

A CHARACTERIZATION OF UNITARY OPERATORS INDUCED
BY NONSINGULAR TRANSFORMATIONS AND ITS APPLICATIONS.

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In this talk we give a necessary and sufficient condition for a unitary operator on an L^2 -space to be induced by a nonsingular transformation and its applications.

Let (X, μ) be a σ -finite measure space, $L^2(\mu)$ be the complex Hilbert space of all square summable functions and $L^\infty(\mu)$ be the algebra of all bounded measurable functions on (X, μ) . Then for every α in $L^\infty(\mu)$ we associate a bounded linear operator $T[\alpha]$ on $L^2(\mu)$ by

$$T[\alpha] : \xi(x) \mapsto \alpha(x)\xi(x), \quad \xi \in L^2(\mu).$$

As is well-known, the correspondence between $L^\infty(\mu)$ and

$$\Lambda(\mu) = \{T[\alpha] ; \alpha \in L^\infty(\mu)\}$$

is isomorphic and $\Lambda(\mu)$ is a commutative von Neumann algebra.

A transformation f of (X, μ) is a nonsingular transformation if it satisfies the following conditions :

(N.1) There exists a null set N such that f is a bijective transformation of $X-N$ onto itself.

(N.2) f is bimeasurable.

(N.3) $m(E) = 0$ if and only if $m \circ f(E) = m(f(E)) = 0$.

We say $[f; q]$ is a nonsingular pair if f is a nonsingular transformation and $q = q(x)$ is a complex measurable function such that

$$|q(x)|^2 = \frac{dm \circ f}{dm}(x), \quad \text{a.e. } (m).$$

For every nonsingular pair $[f; q]$ we define a unitary operator $U[f; q]$ on $L^2(m)$ by

$$U[f; q] : \xi(x) \rightarrow q(x)\xi(f(x)), \quad \xi \in L^2(m).$$

We first prove the following theorem :

THEOREM 1. Let (X, m) be a σ -finite abstract Lebesgue space and U be a unitary operator on $L^2(m)$. Then there exists a nonsingular pair $[f; q]$ such that

$$U = U[f; q]$$

if and only if

$$U^{-1}\Lambda(m)U \subset \Lambda(m).$$

As applications of Theorem 1, we prove the following theorems.

THEOREM 2. Let (X, m) be a non-atomic σ -finite abstract Lebesgue space and U be a unitary operator on $L^2(m)$ such that $U - I$ is a compact operator. Then U is induced by a nonsingular pair if and only if U is the identity operator.

This theorem implies that any non-trivial finite dimensional unitary operator on such an L^2 -space is never induced by a nonsingular pair. Furthermore, let U be a unitary operator on such a space which is represented by a unitary matrix (α_{ij}) such that

$$\sum_{i,j=1}^{\infty} |\alpha_{ij} - \delta_{ij}|^2 < +\infty.$$

Then U is not induced by a nonsingular pair unless U is the identity.

THEOREM 3. Let $[f;q]$ be a nonsingular pair of (R^1, dx) and F be the Fourier transform. Furthermore, assume that $U[f;q]$ is a rotation of S , the nuclear space of all rapidly decreasing functions on the real line. Then $F^{-1}U[f;q]F$ is again induced by a nonsingular pair if and only if $[f;q]$ is given by

$$f(x) = \alpha x + \beta,$$

$$q(x) = \sqrt{|\alpha|} e^{i(\theta x + \tau)},$$

where $\alpha (\neq 0)$, β , θ and τ are real constants.

This theorem enables us to construct two one-parameter unitary groups, one of which is induced by nonsingular pairs and the another is not, and still both of them have the same spectral type.

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