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Complex Classical Mechanics and Quantum Chaos

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Abstruct. The semiclassical coherent state path integral developed by Klauder is evaluated numerically for a chaotic system (the kicked rotator). This evaluation needs the complexfied classical dynamics of the system and involves a proper treatment of the Stokes phenomenon caused by a new caustic, named ⟪ Phase Space Caustic (PSC) ⟫ . It is revealed that the chaotic natute of the dynamics produces PSC's. This investigation suggests that complex classical mechanics has a clear physical reality.

§1. INTRODUCTION

The investigation of classical chaos in Hamiltonian systems has been continued more than one centry and we have already gotten a lot of butiful knowledge on it (see, for example, Poincare [1], Moser [2], Arnold [3], Lichtenberg and Lieberman [4]). This investigation will surely be continued far into the future.

On the other hand, the investigation of quantum chaos in Hamiltonian systems is still in a yong stage and is rapidly growing now. This is partially because quantum mechanics itself has a shorter history than classical mechanics and is partially because huge numerical computations, which became possible in this ten years according to the rapid development of computer technology, are indispensable to inspecting complicated quantum systems.

At the present time, we have no mathematical definition of quantum chaos yet, in contrast with fairly good mathematical definitions of classical chaos, which are due to the number of integrals, Lyapunov exponent and so on. Without the defination of quantum chaos, how we can investigate the ability of quantum systems to produce complexities ? By the usage of the corresponding principle, we can. Namely, if a classical hamiltonian system is chaotic and exhibits complex behaviors, then it is expected that in the limit $\hbar \to 0$ the corresponding quantum Hamiltonian system also exhibits complex behaviors. The numerical computations performed in the past ten years actually confirmed that when a classical system is chaotic, the corresponding quantum system surely exibits complex behaviors shch as the random distributions of energy eigenvalues, the absence of good quantum numbers except energy, the random phase profile of eigenfunctions, the entanglement of time-evolved wave functions, and so on. Thus we can practically define "quantum chaos" as the behavior of the quantum system, with small $\hbar$, corresponding to a classical chaotic system. We have already excelent reviews on quantum chaos (for example, Berry [5]).

In many cases, direct simulations are used to investigate quantum chaos; we execute two simulations of a system in paralell, one is based on classical mechanics and the other is based on quantum mechanics. Then we compare the resuls of the two simulations. This method of investigation is very powerful and enables us to explore unvisted areas of quantum mechanics. However, it is difficult to understand the mechanism of quantum chaotic behaviors through direct simulations alone, because the results of classical and quantum dynamics are obtained directly and separately and there are no internal connections between the results.

We have another method to investigate quantum chaos, namely, the semiclassical evaluation of path integrals. The present research belongs to this class of investivation. As is well known, the paths dominating a path integral in the semiclassical limit are nothing but the classical orbits satisfting the boundary condition of the path integral. Thus, through the semiclassical evaluation of a path integral, we can extract the essential aspects of quantum phenomena from classical information alone; namely, we can build an internal connection between the classical and the quantum chaotic behaviors of a system. On the other hand, this

method has only weaker power of computation than direct simulations. So, we see that these two methods have complement abilities.

This short paper reports the first numerical application of the semiclassical coherent state path integral developed by Klauder [6] to a chaotic system (the kicked rotator). Namely, we aim to understand the quantum dynamics of a chaotic system in terms of the corresponding classical dynamics through the semiclassical method of Klauder. In §2, we explain the model system and its classical, quantum and especially semiclassical dynamics. §3 is devoted to the report of a numerical computation; we compare the phase space distribution functions which are time-evolved according to classical, quantum and semiclassical dynamics. In §4, we summary the obtained results briefly. Details of this research are so complicated that we report them in a separate paper (Adachi [7]).

§2. MODEL SYSTEM AND ITS CLASSICAL/QUANTUM/SEMICLASSICAL DYNAMICS

Model System .

Throughout this report, we use the kicked rotator as the model system. The kicked rotator is described by the Hamiltonian:

$$H(q,p,t) = \frac{1}{2}p^2 + K\cos q \sum_{n=-\infty}^{+\infty} \delta(t-n)\,, \tag{2.1}$$

where q and p are position and momentum conjugate to each other, respectively, and t is time. K is a fixed real parameter.

In this report, we do not impose the usual periodic boundary condition on the position q for the system, since if this periodic boundary condition were imposed, then the interference of wave function according to the boundary condition would prevent the ideal observation of the intrinsic interference due to the $\langle\!\langle$ folding $\rangle\!\rangle$ operation of the chaotic dynamics, as will be seen in §3. Hence, the phase space is $\{(q,p)\} = \mathbf{R}^2$.

Classical Dynamics .

Applying the Hamiltonian equation to (2.1), we get the Classical Standard Map:

$$T : \left\{ \begin{array}{l} q_{n+1} = q_n + p_{n+1} \\ p_{n+1} = p_n + K\sin q_n \end{array} \right\}, \text{for } n \in \mathbf{Z}\,, \tag{2.2}$$

where q_n and p_n are respectively the coordinate and the momentum just before the kick at the time n:

$$q_n = q(t = n-0)\,,\ p_n = p(t = n-0)\,, \text{for } n \in \mathbf{Z}\,. \tag{2.3}$$

When K exceeds the threshold value $K_c (\approx 0.97)$, this classical system shows a diffusion along the momentum direction due to the occurence of global chaos.

The time-evolved *classical distribution function in the phase space* ρ^{CL} at the time n is defined by

$$\rho_n^{\mathrm{CL}}(q_n,p_n) = \int_{-\infty}^{+\infty} dq_0 \int_{-\infty}^{+\infty} dp_0\, \delta\Big(\begin{pmatrix} q_n \\ p_n \end{pmatrix} - T^n \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} \Big)\, \rho_0^{\mathrm{CL}}(q_0,p_0)\,, \tag{2.4}$$

where ρ_0^{CL} denotes the initial distribution function .

Quantum Dynamics .

Applying the Schrödinger equation to (2.1) , we get the Quantum Standard Map:

$$|\psi_{n+1}\rangle = \hat{U}|\psi_n\rangle \tag{2.5}$$

with the unitary operator

$$\hat{U} = e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2} e^{-\frac{i}{\hbar}K\cos\hat{q}}, \tag{2.6}$$

where $|\psi_n\rangle$ is the state vector just before the kick at the time n :

$$|\psi_n\rangle = |\psi(t = n - 0)\rangle . \tag{2.7}$$

It is well known that if we impose the usual periodic boundary condition on the position q for this system, then the diffusion along the momentum direction is limited within a finite time interval even when K exceeds K_c (Casati *et.al.* [8]).

However, in this paper we do not impose the periodic boundary condition on the coordinate q , as mentioned before; we hence expect that the diffusion is not limited because of the absence of the interference due to the condition.

The coherent state representation of the state at the time n is

$$\psi_n^{\mathrm{QM}}(q_n, p_n) = \langle q_n, p_n|\psi_n\rangle . \tag{2.8}$$

Its time evolution is expressed as

$$\begin{aligned}\psi_n^{\mathrm{QM}}(q_n, p_n) &= \langle q_n, p_n|\psi_n\rangle = \langle q_n, p_n|\hat{U}^n|\psi_0\rangle \\ &= \langle q_n, p_n|\hat{U}^n \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dq_0 dp_0}{2\pi\hbar}|q_0, p_0\rangle\langle q_0, p_0|\psi_0\rangle \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dq_0 dp_0}{2\pi\hbar} G_n^{\mathrm{QM}}(q_n, p_n; q_0, p_0)\psi_0^{\mathrm{QM}}(q_0, p_0)\end{aligned} \tag{2.9}$$

where the quantum propagator is given by

$$G_n^{\mathrm{QM}}(q_n, p_n; q_0, p_0) = \langle q_n, p_n|\hat{U}^n|q_0, p_0\rangle . \tag{2.10}$$

Here, we define the *quantum distribution function in the phase space* ρ^{QM} as the following:

$$\rho_n^{\mathrm{QM}}(q_n, p_n) = |\psi_n^{\mathrm{QM}}(q_n, p_n)|^2 . \tag{2.11}$$

This is called the Hushimi representation (Husimi [9] , Takahashi and Saitô [10] , Takahashi [10]) or the Q-representation [10] of the density operator $\hat{\rho}_n = |\psi_n\rangle\langle\psi_n|$, and is nothing but the probability that the state $|\psi_n\rangle$ is observed with the minimum uncertainty wave packet $|q, p\rangle$.

Semiclassical Dynamics .

Corresponding to (2.9), the time evolution of the coherent state representation of state ψ^{SC} according to semiclassical dynamics is described by

$$\psi_n^{\mathrm{SC}}(q_n, p_n) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{dq_0 dp_0}{2\pi\hbar} G_n^{\mathrm{SC}}(q_n, p_n; q_0, p_0)\psi_0^{\mathrm{SC}}(q_0, p_0) . \tag{2.12}$$

According to the Klauder theory described in his paper [6], the semiclassical propagator in the above expression is represented as the follwing:

$$G_n^{\mathrm{SC}}(q_n, p_n; q_0, p_0) = \sum_{\substack{\text{complex} \\ \text{classical orbit}}} E e^{\frac{i}{\hbar}F} \tag{2.13}$$

with the action

$$F = -\frac{1}{2}(p_n\bar{q}_n - q_n\bar{p}_n + \bar{p}_0 q_0 - \bar{q}_0 p_0) - K\sum_{j=0}^{n-1}(\frac{1}{2}\bar{q}_j \sin\bar{q}_j + \cos\bar{q}_j)\,, \tag{2.14}$$

where $(\bar{q}_j, \bar{p}_j)\,(j = 0, 1, 2, \ldots, n)$ is the orbit of the Complex Classical Standard Map such that:

$$\begin{pmatrix} \bar{q}_{j+1} \\ \bar{p}_{j+1} \end{pmatrix} = T \begin{pmatrix} \bar{q}_j \\ \bar{p}_j \end{pmatrix}, \; (j = 0, 1, 2, \ldots, n-1)\,. \tag{2.15}$$

The boundary condition for the complex classical orbit is

$$\bar{q}_0 + i\bar{p}_0 = q_0 + ip_0\,;\; \bar{q}_0, \bar{p}_0 \in \mathbf{C},\; q_0, p_0 \in \mathbf{R}, \tag{2.16}$$

$$\bar{q}_n - i\bar{p}_n = q_n - ip_n\,;\; \bar{q}_n, \bar{p}_n \in \mathbf{C},\; q_n, p_n \in \mathbf{R}. \tag{2.17}$$

If there is more than one orbit satisfying this condition, the summation over them is necessary as expressed in (2.13) . In this case, each of these orbit is labeled by a different value of the complex parameter w which represents the position on the initial complex Lagrangian manifold:

$$\left\{ \begin{array}{l} \bar{q}_0 = q_0 + w \\ \bar{p}_0 = p_0 + iw \end{array} \right\}. \tag{2.18}$$

Next, the amplitude factor E is

$$E = \left\{ \begin{bmatrix} i & 1 \end{bmatrix} M(\bar{q}_{n-1}) M(\bar{q}_{n-2}) \cdots M(\bar{q}_0) \begin{bmatrix} -i/2 \\ 1/2 \end{bmatrix} \right\}^{-1/2} \tag{2.19}$$

with

$$M(\bar{q}_j) = \begin{bmatrix} 1 + K\cos\bar{q}_j & 1 \\ K\cos\bar{q}_j & 1 \end{bmatrix}. \tag{2.20}$$

Corresponding to (2.11), we define the *semiclassical distribution function in the phase space* ρ^{SC} as the following :

$$\rho_n^{\mathrm{SC}}(q_n, p_n) = \left|\psi_n^{\mathrm{SC}}(q_n, p_n)\right|^2. \tag{2.21}$$

§3. NUMERICAL CALCULATION

The Setting of the Numerical Calculation.

First, we explain the initial condition for the time evolution process. In this report, we are interested in the most fundamental situation for the theory of semiclassical coherent state path integral. Accordingly, let us choose as the initial state for quantum and semiclassical dynamics one of those coherent states which are in the basis used to represent the propagator. Let $|q_{\rm init}, p_{\rm init}\rangle$ be the initial coherent state. Then

$$\psi_0^{\rm QM}(q_0, p_0) = \psi_0^{\rm SC}(q_0, p_0) = \langle q_0, p_0 | q_{\rm init}, p_{\rm init}\rangle \,. \tag{3.1}$$

With this condition, the coherent state representation of the time-evolved state at the time n becomes the propagator itself according to (2.9) and (2.12):

$$\psi_n^{\rm QM}(q_n, p_n) = G_n^{\rm QM}(q_n, p_n; q_{\rm init}, p_{\rm init}) \,, \tag{3.2}$$

$$\psi_n^{\rm SC}(q_n, p_n) = G_n^{\rm SC}(q_n, p_n; q_{\rm init}, p_{\rm init}) \,. \tag{3.3}$$

For classical dynamics, we choose the initial distribution function to agree with that for quantum and semiclassical dynamics:

$$\begin{aligned} \rho_0^{\rm CL}(q_0, p_0) &= \rho_0^{\rm QM}(q_0, p_0) = \rho_0^{\rm SC}(q_0, p_0) \\ &= |\langle q_0, p_0 | q_{\rm init}, p_{\rm init}\rangle|^2 \\ &= \exp\left[-\frac{1}{2\hbar}\{(q_0 - q_{\rm init})^2 + (p_0 - p_{\rm init})^2\}\right] . \end{aligned} \tag{3.4}$$

Secondly, we choose the values of the parameters as the following:

$$K = 2.0 \,, \tag{3.5}$$

$$(q_{\rm init}, p_{\rm init}) = (3.5, 3.5) \,, \tag{3.6}$$

$$\hbar = 0.405 \,. \tag{3.7}$$

The Result of the Numerical Calculation.

In Fig.1, we show the contour plots of phase space distributions at time $n = 0, 1, 2, 3$ according to classical, quantum and semiclassical dynamics. We will divide the report of the numerical calculation into two parts according to the stage of time evolution. When the time n is $0, 1, 2$, the phase space distribution functions are being only $\langle\!\langle$ stretched $\rangle\!\rangle$ and have not yet $\langle\!\langle$ folded $\rangle\!\rangle$. From Fig.1, we observe that the quantum and the semiclassical distributions agree with each other very much. Then semiclassical calculation is easy and we have no problem. For each exit label (q_n, p_n) of the semiclassical propagator, there is only one complex classical orbit and the imaginary part of the action along the orbit $\Im F$ is equal to or greater than 0. Moreover, the amplitude factore E is never equal to 0.

On the other hand, when the time evolution enters the next stage ($n = 3$), we have much difficulties in the semiclassical calculation. Let us observe Fig.1. On this time stage, the distribution functions are not only $\langle\!\langle$ stretched $\rangle\!\rangle$ but also $\langle\!\langle$ folded $\rangle\!\rangle$. The branches of the $\langle\!\langle$ folded $\rangle\!\rangle$ wave functions interfere and cause the beat pattern of the distribution functions. In order to express this interference in the semiclassical description, the complex classical orbits contributing to a point in the concave side of the $\langle\!\langle$ folded $\rangle\!\rangle$ wave function should be multiple. Moreover, there are several points on the phase space at which two of such

multiple complex orbits degenerate and the corresponding amplitude factor E diverges. Namely, these points are nothing but caustics. We name this caustic 《 Phase Space Caustic (PSC) 》. As is well known in asymptotic analysis, the appearence of a caustic causes that not all the saddle point solutions contribute to the result. In our case, on the regions near PSCs, not all the complex classical orbits satisfying the boundary condition (2.16) and (2.17) contribute to the semiclassical propagator (2.12). Asymptotic analysis tells us that two curves called Stokes lines run from each PSC and that when we go across a Stokes line, the number of "contributing" complex clasical orbits changes by 1 . Moreover, if we took "non-contributing" complex classical orbits into account of the evaluation of the semiclassical propagator, then the propagator would diverge unphysically. Thus we surely need the criterion to judge whether a complex clasical orbit is "contributing" or "non-contributing". Namely, we need the precise location of Stokes lines. In Fig.2, we show Stokes lines calculated by the so called "principle of exponential dominance" [12]. After getting all the Stokes lines running from PSCs, we can calculate the semiclassical distribution functions as shown in Fig.1. Finally, we will compare the semiclassical distributions with quantum ones. The agreement is very good except on the neighborhoods of PSCs where the amplitude factor E of the semiclassical propagator diverges.

§4. SUMMARY

The 《 folding 》 operation of a chaotic dynamics makes the semiclassical theory of the coherent state path integral be not free from the problem of caustic (《 Phase Space Caustic (PSC) 》). In order to overcome the difficulty induced by PSC, we need the precise location of Stokes lines running from PSCs. We propose to use the "principle of exponential dominance" numerically to determine the location of Stokes lines. By this prescription, the time evolution of a quantum chaotic system is calculatable from the time evolution of the corresponding complex classical system. In this sense, complex classical mechanics has a clear physical reality.

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Figure Captions

Fig.1 **Time-evolved distribution functions in the phase space due to classical/quantum/semiclassical dynamics**
Contour lines are drawn so that the net probability inside each counter equals 0.1, 0.3, 0.5, 0.7 and 0.9, respectively. In each figure, the horizontal axis and the vertical axis are the q_n-axis and the p_n-axis, respectively. Each square bounded by dotted lines is an area of $2\pi \times 2\pi$.

Fig.2 **Phase Space Caustics and Stokes Lines**
The magnification of the figure (c,$n = 3$) of Fig.1. with zero points, PSC's and Stokes lines.

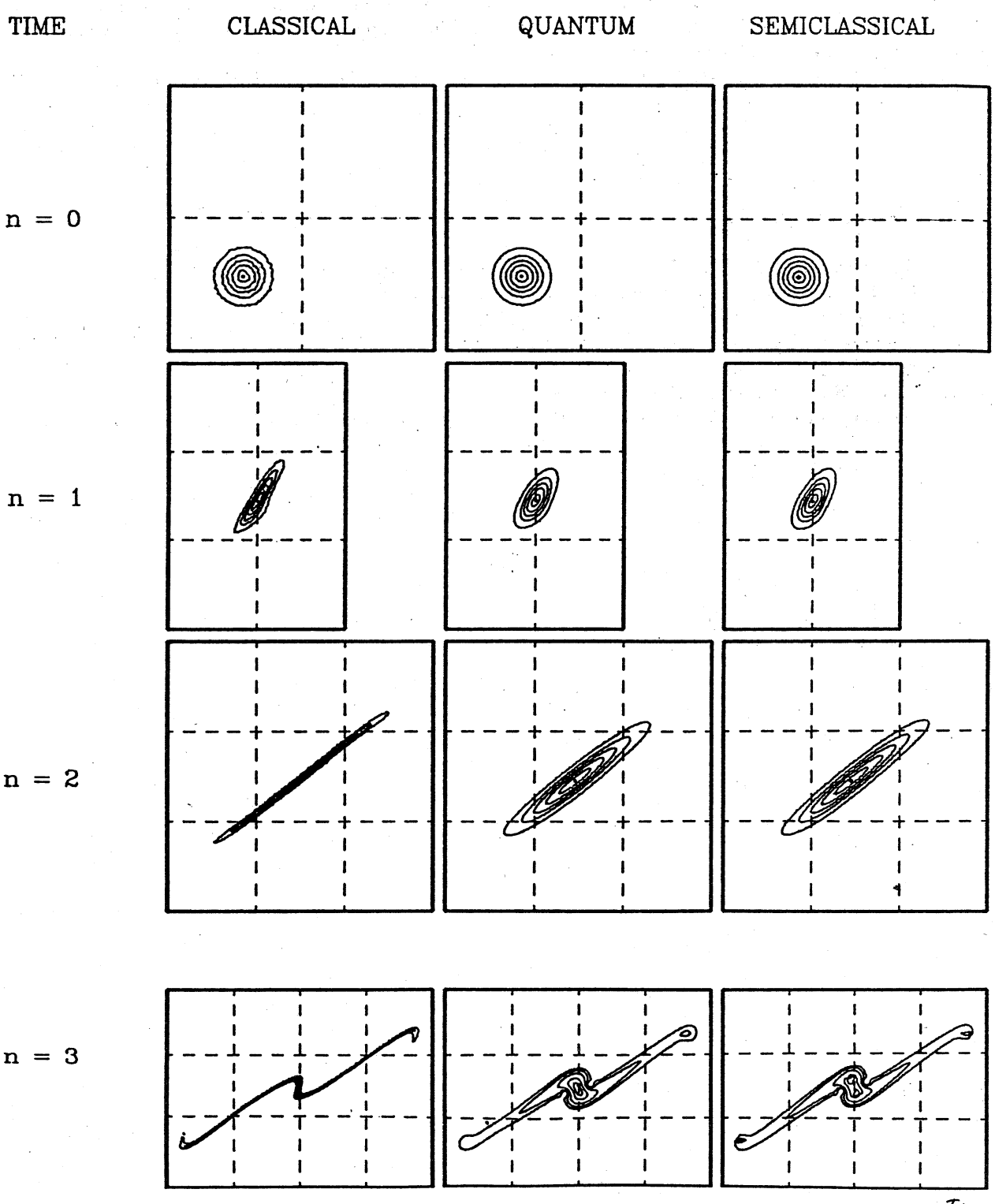

Fig. 1

SEMI-CLASSICAL DISTRIBUTION

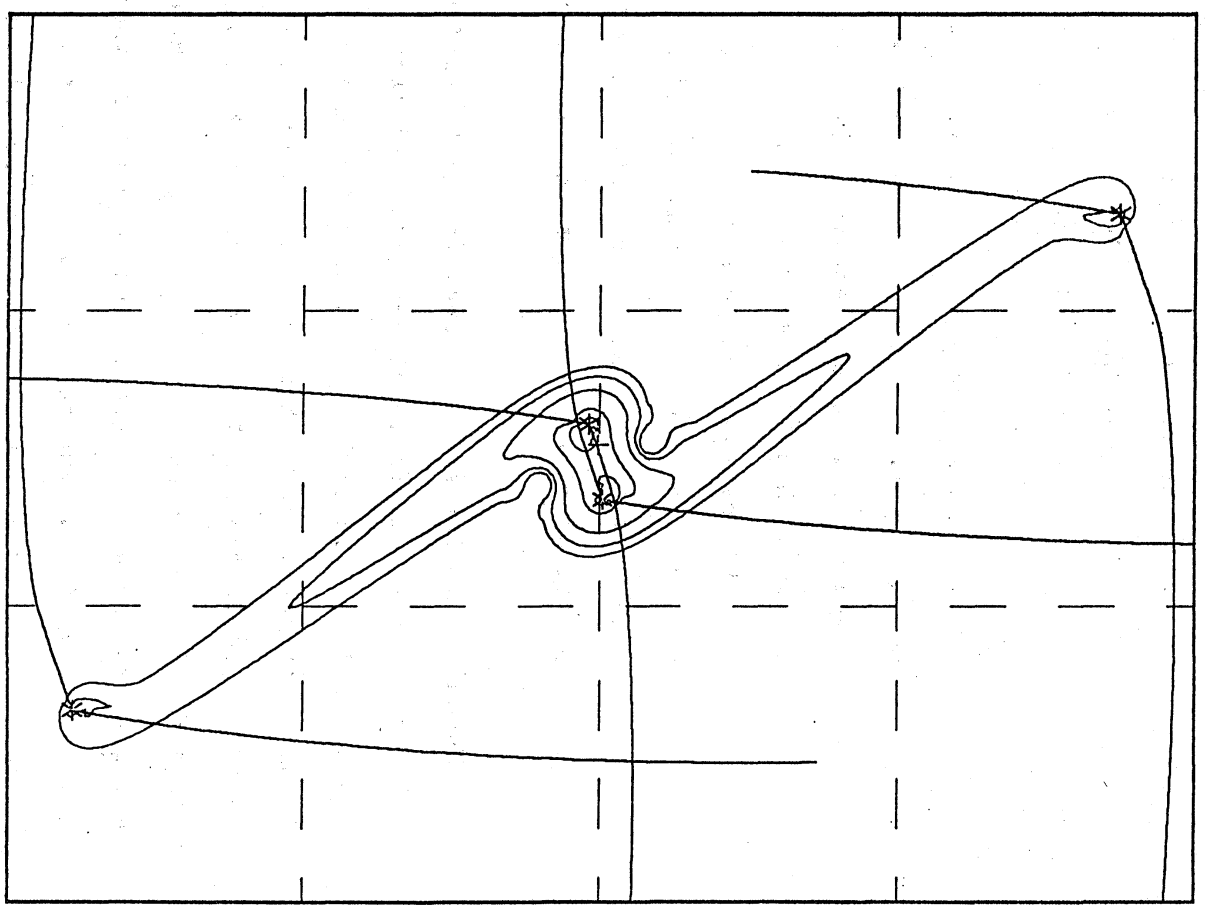

Fig. 2