

Fourier expansion of Arakawa lifting

Atsushi Murase and Hiro-aki Narita

Abstract

This note is a report based on our talk at the conference on automorphic forms held at RIMS during January 21th-25th, 2008. We announce our recent results about Fourier coefficients of Arakawa lifting, i.e. a theta lifting to a cusp form on the quaternion unitary group $GSp(1,1)$ from a pair consisting of an elliptic cusp form f and an automorphic form f' on a definite quaternion algebra over $\mathbb{Q}$. We provide an explicit formula for the Fourier coefficients in terms of toral integrals of f and f' . As an application, we show the existence of non-vanishing Arakawa lifts.

0 Introduction

To explain the background of our study we start with reviewing Böcherer's conjecture on Fourier coefficients of holomorphic Siegel modular forms of degree two. We let F be a Hecke-eigen holomorphic Siegel cusp form of weight k with respect to $Sp(2, \mathbb{Z})$. Its Fourier expansion is described as

$$F(Z) = \sum_{T \in \text{Sym}_2(\mathbb{Z})^*, T > 0} C(T) e^{2\pi\sqrt{-1} \text{Tr} TZ},$$

where $\text{Sym}_2(\mathbb{Z})^*$ denotes the set of half-integral symmetric matrices of degree 2, and $T > 0$ means that T is positive definite. Now let $-D$ be a fundamental discriminant with $D > 0$. For such D we consider the average $A(D)$ of the Fourier coefficients of F as follows:

$$A(D) := \sum_{S \in \{T | \det T = D/4\} / SL_2(\mathbb{Z})} \frac{C(S)}{\epsilon(S)}.$$

Here we put $\epsilon(S) = \#\{\gamma \in SL_2(\mathbb{Z}) \mid {}^t\gamma S \gamma = S\}$. We let $L_{\text{spin}}(F, \left(\frac{D}{*}\right), s)$ be the quadratic twist of the spinor L -function for F . Then Böcherer's conjecture [B] is formulated as

$$|A(D)|^2 = C_F D^{k-1} L_{\text{spin}}(F, \left(\frac{D}{*}\right), k-1)$$

with a constant C_F depending only on F . There are several evidences of this conjecture (cf. [B], [B-S], [K-K]). We note that, in the conjecture, the spinor L -function is evaluated at its central point $s = k - 1$. This conjecture can be regarded as a generalization of the formula by Waldspurger-Kohnen-Zagier (cf. [Wa-1], [K-Z]), which says that the twisted central L -value of an integral weight elliptic cusp form f is proportional to the square of a Fourier coefficient of a half-integral weight elliptic cusp form associated with f by Shimura correspondence. Furusawa and Shalika have made a further expectation that Böcherer conjecture would also hold for an inner form G of $GSp(2)$:

$$\{g \in M_2(B) \mid {}^t\bar{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \nu(g) \in \mathbb{Q}^\times\},$$

where B is a quaternion algebra over $\mathbb{Q}$. This expectation is based on their conjectural relative trace formula for G (cf. [F-S]). In this note, we consider the case where B is definite, i.e. $G = GSp(1, 1)$.

Our target is the “Arakawa lifting”, which is a theta lifting from a pair of elliptic cusp form f and an automorphic form f' on $B_{\mathbb{A}_\mathbb{Q}}^\times$ to a vector-valued cusp form $\mathcal{L}(f, f')$ on $GSp(1, 1)_{\mathbb{A}_\mathbb{Q}}$. Its representation type at the Archimedean place is a quaternionic discrete series representation (for the definition see [G-W]). Our result (Theorem 2.2) says that a certain average of the Fourier coefficients of $\mathcal{L}(f, f')$ (an analogue of $A(D)$) is explicitly written in terms of a product of toral integrals of (f, f') .

Our formula leads us to two directions of further research. One is to show the existence of non-vanishing lifts, which is discussed in §3. In fact, if (f, f') are Hecke eigenforms with non-vanishing toral integrals, we have $\mathcal{L}(f, f') \not\equiv 0$ in view of our formula. Another direction is to find an explicit formula for the constant of proportionality relating the square norms of the averages of the Fourier coefficients to central L -values. Indeed, Furusawa-Shalika [F-S] expects that such square norm is proportional to the central value of a “Rankin-Selberg L -function” of the Arakawa lift. Waldspurger [Wa-1, Proposition 7] and our theorem tell us that the square norm of the average is proportional to a product of central L -values for the quadratic base changes of the Jacquet-Langlands lifts of f and f' twisted by a Hecke character. Such a product is expected to be a central L -value of a (twisted) Rankin-Selberg L -function of the Arakawa lift. We leave the study in this direction to our further research.

1 Arakawa lifting

Let B be a definite quaternion algebra over $\mathbb{Q}$ with the discriminant d_B . Let $x \mapsto \bar{x}$ be the main involution of B . We fix a maximal order $\mathcal{O}$ of B . We denote by $G = GSp(1, 1)$ the $\mathbb{Q}$ -algebraic group defined by

$$\{g \in M_2(B) \mid {}^t\bar{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \nu(g) \in \mathbb{Q}^\times\}.$$

From now on we assume that every automorphic form dealt with in this note has the trivial central character. Let D be a divisor of d_B and $\mathcal{S}_\kappa(\Gamma_0(D))$ the space of elliptic cusp forms of weight κ and level D . We regard each element of $\mathcal{S}_\kappa(\Gamma_0(D))$ as an automorphic form on $GL_2(\mathbb{A}_\mathbb{Q})$. We further denote by $\mathcal{A}_\kappa = \mathcal{A}_\kappa(B_{\mathbb{A}_\mathbb{Q}}^\times)$ the space of automorphic forms on $B_{\mathbb{A}_\mathbb{Q}}^\times$ of weight $(\sigma_\kappa = \text{Sym}_\kappa, V_\kappa)$ and right $\prod_{v < \infty} \mathcal{O}_v^\times$ -invariant.

Let r be the metaplectic representation of $G_{\mathbb{A}_\mathbb{Q}} \times (GL_2(\mathbb{A}_\mathbb{Q}) \times B_{\mathbb{A}_\mathbb{Q}}^\times)$ introduced in [M-N-1, §3]. We then define a theta series on $G_{\mathbb{A}_\mathbb{Q}} \times (GL_2(\mathbb{A}_\mathbb{Q}) \times B_{\mathbb{A}_\mathbb{Q}}^\times)$ by

$$\Theta_\kappa(g, h, h') := \sum_{(X, t) \in B^2 \times \mathbb{Q}^\times} (r(g, h, h')\Phi)(X, t).$$

Here we put $\Phi := \prod_{v \leq \infty} \Phi_v$ with

$$\Phi_v(X, t) := \begin{cases} \text{ch}(\mathcal{O}_v^2 \times \mathbb{Z}_v^\times)(X, t) & (v \nmid D^{-1}d_B), \\ \text{ch}((\mathcal{O}_v \oplus \mathfrak{P}_v^{-1}) \times \mathbb{Z}_v^\times)(X, t) & (v \mid D^{-1}d_B), \\ \text{ch}(t \in \mathbb{R}_+^\times) t^{\frac{\kappa+3}{2}} \sigma_\kappa(X_1 + X_2) e^{-2\pi i t \bar{X} X} & (v = \infty), \end{cases}$$

where $\mathfrak{P}_v$ is a maximal ideal at v and $\text{ch}(S)$ denotes the characteristic function for a set S . Then the Arakawa lifting is defined as follows:

$$\mathcal{S}_\kappa(\Gamma_0(D)) \times \mathcal{A}_\kappa(B_{\mathbb{A}_\mathbb{Q}}^\times) \ni (f, f') \mapsto \mathcal{L}(f, f')(g) := \iint_{(\mathbb{R}_+^\times)^2 (GL_2 \times B^\times)_\mathbb{Q} \backslash (GL_2 \times B^\times)_{\mathbb{A}_\mathbb{Q}}} \overline{f(\bar{h})} \Theta_\kappa(g, h, h') f'(h') dh dh'.$$

This is a cusp form on $G_{\mathbb{A}_\mathbb{Q}}$ belonging to the minimal K_∞ -type of a quaternionic discrete series representation at infinity (cf. [M-N-2, Theorem 3.3.2]), where K_∞ denotes a maximal compact subgroup of the real points of $Sp(1, 1)$.

2 Main result

2.1

In general, a cuspidal automorphic form F on $G_{\mathbb{A}_\mathbb{Q}}$ admits a Fourier expansion as follows:

$$F(g) = \sum_{\xi \in B^- \setminus \{0\}} F_\xi(g) = \sum_{\xi \in B^- \setminus \{0\}} \sum_{\chi \in X_\xi} F_\xi^\chi(g).$$

Here

$$F_\xi(g) := \int_{B^- \setminus B_{\mathbb{A}_\mathbb{Q}}^-} F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-\text{tr}(\xi x)) dx, \quad F_\xi^\chi(g) := \int_{\mathbb{R}_+^\times \mathbb{Q}(\xi)^\times \backslash \mathbb{A}_{\mathbb{Q}(\xi)}^\times} F_\xi(s 1_2 \cdot g) \chi(s)^{-1} ds,$$

where $B^- = \{x \in B \mid \bar{x} = -x\}$, ψ is the standard additive character on $\mathbb{Q} \backslash \mathbb{A}_\mathbb{Q}$ and X_ξ denotes the set of Hecke characters of $\mathbb{A}_{\mathbb{Q}}^\times \mathbb{Q}(\xi)^\times \backslash \mathbb{A}_{\mathbb{Q}(\xi)}^\times$. Our main result is an explicit formula for F_ξ^χ when $F = \mathcal{L}(f, f')$.

2.2

To state the main theorem, we let $(f, f') \in S_\kappa(D) \times \mathcal{A}_\kappa$ be Hecke eigenforms. We further assume that f and f' are eigenforms for the “Atkin-Lehler involution”: For every $p|D$,

$$f\left(h \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}\right) = \epsilon_p f(h), \quad f'(h'\Pi_p) = \epsilon'_p f'(h')$$

with $\epsilon_p, \epsilon'_p \in \{\pm 1\}$. Here Π_p is a prime element of B_p . Note that $\mathcal{L}(f, f') \equiv 0$ unless $\epsilon_p = \epsilon'_p$ for any $p|D$.

For $p < \infty$, let $\mathfrak{a}_p := \begin{cases} \mathcal{O}_p & (p \nmid d_B \text{ or } p|D) \\ \mathfrak{P}_p & (p|D^{-1}d_B) \end{cases}$. We say that $\xi \in B^- \setminus \{0\}$ is *primitive* if

$\xi \in \mathfrak{a}_p \setminus p\mathfrak{a}_p$ for each finite prime p . We note that a Fourier coefficient F_ξ of an automorphic form F on $G_{\mathbb{A}_Q}$ satisfies

$$F_\xi\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right) = F_{t\xi}(g) \quad (t \in \mathbb{Q}^\times).$$

It follows that the calculation of the Fourier expansion of F is reduced to that of F_ξ for primitive ξ .

2.3

We next introduce several notations. Let $\xi \in B^- \setminus \{0\}$ and d_ξ denote the discriminant of an imaginary quadratic field $E := \mathbb{Q}(\xi)$. We put

$$a := \begin{cases} 2\sqrt{-n(\xi)}\sqrt{d_\xi} & (d_\xi \text{ is odd}) \\ \sqrt{-n(\xi)}\sqrt{d_\xi} & (d_\xi \text{ is even}) \end{cases}, \quad b := \xi^2 - \frac{a^2}{4}.$$

With these a and b we define an embedding $\iota_\xi : E^\times \hookrightarrow GL_2(\mathbb{Q})$ by

$$\iota_\xi(x + y\xi) = x \cdot 1_2 + y \cdot \begin{pmatrix} a/2 & b \\ 1 & -a/2 \end{pmatrix} \quad (x, y \in \mathbb{Q}).$$

The completion E_∞ of E at ∞ is identified with $\mathbb{C}$ by

$$\delta_\xi : E_\infty \ni x + y\xi \mapsto x + y\sqrt{-n(\xi)} \in \mathbb{C} \quad (x, y \in \mathbb{R}).$$

For a Hecke character $\chi = \prod_{v \leq \infty} \chi_v$ of E , we define $w_\infty(\chi) \in \mathbb{Z}$ to be

$$\chi_\infty(u) = (\delta_\xi(u)/|\delta_\xi(u)|)^{w_\infty(\chi)} \quad (u \in E_\infty^\times).$$

Furthermore, for each finite prime $p < \infty$, $i_p(\chi)$ denotes the exponent of the conductor of χ at p and

$$\mu_p := \frac{\text{ord}_p(2\xi)^2 - \text{ord}_p(d_\xi)}{2}.$$

We then have the following (cf. [M-N-2, Theorem 5.1.1]):

Proposition 2.1. $\mathcal{L}(f, f')_\xi^\chi \equiv 0$ unless

$$i_p(\chi) = 0 \text{ for any } p|d_B \text{ and } w_\infty(\chi) = -\kappa. \quad (1)$$

2.4 Statement of the main theorem.

In what follows, we assume that (1) holds. We need further notations to state the main theorem.

Define $\gamma_0 = (\gamma_{0,p})_{p \leq \infty} \in GL_2(\mathbb{A}_\mathbb{Q})$ and $\gamma'_0 = (\gamma'_{0,p})_{p < \infty} \in B_{\mathbb{A}_{\mathbb{Q},f}}^\times$ as follows:

$$\gamma_{0,p} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p + i_p(\chi)} \end{pmatrix} & (p \nmid D), \\ 1_2 & (p|D \text{ and } p \text{ is inert in } E), \\ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & (p|D \text{ and } p \text{ ramifies in } E), \\ \begin{pmatrix} 1 & a/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N(\xi)^{1/4} & 0 \\ 0 & N(\xi)^{-1/4} \end{pmatrix} & (p = \infty), \end{cases}$$

$$\gamma'_{0,p} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & p^{-\mu_p + i_p(\chi)} \end{pmatrix} & (p \nmid d_B), \\ \Pi_{B,p}^{-1} & (p|d_B). \end{cases}$$

Here $\Pi_{B,p}$ is a prime element of B_p for $p|d_B$. We furthermore define the following local constants:

$$C_p(f, \xi, \chi) := \begin{cases} p^{2\mu_p - i_p(\chi)} (1 - \delta(i_p(\chi) > 0) e_p(E) p^{-1}) & (p \nmid d_B), \\ 1 & (p|D^{-1} d_B), \\ 2\epsilon_p & (p|D \text{ and } p \text{ is inert in } E), \\ (p+1)^{-1} & (p|D \text{ and } p \text{ ramifies in } E), \end{cases}$$

where $\delta(P) = 1$ (resp. 0) if a condition P holds (resp. does not hold), and

$$e_p(E) = \begin{cases} -1 & (p \text{ is inert in } E), \\ 0 & (p \text{ ramifies in } E), \\ 1 & (p \text{ splits in } E). \end{cases}$$

For $(f, f') \in S_\kappa(D) \times \mathcal{A}_\kappa$ we introduce their toral integrals with respect to a Hecke character χ of E :

$$P_\chi(f; h) := \int_{\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E^\times} f(\iota_\xi(s)h) \chi(s)^{-1} ds, \quad P_\chi(f'; h') := \int_{\mathbb{R}_+^\times E^\times \backslash \mathbb{A}_E^\times} f(sh') \chi(s)^{-1} ds,$$

where $(h, h') \in GL_2(\mathbb{A}_{\mathbf{Q}}) \times B_{\mathbb{A}_{\mathbf{Q}}}^{\times}$. Here we normalize the measure ds of $\mathbb{A}_E^{\times}$ so that

$$\text{vol}(\mathcal{O}_{E_p}^{\times}) = \text{vol}(E_{\infty}^{(1)}) = 1$$

for each finite prime p , where $\mathcal{O}_{E_p}$ is the p -adic completion of the integer ring of E and $E_{\infty}^{(1)}$ denotes the group of elements in E_{∞} with norm 1.

We denote by $h(E)$ and $w(E)$ the class number of E and the number of roots of unity in E respectively. We are now able to state our main result (cf. [M-N-2, Proposition 2.4.1, Theorem 5.2.1]).

Theorem 2.2. (1) When $\xi = 0$, $\mathcal{L}(f, f')_{\xi} \equiv 0$.

(2) Let ξ be a primitive element in $B^{-} \setminus \{0\}$. Suppose that χ satisfies (1) and that $\epsilon_p = \epsilon'_p$ for any $p|D$. We then have the following formula:

$$\begin{aligned} & \mathcal{L}(f, f')_{\xi}^{\chi} \left(g_0 \begin{pmatrix} \sqrt{\eta_{\infty}} & 0 \\ 0 & \sqrt{\eta_{\infty}}^{-1} \end{pmatrix} \right) \\ &= 2^{\kappa-1} N(\xi)^{\kappa/4} \frac{w(E)}{h(E)} \cdot \left(\prod_{p < \infty} C_p(f, \xi, \chi) \right) \eta_{\infty}^{\kappa/2+1} \exp(-4\pi \sqrt{N(\xi) \eta_{\infty}}) \overline{P_{\chi}(f; \gamma_0)} P_{\chi}(f'; \gamma'_0). \end{aligned}$$

Here $\eta_{\infty} \in \mathbb{R}_{+}^{\times}$ and $g_0 = (g_{0,p})_{p < \infty} \in G_{\mathbb{A}_{\mathbf{Q}},f}$ is given by

$$g_{0,p} := \begin{cases} \text{diag}(p^{i_p(\chi) - \mu_p}, p^{2(i_p(\chi) - \mu_p)}, 1, p^{i_p(\chi) - \mu_p}) & (p \nmid d_B), \\ 1_2 & (p|d_B). \end{cases}$$

Remark 2.3. (1) $\mathcal{L}(f, f')_{\xi}^{\chi}$ is determined by the value at $g_0 \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}$ due to Sugano's result ([Su, Proposition 2-5]).

(2) Murase and Sugano have obtained a similar formula for "Kudla lifting", i.e. a theta lift from $U(1, 1)$ to $U(2, 1)$ (cf. [M-S]).

Remark 2.4. Let Π (resp. Π') be the base change to $GL_2(\mathbb{A}_E)$ of the Jacquet-Langlands lift π_f (resp. $\pi_{f'}$) of the automorphic representation attached to f (resp. f'). Waldspurger [Wa-2, Proposition 7] proved the following formula:

$$\begin{aligned} \frac{\|P_{\chi}(f; \gamma_0)\|^2}{\langle f, f \rangle} &= C_{f,\chi} \cdot L(\Pi \otimes \chi^{-1}, \frac{1}{2}), \\ \frac{\|P_{\chi}(f'; \gamma'_0)\|^2}{\langle f', f' \rangle} &= C_{f',\chi} \cdot L(\Pi' \otimes \chi^{-1}, \frac{1}{2}), \end{aligned}$$

where

$$C_{\varphi,\chi} = \frac{\sqrt{|d_{\xi}|}}{4\pi} \cdot \frac{\zeta(2)}{2L(\pi_{\varphi}, \text{Ad}, 1)} \prod_{v: \text{"bad"}} C_{\varphi,\chi,v}$$

for $\varphi = f$ or f' , with the adjoint L -function $L(\pi_\varphi, \text{Ad}, s)$ of $\varphi = f$ or f' and where $C_{\varphi, \chi, v}$ is a ratio of a local period and L -values. We now remark that there does not appear $\frac{\sqrt{|d_\xi|}}{4\pi}$ in Waldspurger's formula [Wa-2, Proposition 7]. This is due to the difference between normalizations of Waldspurger's measure and ours for $\mathbb{A}_E^\times$.

Our theorem and Waldspurger's formula then imply

$$\frac{||\mathcal{L}(f, f')_\xi(g_{0,f})||^2}{\langle f, f \rangle \langle f', f' \rangle} = C_{f, f', \chi} L(\Pi \otimes \chi^{-1}, \frac{1}{2}) L(\Pi' \otimes \chi^{-1}, \frac{1}{2})$$

with

$$C_{f, f', \chi} := 2^{2(\kappa-1)} N(\xi)^{\frac{\kappa}{2}} \frac{w(E)}{h(E)} \left| \prod_{p < \infty} C_p(f, \xi, \chi) \right|^2 \exp(-8\pi \sqrt{N(\xi)}) \cdot C_{f, \chi} \cdot C_{f', \chi}.$$

It would be interesting to find a more explicit form of the constant $C_{f, f', \chi}$

3 Application (Non-vanishing lifts)

A general approach to verify the non-vanishing of theta lifts is to study their Petersson inner products. This technique is due to S. Rallis [R] and J. S. Li [L] etc. Via the Siegel-Weil formula (cf. [We]), it reduces the problem to the non-vanishing of a special value of the standard L -function for the preimages of the theta lifts. This method is useful when the Siegel-Weil formula is available, but this is not the case for our theta lifts.

Our approach to show the existence of the non-vanishing Arakawa lifts is to find examples of (f, f') with non-vanishing toral integrals involved in our formula for Fourier coefficients of the lifts (Theorem 2.2).

3.1 Result

We now specialize the situation. Let $B = \mathbb{Q} + \mathbb{Q} \cdot i + \mathbb{Q} \cdot j + \mathbb{Q} \cdot ij$ with $i^2 = j^2 = -1$ and $ij = -ji$. It is known that $d_B = 2$ and the class number of B is one. We note that $D = 1$ or 2 . Let $\mathcal{O} = \mathbb{Z}1 + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}(1+i+j+k)/2$, and put $\xi = \frac{1}{2}i$, which is primitive. Suppose that the Hecke character χ of E is unramified at all finite places. This assumption implies that $w_\infty(\chi)$ is divisible by 4.

Proposition 3.1. *Let B , ξ and χ be as above.*

Then there exist Hecke eigenforms (f, f') such that

$$\overline{P_\chi(f; \gamma_0)} P_\chi(f'; \gamma'_0) \neq 0$$

for every $\kappa \geq \begin{cases} 12 & (D = 1) \\ 8 & (D = 2) \end{cases}$ with $4|\kappa$.

Theorem 3.2. *Let (f, f') be as above. Then $\mathcal{L}(f, f') \neq 0$.*

3.2 Outline of the proof

Theorem 3.2 is a direct consequence of Proposition 3.1 and Theorem 2.2. This subsection is thus devoted to the outline of our proof of Proposition 3.1. If one finds a pair (f, f') such that $\overline{P_\chi(f; \gamma_0)} P_\chi(f'; \gamma'_0) \neq 0$, there exists a pair of Hecke eigenforms with the same property. This follows from the fact that $\mathcal{S}_\kappa(\Gamma_0(D))$ and $\mathcal{A}_\kappa(B_{\mathbf{A}_Q}^\times)$ have basis consisting of Hecke eigenforms.

To begin with, we find $f' \in \mathcal{A}_\kappa(B_{\mathbf{A}_Q}^\times)$ such that $P_\chi(f'; \gamma'_0) \neq 0$. Eichler's trace formula of Brandt matrices (cf. [E, Theorem 5]) says that

$$\dim_{\mathbf{C}} \mathcal{A}_\kappa(B_{\mathbf{A}_Q}^\times) = \begin{cases} \frac{\kappa+12}{12} & (\kappa \equiv 0 \pmod{12}), \\ \frac{\kappa-4}{12} & (\kappa \equiv 4 \pmod{12}), \\ \frac{\kappa+4}{12} & (\kappa \equiv 8 \pmod{12}) \end{cases}$$

and hence $\dim_{\mathbf{C}} \mathcal{A}_\kappa(B_{\mathbf{A}_Q}^\times) \neq 0$ if $\kappa \geq 8$. By a direct calculation we see that $P_\chi(f'; \gamma'_0) = \pm 1 \times \langle f'(1), v_\kappa^* \rangle v_\kappa$, where v_κ is a highest weight vector of V_κ . Since the class number of B is one, $f' \mapsto f'(1)$ induces an isomorphism $\mathcal{A}_\kappa(B_{\mathbf{A}_Q}^\times) \simeq V_\kappa^{\mathcal{O}^\times}$. Let f' be an element of $\mathcal{A}_\kappa(B_{\mathbf{A}_Q}^\times)$ corresponding to $\sum_{u \in \mathcal{O}^\times} \sigma_\kappa(u) v_\kappa$. We then have $P_\chi(f'; \gamma'_0) \neq 0$.

Next let us find $f \in \mathcal{S}_\kappa(\Gamma_0(D))$ such that $P_\chi(f; \gamma_0) \neq 0$. We view f as a modular form on the complex upper half plane. A direct calculation shows that the non-vanishing of $P_\chi(f; \gamma_0)$ is reduced to that of

$$\begin{cases} f(\sqrt{-1}) & (D = 1), \\ f(\frac{1+\sqrt{-1}}{2}) & (D = 2). \end{cases}$$

When $D = 1$, set

$$f = \begin{cases} \Delta^{\kappa/12} & (\kappa \equiv 0 \pmod{12}), \\ \Delta^{(\kappa-4)/12} E_4 & (\kappa \equiv 4 \pmod{12}), \\ \Delta^{(\kappa-8)/12} E_4^2 & (\kappa \equiv 8 \pmod{12}), \end{cases}$$

where Δ denotes the Ramanujan delta function and E_4 the Eisenstein series of weight 4. We then have $P_\chi(f; \gamma_0) \neq 0$. When $D = 2$, set

$$f = \left(\frac{\eta^{16}(2z)}{\eta^8(z)} \right)^{\kappa/4}$$

with the Dedekind eta function η . Since $\eta^{16}(2z)/\eta^8(z) \in S_4(\Gamma_0(2))$ (cf. [C, §2.1]) and $\eta(z)$ has no zero on the upper half plane, we have $f \in \mathcal{S}_\kappa(\Gamma_0(2))$ and $P_\chi(f; \gamma_0) \neq 0$.

Remark 3.3. The level raising of the modular forms of level $D = 1$ introduced above to forms of level 2 also yields modular forms with non-vanishing toral integrals.

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Atsushi Murase:

Department of Mathematical Sciences, Faculty of Science,
Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto 603-8555, Japan.
E-mail: murase@cc.kyoto-su.ac.jp

Hiro-aki Narita:

Department of Mathematics, Kumamoto University,
Kurokami, Kumamoto 860-8555, Japan
E-mail: narita@sci.kumamoto-u.ac.jp