

Diophantine Frobenius problems from semigroup's series and identities for zeta functions

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1 Introduction

The *Frobenius Problem* is to determine the largest positive integer that is NOT representable as a nonnegative integer combination of given positive integers that are coprime (see [11] for general references).

Given positive integers $d_1, \dots, d_m$ with $\gcd(d_1, \dots, d_m) = 1$, it is well-known that all sufficiently large b the equation

$$d_1x_1 + \dots + d_mx_m = b \quad (1)$$

has a solution with nonnegative integers $x_1, \dots, x_m$.

The *Frobenius number* $F(d_1, \dots, d_m)$ is the LARGEST integer b such that (1) has no solution in nonnegative integers. For $m = 2$, we have

$$F(d_1, d_2) = (d_1 - 1)(d_2 - 1) - 1$$

(Sylvester (1884) [15]). For $m \geq 3$, exact determination of the Frobenius number is difficult. The Frobenius number cannot be given by closed formulas of a certain type (Curtis 1990 [4]), the problem to determine $F(d_1, \dots, d_m)$ is NP-hard under Turing reduction (see, e.g., Ramírez Alfonsín [11]).

Some formulae for the Frobenius number in three variables can be seen in [17].

Proposition 1. *Let*

$$q := \left\lfloor \frac{a}{a-\ell} \right\rfloor \quad \text{and} \quad r := a - q(a-\ell) = (a-\ell) \left\{ \frac{a}{a-\ell} \right\}.$$

If $\ell > k$ and $br < cq$, then

$$F(a, b, c) = \begin{cases} -a + b((\lambda + 1)(a - \ell) + r - 1) & \text{if } \lambda \geq \frac{c(q-1)-br}{b(a-\ell)+c}; \\ -a + b(a - \ell - 1) + c(q - \lambda - 1) & \text{if } \lambda \leq \frac{c(q-1)-br}{b(a-\ell)+c}, \end{cases}$$

where $\lambda := \left\lfloor \frac{cq-br}{b(a-\ell)+c} \right\rfloor$.

Proposition 2. *Let*

$$\bar{q} := \left\lfloor \frac{a}{\ell} \right\rfloor \quad \text{and} \quad \bar{r} := a - \bar{q}\ell = \ell \left\{ \frac{a}{\ell} \right\}.$$

If $\ell > k$ and $b(\ell - \bar{r}) < c(\bar{q} + 1)$, then

$$F(a, b, c) = \begin{cases} -a + b(\ell - 1) + c(\bar{q} - 1) & \text{if } 0 \leq \bar{r} < \ell - k; \\ -a + b(\bar{r} - 1) + c\bar{q} & \text{if } \ell - k \leq \bar{r} < \ell. \end{cases}$$

Consider the number of solutions. Sylvester (1882) gave the number of positive integers with no nonnegative integer representation by d_1 and d_2 by

$$g(d_1, d_2) = \frac{(d_1 - 1)(d_2 - 1)}{2}. \quad (2)$$

The number of solutions of the equation (1) in nonnegative integers $x_1, \dots, x_m$, denoted by $N(d_1, \dots, d_m; b)$. For $m = 2$, there exists an explicit formula for the number of solutions.

Proposition 3. *Tripathi (2000) [16]*

$$N(d_1, d_2; b) = \frac{b + d_1 d'_1 + d_2 d'_2}{d_1 d_2} - 1,$$

where $d'_1 \equiv -bd_1^{-1} \pmod{d_2}$, $d'_2 \equiv -bd_2^{-1} \pmod{d_1}$ with $1 \leq d'_1 \leq d_2$ and $1 \leq d'_2 \leq d_1$.

But, the problem becomes fairly hard if $m \geq 3$.

We give the method for computing the desired number. For the set $\{a_1, \dots, a_n\} \subset \{1, 2, \dots\}$ with $\gcd(a_1, \dots, a_n) = 1$, we have

$$\begin{aligned} \mathcal{N}(x) &:= \sum_{b=0}^{\infty} N(d_1, \dots, d_m; b) x^b = \frac{1}{(1 - x^{d_1}) \cdots (1 - x^{d_m})} \\ &= \frac{c_1}{1 - x} + \cdots + \frac{c_m}{(1 - x)^m} \\ &\quad + \sum_{k=1}^{d_1-1} \frac{A_{d_1}(k)}{1 - \zeta_{d_1}^{-k} x} + \cdots + \sum_{k=1}^{d_m-1} \frac{A_{d_m}(k)}{1 - \zeta_{d_m}^{-k} x}, \end{aligned} \quad (3)$$

where $\zeta_{d_l} = e^{2\pi i/d_l}$ ($l = 1, 2, \dots, m$). For the first decomposition into ordinary partial fractions, putting

$$\sum_{t=0}^{\infty} P_A(t)x^t = \frac{c_1}{1-x} + \dots + \frac{c_m}{(1-x)^m},$$

we know that

$$P_A(t) = \sum_{l=1}^{\infty} c_l \binom{b+l-1}{b},$$

where we take $c_l = 0$ for $l > n$.

Then, we have the following expression ([2]).

Theorem 1.

$$\begin{aligned} P_A(t) &= \frac{1}{d_1 \dots d_m} \sum_{l=0}^{m-1} \frac{(-1)^l}{(m-l-1)!} \\ &\quad \times \sum_{k_1+\dots+k_m=l} d_1^{k_1} \dots d_m^{k_m} \frac{B_{k_1} \dots B_{k_m}}{k_1! \dots k_m!} b^{m-l-1} \\ &= \frac{1}{d_1 \dots d_m} \sum_{l=0}^{m-1} \frac{(-1)^l}{(m-l-1)!} \sum_{k_1+2k_2+\dots+lk_l=l} \frac{(-1)^{k_2+\dots+k_l}}{k_1! \dots k_l!} \\ &\quad \times \left(\frac{B_1 S_1}{1 \cdot 1!} \right)^{k_1} \dots \left(\frac{B_l S_l}{l \cdot l!} \right)^{k_l} b^{m-l-1}, \end{aligned}$$

where $S_j = d_1^j + \dots + d_m^j$ and B_m is the m -th Bernoulli number.

If we write

$$P_A(t) = \sum_{l=1}^{\infty} c_l \binom{b+l-1}{b} = \sum_{j=0}^{m-1} d_j b^j,$$

d_j can be expressed as follows ([9]).

Theorem 2. For $l \geq 0$ we have

$$d_{m-l-1} = \frac{(-1)^l}{(m-l-1)!l!P} \mathbf{Y}_l \left(B_1 S_1, -\frac{B_2 S_2}{2}, \dots, (-1)^{l+1} \frac{B_l S_l}{l} \right),$$

where $P = \prod_{j=1}^m d_j$, $S_n = \sum_{j=1}^m d_j^n$, B_n is the n -th Bernoulli number, and $\mathbf{Y}_n(y_1, \dots, y_n)$ are Bell polynomials defined by

$$\exp\left(\sum_{k=1}^{\infty} y_k \frac{x^k}{k!}\right) = \sum_{n=0}^{\infty} \mathbf{Y}_n(y_1, \dots, y_n) \frac{x^n}{n!}$$

with $\mathbf{Y}_0 = 1$, and expressed as

$$\mathbf{Y}_n(y_1, \dots, y_n) = \sum_{\substack{k_1+2k_2+\dots+nk_n=n \\ k_1, k_2, \dots, k_n \geq 0}} \prod_{i=1}^n \frac{n! y_i^{k_i}}{k_i! (i!)^{k_i}}.$$

For the second decomposition including the periodic sequences in (3), we know that for $l = 1, 2, \dots, m$

$$A_{d_l}(k) = \frac{1}{d_l} \frac{1}{(1 - \zeta_{d_l}^{d_1 k}) \dots (1 - \zeta_{d_l}^{d_{l-1} k}) (1 - \zeta_{d_l}^{d_{l+1} k}) \dots (1 - \zeta_{d_l}^{d_m k})}.$$

2 Numerical semigroups

A *numerical semigroup* $S(\mathbf{d}^m) = \langle d_1, \dots, d_m \rangle$ is said to be generated by a minimal set of natural numbers $\mathbf{d}^m = \{d_1, \dots, d_m\}$ with $\gcd(d_1, \dots, d_m) = 1$ if neither of its elements is linearly representable by the rest of them. Namely,

$$S(\mathbf{d}^m) = \left\{ s \in \mathbb{N} \cup \{0\} \mid s = \sum_{i=1}^m x_i d_i, x_i \in \mathbb{N} \cup \{0\} \right\}.$$

Here, $d_1, \dots, d_m$ are called generators. Put $\pi_m = \prod_{i=1}^m d_i$ and $\sigma_m = \sum_{i=1}^m d_i$. $\mu = \min\{d_1, \dots, d_m\}$ is called multiplicity.

$G(\mathbf{d}^m) = \mathbb{N} \setminus S(\mathbf{d}^m)$: set of gaps of semigroup

$F(\mathbf{d}^m) = \max\{|G(\mathbf{d}^m)|\}$: Frobenius number

$g(\mathbf{d}^m) = \#G(\mathbf{d}^m)$: genus of semigroup

$c(\mathbf{d}^m) = 1 + F(\mathbf{d}^m)$: conductor of semigroup, so that $c(\mathbf{d}^m) \leq 2g(\mathbf{d}^m)$

$\rho(\mathbf{d}^m) = 1 - \frac{g(\mathbf{d}^m)}{c(\mathbf{d}^m)}$: density of non-gaps

$H(\mathbf{d}^m; z) := \sum_{s \in S(\mathbf{d}^m)} z^s$: Hilbert series

$\Phi(\mathbf{d}^m; z) := \sum_{s \in G(\mathbf{d}^m)} z^s$: Generating function of gaps, so that $H(\mathbf{d}^m; z) +$

$$\Phi(\mathbf{d}^m; z) = \frac{1}{1-z}$$

Several special numerical semigroups $S(\mathbf{d}^m)$ are as follows.

Proposition 4 (Roberts (1956) Arithmetic sequence [12]). For $\mathbf{d}^m = \{a, a + d, \dots, a + (m - 1)d\}$

$$F(\mathbf{d}^m) = a \left\lfloor \frac{a - 2}{m - 1} \right\rfloor + d(a - 1)$$

Proposition 5 (Selmer (1997) [14]; Rödseth (1994) [13] Almost arithmetic sequence). For $\mathbf{d}^m = \{a, ha + d, ha + 2d, \dots, ha + (m - 1)d\}$

$$F(\mathbf{d}^m) = ha \left\lfloor \frac{a - 2}{m - 1} \right\rfloor + a(h - 1) + d(a - 1)$$

Proposition 6 (Selmer (1997) [14]; Rödseth (1994) [13]) Almost arithmetic sequence). For $\mathbf{d}^m = \{a, a + 1, a + 2, a + 2^2, \dots, a + 2^{m-2}\}$

$$F(\mathbf{d}^m) = \frac{a(a + 1)}{2^{m-2}} + \sum_{k=0}^{m-3} 2^k \left\lfloor \frac{a + 2^k}{2^{m-2}} \right\rfloor + a(m - 4) - 1$$

Proposition 7 (Ong & Ponomarenko (2008) Geometric sequence [10]). For $\mathbf{d}^m = \{a^{m-1}, a^{m-2}b, a^{m-3}b^2, \dots, b^{m-1}\}$

$$F(\mathbf{d}^m) = b^{m-2}(ab - a - b) + \frac{(b - 1)a^2(a^{m-2} - b^{m-2})}{a - b}$$

A semigroup $S(\mathbf{d}^m)$ is called *symmetric* if for any integer s

$$s \in S(\mathbf{d}^m) \implies F(\mathbf{d}^m) - s \notin S(\mathbf{d}^m).$$

In fact, we have

$$c(\mathbf{d}^m) = 2g(\mathbf{d}^m), \quad \rho(\mathbf{d}^m) = \frac{1}{2}$$

Otherwise, $S(\mathbf{d}^m)$ is called nonsymmetric.

Proposition 8 (Watanabe (1973) [18]). Let $H_1 = \langle d_1, \dots, d_m \rangle$ be a semigroup. For positive integers a and b , satisfying $a \in H_1 \setminus \{d_1, \dots, d_m\}$ and $\gcd(a, b) = 1$, denote $H := \langle a, bH_1 \rangle = \langle a, bd_1, \dots, ad_m \rangle$. Then

$$H \text{ is symmetric} \iff H_1 \text{ is symmetric}.$$

Proposition 9 (Johnson (1960), [8]).

$$F(H) = bF(H_1) + (b - 1)a.$$

The semigroup $S(\mathbf{d}^2)$ is always symmetric.

Proposition 10 (Sylvester (1884) [15], Rödseth (1994) [13]).

$$F(\mathbf{d}^2; z) = d_1 d_2 - d_1 - d_2,$$

$$H(\mathbf{d}^2; z) = \frac{1 - z^{d_1 d_2}}{(1 - z^{d_1})(1 - z^{d_2})}.$$

However, the Hilbert series $H(\mathbf{d}^3; z)$ and the power sum $g_n(\mathbf{d}^3; z)$ are not so simple. For given $\mathbf{d}^3 = (d_1, d_2, d_3)$, Johnson's *minimal relations* (1960) [8] are constructed as follows.

$$a_{11}d_1 = a_{12}d_2 + a_{13}d_3, \quad a_{22}d_2 = a_{21}d_1 + a_{23}d_3, \quad a_{33}d_3 = a_{31}d_1 + a_{32}d_2,$$

where

$$a_{11} = \min\{v_{11} | v_{11} \geq 2, \quad v_{11}d_1 = v_{12}d_2 + v_{13}d_3, \quad v_{12}, v_{13} \in \mathbb{N} \cup \{0\}\},$$

$$a_{22} = \min\{v_{22} | v_{22} \geq 2, \quad v_{22}d_2 = v_{21}d_1 + v_{23}d_3, \quad v_{21}, v_{23} \in \mathbb{N} \cup \{0\}\},$$

$$a_{33} = \min\{v_{33} | v_{33} \geq 2, \quad v_{33}d_3 = v_{31}d_1 + v_{32}d_2, \quad v_{31}, v_{32} \in \mathbb{N} \cup \{0\}\}.$$

The auxiliary invariants a_{ij} ($i \neq j$) are uniquely determined by this definition and

$$\gcd(a_{11}, a_{12}, a_{13}) = \gcd(a_{22}, a_{21}, a_{23}) = \gcd(a_{33}, a_{31}, a_{32}) = 1.$$

The denominator of the Hilbert series is given by $(1 - z^{d_1})(1 - z^{d_2})(1 - z^{d_3})$.

The numerator of the Hilbert series $Q(\mathbf{d}^3; z)$ for *nonsymmetric* semigroups $S(\mathbf{d}^3)$ is given by the following.

$$Q(\mathbf{d}^3; z) = 1 - (z^{a_{11}d_1} + z^{a_{22}d_2} + z^{a_{33}d_3})$$

$$+ z^{1/2(\langle \mathbf{a}, \mathbf{d} \rangle - J(\mathbf{d}^3))} + z^{1/2(\langle \mathbf{a}, \mathbf{d} \rangle + J(\mathbf{d}^3))},$$

where

$$\langle \mathbf{a}, \mathbf{d} \rangle = a_{11}d_1 + a_{22}d_2 + a_{33}d_3$$

and

$$J(\mathbf{d}^3) = \sqrt{\langle \mathbf{a}, \mathbf{d} \rangle - 4 \sum_{i>j} a_{ii}a_{jj}d_i d_j + 4d_1 d_2 d_3}.$$

The numerator of the Hilbert series for symmetric semigroup $S(\mathbf{d}^3)$ is given by

$$Q(\mathbf{d}^3; z) = (1 - z^{a_{22}d_2})(1 - z^{a_{33}d_3}).$$

3 Semigroup's series for negative degrees of the gaps values

We derive an explicit form for an inverse power series over values of gaps of numerical semigroups generated by two integers.

Let $S_m = \langle d_1, \dots, d_m \rangle$ be the semigroup generated by a set of integers $\{d_1, \dots, d_m\}$ such that

$$1 < d_1 < \dots < d_m, \quad \gcd(d_1, \dots, d_m) = 1.$$

This sum of integer powers of values the gaps in numerical semigroups $S_m = \langle d_1, \dots, d_m \rangle$ is referred often as semigroup's series

$$g_n(S_m) = \sum_{s \in \mathbb{N} \setminus S_m} s^n \quad (n \in \mathbb{Z}),$$

and $g_0(S_m)$ is known as a genus of S_m .

For $n \geq 0$, an explicit expression of $g_n(S_2)$ was given.

Proposition 11. *Rödseth (1994) [13]) For $n \geq 0$,*

$$g_n(S_2) = \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n+1} \sum_{l=0}^{n+1-k} \binom{n+2}{k} \binom{n+2-k}{l} B_k B_l d_1^{m+1-k} d_2^{n+1-l} - \frac{B_{n+1}}{n+1},$$

where B_n is n -th Bernoulli number.

Remark. For $n = 0$, it is reduced to Sylvester's expression [15]:

$$g_0(S_2) = \frac{(d_1 - 1)(d_2 - 1)}{2}.$$

For $n = 1$, the result was given by Brown and Shiue in 1993 [3].

$$g_1(S_2) = \frac{g_0(S_2)}{6} (2d_1 d_2 - d_1 - d_2 - 1).$$

An implicit expression of $g_n(S_3)$ was given by Fel and Rubinstein in 2007 [6].

We derive a formula for semigroup series

$$g_{-n}(S_2) = \sum_{s \in \mathbb{N} \setminus S_2} s^{-n} \quad (n \geq 1).$$

Consider the numerical semigroup $S_2 = \langle d_1, d_2 \rangle$, where $d_1, d_2 \geq 2$. We introduce the Hilbert series $H(z; S_2)$ and the gaps generating function $\Phi(z; S_2)$ are given by

$$H(z; S_2) = \sum_{s \in S_2} z^s \quad \text{and} \quad \Phi(z; S_2) = \sum_{s \in \mathbb{N} \setminus S_2} z^s,$$

so that

$$H(z; S_2) + \Phi(z; S_2) = \frac{1}{1-z} \quad (z < 1). \quad (4)$$

Here, $\min\{\mathbb{N} \setminus S_2\} = 1$. $\max\{\mathbb{N} \setminus S_2\} = d_1 d_2 - d_1 - d_2$ is exactly the same as Frobenius number.

The rational representation of $H(z; S_2)$ is given by

$$H(z; S_2) = \frac{1 - z^{d_1 d_2}}{(1 - z^{d_1})(1 - z^{d_2})}. \quad (5)$$

Introduce a new generating function $\Psi_1(z; S_2)$ by

$$\Psi_1(z; S_2) = \int_0^z \frac{\Phi(t; S_2)}{t} dt = \sum_{s \in \mathbb{N} \setminus S_2} \frac{z^s}{s}.$$

Hence,

$$\Psi_1(1; S_2) = \sum_{s \in \mathbb{N} \setminus S_2} \frac{1}{s} = g_{-1}(S_2). \quad (6)$$

Substituting (4) into (6), we obtain

$$\Psi_1(z; S_2) = \int_0^z \left(\frac{1}{1-t} - H(t; S_2) \right) \frac{dt}{t}. \quad (7)$$

Present an integral in (7) as follows.

$$\Psi_1(z; S_2) = \int_0^z \left(\sum_{k=0}^{\infty} t^{k-1} - \frac{H(t; S_2)}{t} \right) dt,$$

$$\frac{H(t; S_2)}{t} = \sum_{j=0}^2 h_j(t; S_2), \quad (8)$$

$$h_0(t; S_2) = \frac{1}{t}, \quad h_1(t; S_2) = \sum_{k_1=1}^{d_2-1} t^{k_1 d_1 - 1},$$

$$h_2(t; S_2) = \sum_{k_1=0}^{d_2-1} \sum_{k_2=1}^{\infty} t^{k_1 d_1 + k_2 d_2 - 1}. \quad (9)$$

Perform integration in (8) as

$$\Psi_1(z; S_2) = \sum_{k=1}^{\infty} \frac{z^k}{k} - \frac{1}{d_1} \sum_{k_1=1}^{d_2-1} \frac{z^{k_1 d_1}}{k_1} - \sum_{k_1=0}^{d_2-1} \sum_{k_2=1}^{\infty} \frac{z^{k_1 d_1 + k_2 d_2}}{k_1 d_1 + k_2 d_2},$$

so by (6) we obtain

$$g_{-1}(S_2) = \sum_{k=1}^{\infty} \frac{1}{k} - \sum_{k_1=0}^{d_2-1} \sum_{k_2=1}^{\infty} \frac{1}{k_1 d_1 + k_2 d_2} - \frac{1}{d_1} \sum_{k_1=1}^{d_2-1} \frac{1}{k_1}.$$

4 A sum of the negative degrees of the gaps values $g_{-n}(S_2)$

We can have a general formula as $g_{-1}(S_2)$ by introducing of a new generating function $\Psi_n(z; S_2)$ ($n \geq 2$) by

$$\begin{aligned} \Psi_n(z; S_2) &= \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \cdots \int_0^{t_{n-1}} \frac{dt_n}{t_n} \Phi(t_n; S_2) \\ &= \sum_{s \in \mathbb{N} \setminus S_2} \frac{z^s}{s^n}, \quad \text{so,} \quad \Psi_n(1; S_2) = g_{-n}(S_2), \end{aligned} \quad (10)$$

satisfying the recursive relation:

$$\Psi_{k+1}(t_{n-k-1}; S_2) = \int_0^{t_{n-k-1}} \frac{dt_{n-k}}{t_{n-k}} \Psi_k(t_{n-k}; S_2) \quad (k \geq 0)$$

with $\Psi_0(t_n; S_2) = \Phi(t_{n-1}; S_2)$ and $t_0 = z$.

Hence,

$$\begin{aligned}\Psi_1(t_{n-1}; S_2) &= \int_0^{t_{n-1}} \frac{dt_n}{t_n} \Psi_0(t_n; S_2), \\ \Psi_2(t_{n-2}; S_2) &= \int_0^{t_{n-2}} \frac{dt_{n-1}}{t_{n-1}} \Psi_1(t_{n-1}; S_2).\end{aligned}$$

Performing integration in (10), we obtain

$$\Psi_n(z; S_2) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} - \frac{1}{d_1^n} \sum_{k_1=1}^{d_2-1} \frac{z^{k_1 d_1}}{k_1^n} - \sum_{k_1, k_2 \in \mathbb{K}_2} \frac{z^{k_1 d_1 + k_2 d_2}}{(k_1 d_1 + k_2 d_2)^n}.$$

Thus, setting $z = 1$, we have for $n \geq 2$

$$g_{-n}(S_2) = \sum_{k=1}^{\infty} \frac{1}{k^n} - \sum_{k_1=0}^{d_2-1} \sum_{k_2=1}^{\infty} \frac{1}{(k_1 d_1 + k_2 d_2)^n} - \frac{1}{d_1^n} \sum_{k_1=1}^{d_2-1} \frac{1}{k_1^n}.$$

Define a ratio $\delta = d_1/d_2$ and represent the last expression as follows.

$$g_{-n}(S_2) = \sum_{k=1}^{\infty} \frac{1}{k^n} - \frac{1}{d_2^n} \sum_{k_2=1}^{\infty} \frac{1}{k_2^n} - \frac{1}{d_2^n} \sum_{k_1=1}^{d_2-1} \sum_{k_2=1}^{\infty} \frac{1}{(k_1 \delta + k_2)^n} - \frac{1}{d_1^n} \sum_{k_1=1}^{d_2-1} \frac{1}{k_1^n}.$$

Making use of the Hurwitz $\zeta(n, q) = \sum_{k=0}^{\infty} (k + q)^{-n}$ and Riemann zeta functions $\zeta(n) = \zeta(n, 1)$, we obtain

$$g_{-n}(S_2) = \left(1 - \frac{1}{d_2^n}\right) \zeta(n) - \frac{1}{d_2^n} \sum_{k_1=1}^{d_2-1} \zeta(n, k_1 \delta) \quad (n \geq 2). \quad (11)$$

Interchanging d_1 and d_2 in (11), we get an alternative expression for $g_{-n}(S_2)$:

$$g_{-n}(S_2) = \left(1 - \frac{1}{d_1^n}\right) \zeta(n) - \frac{1}{d_1^n} \sum_{k_2=1}^{d_1-1} \zeta\left(n, \frac{k_2}{\delta}\right). \quad (12)$$

5 Identities for Hurwitz zeta functions

Combining formulas (11) and (12), we get the identity

$$\delta^n \sum_{k=1}^{d_2-1} \zeta(n, k\delta) = (1 - \delta^n) \zeta(n) + \sum_{k=1}^{d_1-1} \zeta\left(n, \frac{k}{\delta}\right).$$

Another spinoff of formulas (11) and (12) is a set of identities for Hurwitz zeta functions.

For example, consider the numerical semigroup $\langle 3, 4 \rangle$ with three gaps $\mathbb{N} \setminus \langle 3, 4 \rangle = \{1, 2, 5\}$. Substituting it into (11) and (12), we have

$$\zeta\left(n, \frac{3}{4}\right) + \zeta\left(n, \frac{6}{4}\right) + \zeta\left(n, \frac{9}{4}\right) = (4^n - 1)\zeta(n) - \left(4^n + 2^n + \left(\frac{4}{5}\right)^n\right)$$

and

$$\zeta\left(n, \frac{4}{3}\right) + \zeta\left(n, \frac{8}{3}\right) = (3^n - 1)\zeta(n) - \left(3^n + \left(\frac{3}{2}\right)^n + \left(\frac{3}{5}\right)^n\right),$$

respectively.

We¹ shall show the identity (11) can be reduced to the multiplication theorem in Hurwitz zeta functions (see, e.g., [1, p.249], [5, (16), p.71]). It is similar for (12).

Since $\gcd(d_1, d_2) = 1$, if $k_1 d_1 \equiv k_2 d_1 \pmod{d_2}$ then $k_1 \equiv k_2 \pmod{d_2}$. Therefore,

$$\begin{aligned} & \zeta\left(n, \left\{\frac{d_1}{d_2}\right\}\right) + \zeta\left(n, \left\{\frac{2d_1}{d_2}\right\}\right) + \cdots + \zeta\left(n, \left\{\frac{(d_2-1)d_1}{d_2}\right\}\right) \\ &= \zeta\left(n, \frac{1}{d_2}\right) + \zeta\left(n, \frac{2}{d_2}\right) + \cdots + \zeta\left(n, \frac{d_2-1}{d_2}\right), \end{aligned} \quad (13)$$

where $\{x\}$ denotes the fractional part of a real number x . There exists a nonnegative integer a such that

$$\frac{ad_1}{d_2} < 1 < \frac{(a+1)d_1}{d_2}.$$

Then for any integer k' with $a < k' \leq d_2 - 1$ there exists a positive integer l' such that $1 \leq k'd_1 - l'd_2 < d_2$, and

$$\begin{aligned} \zeta\left(n, \frac{k'd_1}{d_2}\right) &= \zeta\left(n, \frac{k'd_1 - l'd_2}{d_2}\right) - \left(\frac{d_2}{k'd_1 - l'd_2}\right)^n \\ &\quad - \left(\frac{d_2}{k'd_1 - (l'-1)d_2}\right)^n - \cdots - \left(\frac{d_2}{k'd_1 - d_2}\right)^n, \end{aligned} \quad (14)$$

¹This part was suggested by Dr. Ade Irma Suriajaya (RIKEN) in February 2018.

where

$$\frac{k'd_1 - l'd_2}{d_2} = \left\{ \frac{k'd_1}{d_2} \right\}.$$

For any positive integer r , there exist integers x and y such that $r = xd_1 + yd_2$. If $0 \leq x < d_2$, then r can be expressed uniquely. Thus, if $y \geq 0$, then $r \in S_2$. If $y < 0$, then $r \notin S_2$. The largest integer is given by $(d_2 - 1)d_1 - d_2$, that is exactly the same as Frobenius number $F(d_1, d_2)$. Thus, $k'd_1 - l''d_2 \notin S_2$ for all l'' with $1 \leq l'' \leq l'$ in (14). In addition, if $k_1d_1 - l_1d_2 = k_2d_1 - l_2d_2$, then by $\gcd(d_1, d_2) = 1$ we have $d_1|(k_1 - k_2)$ and $d_2|(l_1 - l_2)$. As $0 < k_1, k_2 < d_2$ and $0 < l_1, l_2 < d_1$, we get $k_1 = k_2$ and $l_1 = l_2$. Thus, all such numbers of the form $kd_1 - ld_2 \notin S_2$ are different.

In [7, (3.32)] for a real ξ and $d = \gcd(d_1, d_2)$

$$\sum_{k=0}^{d_2-1} \left\lfloor \frac{kd_1 + \xi}{d_2} \right\rfloor = d \left\lfloor \frac{\xi}{d} \right\rfloor + \frac{(d_1 - 1)(d_2 - 1)}{2} + \frac{d - 1}{2}. \quad (15)$$

Hence, by (15) with $d = 1$ and $\xi = 0$, the total number of non-representable positive integers of the form $kd_1 - ld_2$ ($a < k < d_2$, $l = 1, 2, \dots, \lfloor kd_1/d_2 \rfloor - 1$) is

$$\sum_{k=1}^{d_2-1} \left\lfloor \frac{kd_1}{d_2} \right\rfloor = \frac{(d_1 - 1)(d_2 - 1)}{2},$$

that is exactly the same as the number of integers without non-negative integer representations by d_1 and d_2 in (2). Therefore, the right-hand side of (11) is

$$\begin{aligned} & \left(1 - \frac{1}{d_2^n}\right) \zeta(n) - \frac{1}{d_2^n} \sum_{k_1=1}^{d_2-1} \zeta\left(n, \frac{k_1d_1}{d_2}\right) \\ &= \left(1 - \frac{1}{d_2^n}\right) \zeta(n) - \frac{1}{d_2^n} \left(\sum_{k_1=1}^{d_2-1} \zeta\left(n, \left\{\frac{k_1d_1}{d_2}\right\}\right) - d_2^n \sum_{s \in \mathbb{N} \setminus S_2} s^{-n} \right) \\ &= \left(1 - \frac{1}{d_2^n}\right) \zeta(n) - \frac{1}{d_2^n} \sum_{k_1=1}^{d_2-1} \zeta\left(n, \frac{k}{d_2}\right) + \sum_{s \in \mathbb{N} \setminus S_2} s^{-n}. \end{aligned}$$

On the other hand, the left-hand side of (11) is

$$g_{-n}(S_2) = \sum_{s \in \mathbb{N} \setminus S_2} s^{-n}.$$

Therefore, we obtain that

$$\sum_{k=1}^{d_2} \zeta \left(n, \frac{k}{d_2} \right) = d_2^n \zeta(n),$$

that is the multiplication theorem in Hurwitz zeta functions.

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