

Analysis of a pressure-stabilized characteristics finite element scheme for a linearized Navier-Stokes equations

Hirofumi Notsu *

Waseda Institute for Advanced Study, Waseda University

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$), T be a positive constant, $u : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ be a velocity and $\phi : \Omega \times (0, T) \rightarrow \mathbb{R}$ be a scalar function. We consider a trajectory of fluid particle, which is important for flow problems. Let $X = X(\cdot; x, t^n) : (0, T) \rightarrow \mathbb{R}^d$ be a solution of the ordinary differential equation,

$$\frac{dX}{dt} = u(X, t) \quad (1)$$

with a condition $X(t^n) = x$, where $n \in \mathbb{N} \cup \{0\}$, Δt is a time increment and $t^n \equiv n\Delta t$. Then, it holds that

$$\frac{D\phi}{Dt}(x, t) = \frac{d}{dt}\phi(X(t), t), \quad (2)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u \cdot \nabla \quad (3)$$

is a material derivation. Therefore, we can consider a first order approximation of the material derivative at $t = t^n$ ($n \geq 1$) as follows;

$$\begin{aligned} \frac{D\phi}{Dt}(x, t^n) &\approx \frac{\phi(X(t^n), t^n) - \phi(X(t^{n-1}), t^{n-1})}{\Delta t} \\ &\approx \frac{\phi(x, t^n) - \phi(X_1(u^n, \Delta t)(x), t^{n-1})}{\Delta t} \\ &= \frac{\phi^n - \phi^{n-1} \circ X_1(u^n, \Delta t)}{\Delta t}(x), \end{aligned} \quad (4)$$

where we have used notations, $\phi^n \equiv \phi(\cdot, t^n)$, for $w : \Omega \rightarrow \mathbb{R}^d$,

$$X_1(w, \Delta t)(x) \equiv x - w(x)\Delta t, \quad (5)$$

and, for $\psi : \Omega \rightarrow \mathbb{R}$,

$$\psi \circ X_1(w, \Delta t)(x) \equiv \psi(X_1(w, \Delta t)(x)). \quad (6)$$

*Email: h.notsu@aoni.waseda.jp, Web: <http://scheme.hn/>

The approximation (4) of $D\phi/Dt$ is a basic idea to devise numerical schemes based on the method of characteristics. The idea can be combined with finite element and difference methods (e.g., see [5, 10, 12, 13, 15]). In this paper, we introduce a finite element scheme based on the method of characteristics for a linearized Navier-Stokes equations, and give stability and convergence results of the scheme. Although the system of the equations is linear and simpler than one of the Navier-Stokes equations, the results to be shown are useful for an analysis of a scheme for the Navier-Stokes equations [9, 11].

2 Statement of the problem and a characteristics finite element scheme

We consider a linearized Navier-Stokes problem; find $(u, p) : \Omega \times (0, T) \rightarrow \mathbb{R}^d \times \mathbb{R}$ such that

$$\begin{cases} \frac{Du}{Dt_w} - \nabla(2\nu D(u)) + \nabla p = f, & \text{in } \Omega \times (0, T), \\ \nabla \cdot u = 0, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \Gamma \times (0, T), \\ u = u^0, & \text{in } \Omega, \text{ at } t = 0, \end{cases} \quad (7)$$

where u is the velocity, p is the pressure, $f : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ is a given external force, $u^0 : \Omega \rightarrow \mathbb{R}^d$ is a given initial velocity, $\nu(>0)$ is a viscosity, $D(u)$ is a strain-rate tensor defined by

$$D_{ij}(u) \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i, j = 1, \dots, d), \quad (8)$$

D/Dt_w is a material derivation defined by

$$\frac{D}{Dt_w} \equiv \frac{\partial}{\partial t} + w \cdot \nabla \quad (9)$$

and $w : \Omega \times (0, T) \rightarrow \mathbb{R}^d$ is a given velocity. If w is replaced by u , (7) is the Navier-Stokes problem.

We use the same notation $(\cdot, \cdot)$ to represent $L^2(\Omega)$ inner product for scalar-, vector- and matrix-valued functions, and define a bilinear forms a on $H^1(\Omega)^d \times H^1(\Omega)^d$ and b on $H^1(\Omega)^d \times L^2(\Omega)$ by

$$a(u, v) \equiv 2\nu(D(u), D(v)), \quad \text{and} \quad b(v, q) \equiv -(\nabla \cdot v, q), \quad (10)$$

respectively. The weak form of the problem (7) is written as follows; find $\{(u, p)(t)\}_{t \in (0, T)} \subset V \times Q$ such that

$$\begin{cases} \left(\frac{Du}{Dt_w}, v \right) + a(u, v) + b(v, p) + b(u, q) = (f, v), & \forall (v, q) \in V \times Q, \\ u(0) = u^0, \end{cases} \quad (11)$$

where $V \equiv H_0^1(\Omega)^d$ and $Q \equiv L_0^2(\Omega)$.

We state a characteristics finite element scheme for (7). Let $\mathcal{T}_h = \{K\}$ be a triangulation,

$$\Omega_h \equiv \text{int} \left(\bigcup_{K \in \mathcal{T}_h} K \right) \quad \text{and} \quad \Gamma_h \equiv \partial \Omega_h. \quad (12)$$

We define function spaces X_h , M_h , V_h and Q_h by

$$X_h \equiv \{v_h \in C^0(\bar{\Omega}_h)^d; v_h|_K \in P_1(K)^d, \forall K \in \mathcal{T}_h\}, \quad (13)$$

$$M_h \equiv \{q_h \in C^0(\bar{\Omega}_h); q_h|_K \in P_1(K), \forall K \in \mathcal{T}_h\}, \quad (14)$$

$$V_h \equiv X_h \cap H_0^1(\Omega) \quad \text{and} \quad Q_h \equiv M_h \cap L_0^2(\Omega_h), \quad (15)$$

respectively, where $P_1(K)$ is a function space of piecewise linear functions on an element $K \in \mathcal{T}_h$, P means a nodal point on Γ_h , and $L_0^2(\Omega)$ is a subspace of $L^2(\Omega)$ defined by

$$L_0^2(\Omega) \equiv \{q \in L^2(\Omega); (q, 1) = 0\}. \quad (16)$$

Let $N_T \equiv [T/\Delta t]$ be a total number of time steps, δ be a positive constant, h_K be a diameter of K and $(\cdot, \cdot)_K$ be an inner product in $L^2(K)^d$. For $u, \tilde{u}$ and $w \in H^1(\Omega)^d$, we define a linear form $\mathcal{M}_h(u, \tilde{u}; \Delta t, w)$ on V_h and a bilinear form $\mathcal{E}_h$ on $H^1(\Omega) \times H^1(\Omega)$ by

$$\langle \mathcal{M}_h(u, \tilde{u}; \Delta t, w), v_h \rangle \equiv \left(\frac{u - \tilde{u} \circ X_1(w, \Delta t)}{\Delta t}, v_h \right) \quad \text{and} \quad \mathcal{E}_h(p, q) \equiv -\delta \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla p, \nabla q)_K, \quad (17)$$

respectively. Let an approximate function $u_h^0 \in V_h$ of u^0 be given. A pressure-stabilized characteristics finite element scheme for (7) is to find $\{(u_h^n, p_h^n)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that, for $n = 1, \dots, N_T$,

$$\begin{aligned} \langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle + a(u_h^n, v_h) + b(v_h, p_h^n) + b(u_h^n, q_h) + \mathcal{E}_h(p_h^n, q_h) \\ = (f^n, v_h), \quad \forall (v_h, q_h) \in V_h \times Q_h, \end{aligned} \quad (18)$$

The scheme (18) can deal with high Reynolds number (small viscosity) problems by the method of characteristics. The material derivative term $(Du/Dt_w, v)$ in (11) is approximated by $\langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle$. When we find (u_h^n, p_h^n) in the scheme (18), a composite function $u_h^{n-1} \circ X_1(w_h^{n-1}, \Delta t)$ is a known function and a coefficient matrix of the system of the linear equations is symmetric. The advantage enables us to use symmetric linear iterative solvers, i.e., CG, CR, MINRES [2]. Since the coefficient matrix is independent of step number n , it is enough to make the matrix at only the first time step. The scheme employs a cheap element P1/P1, it is useful for large scale computation, especially in 3D. Although P1/P1 element does not satisfy the inf-sup condition [8], the scheme works by a pressure-stabilization term $\mathcal{E}_h$.

We impose assumption for a given velocity w and review a proposition in [15].

Hypothesis 2.1. A function w satisfies

$$\begin{cases} w \in C^0([0, T]; W^{1,\infty}(\Omega)), \\ w = 0 \text{ on } \Gamma, \\ \operatorname{div} w = 0 \text{ in } \Omega. \end{cases} \quad (19)$$

Proposition 2.2 ([15]). Under Hypothesis 2.1 and an inequality;

$$\Delta t < \frac{1}{\|w\|_{C^0(W^{1,\infty})}}, \quad (20)$$

it holds that, for any $t \in [0, T]$,

$$X_1(w(\cdot, t), \Delta t)(\Omega) = \Omega. \quad (21)$$

In the following sections, we assume that Hypothesis 2.1 and the inequality (20) hold, and that, for the sake of simplicity, Ω is convex polygonal domain ($\Omega = \Omega_h$).

3 Stability and convergence

In this section, we consider stability and convergence of the scheme (18). We use c to represent the generic positive constant independent of discretization parameters and solutions, which can take different values at different places, and $c(A)$ is a positive constant, which depends on A . Constants c_0 , c_1 and c_2 have particular meanings in this paper,

$$c_0 = c_0(\|w\|_{C^0(L^\infty)}), \quad c_1 = c_1(\|w\|_{C^0(W^{1,\infty})}) \quad \text{and} \quad c_2 = c_2(\|w\|_{C^0(W^{1,\infty}) \cap C^1(L^\infty)}), \quad (22)$$

respectively.

3.1 Stability

This subsection is devoted to the stability of the scheme (18). We use norms and seminorms, $\|\cdot\|_k \equiv \|\cdot\|_{H^k(\Omega)}$ ($k = 0, 1, 2$), $\|\cdot\|_V \equiv \|\cdot\|_1$, $\|\cdot\|_Q \equiv \|\cdot\|_0$,

$$\begin{aligned} \|(v, q)\|_{V \times Q} &\equiv \{\|v\|_V^2 + \|q\|_Q^2\}^{1/2}, \quad |q|_h \equiv \left\{ \sum_{K \in \mathcal{T}_h} h_K^2 (\nabla q, \nabla q)_K \right\}^{1/2}, \\ \|u\|_{l_{(m)}^\infty(L^2)} &\equiv \max_{n=m, \dots, N_T} \|u^n\|_0, \quad \|u\|_{l_{(m)}^\infty(H^1)} \equiv \max_{n=m, \dots, N_T} \|u^n\|_1, \quad \|u\|_{l_{(0)}^\infty(X)} \equiv \|u\|_{l_{(0)}^\infty(X)} \quad (X = L^2, H^1), \\ \|u\|_{l_{(m)}^2(L^2)} &\equiv \left\{ \Delta t \sum_{n=m}^{N_T} \|u^n\|_0^2 \right\}^{1/2}, \quad \|u\|_{l_{(m)}^2(H^1)} \equiv \left\{ \Delta t \sum_{n=m}^{N_T} \|u^n\|_1^2 \right\}^{1/2}, \quad \|u\|_{l^2(X)} \equiv \|u\|_{l_{(1)}^2(X)} \quad (X = L^2, H^1), \\ |p|_{l_{(m)}^2(M_h)} &\equiv \left\{ \Delta t \sum_{n=m}^{N_T} |p^n|_h^2 \right\}^{1/2} \quad \text{and} \quad |p|_{l^2(M_h)} \equiv |p|_{l_{(1)}^2(M_h)}. \end{aligned} \quad (23)$$

Setting a bilinear form $\mathcal{A}_h$ on $(V_h \times Q_h) \times (V_h \times Q_h)$ by

$$\mathcal{A}_h((u, p); (v, q)) \equiv a(u, v) + b(v, p) + b(u, q) + \mathcal{C}_h(p, q), \quad (24)$$

we have another representation of the scheme (18);

$$\langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle + \mathcal{A}_h((u_h^n, p_h^n); (v_h, q_h)) = (f^n, v_h), \quad \forall (v_h, q_h) \in V_h \times Q_h. \quad (25)$$

We prepare two lemmas for the stability. First one is an estimate of a composite function appearing $\mathcal{M}_h$;

Lemma 3.1. *For any $v \in L^2(\Omega)^d$ and $t \in [0, T]$, it holds that*

$$\|v \circ X_1(w(\cdot, t), \Delta t)\|_0 \leq (1 + c_1 \Delta t) \|v\|_0. \quad (26)$$

We omit the proof, because it is similar to the proof of Lemma 1 in [15].

P1/P1 element is employed in the scheme (18), i.e., both finite dimensional function spaces V_h and Q_h consist of P1 element, and P1/P1 element does not satisfy the inf-sup condition [8];

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq \beta^* > 0. \quad (27)$$

However, a , b and $\mathcal{C}_h$ satisfies the following inf-sup condition.

Lemma 3.2 ([7]). *There is a positive constant γ such that*

$$\inf_{(u_h, p_h) \in V_h \times Q_h} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{\mathcal{A}_h((u_h, p_h); (v_h, q_h))}{\|(u_h, p_h)\|_{V \times Q} \|(v_h, q_h)\|_{V \times Q}} \geq \gamma. \quad (28)$$

From the above lemmas, we obtain a stability result.

Theorem 3.3 (stability). *Let $\Delta t_0 (< 1)$ be a fixed number. Assume $\Delta t \in (0, \Delta t_0]$.*

(i) *It holds that*

$$\|u_h\|_{l^\infty(L^2)} + \sqrt{v}\|u_h\|_{l^2(H^1)} + \sqrt{\delta}\|p_h\|_{l^2(M_h)} \leq c_1(\|u_h^0\|_0 + \|f\|_{l^2(L^2)}). \quad (29)$$

(ii) *The following two inequalities hold:*

$$\begin{aligned} \sqrt{v}\|u_h\|_{l_{(1)}^\infty(H^1)} + \sqrt{\delta}\|p_h\|_{l_{(1)}^\infty(M_h)} + \|\bar{D}_{\Delta t}u_h\|_{l_{(2)}^2(L^2)} + \sqrt{v\Delta t}\|\bar{D}_{\Delta t}u_h\|_{l_{(2)}^2(H^1)} + \sqrt{\delta\Delta t}\|\bar{D}_{\Delta t}p_h\|_{l_{(2)}^2(M_h)} \\ \leq c_1(\|u_h^1\|_1 + \|p_h^1\|_h + \|f\|_{l_{(2)}^2(L^2)}), \end{aligned} \quad (30)$$

$$\|p_h\|_{l_{(2)}^2(L^2)} \leq c_1(\|u_h^1\|_1 + \|p_h^1\|_h + \|f\|_{l_{(2)}^2(L^2)}). \quad (31)$$

Proof. First we prove (i). Substituting $(u_h^n, -p_h^n) \in V_h \times Q_h$ into (v_h, q_h) in the scheme (18), we have

$$\langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), u_h^n \rangle + a(u_h^n, u_h^n) + \mathcal{C}_h(p_h^n, -p_h^n) = (f^n, u_h^n). \quad (32)$$

We evaluate the four terms;

$$\begin{aligned} \langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), u_h^n \rangle &= \frac{1}{\Delta t} \left\{ \frac{1}{2}(\|u_h^n\|_0^2 - \|u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)\|_0^2) + \frac{1}{2}\|u_h^n - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)\|_0^2 \right\} \\ &\quad (\because (a-b)a = (a^2 - b^2)/2 + (a-b)^2/2) \\ &\geq \frac{1}{\Delta t} \left(\frac{1}{2}\|u_h^n\|_0^2 - \frac{1}{2}\|u_h^{n-1}\|_0^2 \right) - c_1\|u_h^{n-1}\|_0^2 \quad (\because \text{Lemma 3.1}) \\ &= \bar{D}_{\Delta t} \left(\frac{1}{2}\|u_h^n\|_0^2 \right) - c_1\|u_h^{n-1}\|_0^2, \end{aligned} \quad (33a)$$

$$a(u_h^n, u_h^n) = 2v\|D(u_h^n)\|_0^2 \geq cv\|u_h^n\|_1^2 \quad (\because \text{Korn's inequality [6]}), \quad (33b)$$

$$\mathcal{C}_h(p_h^n, -p_h^n) = \delta\|p_h^n\|_h^2, \quad (33c)$$

and

$$(f^n, u_h^n) \leq \frac{1}{2}(\|f^n\|_0^2 + \|u_h^n\|_0^2) \quad (\because ab \leq (a^2 + b^2)/2), \quad (33d)$$

where $\bar{D}_{\Delta t}a^n \equiv (a^n - a^{n-1})/\Delta t$. The inequalities (33) implies

$$\bar{D}_{\Delta t} \left(\frac{1}{2}\|u_h^n\|_0^2 \right) + cv\|u_h^n\|_1^2 + \delta\|p_h^n\|_h^2 \leq \frac{1}{2}(\|f^n\|_0^2 + \|u_h^n\|_0^2) \quad (n = 1, \dots, N_T), \quad (34)$$

and (29) holds from the discrete Gronwall lemma [20].

Next, we prove (30) of (ii). Substituting $0 \in V_h$ into v_h in the scheme (18), we have

$$b(u_h^n, q_h) + \mathcal{C}_h(p_h^n, q_h) = 0 \quad (\forall q_h \in Q_h, n = 1, \dots, N_T), \quad (35)$$

and, then,

$$b(\bar{D}_{\Delta t}u_h^n, q_h) + \mathcal{C}_h(\bar{D}_{\Delta t}p_h^n, q_h) = 0 \quad (\forall q_h \in Q_h, n = 2, \dots, N_T). \quad (36)$$

Substituting $(\bar{D}_{\Delta t}u_h^n, 0) \in V_h \times Q_h$ into (v_h, q_h) in the scheme (18), we have

$$\langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), \bar{D}_{\Delta t}u_h^n \rangle + a(u_h^n, \bar{D}_{\Delta t}u_h^n) + b(\bar{D}_{\Delta t}u_h^n, p_h^n) = (f^n, \bar{D}_{\Delta t}u_h^n) \quad (n = 1, \dots, N_T), \quad (37)$$

and, by (36),

$$\langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), \bar{D}_{\Delta t} u_h^n \rangle + a(u_h^n, \bar{D}_{\Delta t} u_h^n) - \mathcal{C}_h(\bar{D}_{\Delta t} p_h^n, p_h^n) = (f^n, \bar{D}_{\Delta t} u_h^n) \quad (n = 2, \dots, N_T). \quad (38)$$

We evaluate the four terms in (38);

$$\begin{aligned} \langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), \bar{D}_{\Delta t} u_h^n \rangle &= \left(\bar{D}_{\Delta t} u_h^n + \frac{1}{\Delta t} (u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)), \bar{D}_{\Delta t} u_h^n \right) \\ &= \|\bar{D}_{\Delta t} u_h^n\|_0^2 + \frac{1}{\Delta t} (u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t), \bar{D}_{\Delta t} u_h^n) \\ &\geq \|\bar{D}_{\Delta t} u_h^n\|_0^2 - \frac{1}{\Delta t} \|u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)\|_0 \|\bar{D}_{\Delta t} u_h^n\|_0 \\ &\geq \|\bar{D}_{\Delta t} u_h^n\|_0^2 - c_0 |u_h^{n-1}|_1 \|\bar{D}_{\Delta t} u_h^n\|_0 \\ &\geq \|\bar{D}_{\Delta t} u_h^n\|_0^2 - \left(c_0 |u_h^{n-1}|_1^2 + \frac{1}{4} \|\bar{D}_{\Delta t} u_h^n\|_0^2 \right) \\ &\geq \frac{3}{4} \|\bar{D}_{\Delta t} u_h^n\|_0^2 - c_0 \|D(u_h^{n-1})\|_0^2 \quad (\because \text{Korn の不等式}), \end{aligned} \quad (39a)$$

$$\begin{aligned} a(u_h^n, \bar{D}_{\Delta t} u_h^n) &= \bar{D}_{\Delta t} \left(\frac{1}{2} a(u_h^n, u_h^n) \right) + \frac{\Delta t}{2} a(\bar{D}_{\Delta t} u_h^n, \bar{D}_{\Delta t} u_h^n) \\ &= \bar{D}_{\Delta t} \left(v \|D(u_h^n)\|_0^2 \right) + v \Delta t \|D(\bar{D}_{\Delta t} u_h^n)\|_0^2, \end{aligned} \quad (39b)$$

$$\begin{aligned} -\mathcal{C}_h(\bar{D}_{\Delta t} p_h^n, p_h^n) &= \bar{D}_{\Delta t} \left(-\frac{1}{2} \mathcal{C}_h(p_h^n, p_h^n) \right) - \frac{\Delta t}{2} \mathcal{C}_h(\bar{D}_{\Delta t} p_h^n, \bar{D}_{\Delta t} p_h^n) \\ &= \bar{D}_{\Delta t} \left(\frac{\delta}{2} |p_h^n|_h^2 \right) + \frac{\delta \Delta t}{2} |\bar{D}_{\Delta t} p_h^n|_h^2, \end{aligned} \quad (39c)$$

and

$$(f^n, \bar{D}_{\Delta t} u_h^n) = \|f^n\|_0^2 + \frac{1}{4} \|\bar{D}_{\Delta t} u_h^n\|_0^2. \quad (39d)$$

Combining (39) with (38), we have

$$\begin{aligned} \bar{D}_{\Delta t} \left(v \|D(u_h^n)\|_0^2 + \frac{\delta}{2} |p_h^n|_h^2 \right) + \frac{1}{2} \|\bar{D}_{\Delta t} u_h^n\|_0^2 + v \Delta t \|D(\bar{D}_{\Delta t} u_h^n)\|_0^2 + \frac{\delta \Delta t}{2} |\bar{D}_{\Delta t} p_h^n|_h^2 \\ \leq \|f^n\|_0^2 + c_0 \|D(u_h^{n-1})\|_0^2 \quad (n = 2, \dots, N_T), \end{aligned} \quad (40)$$

which implies (30) by the discrete Gronwall lemma.

Finally we prove (31) of (ii). From Lemma 3.2, $\|p_h^n\|_0$ is evaluated as follows;

$$\begin{aligned} \|p_h^n\|_0 &\leq \|(u_h^n, p_h^n)\|_{V \times Q} \\ &\leq \frac{1}{\gamma} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{\mathcal{A}_h((u_h^n, p_h^n); (v_h, q_h))}{\|(v_h, q_h)\|_{V \times Q}} \\ &\leq \frac{1}{\gamma} \sup_{(v_h, q_h) \in V_h \times Q_h} \frac{(f^n, v_h) - \langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle}{\|(v_h, q_h)\|_{V \times Q}} \\ &\leq \frac{1}{\gamma} \left(\|f^n\|_{V_h'} + \|\mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1})\|_{V_h'} \right) \\ &\leq \frac{1}{\gamma} \left(\|f^n\|_0 + \|\bar{D}_{\Delta t} u_h^n\|_0 + c_0 |u_h^{n-1}|_1 \right), \end{aligned} \quad (41)$$

where we have used the following inequality in the last inequality of (41);

$$\begin{aligned}
\|\mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1})\|_{V_h'} &\leq \frac{1}{\Delta t} \|u_h^n - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)\|_0 \\
&= \|\bar{D}_{\Delta t} u_h^n + \frac{1}{\Delta t} (u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t))\|_0 \\
&\leq \|\bar{D}_{\Delta t} u_h^n\|_0 + \frac{1}{\Delta t} \|u_h^{n-1} - u_h^{n-1} \circ X_1(w^{n-1}, \Delta t)\|_0 \\
&\leq \|\bar{D}_{\Delta t} u_h^n\|_0 + c_0 \|u_h^{n-1}\|_1.
\end{aligned} \tag{42}$$

Combining (41) with (30), we have

$$\begin{aligned}
\|p_h\|_{l_{(2)}^2(L^2)} &\leq c_0 (\|f\|_{l_{(2)}^2(L^2)} + \|\bar{D}_{\Delta t} u_h^n\|_{l_{(2)}^2(L^2)} + \|u_h\|_{l_{(1)}^2(H^1)}) \\
&\leq c_1 (\|u_h^1\|_1 + |p_h^1|_h + \|f\|_{l_{(2)}^2(L^2)}).
\end{aligned} \tag{43}$$

□

Remark 3.4. In the right hand side of (31), $\|u_h^1\|_1$ can be estimated by (30), and, for $|p_h^1|_h$, a detailed evaluation is required.

3.2 Convergence

This subsection is devoted to convergence of the scheme (18). At first we define a Stokes projection.

Definition 3.5 (Stokes projection). For $(u, p) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^1(\Omega))$, $(w_h, r_h) \in V_h \times Q_h$ is a Stokes projection of (u, p) provided

$$a(w_h, v_h) + b(v_h, r_h) + b(w_h, q_h) + \mathcal{C}_h(r_h, q_h) = \langle \mathcal{F}_h, (v_h, q_h) \rangle, \quad \forall (v_h, q_h) \in V_h \times Q_h, \tag{44}$$

where $\mathcal{F}_h$ is a linear form on $V_h \times Q_h$ defined by

$$\langle \mathcal{F}_h, (v_h, q_h) \rangle \equiv (g, v_h) - \delta \sum_{K \in \mathcal{T}_h} h_K^2 (g, \nabla q_h)_K \quad \text{with} \quad g \equiv -2\nu \nabla D(u) + \nabla p. \tag{45}$$

For the Stokes projection, the following error estimate holds.

Proposition 3.6. Suppose $(u, p) \in (V \cap H^2(\Omega)^d) \times (Q \cap H^2(\Omega))$, and u satisfies

$$b(u, q) = 0 \quad (q \in Q). \tag{46}$$

Let (w_h, r_h) be a Stokes projection of (u, p) by (44). Then, there exists a positive constant c_s , independent of h , such that

$$\|u - w_h\|_1 + \|p - r_h\|_0 \leq c_s h (\|u\|_2 + h \|p\|_2). \tag{47}$$

We omit the proof, because papers [3] and [7] give the result.

Theorem 3.7. Let $\Delta t_0 (< 1)$ be a fixed positive number, $\Delta t \in (0, \Delta t_0]$, and (u, p) and (u_h, p_h) be solutions of (11) and (18), respectively, where u_h^0 is a first component of the Stokes projection of $(u^0, 0)$. Suppose $u \in C^0([0, T]; V \cap H^2) \cap H^2(0, T; L^2) \cap H^1(0, T; H^1)$ and $p \in C^0([0, T]; Q \cap H^1) \cap H^1(0, T; H^2)$. Then, it holds that

$$\|u - u_h\|_{l^\infty(L^2)} + \sqrt{\nu} \|u - u_h\|_{l^2(H^1)} + \sqrt{\delta} \|p - p_h\|_{l^2(M_h)}$$

$$\leq c_2 \left\{ \Delta t (\|u\|_{H^2(0,T;L^2)} + \|u\|_{H^1(0,T;H^1)} + \|u\|_{L^2(0,T;H^2)}) \right. \\ \left. + h \left(\|u^0\|_2 + \sqrt{\delta} (\nu \|u\|_{l^2(H^2)} + \|p\|_{l^2(H^1)}) + \|(u, p)\|_{H^1(0,T; H^2 \times H^2)} \right) \right\}. \quad (48)$$

Proof. Let $(\hat{u}_h, \hat{p}_h)(t)$ be the Stokes projection of $(u, p)(t) \in H^2(\Omega)^d \times H^2(\Omega)$,

$$e_h^n \equiv u_h^n - \hat{u}_h^n, \quad \varepsilon_h^n \equiv p_h^n - \hat{p}_h^n \quad \text{and} \quad \eta_h(t) \equiv (u - \hat{u}_h)(t). \quad (49)$$

For any $(v_h, q_h) \in V_h \times Q_h$, it holds that, from (11) and (18),

$$\begin{aligned} & \langle \mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle + a(e_h^n, v_h) + b(v_h, \varepsilon_h^n) + b(e_h^n, q_h) + \mathcal{C}_h(\varepsilon_h^n, q_h) \\ &= (f^n, v_h) - (-2\nu \nabla D(u^n) + \nabla p^n, v_h) + \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla q_h)_K \\ &= \left(\frac{Du^n}{Dt_w}, v_h \right) + \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla q_h)_K, \end{aligned} \quad (50)$$

and from an identity

$$\mathcal{M}_h(u_h^n, u_h^{n-1}; \Delta t, w^{n-1}) = \mathcal{M}_h(e_h^n, e_h^{n-1}; \Delta t, w^{n-1}) - \mathcal{M}_h(\eta_h^n, \eta_h^{n-1}; \Delta t, w^{n-1}) + \mathcal{M}_h(u^n, u^{n-1}; \Delta t, w^{n-1}), \quad (51)$$

we have

$$\begin{aligned} & \langle \mathcal{M}_h(e_h^n, e_h^{n-1}; \Delta t, w^{n-1}), v_h \rangle + a(e_h^n, v_h) + b(v_h, \varepsilon_h^n) + b(e_h^n, q_h) + \mathcal{C}_h(\varepsilon_h^n, q_h) \\ &= \langle \mathcal{M}_h(\eta_h^n, \eta_h^{n-1}; \Delta t, w^{n-1}) - \mathcal{M}_h(u^n, u^{n-1}; \Delta t, w^{n-1}), v_h \rangle + \left(\frac{Du^n}{Dt_w}, v_h \right) + \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla q_h)_K \\ &= \left(\frac{\eta_h^n - \eta_h^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} - \frac{u^n - u^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} + \frac{Du^n}{Dt_w}, v_h \right) + \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla q_h)_K. \end{aligned} \quad (52)$$

Substituting $(e_h^n, -\varepsilon_h^n)$ into (v_h, q_h) in (52), we have

$$\begin{aligned} & \langle \mathcal{M}_h(e_h^n, e_h^{n-1}; \Delta t, w^{n-1}), e_h^n \rangle + a(e_h^n, e_h^n) - \mathcal{C}_h(\varepsilon_h^n, \varepsilon_h^n) \\ &= \left(\frac{\eta_h^n - \eta_h^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} - \frac{u^n - u^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} + \frac{Du^n}{Dt_w}, e_h^n \right) - \delta \sum_K h_K^2 (-2\nu \nabla D(u^n) + \nabla p^n, \nabla \varepsilon_h^n)_K \\ &\equiv I_1^n + I_2^n. \end{aligned} \quad (53)$$

The two terms I_1 and I_2 are evaluated as follows. We have

$$\begin{aligned} \left\| \frac{\eta_h^n - \eta_h^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} \right\|_0 &= \left\| \int_0^1 \left(\frac{\partial \eta_h}{\partial t} + (w^{n-1}(x) \cdot \nabla) \eta_h \right) (x - s w^{n-1}(x) \Delta t, t^n - s \Delta t) ds \right\|_0 \\ &= \left[\int_\Omega \left\{ \int_0^1 \left(\frac{\partial \eta_h}{\partial t} + (w^{n-1}(x) \cdot \nabla) \eta_h \right) (x - s w^{n-1}(x) \Delta t, t^n - s \Delta t) ds \right\}^2 dx \right]^{1/2} \\ &\leq \left\{ \int_0^1 ds \int_\Omega \left(\frac{\partial \eta_h}{\partial t} + (w^{n-1}(x) \cdot \nabla) \eta_h \right) (x - s w^{n-1}(x) \Delta t, t^n - s \Delta t)^2 dx \right\}^{1/2} \\ &\leq c_0 \left[\left\{ \int_0^1 ds \int_\Omega \frac{\partial \eta_h}{\partial t} (x - s w^{n-1}(x) \Delta t, t^n - s \Delta t)^2 dx \right\}^{1/2} \right] \end{aligned}$$

$$\begin{aligned}
& + \left\{ \int_0^1 ds \int_{\Omega} \nabla \eta_h(x - sw^{n-1}(x)\Delta t, t^n - s\Delta t)^2 dx \right\}^{1/2} \\
& \leq c_1 \left[\left\{ \int_0^1 \left\| \frac{\partial \eta_h}{\partial t}(\cdot, t^n - s\Delta t) \right\|_0^2 ds \right\}^{1/2} + \left\{ \int_0^1 \|\nabla \eta_h(\cdot, t^n - s\Delta t)\|_0^2 ds \right\}^{1/2} \right] \\
& \leq \frac{c_1}{\sqrt{\Delta t}} \left[\left\{ \int_{t^{n-1}}^{t^n} \left\| \frac{\partial \eta_h}{\partial t}(\cdot, t) \right\|_0^2 dt \right\}^{1/2} + \left\{ \int_{t^{n-1}}^{t^n} \|\nabla \eta_h(\cdot, t)\|_0^2 dt \right\}^{1/2} \right] \\
& = \frac{c_1}{\sqrt{\Delta t}} \left\{ \left\| \frac{\partial \eta_h}{\partial t} \right\|_{L^2(t^{n-1}, t^n; L^2)} + \|\nabla \eta_h\|_{L^2(t^{n-1}, t^n; L^2)} \right\} \\
& \leq \frac{c_1 h}{\sqrt{\Delta t}} \left\{ \left\| \left(\frac{\partial u}{\partial t}, \frac{\partial p}{\partial t} \right) \right\|_{L^2(t^{n-1}, t^n; H^2 \times H^2)} + \|(u, p)\|_{L^2(t^{n-1}, t^n; H^2 \times H^2)} \right\} \\
& = \frac{c_1 h}{\sqrt{\Delta t}} \|(u, p)\|_{H^1(t^{n-1}, t^n; H^2 \times H^2)}, \tag{54}
\end{aligned}$$

and

$$\left\| \left(\frac{Du}{Dt_w} \right)^n - \frac{u^n - u^{n-1} \circ X_1(w^{n-1}, \Delta t)}{\Delta t} \right\|_0 \leq c_2 \sqrt{\Delta t} \{ \|u\|_{H^2(t^{n-1}, t^n; L^2)} + \|u\|_{H^1(t^{n-1}, t^n; H^1)} + \|u\|_{L^2(t^{n-1}, t^n; H^2)} \}. \tag{55}$$

For any $\alpha_0 > 0$, it holds that, from the inequalities (54) and (55),

$$I_1^n \leq \frac{c_2}{\alpha_0} \{ \Delta t (\|u\|_{H^2(t^{n-1}, t^n; L^2)}^2 + \|u\|_{H^1(t^{n-1}, t^n; H^1)}^2 + \|u\|_{L^2(t^{n-1}, t^n; H^2)}^2) + \frac{h^2}{\Delta t} \|(u, p)\|_{H^1(t^{n-1}, t^n; H^2 \times H^2)}^2 \} + \alpha_0 \|e_h^n\|_0^2. \tag{56}$$

For I_2 , we have

$$\begin{aligned}
I_2^n & \leq \delta \sum_K h_K^2 \|\nabla \varepsilon_h^n\|_{L^2(K)} \left(2\nu \|\nabla D(u^n)\|_{L^2(K)} + \|\nabla p^n\|_{L^2(K)} \right) \\
& \leq 2\nu \delta \sum_K h_K^2 \|\nabla \varepsilon_h^n\|_{L^2(K)} \|u^n\|_{H^2(K)} + \delta \sum_K h_K^2 \|\nabla \varepsilon_h^n\|_{L^2(K)} \|\nabla p^n\|_{L^2(K)} \\
& \leq \nu \delta \sum_K h_K^2 \left(\alpha_1 \|\nabla \varepsilon_h^n\|_{L^2(K)}^2 + \frac{c}{\alpha_1} \|u^n\|_{H^2(K)}^2 \right) + \delta \sum_K h_K^2 \left(\alpha_2 \|\nabla \varepsilon_h^n\|_{L^2(K)}^2 + \frac{c}{\alpha_2} \|\nabla p^n\|_{L^2(K)}^2 \right) \\
& \leq (\alpha_1 \nu + \alpha_2) \delta |\varepsilon_h^n|_h^2 + c \delta h^2 \left(\frac{\nu}{\alpha_1} \|u^n\|_2^2 + \frac{1}{\alpha_2} \|p^n\|_1^2 \right). \tag{57}
\end{aligned}$$

Combining (56) and (57) with (53), we have, for any positive numbers $\delta, \alpha_0, \alpha_1$ and α_2 and $n = 1, \dots, N_T$,

$$\begin{aligned}
& \overline{D}_{\Delta t} \left(\frac{1}{2} \|e_h^n\|_0^2 \right) + \nu \|D(e_h^n)\|_0^2 + \delta |\varepsilon_h^n|_h^2 \\
& \leq \alpha_0 \|e_h^n\|_0^2 + c_1 \|e_h^{n-1}\|_0^2 + (\alpha_1 \nu + \alpha_2) \delta |\varepsilon_h^n|_h^2 + c \delta h^2 \left(\frac{\nu}{\alpha_1} \|u^n\|_2^2 + \frac{1}{\alpha_2} \|p^n\|_1^2 \right) \\
& \quad + \frac{c_2}{\alpha_0} \{ \Delta t (\|u\|_{H^2(t^{n-1}, t^n; L^2)}^2 + \|u\|_{H^1(t^{n-1}, t^n; H^1)}^2 + \|u\|_{L^2(t^{n-1}, t^n; H^2)}^2) + \frac{h^2}{\Delta t} \|(u, p)\|_{H^1(t^{n-1}, t^n; H^2 \times H^2)}^2 \}, \tag{58}
\end{aligned}$$

and, then, for $\alpha_0 = 1/(4\Delta t_0)$, $\alpha_1 = 1/(4\nu)$ and $\alpha_2 = 1/4$,

$$\overline{D}_{\Delta t} \left(\frac{1}{2} \|e_h^n\|_0^2 \right) + \nu \|D(e_h^n)\|_0^2 + \frac{\delta}{2} |\varepsilon_h^n|_h^2$$

$$\begin{aligned}
&\leq \frac{1}{4\Delta t_0} \|e_h^n\|_0^2 + c_1 \|e_h^{n-1}\|_0^2 + c\delta h^2 \left(v^2 \|u^n\|_2^2 + \|p^n\|_1^2 \right) \\
&\quad + c_2 \left\{ \Delta t \left(\|u\|_{H^2(I^{n-1}, I^n; L^2)}^2 + \|u\|_{H^1(I^{n-1}, I^n; H^1)}^2 + \|u\|_{L^2(I^{n-1}, I^n; H^2)}^2 \right) + \frac{h^2}{\Delta t} \|(u, p)\|_{H^1(I^{n-1}, I^n; H^2 \times H^2)}^2 \right\}. \quad (59)
\end{aligned}$$

By the discrete Gronwall inequality, it holds that

$$\begin{aligned}
&\|e_h\|_{l^\infty(L^2)} + \sqrt{v} \|D(e_h)\|_{l^2(L^2)} + \sqrt{\delta} \|e_h\|_{l^2(M_h)} \\
&\leq c_2 \left\{ \|e_h^0\|_0 + \Delta t \left(\|u\|_{H^2(0, T; L^2)} + \|u\|_{H^1(0, T; H^1)} + \|u\|_{L^2(0, T; H^2)} \right) \right. \\
&\quad \left. + h \left(\sqrt{\delta} (v \|u\|_{l^2(H^2)} + \|p\|_{l^2(H^1)}) + \|(u, p)\|_{H^1(0, T; H^2 \times H^2)} \right) \right\} \\
&\leq c_2 \left\{ \Delta t \left(\|u\|_{H^2(0, T; L^2)} + \|u\|_{H^1(0, T; H^1)} + \|u\|_{L^2(0, T; H^2)} \right) \right. \\
&\quad \left. + h \left(\|u^0\|_2 + \sqrt{\delta} (v \|u\|_{l^2(H^2)} + \|p\|_{l^2(H^1)}) + \|(u, p)\|_{H^1(0, T; H^2 \times H^2)} \right) \right\}, \quad (60)
\end{aligned}$$

which implies (48). $\square$

4 Conclusions

We have introduced a characteristics finite element scheme for a linearized Navier-Stokes equations. The scheme employs P1/P1 element, and the coefficient matrix appearing in the scheme is symmetric. These advantages reduces computational time and cost by half. Therefore, the scheme is useful especially for three dimensional computation. We have shown stability and convergence results with the optimal L^2 -error estimate for the velocity. Although the system of the equations is linear, the analysis is useful even in the nonlinear case. We note that the convergence of the pressure can be proved under some assumptions.

References

- [1] K. Baba and M. Tabata, On a conservative upwind finite element scheme for convective diffusion equations, *R.A.I.R.O. Analyse nuérique/Numerical Analysis*, Vol.15 (1981), pp.3–25.
- [2] R. Barrett, M. Berry, T.F., Chan, J. Demmel, J. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine and H. van der Vorst, *Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods*, SIAM, Philadelphia, 1994.
- [3] F. Brezzi and J. Douglas Jr., Stabilized mixed methods for the Stokes problem, *Numerische Mathematik*, Vol.53 (1988), pp.225–235.
- [4] A.N. Brooks and T.J.R. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations, *Computer Methods in Applied Mechanics and Engineering*, Vol.32 (1982), pp.199–259.
- [5] J. Douglas Jr. and T.F., Russell, Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures, *SIAM Journal on Numerical Analysis*, Vol.19 (1982), pp.871–885.
- [6] G. Duvaut and J. L. Lions, *Inequalities in mechanics and physics*, Springer, Berlin, 1976.

- [7] L.P. Franca and R. Stenberg, Error analysis of some Galerkin least squares methods for the elasticity equations, *SIAM Journal on Numerical Analysis*, **28** (1991), pp.1680–1697.
- [8] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, Springer, Berlin, 1986.
- [9] H. Notsu, Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme, *Transactions of Japan Society for Computational Engineering and Science*, Vol.2008, Online ISSN: 1347-8826.
- [10] H. Notsu, H. Rui and M. Tabata, A second order characteristics finite difference method for convection diffusion problems, in preparation.
- [11] H. Notsu and M. Tabata, Navier-Stokes 方程式のための圧力安定化・特性曲線法結合有限要素スキーム, 日本応用数理学会論文誌, Vol.18 (2008), pp.427–445.
- [12] H. Notsu and M. Tabata, A single-step characteristic-curve finite element scheme of second order in time for the incompressible Navier-Stokes equations, *Journal of Scientific Computing*, Vol.38 (2009), pp.1–14.
- [13] O. Pironneau, *Finite Element Methods for Fluids*, John Wiley & Sons, Chichester, 1989.
- [14] O. Pironneau and M. Tabata, Stability and convergence of a Galerkin-characteristics finite element scheme of lumped mass type, *International Journal for Numerical Methods in Fluids*, Vol.64 (2010), pp.1240–1253.
- [15] H. Rui and M. Tabata, A second order characteristic finite element scheme for convection-diffusion problems, *Numerische Mathematik*, Vol.92 (2002), pp.161–177.
- [16] A.H. Stroud, *Approximate calculation of multiple integrals*, Prentice-Hall, Englewood Cliffs, New Jersey, 1971.
- [17] M. Tabata, A finite element approximation corresponding to the upwind finite differencing, *Memoirs of Numerical Mathematics*, Vol.4 (1977), pp.47–63.
- [18] M. Tabata, Discrepancy between theory and real computation on the stability of some finite element schemes, *Journal of Computational and Applied Mathematics*, Vol.199 (2007), pp.424–431.
- [19] M. Tabata and S. Fujima, Robustness of a characteristic finite element scheme of second order in time increment, In C. Groth and D. W. Zingg, editors, *Computational Fluid Dynamics 2004*, Springer, pp.177–182, 2006.
- [20] M. Tabata and D. Tagami, Error estimates for finite element approximations of drag and lift in non-stationary Navier-Stokes flows. *Japan Journal of Industrial and Applied Mathematics*, **17** (2000), pp.371–389.