

On 2-normed spaces

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First we shall recall some definitions needed in the sequel.

Let X be a set. Consider a mapping $\rho : X \times X \times X \rightarrow \mathbb{R}$.

ρ is called a *2-metric* (S. Gähler), if it satisfies

- (1) $\rho(x, y, z) \neq 0$ for any $x, y (x \neq y)$ and some z ,
- (2) $\rho(x, y, z) = 0$, if at least two points of the points x, y, z are equal,
- (3) ρ is symmetric on x, y, z , i. e. $\rho(x, y, z) = \rho(x, y, z) = \dots = \rho(z, y, x)$,
- (4) $\rho(x, y, z) \leq \rho(u, y, z) + \rho(x, u, z) + \rho(x, y, u)$.

By a (*linear*) *2-normed space* X over reals (S. Gähler), we mean a linear space X in which to each pair of points $x, y \in X$, there exists a real number $\|x, y\|$ satisfying the following properties

- (1) $\|x, y\| = 0 \Leftrightarrow x, y$ are linear dependent,
- (2) $\|x, y\| = \|y, x\|$,
- (3) $\|\alpha x, y\| = |\alpha| \|x, y\|$ for any real α ,
- (4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

We assume $\dim L \geq 2$.

On a 2-normed space X ,

$$\rho(x, y, z) = \|y - x, z - x\|$$

defines a 2-metric on X .

A *2-inner product space* (R. Ehret) is a linear space X with a mapping $(\cdot, \cdot | \cdot) : X \times X \times X \rightarrow \mathbb{R}$ satisfying the following conditions:

(1) $(a, a|b) \geq 0$, and $(a, a|b) = 0 \Leftrightarrow a, b$ are linearly dependent,

$$(2) \quad (a, a|b) = (b, a|a),$$

$$(3) \quad (a, b|c) = (b, a|c),$$

$$(4) \quad (\alpha a, b|c) = \alpha(a, b|c) \quad \text{for any real } \alpha,$$

$$(5) \quad (a + a', b|c) = (a, b|c) + (a', b|c).$$

Proposition 1. On a 2-inner product space,

$$\|a, b\| = \sqrt{(a, a|b)}$$

define a 2-norm for which

$$(a, b|c) = \frac{1}{4} (\|a + b, c\|^2 - \|a - b, c\|^2)$$

and

$$\|a + b, c\|^2 + \|a - b, c\|^2 = 2(\|a, c\|^2 + \|b, c\|^2).$$

Proposition 2. Let X be a pre-Hilbert space. Then

$$\begin{aligned} (a, b|c) &= \begin{vmatrix} (a, b) & (a, c) \\ (b, c) & (c, c) \end{vmatrix} \\ &= (a, b) \|c\|^2 - (a, c)(b, c) \end{aligned}$$

defines a 2-inner product on X .

2-normed space X is called *strictly convex* if for $a, b \neq 0$, $\|a + b, c\| = \|a, c\| + \|b, c\|$ and $\|a, c\| = \|b, c\| = 1$, where c is linearly independent to a, b , implies $a = b$, equivalently $\|a, c\| = \|b, c\| = \frac{1}{2} \|a + b, c\| = 1$, where c is linear independent to a, b , implies $a = b$.

A 2-normed space X is said to be *strictly 2-convex*, if $\|a, b\| = \|a, c\| = \|b, c\| = \frac{1}{3} \|a + c, b + c\| = 1$ implies $c = a + b$.

Let c be a fixed non-zero element of a 2-normed space X , and let $V(c)$ be the linear subspace of X generated by c .

Then we obtain the quotient space $X/V(c) = L_c$. We put

$$\|x\|_c = \|x, c\|,$$

then $\|\cdot\|_c$ is well-defined on X_c .

Proposition 3. $\|\cdot\|_c$ is a norm on X_c .

Proposition 4. A 2-normed space X is strictly convex if and only if $X_c (c \neq 0)$ is strictly convex in usual sense, i. e.

$$\|x + y\|_c = \|x\|_c + \|y\|_c, \|x\|_c = \|y\|_c = 1 \text{ imply } x = y.$$

Some new characterizations of the strictly convexity by bounded linear 2-functionals in some sense are recently given by Y. Cho, K. Ha and W. Kim [1].

Quite recently, a wonderful characterization of a 2-inner product space is given by C. Diminnie and A. White [2].

Proposition 5. 2-normed space is a 2-inner product space if and only if

$$\begin{aligned} & \|x + y, y + z\|^2 + \|x + y, y - z\|^2 \\ & + \|x - y, y + z\|^2 + \|x - y, y - z\|^2 \\ & = \|x, y\|^2 + \|x, z\|^2 + \|y, z\|^2 \end{aligned}$$

holds.

Next we concern with some special classes of mappings.

A 2-metric space X is called to be complete, if for any sequence $\{x_n\}$, $\rho(x_m, x_n, a) \rightarrow 0$ for all $a \in X (m, n \rightarrow \infty)$ implies $\rho(x_m, x, a) \rightarrow 0$ for all $a \in X$ and for some $x \in X (m \rightarrow \infty)$.

The following fixed point theorem is obtained.

Proposition 6. Let X be a bounded complete 2-metric space, and let f_n be a sequence of mappings of X into itself. If there are non-negative α, β such that for all $x, y \in X$

$$\begin{aligned} \rho(f_m(x), f_n(y), a) & \leq \alpha(\rho(x, f_m(x), a) \\ & + \rho(y, f_n(y), a)) + \beta\rho(x, y, a) \end{aligned}$$

with $2\alpha + \beta < 1$, then the sequence $\{f_n\}$ has a unique common fixed point (K. Iséki, P. L. Sharma and B. K. Sharma).

Let E be a usual normed space, and let X be a 2-normed space. Then the following three mappings are considered.

(1) $f_1 : E \rightarrow X$ satisfies

$$\|f(x) - f(y)\| = \|x - y, c\|$$

for some fixed $c \in X$.

(2) $f_2 : X \rightarrow E$ satisfies

$$\|f(x) - f(y), c\| = \|x - y\|$$

for some fixed $c \in X$.

(3) $f_3 : X \rightarrow X$ satisfies

$$\|f(x) - f(y), c\| \leq \|x - y, c\|$$

for some fixed $c \in X$.

The first two mappings f_1, f_2 are due to C. Diminnie and A. White. f_3 is discussed by the present author. The mapping f_3 is uniquely determined. Roughly speaking this type is of an affine mapping (Iséki-Diminnie-White).

References

- [1] Y. J. Cho, K. S. Ha and W. S. Kim, Strictly convex linear 2-normed spaces, Math. Japonica, 26(1981), 475-478.
- [2] C. Diminnie and A. White, A characterization of 2-inner product spaces, to appear in Math. Japonica.