

# Simple construction of parameter map germ and its applications

TAKASHI NISHIMURA

西村尚史

Department of Mathematics, Faculty of Education  
Yokohama National University  
Yokohama 240, JAPAN

In this note, we shall construct a simple parameter map germ  $(\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^p, 0)$  under the assumption that there is an  $\mathcal{A}$ -morphism (resp. topological  $\mathcal{A}$ -morphism) from a given deformation  $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of a given map germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  to the trivial deformation  $f : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$ .

This parameter map germ induces a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from  $\Psi$  to the graph deformation of  $f$ .

By this construction, we can prove the following:

**THEOREM D ([M2]):** Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  stable map germs. Suppose there exist a  $C^\infty$  diffeomorphic germ  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)(g \circ s)(x)$ . Then  $f$  and  $g$  are right-left equivalent.

Though our method seems to be close to Martinet's one ([Mr]), we can treat also map germs which are not necessarily  $C^\infty$  stable.

**THEOREM E ([FF]):** Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two MT stable map germs. Suppose there exist a  $C^\infty$  diffeomorphic germ  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map germ  $M : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$  such that  $f(x) = M(x)(g \circ s)(x)$ . Then  $f$  and  $g$  are topologically right-left equivalent.

**THEOREM A:** Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  map germs. Suppose there exist a  $C^\infty$  diffeomorphic germ  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map germ

$$M(x) = (m_1(x), \dots, m_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

such that  $f(x) = M(x)(g \circ s)(x)$ . Suppose furthermore there exists a positive integer  $k$  such that

$$m_i(x) - m_i(0) \in m_x^k \mathcal{E}_x^p \subset tf(m_x \mathcal{E}_x^n) + \omega f(m_y \mathcal{E}_y^p)$$

for any  $i$  ( $1 \leq i \leq p$ ). Then  $f$  and  $g$  are right-left equivalent.

As a corollary of theorem A, we get

COROLLARY A: Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ. Suppose there exist positive integers  $k, l$  such that

$$\mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + \omega f(\mathfrak{m}_y \mathcal{E}_y^p)$$

and

$$\mathfrak{m}_x^l \mathcal{E}_x^p \subset tf(\mathfrak{m}_x^2 \mathcal{E}_x^n) + f^* \mathfrak{m}_y \mathfrak{m}_x^k \mathcal{E}_x^p.$$

Then  $f$  is  $(l-1)$ -determined with respect to right-left equivalence.

Corollary A induces the following Gaffney type estimate of the order of determinacy (c.f. [G]).

COROLLARY B: Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ. Suppose there exist positive integers  $k, l$  such that

$$\mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + \omega f(\mathfrak{m}_y \mathcal{E}_y^p)$$

and

$$\mathfrak{m}_x^l \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + f^* \mathfrak{m}_y \mathcal{E}_x^p.$$

Then  $f$  is  $(k+l-1)$ -determined with respect to right-left equivalence.

Corollary B induces the following du Plessis-Wall's estimate of the order of determinacy.

COROLLARY C ([dP, W]): Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ. Suppose there exist a positive integer  $k$  such that

$$\mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + \omega f(\mathfrak{m}_y \mathcal{E}_y^p).$$

Then  $f$  is  $(2k-1)$ -determined with respect to right-left equivalence.

In [W], we can find an estimate of the order of topological determinacy of an MT stable map germ (corollary D bellow) which is due to T. Gaffney, but without proof. By using of our method, we can give a proof of his estimate.

COROLLARY D (GAFFNEY): Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be an MT stable map germ. Suppose there exist a positive integer  $k$  such that

$$\mathfrak{m}_x^k \mathcal{E}_x^p \subset tf(\mathfrak{m}_x \mathcal{E}_x^n) + f^*(\mathfrak{m}_y^2) \mathcal{E}_x^p.$$

Then  $f$  is  $k$ -determined with respect to topologically right-left equivalence.

For details on these corollaries, refer to [N].

This note is organized in the following way. In §1 and §2, we give several preparations for the proofs of theorem A, a generalized version of Mather's

classification theorem (theorem D in §5) and the theorem of Fukuda-Fukuda (theorem E in §6). §3 treats algebraic argument which we need for the proof of theorem A. Theorem A will be proved in §4. A generalized version of Mather's classification theorem will be proved in §5. In §6, an alternative proof of the theorem of Fukuda-Fukuda will be given.

The results in this paper are all valid in the complex analytic category as well except example (1.5.2).

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#### §1. $\mathcal{K}$ -MORPHISM

##### FROM A GIVEN DEFORMATION TO THE GRAPH DEFORMATION

Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ and  $\Psi_i : (\mathbb{R}^n \times \mathbb{R}^{r(i)}, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  deformation of  $f$  (i.e.  $\Psi_i(x, 0) = f(x)$ ) ( $i = 1, 2$ ).

**DEFINITION (1.1).** We say if there exist  $C^\infty$  (resp. continuous) map germs  $h : (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0))$ ,  $H : (\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0, 0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0, 0, 0))$  and  $\phi : (\mathbb{R}^{r(1)}, 0) \rightarrow (\mathbb{R}^{r(2)}, 0)$  such that the following conditions (1.1.1), (1.1.2), (1.1.3) and (1.1.4) hold, then  $\{h, H, \phi\}$  is a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from  $\Psi_1$  to  $\Psi_2$ .

(1.1.1) the restrictions  $h|_{\mathbb{R}^n \times \{\lambda\}}$  and  $H|_{\mathbb{R}^n \times \{\lambda\} \times \mathbb{R}^p}$  are  $C^\infty$  diffeomorphic (resp. homeomorphic) for any  $\lambda \in \mathbb{R}^{r(1)}$ ,

(1.1.2)  $H(\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^{r(2)} \times \{0\}$ ,

(1.1.3) the following diagram commutes:

$$\begin{array}{ccccc}
 (\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0, 0, 0)) & \xrightarrow{\pi_{*, \lambda}} & (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(1)}, 0) \\
 H \downarrow & & h \downarrow & & \phi \downarrow \\
 (\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0, 0, 0)) & \xrightarrow{\pi_{*, \lambda}} & (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(2)}, 0),
 \end{array}$$

(1.1.4) the following diagram commutes:

$$\begin{array}{ccc}
 (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) & \xrightarrow{(\pi_{\bullet, \lambda}, \Psi_1)} & (\mathbb{R}^n \times \mathbb{R}^{r(1)} \times \mathbb{R}^p, (0, 0, 0)) \\
 h \downarrow & & H \downarrow \\
 (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0)) & \xrightarrow{(\pi_{\bullet, \lambda}, \Psi_2)} & (\mathbb{R}^n \times \mathbb{R}^{r(2)} \times \mathbb{R}^p, (0, 0, 0)).
 \end{array}$$

Here  $\pi_{\bullet, \lambda}$ ,  $\pi_\lambda$  mean the canonical projection to  $\mathbb{R}^n \times \mathbb{R}^{r(i)}$ ,  $\mathbb{R}^{r(i)}$  respectively. We remark that the conditions (1.1.1), (1.1.2) and (1.1.3) in the definition (1.1) imply  $H(\mathbb{R}^n \times \mathbb{R}^{r(1)} \times (\mathbb{R}^p - \{0\})) \subset \mathbb{R}^n \times \mathbb{R}^{r(2)} \times (\mathbb{R}^p - \{0\})$ ; and the condition (1.1.4) implies  $H(\text{graph}(\Psi_1)) \subset \text{graph}(\Psi_2)$ .

DEFINITION (1.2). We say if there exist  $C^\infty$  (resp. continuous) map germs  $h : (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0))$ ,  $H : (\mathbb{R}^p \times \mathbb{R}^{r(1)}, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^{r(2)}, (0, 0))$  and  $\phi : (\mathbb{R}^{r(1)}, 0) \rightarrow (\mathbb{R}^{r(2)}, 0)$  such that the following conditions (1.2.1) and (1.2.2) hold, then  $\{h, H, \phi\}$  is a  $\mathcal{A}$ -morphism (resp. topological  $\mathcal{A}$ -morphism) from  $\Psi_1$  to  $\Psi_2$ .

(1.2.1) the restrictions  $h|_{\mathbb{R}^n \times \{\lambda\}}$  and  $H|_{\mathbb{R}^p \times \{\lambda\}}$   
are  $C^\infty$  diffeomorphic (resp. homeomorphic)  
for any  $\lambda \in \mathbb{R}^{r(1)}$ ,

(1.2.2) the following diagram commutes:

$$\begin{array}{ccccc}
 (\mathbb{R}^n \times \mathbb{R}^{r(1)}, (0, 0)) & \xrightarrow{(\Psi_1, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^{r(1)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(1)}, 0) \\
 h \downarrow & & H \downarrow & & \phi \downarrow \\
 (\mathbb{R}^n \times \mathbb{R}^{r(2)}, (0, 0)) & \xrightarrow{(\Psi_2, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^{r(2)}, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^{r(2)}, 0).
 \end{array}$$

Let  $\mathcal{G}$  be  $\mathcal{K}$  or  $\mathcal{A}$ . A  $\mathcal{G}$ -morphism (resp. topological  $\mathcal{G}$ -morphism)  $\{h, H, \phi\}$  from  $\Psi_1$  to  $\Psi_2$  is said to be *equivalent* (resp. *topologically equivalent*) if  $\phi$  is  $C^\infty$ -diffeomorphic (resp. homeomorphic). Definitions of  $\mathcal{G}$ -morphism and equivalent  $\mathcal{G}$ -morphism are equivalent to those of Martinet's definitions ([Mr]); and definitions of topological  $\mathcal{G}$ -morphism and topologically equivalent topological  $\mathcal{G}$ -morphism are topological analogues of these. If there exists an equivalent  $\mathcal{A}$ -morphism (resp. topologically equivalent topological  $\mathcal{A}$ -morphism) from a given deformation  $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  to the trivial deformation  $f : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$ , then we say  $\Psi$  has a *triviality* (resp. *topological triviality*).

In this chapter, we show if there is a  $\mathcal{A}$ -morphism (resp. topological  $\mathcal{A}$ -morphism) from a given deformation  $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  to the trivial deformation  $f : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$ , then we can directly construct a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from  $\Psi$  to the graph deformation.

Now suppose there exist  $C^\infty$  (resp. continuous) map germs  $h : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^r, (0, 0))$ ,  $H : (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p \times \mathbb{R}^r, (0, 0))$  and  $\phi : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^r, 0)$  such that the following (1.3.1) and (1.3.2) hold:

(1.3.1) the restrictions  $h|_{\mathbb{R}^n \times \{\lambda\}}$  and  $H|_{\mathbb{R}^p \times \{\lambda\}}$   
are  $C^\infty$  diffeomorphic (resp. homeomorphic)  
for any  $\lambda \in \mathbb{R}^r$ ,

(1.3.2) the following diagram commutes:

$$\begin{array}{ccccc} (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(\Psi, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0) \\ h \downarrow & & H \downarrow & & \phi \downarrow \\ (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0). \end{array}$$

By (1.3.2), we can write

$$h = (h_1, \phi) \quad \text{and} \quad H = (H_1, \phi).$$

Then, set  $\phi'_H : (\mathbb{R}^r, 0) \rightarrow (\mathbb{R}^p, 0)$  as

$$\phi'_H(\lambda) = H_1(0, \lambda).$$

Also, set  $h' : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p, (0, 0))$  as

$$h'(x, \lambda) = (h_1(x, \lambda), \phi'_H(\lambda))$$

and set  $H' : (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0, 0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0, 0, 0))$  as

$$H'(x, \lambda, y) = (h'(x, \lambda), H_1(y, \lambda) - H_1(0, \lambda)).$$

Then we have

(1.4.0)  $h'$  and  $H'$  are  $C^\infty$  (resp. continuous) map germs,

(1.4.1) the restrictions  $h'|_{\mathbb{R}^n \times \{\lambda\}}$  and  $H'|_{\mathbb{R}^n \times \{\lambda\} \times \mathbb{R}^p}$   
are  $C^\infty$  diffeomorphic (resp. homeomorphic)  
for any  $\lambda \in \mathbb{R}^r$ ,

$$(1.4.2) \quad H'(\mathbb{R}^n \times \mathbb{R}^r \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^p \times \{0\},$$

(1.4.3) the following diagram commutes:

$$\begin{array}{ccccc} (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0, 0, 0)) & \xrightarrow{\pi_{\bullet, \lambda}} & (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0) \\ H' \downarrow & & h' \downarrow & & \phi'_H \downarrow \\ (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0, 0, 0)) & \xrightarrow{\pi_{\bullet, y}} & (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) & \xrightarrow{\pi_y} & (\mathbb{R}^p, 0). \end{array}$$

Next, we set  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  as

$$F(x, y) = f(x) - y.$$

We call  $F : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  the graph deformation of  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ .

Then, we can see

$$\begin{aligned} F(h'(x, \lambda)) &= F(h_1(x, \lambda), \phi'_H(\lambda)) && \text{(definition of } h') \\ &= f(h_1(x, \lambda)) - \phi'_H(\lambda) && \text{(definition of } F) \\ &= H_1(\Psi(x, \lambda), \lambda) - \phi'_H(\lambda) && (1.3.2) \\ &= H_1(\Psi(x, \lambda), \lambda) - H_1(0, \lambda) && \text{(definition of } \phi'_H). \end{aligned}$$

Hence, we have

(1.4.4) the following diagram also commutes:

$$\begin{array}{ccc} (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(\pi_{\bullet, \lambda}, \Psi)} & (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0, 0, 0)) \\ h' \downarrow & & H' \downarrow \\ (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) & \xrightarrow{(\pi_{\bullet, y}, F)} & (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^p, (0, 0, 0)). \end{array}$$

Therefore,  $\{h', H', \phi'_H\}$  is a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from the given deformation  $\Psi$  to the graph deformation  $F$ .

In particular, by (1.4.2) and (1.4.4) we have

$$(1.4.5) \quad h'(\Psi^{-1}(0)) \subset F^{-1}(0).$$

Furthermore, by (1.4.1) - (1.4.4) and the remark after definition (1.1) we have

$$(1.4.6) \quad h'(\mathbb{R}^n \times \mathbb{R}^r - \Psi^{-1}(0)) \subset \mathbb{R}^n \times \mathbb{R}^p - F^{-1}(0).$$

For the proofs of theorems A, D, E, we need only the properties (1.4.1), (1.4.5) and (1.4.6) (see §. 2).

EXAMPLE (1.5): For any  $C^\infty$  map germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ ,

(1) let  $\Psi_1 : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be its  $C^\infty$  deformation of the form  $\Psi_1(x, \lambda) = f(x) + \lambda$ . Then,  $\{h(x, \lambda) = (x, \lambda), H(y, \lambda) = (y - \lambda, \lambda) \text{ and } \phi(\lambda) = \lambda\}$  gives a triviality of  $\Psi_1$ . In this case,  $\phi'_H(\lambda) = -\lambda$ ,  $h'(x, \lambda) = (x, -\lambda)$  and  $H'(x, \lambda, y) = (x, -\lambda, y)$  as we expect. Of course,  $\{h', H', \phi'_H\}$  is an equivalent  $\mathcal{K}$ -morphism from  $\Psi_1$  to  $F$ .

(2) let  $\Psi_2 : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be the deformation of  $f$  of the form  $\Psi_2(x, \lambda) = f(x) - \lambda^3$ ; where  $\lambda^3 = (\lambda_1^3, \dots, \lambda_p^3)$ . Then  $\{h(x, \lambda) = (x, \lambda), H(y, \lambda) = (y + \lambda^3, \lambda) \text{ and } \phi(\lambda) = \lambda\}$  gives a topological triviality of  $\Psi_2$ . In this case,  $\phi'_H(\lambda) = \lambda^3$ ,  $h'(x, \lambda) = (x, \lambda^3)$  and  $H'(x, \lambda, y) = (x, \lambda^3, y)$ . We see  $\{h', H', \phi'_H\}$  is a topologically equivalent topological  $\mathcal{K}$ -morphism from  $\Psi_2$  to  $F$ .

DEFINITION (1.6). Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ and let  $\Psi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  deformation of  $f$ . We say  $\Psi$  is  $\mathcal{K}$ -versal (resp. topologically  $\mathcal{K}$ -versal) if for any  $C^\infty$  deformation  $\tilde{\Psi} : (\mathbb{R}^n \times \mathbb{R}^t, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$  there exist  $C^\infty$  (resp. continuous) map germs  $h : (\mathbb{R}^n \times \mathbb{R}^t, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^r, (0, 0))$ ,  $H : (\mathbb{R}^n \times \mathbb{R}^t \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^p, (0, 0))$  and  $\phi : (\mathbb{R}^t, 0) \rightarrow (\mathbb{R}^r, 0)$  which give a  $\mathcal{K}$ -morphism (resp. topological  $\mathcal{K}$ -morphism) from  $\tilde{\Psi}$  to  $\Psi$ .

We can define  $\mathcal{A}$ -versality and topological  $\mathcal{A}$ -versality similarly. Let  $\mathcal{G}$  be  $\mathcal{K}$  or  $\mathcal{A}$ . The definition of  $\mathcal{G}$ -versality is equivalent to that of Martinet's definitions ([Mr]); and the definition of topological  $\mathcal{G}$ -versality is its topological analogue.

Since any  $C^\infty$  stable map germ is, when viewed as a  $C^\infty$  deformation of itself,  $\mathcal{A}$ -versal: i.e. any  $C^\infty$  deformation  $\Psi$  of a  $C^\infty$  stable map germ has a triviality; by the above argument we see

THEOREM B (MARTINET([Mr])). For any  $C^\infty$  stable map germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ , its graph deformation  $F(x, y) = f(x) - y$  is  $\mathcal{K}$ -versal.

There are several definitions for topological stable map germs (for instance, [dW]). However, it is well-known that for any MT-stable map germ (map germ multi-transversal to Thom-Mather canonical stratification) any  $C^\infty$  deformation of it has a topological triviality (see [M3] or [GWdL]). Hence, again by the above argument, we see

THEOREM C. For any MT-stable map germ  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ , its graph deformation  $F(x, y) = f(x) - y$  is topologically  $\mathcal{K}$ -versal.

## §2. SPECIAL CASE OF §1

In this chapter, we review a part of Martinet's argument in [Mr]. Let  $f, g :$

$(\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be  $C^\infty$  map germs. Suppose there exist a  $C^\infty$  diffeomorphic (resp. homeomorphic) map germ  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  and a  $C^\infty$  map germ

$$M(x) = (m_1(x), \dots, m_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

such that  $f(x) = M(x)(g \circ s)(x)$ .

We set a  $C^\infty$  map germ  $\Phi : (\mathbb{R}^n \times \mathbb{R}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  as

$$\begin{aligned}\Phi(x, y) &= M(x)((g \circ s)(x) - y) \\ &= f(x) - M(x)y.\end{aligned}$$

Hereafter, we concentrate on studying deformations of this type. Hence, in particular, we assume  $r = p$ . We treat two kinds of  $p$ -dimensional euclidean space  $\mathbb{R}^p$ . When we are considering  $\mathbb{R}^p$  as the target space, we write it  $\mathbb{R}_y^p$ . When we are considering  $\mathbb{R}^p$  as the parameter space, we write it  $\mathbb{R}_\lambda^p$ .

Now suppose there exist  $C^\infty$ -diffeomorphic (resp. homeomorphic) map germs  $h : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0))$ ,  $H : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0))$  and  $\phi : (\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_\lambda^p, 0)$  such that the following diagram commutes:

$$\begin{array}{ccccc}(\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{(\Phi, \pi_\lambda)} & (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}_\lambda^p, 0) \\ h \downarrow & & H \downarrow & & \phi \downarrow \\ (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}_\lambda^p, 0)\end{array}$$

In §1, we defined  $C^\infty$  (resp. continuous) map germs

$$\begin{aligned}\phi'_H &: (\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_y^p, 0) \\ h' &: (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}_y^p, (0, 0)) \\ H' &: (\mathbb{R}^n \times \mathbb{R}_\lambda^p \times \mathbb{R}_y^p, (0, 0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}_y^p \times \mathbb{R}_y^p, (0, 0, 0))\end{aligned}$$

and we saw  $\{h', H', \phi'_H\}$  is a  $\mathcal{K}$ -morphism from  $\Phi$  to  $F$ . By (1.4.5) in §1 and by the form of  $\Phi$ , we have

$$(2.1) \quad f(h_1(x, (g \circ s)(x))) = \phi'_H((g \circ s)(x))$$

as germs at the origin.

We would like to show the following map germ (2.2) is  $C^\infty$  diffeomorphic (resp. homeomorphic) if we assume  $\phi'_H$  is  $C^\infty$  diffeomorphic (resp. homeomorphic).

$$(2.2) \quad x \longmapsto h_1(x, (g \circ s)(x))$$

The map germ (2.2) can be decomposed as follows.

$$(2.3) \quad x \mapsto (x, (g \circ s)(x)) \mapsto h'(x, (g \circ s)(x)) \mapsto h_1(x, (g \circ s)(x)).$$

The first map germ of (2.3) is trivially  $C^\infty$  diffeomorphic. If we assume  $\phi'_H$  is  $C^\infty$  diffeomorphic (resp. homeomorphic), then by (1.4.1) in §1  $h' = (h_1, \phi'_H)$  is  $C^\infty$  diffeomorphic (resp. homeomorphic). Thus, the second map germ of (2.3) is  $C^\infty$  diffeomorphic (resp. homeomorphic). Furthermore, in the case that we assume  $\phi'_H$  is  $C^\infty$  diffeomorphic (resp. homeomorphic), by (1.4.5) and (1.4.6) in §1 we have

$$(2.4) \quad h'(\Phi^{-1}(0)) = F^{-1}(0).$$

By the form of  $\Phi$  and  $F$ , (2.4) means

$$(2.5) \quad \begin{aligned} & \text{the germ of the set } \{h'(x, (g \circ s)(x)) | x \in \mathbb{R}^n\} \\ & = \text{the germ of } F^{-1}(0) \\ & = \text{graph}(f). \end{aligned}$$

By (2.5) and by the form of  $h' = (h_1, \phi'_H)$ , the last map germ of (2.3) is also  $C^\infty$  diffeomorphic.

Therefore, we see

LEMMA (2.6). *If  $\Phi$  has a triviality (resp. topological triviality) and  $\phi'_H$  is  $C^\infty$  diffeomorphic (resp. homeomorphic), then  $f$  and  $g$  are right-left equivalent (resp. topologically right-left equivalent).*

### §3. MODULE

Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be a  $C^\infty$  map germ and let

$$M(x) = (m_1(x), \dots, m_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

be also a  $C^\infty$  map germ. Let  $\Phi : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p, 0)$  be the  $C^\infty$  deformation of  $f$  having the following form:

$$\Phi(x, \lambda) = f(x) - M(x)\lambda.$$

In this chapter, we prove the following lemma.

LEMMA (3.1). *Suppose there exists a positive integer  $k$  such that*

$$m_i(x) - m_i(0) \in m_x^k \mathcal{E}_x^p \subset tf(m_x \mathcal{E}_x^n) + \omega f(m_y \mathcal{E}_y^p)$$

for any  $i$  ( $1 \leq i \leq p$ ). Then  $\mathbf{m}_i(\mathbf{x}) - \mathbf{m}_i(0)$  is included in

$$t\tilde{\Phi}_{\mathbf{x}}(\mathbf{m}_{\mathbf{x},\lambda}\mathcal{E}_{\mathbf{x},\lambda}^n) + \omega(\tilde{\Phi}, \pi_{\lambda})(\mathbf{m}_{\mathbf{y},\lambda}\mathcal{E}_{\mathbf{y},\lambda}^p)$$

for any  $i$  ( $1 \leq i \leq p$ ).

PROOF OF LEMMA (3.1): Since we assumed

$$\mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x}}^p \subset tf(\mathbf{m}_{\mathbf{x}}\mathcal{E}_{\mathbf{x}}^n) + \omega f(\mathbf{m}_{\mathbf{y}}\mathcal{E}_{\mathbf{y}}^p),$$

by Malgrange preparation theorem we have

$$(3.2) \quad \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p \subset tf(\mathbf{m}_{\mathbf{x}}\mathcal{E}_{\mathbf{x},\lambda}^n) + \omega(f, \pi_{\lambda})(\mathbf{m}_{\mathbf{y}}\mathcal{E}_{\mathbf{y},\lambda}^p).$$

We set  $\tilde{\Phi} : (\mathbb{R}^n \times \mathbb{R}_{\lambda}^p, (0, 0)) \rightarrow (\mathbb{R}_{\mathbf{y}}^p, 0)$  as

$$\begin{aligned} \tilde{\Phi}(\mathbf{x}, \lambda) &= \Phi(\mathbf{x}, \lambda) + M(0)\lambda \\ &= f(\mathbf{x}) - (M(\mathbf{x}) - M(0))\lambda. \end{aligned}$$

Since we assumed

$$\mathbf{m}_i(\mathbf{x}) - \mathbf{m}_i(0) \in \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x}}^p$$

for any  $i$  ( $1 \leq i \leq p$ ), the difference

$$\tilde{\Phi}(\mathbf{x}, \lambda) - f(\mathbf{x}) = (M(\mathbf{x}) - M(0))\lambda = \sum_{i=1}^p \lambda_i (\mathbf{m}_i(\mathbf{x}) - \mathbf{m}_i(0))$$

is included in

$$\pi_{\lambda}^* \mathbf{m}_{\lambda} \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p \subset (\tilde{\Phi}, \pi_{\lambda})^* \mathbf{m}_{\mathbf{y},\lambda} \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p.$$

Hence, we can approximate (3.2) as follows.

$$\begin{aligned} (3.3) \quad \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p &\subset t\tilde{\Phi}_{\mathbf{x}}(\mathbf{m}_{\mathbf{x}}\mathcal{E}_{\mathbf{x},\lambda}^n) + \omega(\tilde{\Phi}, \pi_{\lambda})(\mathbf{m}_{\mathbf{y}}\mathcal{E}_{\mathbf{y},\lambda}^p) + (\tilde{\Phi}, \pi_{\lambda})^* \mathbf{m}_{\mathbf{y},\lambda} \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p \\ &\subset t\tilde{\Phi}_{\mathbf{x}}(\mathbf{m}_{\mathbf{x},\lambda}\mathcal{E}_{\mathbf{x},\lambda}^n) + \omega(\tilde{\Phi}, \pi_{\lambda})(\mathbf{m}_{\mathbf{y},\lambda}\mathcal{E}_{\mathbf{y},\lambda}^p) + (\tilde{\Phi}, \pi_{\lambda})^* \mathbf{m}_{\mathbf{y},\lambda} \mathbf{m}_{\mathbf{x}}^k \mathcal{E}_{\mathbf{x},\lambda}^p. \end{aligned}$$

We set

$$C = \mathcal{E}_{\mathbf{x},\lambda}^p / t\tilde{\Phi}_{\mathbf{x}}(\mathbf{m}_{\mathbf{x},\lambda}\mathcal{E}_{\mathbf{x},\lambda}^n),$$

$$A = \text{image of } \omega(\tilde{\Phi}, \pi_{\lambda})(\mathbf{m}_{\mathbf{y},\lambda}\mathcal{E}_{\mathbf{y},\lambda}^p) \text{ by the canonical projection to } C,$$

$$B = \mathbf{m}_{\mathbf{x}}^k.C.$$

Then, by (3.3) we have

$$(3.4) \quad B \subset A + (\tilde{\Phi}, \pi_{\lambda})^* \mathbf{m}_{\mathbf{y},\lambda} B.$$

Since

$$\begin{aligned} & \dim_{\mathbb{R}} B / (\tilde{\Phi}, \pi_{\lambda})^* \mathfrak{m}_{y,\lambda} B \\ &= \dim_{\mathbb{R}} \mathfrak{m}_x^k \mathcal{E}_x^p / \mathfrak{m}_x^k (t f(\mathfrak{m}_x \mathcal{E}_x^p) + f^* \mathfrak{m}_y \mathcal{E}_x^p) < \infty, \end{aligned}$$

by Malgrange preparation theorem we see  $B$  is finitely generated  $\mathcal{E}_{y,\lambda}$ -module via  $(\tilde{\Phi}, \pi_{\lambda})$ . Hence, by Nakayama's lemma (3.4) implies

$$(3.5) \quad B \subset A$$

From the form  $\tilde{\Phi}(x, \lambda) = \Phi(x, \lambda) + M(0)\lambda$ , we see

$$(3.6) \quad \begin{aligned} & t\tilde{\Phi}_x(\mathfrak{m}_{x,\lambda} \mathcal{E}_{x,\lambda}^n) + \omega(\tilde{\Phi}, \pi_{\lambda})(\mathfrak{m}_{y,\lambda} \mathcal{E}_{y,\lambda}^p) \\ &= t\Phi_x(\mathfrak{m}_{x,\lambda} \mathcal{E}_{x,\lambda}^n) + \omega(\Phi, \pi_{\lambda})(\mathfrak{m}_{y,\lambda} \mathcal{E}_{y,\lambda}^p) \end{aligned}$$

(3.5) and (3.6) yields

$$\mathfrak{m}_i(x) - \mathfrak{m}_i(0) \in \mathfrak{m}_x^k \mathcal{E}_{x,\lambda}^p \subset t\Phi_x(\mathfrak{m}_{x,\lambda} \mathcal{E}_{x,\lambda}^n) + \omega(\Phi, \pi_{\lambda})(\mathfrak{m}_{y,\lambda} \mathcal{E}_{y,\lambda}^p)$$

for any  $i$  ( $1 \leq i \leq p$ ). ■

#### §4. PROOF OF THEOREM A

Let  $\Phi : (\mathbb{R}^n \times \mathbb{R}_{\lambda}^p, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  be the  $C^{\infty}$  deformation of  $f$  having the following form:

$$\Phi(x, \lambda) = f(x) - M(x)\lambda.$$

Since

$$\frac{\partial \Phi}{\partial \lambda_i} = -\mathfrak{m}_i(x)$$

for any  $i$  ( $1 \leq i \leq p$ ), by lemma (3.1) we can choose germs of  $C^{\infty}$  vector fields

$$\xi_i \in \mathcal{E}_{x,\lambda}^n \quad \text{and} \quad \eta_i \in \mathcal{E}_{y,\lambda}^p$$

such that

$$(4.1) \quad -\frac{\partial \Phi}{\partial \lambda_i} = \xi_i(\Phi) - \eta_i \circ (\Phi, \pi_{\lambda})$$

$$(4.2) \quad \frac{\partial \Phi}{\partial \lambda_i}(0) = \eta_i(0, 0)$$

for any  $i$  ( $1 \leq i \leq p$ ).

By (4.1), integrating germs of  $C^\infty$  vector fields

$$\xi_1 + \partial/\partial\lambda_1, \dots, \xi_p + \partial/\partial\lambda_p$$

and

$$\eta_1 + \partial/\partial\lambda_1, \dots, \eta_p + \partial/\partial\lambda_p$$

yields  $C^\infty$  diffeomorphic map germs

$$h^{-1} : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0))$$

and

$$H^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0))$$

such that the following diagram commutes.

$$\begin{array}{ccccc} (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{(\Phi, \pi_\lambda)} & (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}_\lambda^p, 0) \\ h^{-1} \uparrow & & H^{-1} \uparrow & & \parallel \\ (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}_\lambda^p, 0) \end{array}$$

Consider the inverse map germ  $H$  of  $H^{-1}$  and

$$\phi'_H : (\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_y^p, 0)$$

associated with  $H$ .

Let  $\Theta_i(t; \mathbf{y})$  be the integral curve of  $\eta_i$  starting from  $\mathbf{y}$  and of time  $t$ . Then we can get the image  $\mathbf{y}(\lambda_1, \dots, \lambda_p) = \phi'_H(\lambda_1, \dots, \lambda_p)$  of  $\lambda = (\lambda_1, \dots, \lambda_p)$  by  $\phi'_H$  as the unique solution of the integral equation

$$(4.3) \quad \Theta_1(\lambda_1; \Theta_2(\lambda_2; \dots; \Theta_p(\lambda_p; \mathbf{y}(\lambda_1, \dots, \lambda_p))) \dots) = 0.$$

We differentiate (4.3) with respect to  $\lambda_i$ . Then we get

$$(4.4) \quad \begin{aligned} & \eta_i(\Theta_{i+1}(\lambda_{i+1}; \dots; \Theta_p(\lambda_p; \mathbf{y})) \dots) \\ & + (d\Theta_1)_y \dots (d\Theta_p)_y \partial \mathbf{y}(\lambda_1, \dots, \lambda_p) / \partial \lambda_i = 0 \end{aligned}$$

for any  $i$  ( $1 \leq i \leq p$ ).

Taking values at  $\lambda = 0$  in (4.4), we get

$$\begin{aligned} \frac{\partial \phi'_H}{\partial \lambda_i}(0) &= \frac{\partial \mathbf{y}}{\partial \lambda_i}(0) \\ &= -\eta_i(0, 0) \quad (\mathbf{y}(0, \dots, 0) = 0) \\ &= -\frac{\partial \Phi}{\partial \lambda_i}(0) \quad (4.2) \\ &= \mathbf{m}_i(0). \quad \left( \frac{\partial \Phi}{\partial \lambda_i} = -\mathbf{m}_i \right) \end{aligned}$$

Since  $(\mathbf{m}_1(0), \dots, \mathbf{m}_p(0)) = M(0)$  is in  $GL(p, \mathbb{R})$ ,  $\phi'_H$  is  $C^\infty$  diffeomorphic. Hence, by lemma (2.3),  $f$  and  $g$  are right-left equivalent. ■

### §5. AN ALTERNATIVE PROOF OF MATHER'S CLASSIFICATION THEOREM

In this chapter, we give a proof of the following theorem D which is a generalized version of Mather's classification theorem.

**DEFINITION (5.1).** Let  $X$  be a Banach space. We say a  $C^\infty$  map germ  $f : (X, 0) \rightarrow (\mathbb{R}^p, 0)$  is  $C^\infty$  stable if for any finite dimensional  $C^\infty$  deformation  $\Phi : (X \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$  there exist  $C^\infty$  diffeomorphic map germs

$$\begin{aligned} h : (X \times \mathbb{R}^r, (0, 0)) &\rightarrow (X \times \mathbb{R}^r, (0, 0)) \\ H : (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) &\rightarrow (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \\ \phi : (\mathbb{R}^r, 0) &\rightarrow (\mathbb{R}^r, 0) \end{aligned}$$

such that the following diagram commutes:

$$\begin{array}{ccccc} (X \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(\Phi, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0) \\ h \downarrow & & H \downarrow & & \phi \downarrow \\ (X \times \mathbb{R}^r, (0, 0)) & \xrightarrow{(f, \pi_\lambda)} & (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) & \xrightarrow{\pi_\lambda} & (\mathbb{R}^r, 0). \end{array}$$

**THEOREM D.** Let  $X$  be a Banach space. Let  $f, g : (X, 0) \rightarrow (\mathbb{R}^p, 0)$  be two  $C^\infty$  stable map germs. Suppose there exists a  $C^\infty$  diffeomorphic germ  $s : (X, 0) \rightarrow (X, 0)$  and a  $C^\infty$  map germ

$$M(x) = (\mathbf{m}_1(x), \dots, \mathbf{m}_p(x)) : (X, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

such that  $f(x) = M(x)(g \circ s)(x)$ . Then  $f$  and  $g$  are isomorphic.

Mather's classification theorem ([M2]) is the case when  $X$  is finite dimensional.

**PROOF:** Let  $M_p(\mathbb{R})$  be the set of all  $(p \times p)$  matrices of real elements and let  $E_p$  be the  $(p \times p)$  unit matrix. For any fixed matrix  $A = (a_1, \dots, a_p) \in M_p(\mathbb{R})$ , define a map germ

$$\Phi_A : (X \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_y^p, 0)$$

as

$$\Phi_A(x, \lambda, B) = f(x) - (A + M(x)B)\lambda.$$

Then  $\Phi_A$  is a finite dimensional  $C^\infty$  deformation of  $f$ . Since  $f$  is  $C^\infty$  stable, for any  $i$  ( $1 \leq i \leq p$ ) and  $A = O$  (zero matrix) we see

$$\sum_{j=1}^p b_{ji} \mathbf{m}_j(\mathbf{x}) = 0 + \sum_{j=1}^p b_{ji} \mathbf{m}_j(\mathbf{x}) = \frac{\partial \Phi_0}{\partial \lambda_i}$$

is included in the set

$$t(\Phi_0)_x(\mathcal{E}_{x,\lambda,B}^n + \omega(\Phi_0, \pi_\lambda, \pi_B)(\mathcal{E}_{y,\lambda,B}^p)).$$

Here we set  $B = [b_{ij}]_{1 \leq i,j \leq p}$ . Since we see trivially

$$t(\Phi_0)_x = t(\Phi_A)_x$$

and

$$\omega(\Phi_0, \pi_\lambda, \pi_B)(\mathcal{E}_{y,\lambda,B}^p) = \omega(\Phi_A, \pi_\lambda, \pi_B)(\mathcal{E}_{y,\lambda,B}^p)$$

for any fixed  $A \in M_p(\mathbb{R})$ , we can choose germs of  $C^\infty$  vector fields

$$\xi_i \in \mathcal{E}_{x,\lambda,B}^n \quad \text{and} \quad \eta_{i,A} \in \mathcal{E}_{y,\lambda,B}^p$$

such that

$$\begin{aligned} -\frac{\partial \Phi_A}{\partial \lambda_i} &= (\mathbf{a}_i + \sum_{j=1}^p b_{ji} \mathbf{m}_j(\mathbf{x})) \\ &= \xi_i(\Phi_A) - \eta_{i,A} \circ (\Phi_A, \pi_\lambda, \pi_B) \end{aligned}$$

for any  $i$  ( $1 \leq i \leq p$ ). Since  $f$  is  $C^\infty$  stable, we can choose germs of  $C^\infty$  vector fields

$$\xi_{jk,A} \in \mathcal{E}_{x,\lambda,B}^n \quad \text{and} \quad \eta_{jk,A} \in \mathcal{E}_{y,\lambda,B}^p$$

such that

$$\begin{aligned} -\frac{\partial \Phi_A}{\partial b_{jk}} &= \lambda_k \mathbf{m}_j(\mathbf{x}) \\ &= \xi_{jk,A}(\Phi_A) - \eta_{jk,A} \circ (\Phi_A, \pi_\lambda, \pi_B) \end{aligned}$$

for any  $j, k$  ( $1 \leq j, k \leq p$ ) and any  $A$  of  $M_p(\mathbb{R})$ .

By integrating

$$\begin{aligned} &\eta_{1,A} + \partial/\partial \lambda_1, \dots, \eta_{p,A} + \partial/\partial \lambda_p, \\ &\eta_{11,A} + \partial/\partial b_{11}, \dots, \eta_{pp,A} + \partial/\partial b_{pp}, \end{aligned}$$

we get a  $C^\infty$  diffeomorphic germ

$$H_A^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)).$$

We consider the map germ

$$\phi'_{H_A} : (\mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, E_p)) \rightarrow (\mathbb{R}_y^p, 0)$$

associated with  $H_A$  and its restriction

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}}$$

for  $B$  sufficiently near the zero matrix.

Let  $\Theta_{i,A}(t; \mathbf{y})$  (resp.  $\Theta_{j,k,A}(t; \mathbf{y})$ ) be the integral curve of  $\eta_{i,A}$  (resp.  $\eta_{j,k,A}$ ) starting from  $\mathbf{y}$  and of time  $t$  for any  $i, j, k$  ( $1 \leq i, j, k \leq p$ ). Then  $\phi'_{H_A}(\lambda_1, \dots, \lambda_p, b_{11}, \dots, b_{pp}) = \mathbf{y}$ , where  $\mathbf{y}$  is the unique solution of the following integral equation

$$\Theta_{1,A}(\lambda_1; \dots; \Theta_{p,A}(\lambda_p; \Theta_{11,A}(b_{11}; \dots (\Theta_{pp,A}(b_{pp}; \mathbf{y}(\lambda_1, \dots, b_{11}, \dots, b_{pp}))) \dots)) = 0.$$

We differentiate this equation with respect to  $\lambda_i$ . Then we have

$$(5.2) \quad \eta_{i,A}(\Theta_{i+1,A}(\lambda_{i+1}; \dots; \Theta_{pp,A}(b_{pp}; \mathbf{y}))) \dots + (d\Theta_{1,A})_{\mathbf{y}} \dots (d\Theta_{pp,A})_{\mathbf{y}} \partial \mathbf{y}(\lambda_1, \dots, \lambda_p, b_{11}, \dots, b_{pp}) / \partial \lambda_i = 0$$

for any  $i$  ( $1 \leq i \leq p$ ).

Taking values at  $\lambda = 0$  and  $B = E_p$  in (5.2), we get

$$(5.3) \quad \begin{aligned} \frac{\partial \phi'_{H_A}}{\partial \lambda_i}(0, E_p) &= \frac{\partial \mathbf{y}}{\partial \lambda_i}(0, E_p) \\ &= -\eta_{i,A}(0, 0, E_p) \\ &= -\frac{\partial \Phi_A}{\partial \lambda_i}(0, 0, E_p) - d(\Phi_A)_x(\xi_i(0, 0, E_p)) \\ &= -\frac{\partial \Phi_A}{\partial \lambda_i}(0, 0, E_p) - df_0(\xi_i(0, 0, E_p)) \\ &= \mathbf{a}_i + \mathbf{m}_i(0) - df_0(\xi_i(0, 0, E_p)) \end{aligned}$$

for any  $i$  ( $1 \leq i \leq p$ ). From (5.3) and since  $\xi_i$  is  $C^\infty$  with respect to  $B = [b_{ij}]$ , we have

LEMMA (5.4). *There exists an open dense subset  $\mathcal{U}$  of  $M_p(\mathbb{R})$  such that for any  $A$  of  $\mathcal{U}$  there exists a neighborhood  $\mathcal{V}_A$  of  $E_p$  in  $M_p(\mathbb{R})$  such that the germ of the restriction*

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}} : (\mathbb{R}_\lambda^p \times \{B\}, (0, B)) \rightarrow (\mathbb{R}_y^p, 0)$$

is  $C^\infty$  diffeomorphic for any  $B$  of  $\mathcal{V}_A$ .

Therefore, by lemmata (2.6) and (5.4), we have

LEMMA (5.5). If we choose  $(p \times p)$  matrix  $A$  of  $\mathcal{U}$  sufficiently near the zero matrix, then  $f(x)$  and  $g_{A,B}(x) = (A + M(x)B)^{-1}f(x)$  are right-left equivalent for any  $B$  of  $\mathcal{V}_A$ .

Next, we take a matrix  $A_0$  of  $\mathcal{U}$  sufficiently near the zero matrix and fix it. We set

$$M(x)^{-1}A_0 = N_{A_0}(x) = (\mathbf{n}_1(x), \dots, \mathbf{n}_p(x)).$$

For any fixed  $B$  of  $\mathcal{V}_{A_0}$ , we define the  $C^\infty$  map germ

$$\tilde{\Phi}_{A_0,B} : (X \times \mathbb{R}_\lambda^p, (0,0)) \rightarrow (\mathbb{R}_y^p, 0)$$

as

$$\tilde{\Phi}_{A_0,B}(x, \lambda) = (N_{A_0}(x) + B)(g_{A_0,B}(x) - \lambda).$$

Then, since

$$\begin{aligned} (g \circ s)(x) &= M(x)^{-1}(A_0 + M(x)B)(A_0 + M(x)B)^{-1}f(x) \\ &= M(x)^{-1}(A_0 + M(x)B)g_{A_0,B}(x) \\ &= (N_{A_0}(x) + B)g_{A_0,B}(x); \end{aligned}$$

we see  $\tilde{\Phi}_{A_0,B}(x, \lambda) = (g \circ s)(x) - (N_{A_0}(x) + B)\lambda$  is a  $C^\infty$  deformation of  $(g \circ s)$ . Since  $(g \circ s)$  is  $C^\infty$  stable, for any  $i$  ( $1 \leq i \leq p$ ) and  $B = E_p$  we see

$$\frac{\partial \tilde{\Phi}_{A_0,E_p}}{\partial \lambda_i} \in t(\tilde{\Phi}_{A_0,E_p})_x(\mathcal{E}_{x,\lambda}^n) + \omega(\tilde{\Phi}_{A_0,E_p}, \pi_\lambda)(\mathcal{E}_{y,\lambda}^p).$$

Since

$$t(\tilde{\Phi}_{A,E_p})_x = t(\tilde{\Phi}_{A,B})_x$$

and

$$\omega(\tilde{\Phi}_{A,E_p}, \pi_\lambda)(\mathcal{E}_{y,\lambda}^p) = \omega(\tilde{\Phi}_{A,B}, \pi_\lambda)(\mathcal{E}_{y,\lambda}^p)$$

for any  $A \in \mathcal{U}$  and  $B \in \mathcal{V}_A$ , we can choose germs of  $C^\infty$  vector fields

$$\tilde{\xi}_i \in \mathcal{E}_{x,\lambda}^n \quad \text{and} \quad \tilde{\eta}_{i,B} \in \mathcal{E}_{y,\lambda}^p$$

such that

$$-\frac{\partial \tilde{\Phi}_{A_0,B}}{\partial \lambda_i} = \tilde{\xi}_i(\tilde{\Phi}_{A_0,B}) - \tilde{\eta}_{i,B} \circ (\tilde{\Phi}_{A_0,B}, \pi_\lambda)$$

for any  $i$  ( $1 \leq i \leq p$ ).

By integrating

$$\tilde{\eta}_{1,B} + \partial/\partial \lambda_1, \dots, \tilde{\eta}_{p,B} + \partial/\partial \lambda_p,$$

we get a  $C^\infty$  diffeomorphic germ

$$H_{A_0, B}^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)).$$

We consider the map germ

$$\phi'_{H_{A_0, B}} : (\mathbb{R}_\lambda^p, 0) \rightarrow (\mathbb{R}_y^p, 0)$$

associated with  $H_{A_0, B}$ . We see

$$\begin{aligned} (5.6) \quad \frac{\partial \phi'_{H_{A_0, B}}}{\partial \lambda_i}(0) &= -\tilde{\eta}_{i, B}(0, 0) \\ &= -\frac{\partial \tilde{\Phi}_{A_0, B}}{\partial \lambda_i}(0) - d(\tilde{\Phi}_{A_0, B})_x(\tilde{\xi}_i(0, 0)) \\ &= -\frac{\partial \tilde{\Phi}_{A_0, B}}{\partial \lambda_i}(0) - d(g \circ s)_0(\tilde{\xi}_i(0, 0)) \\ &= \mathbf{n}_i(0) + \mathbf{b}_i - d(g \circ s)_0(\tilde{\xi}_i(0, 0)) \end{aligned}$$

for any  $i$  ( $1 \leq i \leq p$ ). By (5.6), we can choose a matrix  $B$  of  $\mathcal{V}_{A_0}$  with the property that  $\phi'_{H_{A_0, B}}$  is  $C^\infty$  diffeomorphic. Thus, by lemma (2.6), we have

**LEMMA (5.7).** *We can choose a matrix  $B$  of  $\mathcal{V}_{A_0}$  with the property that  $(g \circ s)$  and  $g_{A_0, B}$  are right-left equivalent.*

Lemmata (5.5) and (5.7) concludes that  $f$  and  $g$  are isomorphic. ■

## §6. AN ALTERNATIVE PROOF OF FUKUDA-FUKUDA'S THEOREM

In this chapter, we give a proof of the following theorem.

**THEOREM E ([FF]).** *Let  $f, g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be two MT-stable map germs. Suppose there exists a  $C^\infty$  diffeomorphic germ  $s : (\mathbb{R}^n, 0) \rightarrow (X, 0)$  and a  $C^\infty$  map germ*

$$M(x) = (\mathbf{m}_1(x), \dots, \mathbf{m}_p(x)) : (\mathbb{R}^n, 0) \rightarrow (GL(p, \mathbb{R}), M(0))$$

*such that  $f(x) = M(x)(g \circ s)(x)$ . Then  $f$  and  $g$  are topologically isomorphic.*

For the definition of MT-stable map germs, refer to [M3] or [GWdL]. For our proof of theorem E, we use only the following fact on MT-stable map germs.

FACT (6.1). Let  $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  be an MT-stable map germ. Then for any  $C^\infty$  deformation  $\Phi : (\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \rightarrow (\mathbb{R}^p, 0)$  of  $f$  there exist Whitney stratifications  $S$  of  $\mathbb{R}^n \times \mathbb{R}^r$  and  $T$  of  $\mathbb{R}^p \times \mathbb{R}^r$  such that the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}^r, (0, 0)) \xrightarrow{(\Phi, \pi_\lambda)} (\mathbb{R}^p \times \mathbb{R}^r, (0, 0)) \xrightarrow{\pi_\lambda} (\mathbb{R}^r, 0)$$

is a Thom sequence with respect to  $S, T$  and  $\{\mathbb{R}^r\}$ .

PROOF OF THEOREM E: As in §5, for any fixed matrix  $A = (a_1, \dots, a_p) \in M_p(\mathbb{R})$ , define a map germ

$$\Psi_A : (X \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_y^p, 0)$$

as

$$\Psi_A(x, \lambda, B) = f(x) - (A + M(x)B)\lambda.$$

Then  $\Psi_A$  is a  $C^\infty$  deformation of  $f$ . Since  $f$  is MT-stable, by (6.1), there exist Whitney stratifications  $S$  of  $\mathbb{R}^n \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R})$  and  $T$  of  $\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R})$  such that the germ of the sequence

$$\begin{array}{c} (\mathbb{R}^n \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \\ \downarrow (\Psi_0, \pi_{\lambda, B}) \\ (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \\ \downarrow \pi_{\lambda, B} \\ (\mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, E_p)) \end{array}$$

is a Thom sequence with respect to  $S, T$  and  $\{\mathbb{R}_\lambda^p \times M_p(\mathbb{R})\}$ .

For any stratum  $T$  of  $T$  and any  $A$  of  $M_p(\mathbb{R})$ , we set

$$T_A = \{(y, \lambda, B) - (A\lambda, \lambda, B) \in \mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}) \mid (y, \lambda, B) \in T\}$$

and

$$T_A = \{T_A\}.$$

Then, since

$$\begin{aligned} \Psi_A(x, \lambda, B) &= \Psi_0(x, \lambda, B) - A\lambda \\ &= \Psi_0(x, \lambda, B) + (\text{family of parallel translation of } \mathbb{R}_y^p) \end{aligned}$$

we see

LEMMA (6.2). For any matrix  $A$  of  $M_p(\mathbb{R})$ , the germ of the sequence

$$\begin{array}{c} (\mathbb{R}^n \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \\ \downarrow (\Psi_A, \pi_{\lambda, B}) \\ (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \\ \downarrow \pi_{\lambda, B} \\ (\mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, E_p)) \end{array}$$

is a Thom sequence with respect to  $S, T_A$  and  $\{\mathbb{R}_\lambda^p \times M_p(\mathbb{R})\}$ .

By (6.2) we see

LEMMA (6.3). There exists an open dense subset  $\mathcal{U}$  of  $M_p(\mathbb{R})$  such that for any  $A$  of  $\mathcal{U}$  there exists a neighborhood  $\mathcal{V}_A$  of  $E_p$  in  $M_p(\mathbb{R})$  such that the subset  $\{0\} \times \mathbb{R}_\lambda^p \times \{B\} (\subset \mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}))$  is transversal to the intersection  $T_A \cap (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times \{B\})$  near  $\{0\} \times \{0\} \times \{0\}$  for any  $B$  of  $\mathcal{V}_A$  and any  $T_A$  of  $\mathcal{T}_A$ .

By lifting vector fields  $\partial/\partial\lambda_1, \dots, \partial/\partial\lambda_p, \partial/\partial b_{11}, \dots, \partial/\partial b_{pp}$ , we get germs of vector fields

$$\begin{aligned} \eta_{1,A} + \partial/\partial\lambda_1, \dots, \eta_{p,A} + \partial/\partial\lambda_p, \\ \eta_{11,A} + \partial/\partial b_{11}, \dots, \eta_{pp,A} + \partial/\partial b_{pp}, \end{aligned}$$

which are stratified with respect to the stratification  $\mathcal{T}_A$  and satisfy the control conditions. By integrating these stratified vector fields, we get a homeomorphic germ

$$H_A^{-1} : (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)) \rightarrow (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, 0, E_p)).$$

We consider the map germ

$$\phi'_{H_A} : (\mathbb{R}_\lambda^p \times M_p(\mathbb{R}), (0, E_p)) \rightarrow (\mathbb{R}_y^p, 0)$$

associated with  $H_A$  and its restriction

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}}$$

for  $B$  sufficiently near the zero matrix.

Let  $\Theta_{i,A}(t; y)$  (resp.  $\Theta_{j,k,A}(t; y)$ ) be the integral curve of  $\eta_{i,A}$  (resp.  $\eta_{j,k,A}$ ) starting from  $y$  and of time  $t$  for any  $i, j, k$  ( $1 \leq i, j, k \leq p$ ). Then

$\phi'_{H_A}(\lambda_1, \dots, \lambda_p, b_{11}, \dots, b_{pp})$  can be given as the unique solution of the following integral equation

$$\Theta_{1,A}(\lambda_1; \dots; \Theta_{p,A}(\lambda_p; \Theta_{11,A}(b_{11}; \dots (\Theta_{pp,A}(b_{pp}; \phi'_{H_A}(\lambda_1, \dots, b_{pp})) \dots)) = 0.$$

Since the germs of vector fields

$$\begin{aligned} \eta_{1,A} + \partial/\partial\lambda_1, \dots, \eta_{p,A} + \partial/\partial\lambda_p, \\ \eta_{11,A} + \partial/\partial b_{11}, \dots, \eta_{pp,A} + \partial/\partial b_{pp}, \end{aligned}$$

are controlled, by lemma (6.3), we see

LEMMA (6.4). *For any  $A$  of  $\mathcal{U}$  and any  $B$  of  $\mathcal{V}_A$ , the germ of the restriction*

$$\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}} : (\mathbb{R}_\lambda^p \times \{B\}, (0, B)) \rightarrow (\mathbb{R}_y^p, 0)$$

*is injective.*

Since  $\phi'_{H_A}|_{\mathbb{R}_\lambda^p \times \{B\}}$  is continuous, injectivity means being homeomorphic. Therefore, by lemma (2.6) we have

LEMMA (6.5). *If we choose  $(p \times p)$  matrix  $A$  of  $\mathcal{U}$  sufficiently near the zero matrix, then  $f(x)$  and  $g_{A,B}(x) = (A + M(x)B)^{-1}f(x)$  are topologically right-left equivalent for any  $B$  of  $\mathcal{V}_A$ .*

Next, we take a matrix  $A_0$  of  $\mathcal{U}$  sufficiently near the zero matrix and fix it. We set

$$M(x)^{-1}A_0 = N_{A_0}(x) = (n_1(x), \dots, n_p(x)).$$

For any fixed  $B$  of  $\mathcal{V}_{A_0}$ , we define the  $C^\infty$  map germ

$$\tilde{\Psi}_{A_0,B} : (\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \rightarrow (\mathbb{R}_y^p, 0)$$

as

$$\tilde{\Psi}_{A_0,B}(x, \lambda) = (N_{A_0}(x) + B)(g_{A_0,B}(x) - \lambda).$$

Then, since

$$\begin{aligned} (g \circ s)(x) &= M(x)^{-1}(A_0 + M(x)B)(A_0 + M(x)B)^{-1}f(x) \\ &= M(x)^{-1}(A_0 + M(x)B)g_{A_0,B}(x) \\ &= (N_{A_0}(x) + B)g_{A_0,B}(x); \end{aligned}$$

we see  $\tilde{\Psi}_{A_0,B}(x, \lambda) = (g \circ s)(x) - (N_{A_0}(x) + B)\lambda$  is a  $C^\infty$  deformation of  $(g \circ s)$ . Since  $g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$  is MT-stable and  $s : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$  is  $C^\infty$

diffeomorphic,  $(g \circ s)(x) = M(x)^{-1}f(x)$  is MT-stable. Thus, by (6.1), there exist Whitney stratifications  $\tilde{S}$  of  $\mathbb{R}^n \times \mathbb{R}_\lambda^p$ ,  $\tilde{T}$  of  $\mathbb{R}_y^p \times \mathbb{R}_\lambda^p$  such that the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \xrightarrow{(\tilde{\Psi}_{A_0, 0}, \pi_\lambda)} (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \xrightarrow{\pi_\lambda} (\mathbb{R}_\lambda^p, 0)$$

is a Thom sequence with respect to  $\tilde{S}, \tilde{T}$  and  $\{\mathbb{R}_\lambda^p\}$ .

For any stratum  $\tilde{T}$  of  $\tilde{T}$  and any  $B$  of  $M_p(\mathbb{R})$ , we set

$$\tilde{T}_B = \{(y, \lambda) - (B\lambda, \lambda) \in \mathbb{R}_y^p \times \mathbb{R}_\lambda^p | (y, \lambda) \in \tilde{T}\}$$

and

$$\tilde{T}_B = \{\tilde{T}_B\}.$$

Then, since

$$\tilde{\Psi}_{A_0, B}(x, \lambda) = \tilde{\Psi}_{A_0, 0}(x, \lambda) - B\lambda$$

we see

LEMMA (6.6). For any matrix  $B$  of  $M_p(\mathbb{R})$ , the germ of the sequence

$$(\mathbb{R}^n \times \mathbb{R}_\lambda^p, (0, 0)) \xrightarrow{(\tilde{\Psi}_{A_0, B}, \pi_\lambda)} (\mathbb{R}_y^p \times \mathbb{R}_\lambda^p, (0, 0)) \xrightarrow{\pi_\lambda} (\mathbb{R}_\lambda^p, 0)$$

is a Thom sequence with respect to  $\tilde{S}, \tilde{T}_B$  and  $\{\mathbb{R}_\lambda^p\}$ .

By lemma (6.6) and by using the same argument as before, we can choose a matrix  $B$  of  $\mathcal{V}_{A_0}$  with the property that  $\phi'_{H_{A_0, B}}$  is homeomorphic. Thus, by lemma (2.6), we have

LEMMA (6.7). We can choose a matrix  $B$  of  $\mathcal{V}_{A_0}$  with the property that  $(g \circ s)$  and  $g_{A_0, B}$  are topologically right-left equivalent.

Lemmata (6.6) and (6.7) concludes that  $f$  and  $g$  are topologically isomorphic. ■

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