

On A Certain Class Of Starlike Functions.II

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ABSTRACT. There are many classes of starlike functions in the unit disc $U=\{z: |z|<1\}$. In this paper we consider a class $S_p^*(\alpha, \beta, \gamma, A, B)$ of starlike functions of the form $f(z)=z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ ($p \in \mathbb{N}$) in the unit disc U and satisfying the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(B-A)\gamma\left(\frac{zf'(z)}{f(z)} - \alpha\right) + A\left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < \beta$$

for some α ($0 \leq \alpha < 1$), β ($0 < \beta \leq 1$), γ ($0 < \gamma \leq 1$) and $-1 \leq A < B \leq 1$, $0 < B \leq 1$.

It is the purpose of this paper to show a representation formula, a distortion theorem and a sufficient condition for the class $S_p^*(\alpha, \beta, \gamma, A, B)$. Moreover we give the radius of convexity for functions in the class $S_p^*(\alpha, \beta, \gamma, A, B)$.

1. Introduction.

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $U = \{z: |z| < 1\}$. A

A function $f(z) \in S$ is said to be starlike of order α ($0 \leq \alpha < 1$) in the unit disc U if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha$$

for some α , and for all $z \in U$. And the above condition is equivalent to

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\left(\frac{zf'(z)}{f(z)} - \alpha\right) - \left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < 1.$$

In this paper, we consider the class $S_p^*(\alpha, \beta, \gamma, A, B)$ of functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} = \{1, 2, \dots\})$$

which are analytic and starlike in the unit disc U satisfying the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(B-A)\gamma\left(\frac{zf'(z)}{f(z)} - \alpha\right) + A\left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < \beta \quad (1.1)$$

for some α ($0 \leq \alpha < 1$), β ($0 < \beta \leq 1$), γ ($0 < \gamma \leq 1$), $-1 \leq A < B \leq 1$, $0 < B \leq 1$, and for all $z \in U$. The classes $S_1^*(\alpha, 1, \frac{1}{2}, -1, 1)$, $S_1^*(0, 1, \frac{2\gamma-1}{2\gamma}, -1, 1)$ ($\gamma > \frac{1}{2}$), $S_1^*(\frac{1-\gamma}{1+\gamma}, 1, \frac{1+\gamma}{2}, -1, 1)$ ($0 < \gamma \leq 1$), $S_1^*(1-\alpha, 1, \frac{1}{2}, -1, 1)$, $S_1^*(\alpha, 1, \gamma, -1, 1)$ and $S_1^*(\alpha, \beta, \gamma, A, B)$ were studied by McCarty [6], Singh [13,14], Padmanabhan [12], Eenigenburg [3], Juneja and Mogra [4] and Aouf and Nunokawa [1], respectively. Also in [8,9,10,11] Owa showed some results for functions in the class $S_1^*(\alpha, \beta, \gamma, -1, 1)$.

2. A representation formula.

First of all, we require the following lemma.

Lemma 1. Let a function

$$H(z) = 1 + b_p z^p + b_{p+1} z^{p+1} + \dots \quad (p \in \mathbb{N})$$

be analytic in the unit disc U . Then $H(z)$ satisfies the condition

$$\left| \frac{H(z) - 1}{(B-A)\gamma(H(z) - \alpha) + A(H(z) - 1)} \right| < \beta \quad (z \in U)$$

for some α ($0 \leq \alpha < 1$), β ($0 < \beta \leq 1$), γ ($0 < \gamma \leq 1$) and $-1 < A \leq B \leq 1$, $0 < B \leq 1$, if and only if there exists an analytic function $\phi(z)$ in the unit disc U such that $|\phi(z)| \leq \beta$ for $z \in U$ and

$$H(z) = \frac{1 + [(B-A)\alpha\delta + A]z^p\phi(z)}{1 + [(B-A)\delta + A]z^p\phi(z)}.$$

Proof. We use a method by Padmanabhan [12]. Assume that a function

$$H(z) = 1 + b_p z^p + b_{p+1} z^{p+1} + \dots \quad (p \in \mathbb{N})$$

satisfies the condition

$$\left| \frac{H(z) - 1}{(B-A)\delta(H(z) - \alpha) + A(H(z) - 1)} \right| < \beta$$

for some $\alpha (0 \leq \alpha < 1)$, $\beta (0 < \beta \leq 1)$, $\delta (0 < \delta \leq 1)$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$.

Let

$$z^{p-1}h(z) = \frac{1 - H(z)}{(B-A)\delta(H(z) - \alpha) + A(H(z) - 1)}.$$

Then we see that the function $h(z)$ is analytic in the unit disc U , and $|h(z)| < \beta$ for $z \in U$ and $h(0) = 0$. Accordingly, by Schwarz's Lemma [7], we have $h(z) = z\phi(z)$, where $\phi(z)$ is an analytic function in the unit disc U and satisfies $|\phi(z)| \leq \beta$ for $z \in U$. Thus we get

$$\begin{aligned} H(z) &= \frac{1 + [(B-A)\alpha\delta + A]z^{p-1}h(z)}{1 + [(B-A)\delta + A]z^{p-1}h(z)} \\ &= \frac{1 + [(B-A)\alpha\delta + A]z^p\phi(z)}{1 + [(B-A)\delta + A]z^p\phi(z)}. \end{aligned}$$

Conversely, if

$$H(z) = \frac{1 + [(B-A)\alpha\delta + A] z^p \phi(z)}{1 + [(B-A)\delta + A] z^p \phi(z)}$$

and $|\phi(z)| < \beta$ for $z \in U$, then $H(z)$ is analytic in the unit disc U . Further, since $|z^p \phi(z)| < \beta |z|^p < \beta$ for $z \in U$, we get

$$\left| \frac{H(z) - 1}{(B-A)\delta(H(z) - \alpha) + A(H(z) - 1)} \right| = |z^p \phi(z)| < \beta$$

for $z \in U$. Hence we have the lemma.

Theorem 1. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N})$$

be analytic in the unit disc U . Then the function $f(z)$ belongs to the class $S_p^*(\alpha, \beta, \delta, A, B)$ if and only if

$$f(z) = z \exp \left\{ -(B-A)\delta(1-\alpha) \int_0^z \frac{t^{p-1} \phi(t)}{1 + [(B-A)\delta + A] t^p \phi(t)} dt \right\}, \quad (2.1)$$

where $\phi(z)$ is analytic in the unit disc U and satisfies $|\phi(z)| < \beta$ for $z \in U$.

Proof. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N})$$

belong to the class $S_p^*(\alpha, \beta, \gamma, A, B)$. Then, since the function $f(z)$ satisfies the condition (1.1), we can write

$$\frac{zf'(z)}{f(z)} = \frac{1 + [(B-A)\alpha\gamma + A]z^p\phi(z)}{1 + [(B-A)\gamma + A]z^p\phi(z)}$$

by using Lemma 1. Hence we get

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = - \frac{(B-A)\gamma(1-\alpha)z^{p-1}\phi(z)}{1 + [(B-A)\gamma + A]z^p\phi(z)}.$$

On integrating both sides of the above equality from 0 to z , we obtain the representation formula (2.1).

On the other hand, if $f(z)$ has the representation (2.1), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{1 + [(B-A)\alpha\gamma + A]z^p\phi(z)}{1 + [(B-A)\gamma + A]z^p\phi(z)}$$

holds with $\phi(z)$ as in Lemma 1. Therefore we see that $f(z)$ is in the class $S_p^*(\alpha, \beta, \gamma, A, B)$ by using Lemma 1.

3. A distortion theorem.

Lemma 2. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N})$$

belong to the class $S_p^*(\alpha, \beta, \gamma, A, B)$. Then we have

$$\frac{1 + [(B-A)\alpha\gamma + A]\beta|z|^p}{1 + [(B-A)\gamma + A]\beta|z|^p} \leq \operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \leq \frac{1 - [(B-A)\alpha\gamma + A]\beta|z|^p}{1 - [(B-A)\gamma + A]\beta|z|^p}$$

for $z \in U$.

Proof. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N})$$

be in the class $S_p^*(\alpha, \beta, \gamma, A, B)$. Then, by using Schwarz's Lemma [7], the condition (1.1) implies that $\frac{zf'(z)}{f(z)}$ assumes values lying in the disc obtained by taking the line segment joining two points

$$\frac{1 + [(B-A)\alpha\gamma + A]\beta|z|^p}{1 + [(B-A)\gamma + A]\beta|z|^p}$$

and

$$\frac{1 - [(B-A)\alpha\gamma + A]\beta|z|^p}{1 - [(B-A)\gamma + A]\beta|z|^p}$$

as diameter. Consequently we have the lemma.

Theorem 2. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N})$$

be analytic in the unit disc U and suppose $f(z) \in S_p^*(\alpha, \beta, \gamma, A, B)$. Then we have

$$|f(z)| \geq \frac{|z|}{\frac{(B-A)\gamma(1-\alpha)}{\{1 + [(B-A)\gamma+A]\beta|z|^p\}^{[(B-A)\gamma+A]p}}}$$

and

$$|f(z)| \leq \frac{|z|}{\frac{(B-A)\gamma(1-\alpha)}{\{1 - [(B-A)\gamma+A]\beta|z|^p\}^{[(B-A)\gamma+A]p}}}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $\gamma \neq \frac{-A}{(B-A)\gamma}$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $z \in U$. Moreover, for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\gamma = \frac{-A}{(B-A)\gamma}$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $z \in U$, we have

$$|z| \exp \left\{ \frac{-A\beta(\alpha-1)}{p} |z|^p \right\} \leq |f(z)| \leq |z| \exp \left\{ \frac{-A\beta(1-\alpha)}{p} |z|^p \right\}.$$

Proof. Since the function $f(z)$ is in the class $S_p^*(\alpha, \beta, \gamma, A, B)$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1 + [(B-A)\alpha\gamma+A]z^p\phi(z)}{1 + [(B-A)\gamma+A]z^p\phi(z)},$$

where $\phi(z)$ is an analytic function in the unit disc U and $|\phi(z)| \leq \beta$ for $z \in U$. Consequently we obtain

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{(B-A)\gamma(1-\alpha)z^{p-1}\phi(z)}{1+[(B-A)\gamma+A]z^{p-1}\phi(z)}. \quad (3.1)$$

On integrating both sides of (3.1) from 0 to z and taking real parts of both sides of the resulting equation, we have

$$\begin{aligned} \log \left| \frac{f(z)}{z} \right| &= \operatorname{Re} \left\{ \log \left(\frac{f(z)}{z} \right) \right\} = \operatorname{Re} \int_0^z \left\{ \frac{f'(t)}{f(t)} - \frac{1}{t} \right\} dt \\ &= \operatorname{Re} \int_0^z \frac{-(B-A)\gamma(1-\alpha)t^{p-1}\phi(t)}{1+[(B-A)\gamma+A]t^p\phi(t)} dt \\ &\leq \int_0^{|z|} \frac{(B-A)\gamma(1-\alpha)|\phi(te^{i\theta})|t^{p-1}}{|1+[(B-A)\gamma+A]t^p e^{ip\theta}\phi(te^{i\theta})|} dt. \end{aligned}$$

Hence

$$\begin{aligned} \log \left| \frac{f(z)}{z} \right| &\leq \int_0^{|z|} \frac{(B-A)\beta\gamma(1-\alpha)t^{p-1}}{1-[(B-A)\gamma+A]\beta t^p} dt \\ &= -\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p} \log \left\{ 1-[(B-A)\gamma+A]\beta|z|^p \right\} \\ &= -\log \left\{ 1-[(B-A)\gamma+A]\beta|z|^p \right\}^{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}} \end{aligned}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $\gamma \neq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $z \in U$. Furthermore, for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\gamma = \frac{-A}{(B-A)}$

and $-1 \leq A < B \leq 1$, $0 < B \leq 1$, we obtain

$$\begin{aligned} \log \left| \frac{f(z)}{z} \right| &\leq -A\beta(1-\alpha) \int_0^{|z|} t^{p-1} dt \\ &= \frac{-A\beta(1-\alpha)}{p} |z|^p. \end{aligned}$$

Therefore we see that

$$|f(z)| \leq \frac{|z|}{\left\{ 1 - [(B-A)\delta + A]\beta |z|^p \right\}^{\frac{(B-A)\delta(1-\alpha)}{[(B-A)\delta + A]p}}}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\delta \neq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$, and that

$$|f(z)| \leq |z| \exp \left\{ \frac{-A\beta(1-\alpha)}{p} |z|^p \right\}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\delta = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$.

On the other hand, by using Lemma 2, we get

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \frac{1 + [(B-A)\alpha\delta + A]\beta |z|^p}{1 + [(B-A)\delta + A]\beta |z|^p}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \delta \leq 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $z \in U$. From this, we obtain

$$\begin{aligned}
r \operatorname{Re} \left\{ \frac{\partial}{\partial r} \left(\log \frac{f(z)}{z} \right) \right\} &= \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} - 1 \right\} \\
&\geq \frac{1 + [(B-A)\alpha\delta + A]\beta r^p}{1 + [(B-A)\delta + A]\beta r^p} - 1 \\
&= - \frac{(B-A)\beta\delta(1-\alpha)r^p}{1 + [(B-A)\delta + A]\beta r^p}
\end{aligned}$$

for $|z| = r$. Thus we see that

$$\begin{aligned}
\log \left| \frac{f(z)}{z} \right| &= \operatorname{Re} \left\{ \log \frac{f(z)}{z} \right\} \\
&\geq \int_0^r \frac{-(B-A)\beta\delta(1-\alpha)t^{p-1}}{1 + [(B-A)\delta + A]\beta t^p} dt.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\log \left| \frac{f(z)}{z} \right| &\geq - \frac{(B-A)\delta(1-\alpha)}{[(B-A)\delta + A]p} \log \left\{ 1 + [(B-A)\delta + A]\beta r^p \right\} \\
&= - \log \left\{ 1 + [(B-A)\delta + A]\beta r^p \right\}^{\frac{(B-A)\delta(1-\alpha)}{[(B-A)\delta + A]p}}
\end{aligned}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \delta \leq 1$, $\delta \neq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$
and

$$\log \left| \frac{f(z)}{z} \right| \geq \frac{-A\beta(1-\alpha)}{p} r^p$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $\gamma = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$.

Consequently

$$|f(z)| \geq \frac{|z|}{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma + A]^p} \{1 + [(B-A)\gamma + A] \beta |z|^p\}}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $\gamma \neq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$,

and

$$|f(z)| \geq |z| \exp \left\{ \frac{-A\beta(\alpha-1)}{p} |z|^p \right\}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\gamma = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$. Finally,

for equality, we may take

$$f(z) = \frac{z}{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma + A]^p} \{1 - [(B-A)\gamma + A] \beta z^p\}}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $\gamma \neq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$,

and

$$f(z) = z \exp \left\{ \frac{-A\beta(1-\alpha)}{p} z^p \right\}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\gamma = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$.

4. A sufficient condition for the class $S_p^*(\alpha, \beta, \gamma, A, B)$.

Theorem 3. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N})$$

be analytic in the unit disc U . If we have

$$\sum_{n=1}^{\infty} \left\{ (p+n-1) + \beta(-A(p+n+1) - (B-A)\gamma p - (B-A)\gamma n - (B-A)\alpha\gamma) \right\} |a_{p+n}| \leq (B-A)\beta\gamma(1-\alpha) \quad (4.1)$$

for some $\alpha (0 \leq \alpha < 1)$, $\beta (0 < \beta \leq 1)$, $\gamma (0 < \gamma \leq \frac{-A}{(B-A)})$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$, then the function $f(z)$ belongs to the class $S_p^*(\alpha, \beta, \gamma, A, B)$.

Proof. We employ the technique used by Clunie and Keogh [2]. We assume that the condition (4.1) holds. Then we obtain

$$\begin{aligned} & \left| z f'(z) - f(z) \right| - \beta \left| (B-A)\gamma (z f'(z) - \alpha f(z)) + A(z f'(z) - f(z)) \right| \\ &= \left| \sum_{n=1}^{\infty} (p+n-1) a_{p+n} z^{p+n} \right| - \beta \left| (B-A)\gamma (1-\alpha) z + \right. \\ & \quad \left. \sum_{n=1}^{\infty} (-A - (B-A)\alpha\gamma) a_{p+n} z^{p+n} - \sum_{n=1}^{\infty} (-A - (B-A)\gamma) (p+n) a_{p+n} z^{p+n} \right| \\ &\leq \sum_{n=1}^{\infty} (p+n-1) |a_{p+n}| |z|^{p+n} - \end{aligned}$$

$$\left\{ \left| (B-A)\beta\gamma(1-\alpha)z + \sum_{n=1}^{\infty} (-A-(B-A)\alpha\gamma)\beta a_{p+n}z^{p+n} \right| \right. \\ \left. - \sum_{n=1}^{\infty} (-A-(B-A)\gamma)\beta(p+n) |a_{p+n}| |z|^{p+n} \right\}$$

$$\leq \sum_{n=1}^{\infty} (p+n-1) |a_{p+n}| |z|^{p+n} -$$

$$\left\{ (B-A)\beta\gamma(1-\alpha) |z| - \sum_{n=1}^{\infty} (-A-(B-A)\alpha\gamma)\beta |a_{p+n}| |z|^{p+n} \right.$$

$$\left. - \sum_{n=1}^{\infty} (-A-(B-A)\gamma)\beta(p+n) |a_{p+n}| |z|^{p+n} \right\}$$

$$\leq \left[\sum_{n=1}^{\infty} \left\{ (p+n-1) + \beta(-A(p+n+1) - (B-A)\gamma p \right. \right.$$

$$\left. - (B-A)\alpha\gamma - (B-A)\gamma n \right\} \cdot |a_{p+n}| - (B-A)\beta\gamma(1-\alpha) \right] |z| \leq 0$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq \frac{-A}{(B-A)}$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $z \in U$.

Hence, by the maximum modulus theorem $f(z)$ is in the

class $S_p^*(\alpha, \beta, \gamma, A, B)$.

5. The radius of convexity for functions in the class

$$S_p^*(\alpha, \beta, \gamma, A, B).$$

Theorem 4. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N})$$

be in the class $S_p^*(\alpha, \beta, \gamma, A, B)$ with $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$. Then the function $f(z)$ maps

$$|z| < \left(\frac{-A + \left(\frac{B-A}{2}\right)\gamma(1-\alpha) - \sqrt{[-A-1 + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)][-A+1 + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)]}}{\beta} \right)^{\frac{1}{p}}$$

on to a convex domain if

$$\begin{aligned} & \left\{ [-A + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)] - \sqrt{[-A-1 + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)][-A+1 + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)]} \right\} \\ & \cdot \left[\left(\frac{B-A}{2}\right)\gamma(1-\alpha) + \sqrt{(B-A)\gamma(1-\alpha)[-A + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)]} \right]^p \\ & \leq \beta \left(\sqrt{[-A-1 + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)][-A+1 + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)]} \right)^p \\ & \leq \left(\sqrt{[-A-1 + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)][-A+1 + \left(\frac{B-A}{2}\right)\gamma(1-\alpha)]} \right)^p. \end{aligned}$$

This result is sharp.

Proof. We employ the technique used by Lakshminarasimhan [5]. Since $f(z)$ belongs to the class $S_p^*(\alpha, \beta, \gamma, A, B)$, by Theorem 1, we get

$$\frac{zf'(z)}{f(z)} = \frac{1 + [(B-A)\alpha\gamma + \Lambda]z^p\phi(z)}{1 + [(B-A)\gamma + \Lambda]z^p\phi(z)}, \quad (5.1)$$

where $\phi(z)$ is analytic in the unit disc U and satisfies

$$|\phi(z)| \leq \beta \text{ for } z \in U. \text{ On differentiating both sides of (5.1)}$$

with respect to z logarithmically, we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + [(B-A)\alpha\delta + A]z^p\phi(z)}{1 + [(B-A)\delta + A]z^p\phi(z)} - \frac{(B-A)\delta(1-\alpha)\{pz^p\phi(z) + z^{p+1}\phi'(z)\}}{\{1 + [(B-A)\delta + A]z^p\phi(z)\}\{1 + [(B-A)\alpha\delta + A]z^p\phi(z)\}}.$$

Furthermore we have

$$\left| \frac{\phi'(z)}{\beta} \right| \leq \frac{1 - \left| \frac{\phi(z)}{\beta} \right|^2}{1 - |z|^2} \quad (5.2)$$

for the analytic function $\phi(z)$ in the unit disc U . Now, since

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{1 + [(B-A)\alpha\delta + A]z^p\phi(z)}{1 + [(B-A)\delta + A]z^p\phi(z)} \right\} \\ &= \frac{1 + [(B-A)\alpha\delta + A][(B-A)\delta + A]|z^p\phi(z)|^2 + [(B-A)\alpha\delta + (B-A)\delta + 2A]\operatorname{Re}(z^p\phi(z))}{|1 + [(B-A)\delta + A]z^p\phi(z)|^2} \\ &\geq \frac{\{1 + [(B-A)\alpha\delta + A]|z^p\phi(z)|\}\{1 + [(B-A)\delta + A]|z^p\phi(z)|\}}{|1 + [(B-A)\delta + A]z^p\phi(z)|^2} \\ &\geq \frac{1 + [(B-A)\alpha\delta + A]|z^p\phi(z)|}{1 + [(B-A)\delta + A]|z^p\phi(z)|} \end{aligned}$$

and

$$\begin{aligned}
& \operatorname{Re} \left\{ \frac{pz^p \phi(z) + z^{p+1} \phi'(z)}{\{1 + [(B-A)\delta + A] |z^p \phi(z)|\} \{1 + [(B-A)\alpha\delta + A] |z^p \phi(z)|\}} \right\} \\
& \leq \frac{p |z^p \phi(z)| + |z^{p+1} \phi'(z)|}{\{1 + [(B-A)\delta + A] |z^p \phi(z)|\} \{1 + [(B-A)\alpha\delta + A] |z^p \phi(z)|\}} \\
& \leq \frac{p |z^p \phi(z)| + |z^{p+1} \phi'(z)|}{\{1 + [(B-A)\delta + A] |z^p \phi(z)|\}^2},
\end{aligned}$$

we obtain

$$\begin{aligned}
& \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \frac{1 + [(B-A)\alpha\delta + A] |z^p \phi(z)|}{1 + [(B-A)\delta + A] |z^p \phi(z)|} \\
& - \frac{(B-A)\delta(1-\alpha) \{p |z^p \phi(z)| + |z^{p+1} \phi'(z)|\}}{\{1 + [(B-A)\delta + A] |z^p \phi(z)|\}^2}.
\end{aligned}$$

If we assume that

$$\begin{aligned}
& 1 + A^2 |z^p \phi(z)|^2 + [2A - (B-A)\delta(1-\alpha)p] |z^p \phi(z)| \\
& - (B-A)\delta(1-\alpha) |z^{p+1} \phi'(z)| > 0,
\end{aligned} \tag{5.3}$$

then we have

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0.$$

Now, in virtue of (5.2), the condition (5.3) will be satisfied

$$1 + |z^p \phi(z)|^2 + [2A - (B-A)\delta(1-\alpha)] |z^p \phi(z)| -$$

$$(B-A)\delta(1-\alpha) |z|^{p+1} \frac{\beta - \frac{|\phi(z)|^2}{\beta}}{1 - |z|^2} > 0.$$

On putting $a = |z|$ and $t = |z^p \phi(z)|$, the above condition can be re-written as

$$(1-a^2) \{1+t^2 + [2A - (B-A)\delta(1-\alpha)]t\} -$$

$$(B-A)\delta(1-\alpha) \left(\beta a^{p+1} - \frac{t^2}{\beta a^{p-1}} \right) > 0,$$

that is,

$$t^2 \left\{ (1-a^2) + \frac{(B-A)\delta(1-\alpha)}{\beta a^{p-1}} \right\} +$$

$$t(1-a^2)[2A - (B-A)\delta(1-\alpha)] + 1-a^2 -$$

$$(B-A)\beta\delta(1-\alpha)a^{p+1} > 0, \quad (5.4)$$

where $0 < a < 1$ and $0 \leq t \leq \beta a^p$. If $G(t)$ denote the left hand side of (5.4), then we see that

$$G'(t) = 2t \left\{ (1-a^2) + \frac{(B-A)\delta(1-\alpha)}{\beta a^{p-1}} \right\} +$$

$$(1-a^2)[2A - (B-A)\delta(1-\alpha)] = 0$$

for

$$t=t_1 = \frac{\beta a^{p-1}(1-a^2) [-A + (\frac{B-A}{2}) \delta (1-\alpha)]}{\beta a^{p-1}(1-a^2) + (B-A)\delta(1-\alpha)}$$

Moreover

$$G''(t) = 2 \left\{ (1-a^2) + \frac{(B-A)\delta(1-\alpha)}{\beta a^{p-1}} \right\} > 0,$$

because $0 < a < 1$. Now $t_1 - \beta a^p$ is positive and negative with

$$\beta a^{2p+1} - \beta a^{2p-1} - [-A + (\frac{B-A}{2}) \delta (1-\alpha)] a^{p+1}$$

$$- (B-A)\delta(1-\alpha)a^p + [-A + (\frac{B-A}{2}) \delta (1-\alpha)] a^{p-1},$$

respectively. Let

$$E(a) = \beta a^{2p+1} - \beta a^{2p-1} - [-A + (\frac{B-A}{2}) \delta (1-\alpha)] a^{p+1}$$

$$- (B-A)\delta(1-\alpha)a^p + [-A + (\frac{B-A}{2}) \delta (1-\alpha)] a^{p-1}$$

and let a_0 be the positive root of $E(a) = 0$ lying in the open interval $(0,1)$. Then $E(a)$ is positive for $0 < a < a_0$ and so $t_1 > \beta a^p$. Hence $G'(t)$ is negative for $0 \leq t \leq \beta a^p$, $G(\beta a^p) < G(t)$ and the condition is satisfied if $G(\beta a^p) > 0$. This is equivalent to

$$\beta^2 a^{2p} (1-a^2) - 2\beta a^p (1-a^2) \left[-A + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] + (1-a^2) > 0,$$

that is,

$$(1-a^2) \left\{ \beta^2 a^{2p} - 2\beta \left[-A + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] a^p + 1 \right\} > 0$$

which holds for

$$a < \left(\frac{\left[-A + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] - \sqrt{\left[-A - 1 + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] \left[-A + 1 + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right]}}{\beta} \right)^{\frac{1}{p}}.$$

Further we can show that

$$a_0 > \left(\frac{\left[-A + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] - \sqrt{\left[-A - 1 + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] \left[-A + 1 + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right]}}{\beta} \right)^{\frac{1}{p}}$$

if

$$\left[-A + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] - \sqrt{\left[-A + 1 + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] \left[-A + 1 + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right]} \leq \beta \leq 1.$$

The condition on β implies that

$$\left(\frac{\left[-A + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] - \sqrt{\left[-A - 1 + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right] \left[-A + 1 + \left(\frac{B-A}{2} \right) \gamma(1-\alpha) \right]}}{\beta} \right)^{\frac{1}{p}} < 1,$$

and so

$$a_1 = \left(\frac{[-A + (\frac{B-A}{2})\delta(1-\alpha)] - \sqrt{[-A-1 + (\frac{B-A}{2})\delta(1-\alpha)][-A+1 + (\frac{B-A}{2})\delta(1-\alpha)]}}{\beta} \right)^{\frac{1}{p}} < a_0$$

if $E(a_1) > 0$. Moreover $E(a_1) > 0$ is satisfied if

$$\begin{aligned} & \sqrt{[-A-1 + (\frac{B-A}{2})\delta(1-\alpha)][-A+1 + (\frac{B-A}{2})\delta(1-\alpha)]} \times \\ & \times \left\{ [-A + (\frac{B-A}{2})\delta(1-\alpha)] - \sqrt{[-A-1 + (\frac{B-A}{2})\delta(1-\alpha)][-A+1 + (\frac{B-A}{2})\delta(1-\alpha)]} \right\}^{\frac{1-\frac{1}{p}}{2}} \beta^{\frac{2}{p}} \\ & - (B-A)\delta(1-\alpha) \left\{ [-A + (\frac{B-A}{2})\delta(1-\alpha)] - \sqrt{[-A-1 + (\frac{B-A}{2})\delta(1-\alpha)][-A+1 + (\frac{B-A}{2})\delta(1-\alpha)]} \right\}^{\frac{1}{p}} \beta^{\frac{1}{p}} \\ & - \sqrt{[-A-1 + (\frac{B-A}{2})\delta(1-\alpha)][-A+1 + (\frac{B-A}{2})\delta(1-\alpha)]} \times \\ & \times \left\{ [-A + (\frac{B-A}{2})\delta(1-\alpha)] - \sqrt{[-A-1 + (\frac{B-A}{2})\delta(1-\alpha)][-A+1 + (\frac{B-A}{2})\delta(1-\alpha)]} \right\}^{1+\frac{1}{p}} \beta^{\frac{1}{p}} \\ & > 0 \end{aligned}$$

which holds if

$$\beta > \frac{[-A + (\frac{B-A}{2})\delta(1-\alpha)] - \sqrt{[-A-1 + (\frac{B-A}{2})\delta(1-\alpha)][-A+1 + (\frac{B-A}{2})\delta(1-\alpha)]}}{\left(\sqrt{[-A-1 + (\frac{B-A}{2})\delta(1-\alpha)][-A+1 + (\frac{B-A}{2})\delta(1-\alpha)]} \right)^p} \times$$

$$\times \left\{ \left(\frac{B-A}{2} \right) \delta(1-\alpha) + \sqrt{(B-A) \delta(1-\alpha) \left[-A + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right]} \right\}^p.$$

Let $C_p(\alpha, \beta, \delta, A, B)$ denote the right hand member of the above inequality. If $\beta = C_p(\alpha, \beta, \delta, A, B)$, then we see that

$$a_0 = \left(\frac{\left[-A + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right] - \sqrt{\left[-A - 1 + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right] \left[-A + 1 + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right]}}{\beta} \right)^{\frac{1}{p}}.$$

This shows that

$$|z| < \left(\frac{\left[-A + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right] - \sqrt{\left[-A - 1 + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right] \left[-A + 1 + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right]}}{\beta} \right)^{\frac{1}{p}}$$

is mapped on to a convex domain by $f(z)$ provided $C_p(\alpha, \beta, \delta, A, B) \leq \beta \leq 1$. To show that the estimate is sharp, we choose

$$f(z) = \frac{z}{\left\{ 1 - \left[(B-A)\delta + A \right] \beta z^p \right\}^{\frac{(B-A)\delta(1-\alpha)}{[(B-A)\delta + A]p}}}$$

so that $f(z) \in S_p^*(\alpha, \beta, \delta, A, B)$ while

$$1 + \frac{zf''(z)}{f'(z)} = 0$$

when

$$z = \left(\frac{\left[-A + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right] - \sqrt{\left[-A - 1 + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right] \left[-A + 1 + \left(\frac{B-A}{2} \right) \delta(1-\alpha) \right]}}{\beta} \right)^{\frac{1}{p}},$$

$0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \delta < \frac{-A}{(B-A)}$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, so that $f(z)$ is not convex in any disc $|z| < R$ if R exceeds

$$\left(\frac{[-A + (\frac{B-A}{2})\delta(1-\alpha)] - \sqrt{[-A-1 + (\frac{B-A}{2})\delta(1-\alpha)][-A+1 + (\frac{B-A}{2})\delta(1-\alpha)]}}{\beta} \right)^{\frac{1}{p}}.$$

Furthermore, for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\delta = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$, we ought to choose

$$f(z) = z \exp \left\{ \frac{-A\beta(\alpha-1)}{p} z \right\}.$$

This completes the proof of the theorem.

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