v \end{bmatrix} & \text{if } n \text{ is odd} \end{cases}$$

where $A_o = \begin{bmatrix} 1 & & 0 & 1 \\ & \ddots & & \vdots \\ 0 & & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in GL_v(F)$ and $\eta = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in F^v$. The following result is the "Cartan decomposition" for $\mathcal{X}$.

Proposition 2 *We have*

$$G = \coprod_{m \in \Lambda_v,\, m' \in \Lambda_{v'}} K' g(m, m') K \qquad \textit{(disjoint union)}$$

where $g(m, m') = \Pi'_{m'}\, g_o\, \Pi_m \in G$.

Put

$$Q_n(q) = \prod_{i=1}^{v-1} (1 - q^{-2i}) \times \begin{cases} 1 & \text{if } n \text{ is even} \\ (1 - q^{-v}) & \text{if } n \text{ is odd.} \end{cases}$$

Let $\chi = (\chi_1, \dots, \chi_v) \in X_{unr}(F^\times)^v$ and $\chi' = (\chi'_1, \dots, \chi'_{v'}) \in X_{unr}(F^\times)^{v'}$. To simplify notation, we often write χ'_i and χ_j for the values $\chi'_i(\pi)$ and $\chi_j(\pi)$ respectively. Define

$$\mathcal{D}(\chi, \chi') = \frac{\prod_{1 \leq i \leq v', 1 \leq j \leq v} (1 - q^{-1/2} (\chi_i'^{-1} \chi_j)^{\varepsilon_{ij}}) (1 - q^{-1/2} (\chi_i' \chi_j)^{-1})}{\Delta_G(\chi) \cdot \Delta_{G'}(\chi')}$$

where

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } i < j \\ -1 & \text{if } i \geq j \end{cases}$$

and

$$\Delta_G(\chi) = \prod_{1 \leq k \leq \ell \leq v} (1 - \chi_k^{-1} \chi_\ell) (1 - \chi_k^{-1} \chi_\ell^{-1}) \times \begin{cases} 1 & \text{if } n \text{ is even} \\ \prod_{1 \leq k \leq v} (1 - \chi_k^{-2}) & \text{if } n \text{ is odd} \end{cases}$$

($\Delta_{G'}(\chi')$ is similarly defined).

Theorem 3 *Let* $W_{\chi,\chi'} \in \mathrm{Sh}(\chi, \chi')$ *be as in Theorem 1. Then, for* $(m, m') \in \Lambda_v \times \Lambda_{v'}$, *we have*

$$W_{\chi,\chi'}(g(m, m'))$$
$$= \frac{1}{Q_n(q)} \sum_{\substack{w \in W_G \\ w' \in W_{G'}}} \mathcal{D}(w\chi, w'\chi') (w\chi \cdot \delta_P^{1/2})(\Pi_m) (w'\chi' \cdot \delta_{P'}^{1/2})(\Pi'_{m'}),$$

where W_G *(resp.* $W_{G'}$*) is the Weyl group of* G *(resp.* G'*) and* δ_P *(resp.* $\delta_{P'}$*) is the module of the standard minimal parabolic subgroup* P *(resp.* P'*) of* G *(resp.* G'*).*

§3. Sketch of proof

The existence part of Theorem 1 is proved by using an integral expression of Shintani functions similar to that of [MS2]. We can prove Theorem 3 following the method of [KM], where an explicit formula for local Shintani functions on $GL(n)$ is shown.

We now give an outline of the proof of the uniqueness part of Theorem 1. For $(m, m') \in \Lambda = \Lambda_v \times \Lambda_{v'}$, we define an element $\{m, m'\}$ of $\mathbb{Z}^{v+v'}$ by

$$\{m, m'\} = \begin{cases} (m_1, m_1', m_2, m_2', \ldots, m_{\nu'}, m_{\nu'}', m_\nu) & \text{if } n \text{ is even (in this case } \nu = \nu'+1) \\ (m_1, m_1', m_2, m_2', \ldots, m_{\nu'}, m_{\nu'}') & \text{if } n \text{ is odd (in this case } \nu = \nu'). \end{cases}$$

We define a total ordering of Λ as follows: $(\ell, \ell') \prec (m, m')$ if and only if $\{\ell, \ell'\} < \{m, m'\}$ (in the usual lexicographic ordering of $\mathbf{Z}^{\nu+\nu'}$). The proof of the uniqueness of Shintani functions is reduced to the following:

Proposition 4 *Let* $W \in Sh(\chi, \chi')$ *and* $(m, m') \in \Lambda$. *Then we have*

$$W(g(m, m')) = \sum c_{\ell,\ell'}(\chi, \chi')\, W(g(\ell, \ell')),$$

where the summation is over $(\ell, \ell') \in \Lambda$ *with* $(\ell, \ell') \prec (m, m')$, *and* $c_{\ell,\ell'}(\chi, \chi')$ *is an element of* $\mathbf{C}[\chi_1^{\pm 1}, \ldots, \chi_\nu^{\pm 1}, (\chi_1')^{\pm 1}, \ldots, (\chi_{\nu'}')^{\pm 1}]$ *depending only on* (ℓ, ℓ') *and* (χ, χ') *and not on* W.

The proposition follows from the next result, which is an analogue of Proposition (4.4.4) in [BT].

Key lemma *Let* $(m, m'), (\ell, \ell') \in \Lambda$ *and* $k \in K$ *, and suppose that* $\Pi'_{m'} k\, \Pi_m \in K' g(\ell, \ell') K$. *Then we have* $(\ell, \ell') \preceq (m, m')$.

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