

Algebras, Representations and Quantum Mechanics

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代数と表現と量子力学

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可積分性という概念は古典力学では明確に定義されているが、量子力学における対応概念は不明瞭である。また、古典力学との対応を求めず、量子力学のハイゼンベルグ方程式を解くという目的に限定しても、作用角変数に匹敵するような強力かつ柔軟な方法論の構築が望ましい。本講演ではこれらの問題点を指摘し、解決の糸口を代数の表現論による多様体上の量子力学の定式化に探ってみた。

1 Introduction

Integrability in quantum mechanics has been intensively investigated from various aspects. Here we discuss genuine integrability of the Heisenberg equation of motion.

2 Integrability

2.1 Integrability in classical mechanics

Let $(p, q) = (p_1, \dots, p_n, q_1, \dots, q_n)$ be the canonical coordinates of the phase space. If the system admits an involutive system $\{\Phi_j\}_{j=1, \dots, n}$ ($\Phi_1 = H$: Hamiltonian), which are functions commuting mutually with respect to the Poisson bracket, the system is called integrable. If the level set $\cap_j \Phi_j^{-1}(c_j)$ for constants $(c_1, \dots, c_n)$ is compact, it is isomorphic to an n -dimensional torus. Then action variables are defined by integration $I_j = \frac{1}{2\pi} \int_{\gamma_j} p \cdot dq$ along cycles γ_j in the torus. Angle variables are defined by $\{\theta_j, I_k\}_{\text{PB}} = \delta_{jk}$. Then the Hamiltonian $H = H(I)$ generates a multi-periodic motion in the torus: $\frac{d\theta_j}{dt} = \frac{\partial H}{\partial I_j} = \omega_j(I)$.

2.2 Integrability in quantum mechanics

Although integrability of the Hamilton equation of classical mechanics has a clear geometric picture and is useful for solving dynamics, integrability of the Heisenberg equation of quantum mechanics still remains vague. Even for such a simple system as the harmonic oscillator the action-angle variable method fails to make sense as seen below. Of course, the Heisenberg equation of the harmonic oscillator

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2, \quad \frac{d\hat{p}}{dt} = \frac{1}{i\hbar}[\hat{p}, \hat{H}] = -\omega^2\hat{q}, \quad \frac{d\hat{q}}{dt} = \frac{1}{i\hbar}[\hat{q}, \hat{H}] = \hat{p}$$

is solved as

$$\hat{p}(t) = \hat{p}(0) \cos \omega t - \omega \hat{q}(0) \sin \omega t, \quad \omega \hat{q}(t) = \hat{p}(0) \sin \omega t + \omega \hat{q}(0) \cos \omega t.$$

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Let us try to define action and angle operators, $\hat{I}$ and $\hat{\theta}$, naively by

$$\hat{p} = \sqrt{2\omega\hat{I}} \cos \hat{\theta}, \quad \hat{q} = \sqrt{\frac{2\hat{I}}{\omega}} \sin \hat{\theta}, \quad [\hat{\theta}, \hat{I}] = i\hbar \quad (\text{later revealed to be wrong equations}).$$

In the context of classical mechanics, the action represents the squared radius of a circle in the phase space and the angle represents the cyclic coordinate of the circle. For the above equations make sense, the action operator $\hat{I}$ must have non-negative spectrum. Moreover, the eigenstate $|\theta\rangle$ of the angle operator $\hat{\theta}$ must coincide with the state $|\theta + 2\pi\rangle$. But in fact the eigenvalues of $\hat{I}$ range over all real numbers and $|\theta + 2\pi\rangle \neq |\theta\rangle$ because the commutation relation $[\hat{\theta}, \hat{I}] = i\hbar$ is just a rewriting of the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar$.

We need a correct quantization scheme for a circle. The answer was given by Ohnuki and Kitakado. The operators for quantization in a circle are a unitary operator (angle operator) $\hat{U}$ and an Hermitian operator (action operator) $\hat{I}$. The correct algebra is defined by the commutator

$$[\hat{I}, \hat{U}] = \hbar \hat{U}.$$

Although this relation can be deduced from $[\hat{I}, e^{i\hat{\theta}}] = \hbar e^{i\hat{\theta}}$ and $e^{i\hat{\theta}} = \hat{U}$, we do not write $\hat{\theta}$ explicitly. Instead, we regard the unitary operator $\hat{U}$ as a fundamental object of quantum mechanics in the circle. To complete quantization we must have a representation of the algebra. Representation means a Hilbert space on which the algebra acts. In our case, the Hilbert space is the space of functions in the circle. Namely, on a periodic wave function such that $\psi(\theta + 2\pi) = \psi(\theta)$ the operators act as

$$\hat{U}\psi(\theta) = e^{i\theta} \psi(\theta), \quad \hat{I}\psi(\theta) = -i\hbar \left(\frac{\partial}{\partial \theta} + i\alpha \right) \psi(\theta).$$

Here α is a real parameter labeling inequivalent representations. Then we can calculate the spectrum of the action operator $\hat{I}$. Eigenfunctions and eigenvalues of $\hat{I}$ are obtained as

$$\chi_n(\theta) = \langle \theta | n \rangle = e^{in\theta}, \quad \hat{I}\chi_n(\theta) = -i\hbar \left(\frac{\partial}{\partial \theta} + i\alpha \right) e^{in\theta} = \hbar(n + \alpha) \chi_n(\theta).$$

This spectrum reminds us the Wilson-Sommerfeld-Ishihara quantization condition in old quantum theory:

$$\left. \begin{aligned} I &= \frac{1}{2\pi} \oint p dq \\ \hat{I}|n\rangle &= \hbar(n + \alpha)|n\rangle \end{aligned} \right\} \rightarrow \oint p dq = 2\pi\hbar(n + \alpha).$$

Can we use the action and angle operators defined above for solving the harmonic oscillator? In the context of classical mechanics the Hamiltonian is related to the action variable as $H = \omega I$. If we put $\hat{H} = \omega \hat{I}$, its spectrum is given as

$$\hat{H}|n\rangle = \hbar\omega(n + \alpha)|n\rangle \quad (n = \dots, -2, -1, 0, 1, 2, \dots).$$

This resembles the spectrum of the harmonic oscillator but it is not bound below. So the action-angle variable method fails to reproduce the correct spectrum of the harmonic oscillator. The reason of this failure is that the circle $\{(p, q) | \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 = \omega I\}$, which is parametrized by $I \geq 0$, shrinks to a point at $I = 0$ and the action-angle variables become bad coordinates there. Therefore the harmonic oscillator cannot be described globally by quantum mechanics in circle.

3 Quantization in manifolds

3.1 Algebra and representation

In this section we seek for a flexible scheme of quantization that is applicable to a manifold with various topology. This scheme is based on representation theory of algebras.

A set $\mathcal{A}$ is called an algebra when it is equipped with three kinds of operations: (1) scalar multiplication λX , (2) sum $X + Y$, (3) product XY of $X, Y \in \mathcal{A}$ and $\lambda \in \mathbf{C}$. We assume the associativity law and the distribution law. The unit element $\mathbf{1} \in \mathcal{A}$ satisfies $\mathbf{1}X = X\mathbf{1} = X$ for any $X \in \mathcal{A}$. A commutator of X and Y is defined as $[X, Y] := XY - YX$. A well-known example of algebras is the $SU(2)$ algebra, which is defined in terms of Hermitian generators $\{J_1, J_2, J_3\}$ and commutation relations $[J_j, J_k] = i\varepsilon_{jkl}J_l$.

A representation ρ of an algebra $\mathcal{A}$ is a mapping that assigns an $n \times n$ complex matrix $\rho(X)$ to each $X \in \mathcal{A}$ and satisfies $\rho(\lambda X) = \lambda\rho(X)$, $\rho(X + Y) = \rho(X) + \rho(Y)$, $\rho(XY) = \rho(X)\rho(Y)$ for $\lambda \in \mathbf{C}$ and $Y \in \mathcal{A}$. Then $\rho(X)$ is called an n -dimensional representation of X . The vector space on which these matrices act is called a representation space.

An example of representations of the $SU(2)$ algebra is a mapping ρ_1 that assigns zero to every element X as $\rho_1(X) = 0$. This is a one-dimensional representation. Another example of representations of the $SU(2)$ algebra is a mapping ρ_2 that assigns

$$\rho_2(J_1) = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2(J_2) = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_2(J_3) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This is a two-dimensional representation. We can make another representation $\tilde{\rho}_2$ that assigns

$$\tilde{\rho}_2(J_1) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{\rho}_2(J_2) = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\rho}_2(J_3) = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

However, $\tilde{\rho}_2$ is equivalent to ρ_2 . Another inequivalent representation is, for example,

$$\rho_3(J_1) = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho_3(J_2) = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \rho_3(J_3) = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3.2 Algebra of quantum mechanics in Euclidean space

The algebra of quantum mechanics in an n -dimensional Euclidean space is defined in terms of Hermitian generators $\{\hat{p}_1, \dots, \hat{p}_n, \hat{q}_1, \dots, \hat{q}_n\}$ and the canonical commutation relations (CCR), $[\hat{p}_j, \hat{p}_k] = 0$, $[\hat{q}_j, \hat{q}_k] = 0$, $[\hat{q}_j, \hat{p}_k] = i\hbar \delta_{jk} \mathbf{1}$. The CCR algebra is represented on the space of wave functions $\psi(\mathbf{q})$ in the Euclidean space $\mathbf{R}^n$ as

$$\hat{q}_j \psi(\mathbf{q}) = q_j \psi(\mathbf{q}), \quad \hat{p}_j \psi(\mathbf{q}) = -i\hbar \frac{\partial}{\partial q_j} \psi(\mathbf{q}).$$

Actually, this is an infinite-dimensional representation. The von Neumann-Stone theorem states that every irreducible representation of the CCR is unitarily equivalent. This theorem tells that

if we start from the CCR algebra, we inevitably reach the Euclidean space. The contraposition tells that for quantization in a manifold other than the Euclidean space we need to introduce an algebra other than the CCR. Here we notice interrelation between topology and algebra.

3.3 Algebra of quantum mechanics in sphere

For quantization in a two-dimensional sphere Ohnuki and Kitakado defined a new algebra in terms of Hermitian generators $\{\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{J}_1, \hat{J}_2, \hat{J}_3\}$ and the relations

$$[\hat{x}_j, \hat{x}_k] = 0, \quad [\hat{J}_j, \hat{J}_k] = i \varepsilon_{jkl} \hat{J}_l, \quad [\hat{J}_j, \hat{x}_k] = i \varepsilon_{jkl} \hat{x}_l.$$

The representation space is a space of wave functions $\psi(\theta, \phi)$ on a sphere of a radius r . Actually the wave function on the sphere is a pair of functions (ψ_+, ψ_-) ; each function is defined in northern or southern hemisphere and they are related by the gauge transformation $\psi_-(\theta, \phi) = e^{-im\phi} \psi_+(\theta, \phi)$. Then the generators $\{\hat{J}_1, \hat{J}_2, \hat{J}_3\}$ act on the wave function as

$$\begin{aligned} \hat{J}_1 \psi_{\pm} &= i \left[\sin \phi \frac{\partial}{\partial \theta} + \frac{\cos \phi}{\sin \theta} \left(\cos \theta \frac{\partial}{\partial \phi} + i \frac{m}{2} (1 \mp \cos \theta) \right) \right] \psi_{\pm}, \\ \hat{J}_2 \psi_{\pm} &= i \left[-\cos \phi \frac{\partial}{\partial \theta} + \frac{\sin \phi}{\sin \theta} \left(\cos \theta \frac{\partial}{\partial \phi} + i \frac{m}{2} (1 \mp \cos \theta) \right) \right] \psi_{\pm}, \\ \hat{J}_3 \psi_{\pm} &= i \left[-\frac{\partial}{\partial \phi} \pm i \frac{m}{2} \right] \psi_{\pm}. \end{aligned}$$

They are angular momenta in a monopole magnetic field. Irreducible representation is labeled by the real number $r \geq 0$ and the monopole number $m = 0, \pm 1, \pm 2, \dots$.

4 Summary and future prospect

Integrability in quantum mechanics, does not have a clear meaning or picture. The action-angle operators fail to describe a correct topology and fail to produce a correct spectrum. To formulate quantum mechanics in manifolds with various topology we need new algebras and representations. It is strongly desired to develop a strategy for solving the Heisenberg equation. Probably, we may need a more flexible scheme of quantization that is applicable to foliation or stratification with varying topology.

References

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