

THE ENDOMORPHISM SEMIGROUP OF A SEMIGROUP AND ITS APPLICATION

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An endomorphism of a semigroup is one of important tools in the study of semigroups. In this paper, we represent the endomorphism semigroup by means of a wreath product and the greatest semilattice homomorphic image. And we apply to a left inverse extension of certain left regular band by a semilattice of groups.

1. The endomorphism semigroup of semigroups.

Let S be a semigroup. The set $\text{End}(S)$ of all endomorphisms of S is a semigroup under the composition. The set $\text{Aut}(S)$ of all automorphisms of S is a subgroup of $\text{End}(S)$. $\text{End}(S)$ [$\text{Aut}(S)$] is called the endomorphism semigroup [the automorphism group] of S . Now, let $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$ be the greatest semilattice decomposition of S . Let $H(S) = \bigcup \{\text{Hom}(S_\alpha, S_\beta) : \alpha, \beta \in \Gamma\} \cup \{\emptyset\}$ be a semigroup under composition, where $\text{Hom}(S_\alpha, S_\beta)$ is the set of all homomorphisms of S_α to S_β and $\emptyset$ is the empty mapping. Consider the set $H(S) \text{ wr } \text{End}(\Gamma) = \{(k, \theta) : k : \Gamma \rightarrow H(S), \theta \in \text{End}(\Gamma)\}$ and define a multiplication in $H(S) \text{ wr } \text{End}(\Gamma)$ as follows:

$$(k, \theta)(k', \theta') = (k \cdot \theta k', \theta \theta'),$$

where $\gamma k = k_\gamma$ and $(k \cdot \theta k') = k_\gamma k'_\theta$ for all $\gamma \in \Gamma$. Then

$H(S) \text{ wr } \text{End}(\Gamma)$ is a semigroup and we call it the wreath product of $H(S)$ and $\text{End}(\Gamma)$.

For each $\phi \in \text{End}(S)$, we define mappings $k : \Gamma \rightarrow H(S)$ and $\theta : \Gamma \rightarrow \Gamma$ as follows:

$$k_\alpha = \phi|_{S_\alpha}, \quad S_\alpha \phi \subseteq S_{\alpha\theta} \quad \text{for all } \alpha \in \Gamma.$$

Then a mapping $f : \phi \mapsto (k, \theta)$ is an isomorphism of $\text{End}(S)$ into

$H(S) \text{ wr } \text{End}(\Gamma)$. Thus we have the following lemma.

Lemma 1. $\text{End}(S) \cong_{\text{in}} H(S) \text{ wr } \text{End}(\Gamma)$.

Next, we shall consider the condition that $(k, \theta) \in H(S) \text{ wr } \text{End}(\Gamma)$ is contained in the image $\text{Im}(f)$ of f .

Lemma 2. Let $(k, \theta) \in H(S) \text{ wr } \text{End}(\Gamma)$. Then $(k, \theta) \in \text{Im}(f)$ if and only if

- (i) $k_\alpha \in \text{Hom}(S_\alpha, S_{\alpha\theta})$ for all $\alpha \in \Gamma$,
- (ii) $(xy) \cdot k_{\alpha\beta} = [x \cdot k_\alpha][y \cdot k_\beta]$ for all $x \in S_\alpha, y \in S_\beta$.

Now, we shall call this condition an H-property and let $H(S) \widetilde{\text{wr}} \text{End}(\Gamma) = \{(k, \theta) \in H(S) \text{ wr } \text{End}(\Gamma) : (k, \theta) \text{ has an H-property}\}$. By above lemmas we obtain the following theorem.

Theorem 3. Let $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$ be the greatest semilattice decomposition of a semigroup S . Then

$$\text{End}(S) \cong H(S) \widetilde{\text{wr}} \text{End}(\Gamma).$$

From now on, we identify elements of $\text{End}(S)$ with elements of $H(S) \widetilde{\text{wr}} \text{End}(\Gamma)$.

Remark 1. Since $H(S) \text{ wr } \mathcal{I}_\Gamma$ is isomorphic with the semigroup of all row monomial $\Gamma \times \Gamma$ -matrices over $H(S)$, $\text{End}(S)$ can represent by row monomial matrices.

2. Let $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$ be a strong semilattice of semigroups S_α determined by a transitive system $\{\phi_{\alpha, \beta} : \alpha \geq \beta\}$ of homomorphisms. If each S_α satisfy certain conditions (e.g., S_α has an identity and no other idempotent or S_α is weakly cancellative), then an H-property is equivalent to that the following diagram is commutative:

$$\begin{array}{ccc} S_\alpha & \xrightarrow{k_\alpha} & S_{\alpha\theta} \\ \phi_{\alpha, \beta} \downarrow & & \downarrow \phi_{\alpha\theta, \beta\theta} \\ S_\beta & \xrightarrow{k_\beta} & S_{\beta\theta} \end{array} \quad (\alpha \geq \beta, \alpha, \beta \in \Gamma)$$

(c.f., Petrich[2]).

2. The endomorphism semigroup of a successively annihilating chain of semigroups.

To each α in a chain Y assign a pairwise disjoint semigroup S_α , and on $S = \bigcup \{S_\alpha : \alpha \in Y\}$ define a multiplication $\cdot$ by

$$a \cdot b = b \cdot a = b \quad \text{if } a \in S_\alpha, b \in S_\beta, \alpha > \beta,$$

$$a \cdot b = ab \quad \text{if } a, b \in S_\alpha.$$

Then an semigroup $S(\cdot)$ is called a successively annihilating chain Y of semigroups S_α (c.f., Petrich [2]).

We shall first consider an endomorphism of a successively annihilating chain of completely simple semigroups.

Let Y be a chain. To each $\alpha \in Y$ assign a pairwise disjoint completely simple semigroup $S_\alpha = M(G_\alpha : I_\alpha, \Lambda_\alpha : P_\alpha)$ where $P_\alpha = (p_{\lambda, i}^{(\alpha)})$ has been normalized. Let $S = \bigcup \{S_\alpha : \alpha \in Y\}$ be a successively annihilating chain of S_α .

Theorem 4. Let θ be an endomorphism of Y (i.e., an isotone transformation of Y). We define a mapping $k: Y \rightarrow H(S)$ as follows:

(i) The case $|\alpha\theta\theta^{-1}| > 1$.

Let i_0 and λ_0 be fixed elements of $I_{\alpha\theta}$ and $\Lambda_{\alpha\theta}$, respectively.

Let β be an element of Y such that $\beta\theta = \alpha\theta$. Let ω be a homomorphism of G_β in $G_{\alpha\theta}$ such that $p_{\lambda, i}^{(\beta)}\omega = e_{\alpha\theta}$, the identity element of $G_{\alpha\theta}$, for all $i \in I_\beta$ and $\lambda \in \Lambda_\beta$. Define $k_\beta: S_\beta \rightarrow S_{\alpha\theta}$ by, for all $i_\beta \in I_\beta$, $\lambda_\beta \in \Lambda_\beta$ and $g_\beta \in G_\beta$,

$$(i_\beta, g_\beta, \lambda_\beta) \cdot k_\beta = \begin{cases} (i_0, (g_\beta \omega) (p_{\lambda_0, i_0}^{(\alpha\theta)})^{-1}, \lambda_0) & \text{if } \beta \text{ is the least element} \\ & \text{in } \alpha\theta\theta^{-1}, \\ (i_0, (p_{\lambda_0, i_0}^{(\alpha\theta)})^{-1}, \lambda_0) & \text{otherwise.} \end{cases}$$

(ii) Other case (i.e., $|\alpha\theta\theta^{-1}| = 1$).

Define k_α by any homomorphism of S_α in $S_{\alpha\theta}$ (c.f., [1]).

Then $(k, \theta) \in \text{End}(S) = H(S) \widetilde{\text{wr}} \text{End}(Y)$ and conversely every endo-

morphism of S can be obtained in this fashion.

We shall use the following corollary in next section.

Corollary 5. Let $L = \bigcup \{L_\alpha : \alpha \in Y\}$ be a successively annihilating chain Y of left zero semigroups L_α . Let θ be an endomorphism of Y . We define a mapping $k: Y \rightarrow H(L)$ as follows:

(i) Case $|\alpha\theta\theta^{-1}| > 1$.

Let l_0 be a fixed element of $L_{\alpha\theta}$. Let β be an element of Y such that $\beta\theta = \alpha\theta$. Define $k_\beta: L_\beta \rightarrow L_{\alpha\theta}$ by

$$xk_\beta = l_0 \quad \text{for all } x \in L_\beta.$$

(ii) Case $|\alpha\theta\theta^{-1}| = 1$.

Define k_α by any mapping of L_α in $L_{\alpha\theta}$.

Then $(k, \theta) \in \text{End}(L) = H(L) \widetilde{\text{wr}} \text{End}(Y)$ and conversely every endomorphism of L can be constructed in this manner.

Under some condition we next show that the endomorphism semigroup is a direct product of endomorphism semigroups of subsemigroups.

Let $\Gamma = \bigcup \{\Gamma_i : i \in Y\}$ be a semilattice which is a chain Y of semilattice Γ_i . Let $\{T_i : i \in Y\}$ be a family of semilattice indecomposable semigroups T_i such that $\text{Hom}(T_i, T_j) = \emptyset$ $[T_i \not\cong T_j]$ for all $i \neq j \in Y$. Let $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$ be a semigroup such that Γ is the structure semilattice of S and $S_\alpha \cong T_i$ if $\alpha \in \Gamma_i$.

In Proposition 6 and Corollary 7, let $S = \bigcup \{S_i : i \in Y\}$ be a chain of semigroups S_i where $S_i = \bigcup \{S_\alpha : \alpha \in \Gamma_i\}$, and assume that T_i 's satisfy conditions in the brackets for a statement on an automorphism group.

Proposition 6. $\text{End}(S) \cong_{\text{in}} \prod_{i \in Y} \text{End}(S_i)$.

$$[\text{Aut}(S) \cong_{\text{in}} \prod_{i \in Y} \text{Aut}(S_i)].$$

Corollary 7. Moreover, let S be a successively annihilating chain Y of semigroups S_i . Then

$$\begin{aligned} \text{End}(S) &\cong \prod_{i \in Y} \text{End}(S_i). \\ [\text{Aut}(S) &\cong \prod_{i \in Y} \text{Aut}(S_i)]. \end{aligned}$$

By the above result we need to consider the endomorphism semigroup as follows. Let S be a semigroup such that $S = \bigcup \{S_\alpha : \alpha \in \Gamma\}$ and $\phi_\alpha : S_\alpha \cong S_0$ for all $\alpha \in \Gamma$, where S_0 is a semi-lattice indecomposable semigroup.

Let k be a mapping of Γ into $\text{End}(S_0)$ and let $\theta \in \text{End}(\Gamma)$. Then we say that (k, θ) has an H-property if (k, θ) satisfy the following condition:

$$[a(\phi_\alpha k_\alpha \phi_{\alpha\theta}^{-1})][b(\phi_\beta k_\beta \phi_{\beta\theta}^{-1})] = (ab)(\phi_{\alpha\beta} k_{\alpha\beta} \phi_{(\alpha\beta)\theta}^{-1})$$

for all $a \in S_\alpha, b \in S_\beta$,

and $\text{End}(S_0) \overline{\text{wr}} \text{End}(\Gamma) = \{(k, \theta) \in \text{End}(S_0) \overline{\text{wr}} \text{End}(\Gamma) : (k, \theta) \text{ has an H-property}\}.$

Proposition 8. Let S be the above semigroup. Then

$$\begin{aligned} \text{End}(S) &\cong \text{End}(S_0) \overline{\text{wr}} \text{End}(\Gamma) \text{ and} \\ \text{Aut}(S) &\cong \text{Aut}(S_0) \overline{\text{wr}} \text{Aut}(\Gamma). \end{aligned}$$

Corollary 9. Let S_0 be a weakly cancellative semigroup or S_0 has an identity and no other idempotent. Then

$$\begin{aligned} \text{End}(S_0 \times \Gamma) &\cong \text{End}(S_0) \times \text{End}(\Gamma) \text{ and} \\ \text{Aut}(S_0 \times \Gamma) &\cong \text{Aut}(S_0) \times \text{Aut}(\Gamma). \end{aligned}$$

3. An application to left inverse semigroup

A regular semigroup S is called a left inverse semigroup if the set E of all idempotents in S satisfy the identity $xyx = xy$. A structure theorem for left inverse semigroups have been given by M. Yamada [3] as follows:

Structure Theorem. Let Ω be an inverse semigroup and Y its

basic semilattice. Let $L = \bigcup \{L_\alpha : \alpha \in Y\}$ be a left regular band (i.e., L is a semilattice Y of left zero semigroups L_α). Let φ be a mapping of Ω into $\text{End}(L)$ such that the family $\{\varphi_\omega : \omega \in \Omega\}$, $\varphi_\omega = \varphi(\omega)$, satisfies the following (C_1) and (C_2) :

(C_1) Each φ_ω is an endomorphism on L such that $\varphi_\omega(L_\alpha) \subseteq L_{\omega\alpha(\omega\alpha)^{-1}}$ for all $\alpha \in Y$. In particular, for $\tau \in Y$, φ_τ is an inner endomorphism on L .

(C_2) $\delta_e \delta_f \varphi_\alpha \varphi_\beta = \delta_e \delta_f \varphi_{\alpha\beta}$ for $e \in L_{\alpha\alpha^{-1}}$, $f \in L_{(\alpha\beta)(\alpha\beta)^{-1}}$, $\alpha, \beta \in \Omega$ (where δ_e is the inner endomorphism on L induced by e).

Consider the set $L \otimes_{\varphi} \Omega = \{(e, \omega) : \omega \in \Omega, e \in L_{\omega\omega^{-1}}\}$ and define multiplication in $L \otimes_{\varphi} \Omega$ as follows:

$$(e, \omega)(f, \tau) = (e \varphi_\omega(f), \omega\tau).$$

Then $L \otimes_{\varphi} \Omega$ is a left inverse semigroup and conversely every left inverse semigroup can be constructed in this manner.

We shall call this $L \otimes_{\varphi} \Omega$ the left inverse extension of L by Ω and a mapping $\varphi : \Omega \rightarrow \text{End}(L)$ which satisfies (C_1) and (C_2) is called a factor system of Ω into L and we shall denote the set of all factor systems of Ω into L by $\Phi(\Omega, L)$. We shall define a relation $\sim$ on $\Phi(\Omega, L)$ as follows; let $\varphi_1, \varphi_2 \in \Phi(\Omega, L)$,

$$\varphi_1 \sim \varphi_2 \text{ if and only if } L \otimes_{\varphi_1} \Omega \cong L \otimes_{\varphi_2} \Omega.$$

The purpose of this section is to investigate $\Phi(\Omega, L)/\sim$ in case Ω is a chain of groups and L is a successively annihilating chain of left zero semigroups.

Let $L = \bigcup \{L_\alpha : \alpha \in Y\}$ be a successively annihilating chain Y of left zero semigroups L_α . Let $\Omega = \bigcup \{G_\alpha : \alpha \in Y\}$ be a chain Y of groups G_α determined by a transitive system $\{f_{\beta, \alpha} : \beta \leq \alpha\}$ of a homomorphism $f_{\beta, \alpha} : G_\alpha \rightarrow G_\beta$.

First of all, we shall investigate a factor system of Ω in

L.

Theorem 10. Let $k = (k_{\beta, \alpha})$ be an element of $\prod_{\beta < \alpha} \text{Hom}(G_\alpha, S(L_\beta))$ such that if $\gamma < \beta < \alpha$ then $k_{\gamma, \beta} \circ f_{\beta, \alpha} = k_{\gamma, \alpha}$ where $S(L_\beta)$ is the symmetric group on L_β . Let ϕ be a mapping of Ω in L such that $\phi(G_\alpha) \subseteq L_\alpha$ for all $\alpha \in Y$ where if α is the greatest element in Y then assume that $\phi(G_\alpha) \subseteq \mathcal{I}(L_\alpha)$, the full transformation semigroup on L_α and $\phi(e_\alpha)$ is a constant mapping (e_α is an identity of G_α). Define $\mathcal{V}: \Omega \rightarrow \text{End}(L)$ by

$$x \cdot \mathcal{V}(g_\alpha) = \begin{cases} x \cdot k_{\beta, \alpha}(g_\alpha) & \text{if } \beta < \alpha, \\ \phi(g_\alpha) & \text{if } \beta \geq \alpha, \alpha \text{ is not greatest,} \\ x \cdot \phi(g_\alpha) & \text{if } \beta = \alpha, \alpha \text{ is greatest,} \end{cases}$$

for all $x \in L_\beta$, $g_\alpha \in G_\alpha$, $\alpha, \beta \in \Omega$.

Then $\mathcal{V}$ is a factor system of Ω in L. Conversely any factor system of Ω in L can be obtained in this fashion.

Hence we shall denote a factor system $\mathcal{V}$ by $\mathcal{V} = ((k, \phi))$.

Theorem 11. Let $((k^{(1)}, \phi^{(1)}))$ and $((k^{(2)}, \phi^{(2)}))$ be factor systems of Ω in L. Then

$((k^{(1)}, \phi^{(1)})) \sim ((k^{(2)}, \phi^{(2)}))$ if and only if there are an automorphism $(\psi, \theta) \in \text{Aut}(\Omega)$ and bijections $p_\beta: L_\beta \rightarrow L_{\beta\theta}$ ($\beta \in Y$) such that $k_{\beta, \alpha}^{(1)} = \rho_{p_\beta} k_{\beta\theta, \alpha\theta}^{(2)} \psi_\alpha$ for any $\alpha, \beta \in Y$ such that $\beta < \alpha$, where $\rho_{p_\beta}: S(L_{\beta\theta}) \rightarrow S(L_\beta)$ is defined by $\rho_{p_\beta}(p) = p_\beta^{-1} p p_\beta$.

Corollary 12. Let $\Omega = G \times Y$ be a direct product of a group G and a well-ordered chain Y. Let $L = \bigcup \{L_\alpha: \alpha \in Y\}$ be a successively annihilating chain Y of left zero semigroups L_α .

On the set $\prod_{\beta \in Y^*} \text{Hom}(G, S(L_\beta))$, define a relation $\approx$ by,

$k^{(1)} \approx k^{(2)}$ if and only if there are an automorphism $\psi \in \text{Aut}(G)$ and $p_\beta \in S(L_\beta)$ ($\beta \in Y^*$) such that $k_\beta^{(1)} = \rho_{p_\beta} k_\beta^{(2)} \psi$ for all $\beta \in Y^*$

where $Y^* = Y \setminus \alpha_1$, α_1 is the greatest element of Y . Then there is a one-to-one correspondence between the classes of isomorphic left inverse extensions of L by Ω and the elements of $\prod_{\beta \in Y^*} \text{Hom}(G, S(L_\beta)) / \approx$.

Example. Let G be a cyclic group of order 2, $Y = \{0, 1\}$ and $L = L_0 \cup L_1$. If $|L_0| = 2n$, then $|\Xi(\Omega, L) / \sim| = n+1$. If $|L_0| = \infty$ then $|\Xi(\Omega, L) / \sim| = \infty$.

Corollary 13. Let Y be a dense chain (i.e., if $\alpha > \beta$, there is an element γ such that $\alpha > \gamma > \beta$). Let Ω and L be successively annihilating chains Y of groups G_α and left zero semigroups L_α , respectively. Then there exist the only one left inverse extension of L by Ω , up to isomorphism.

References

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